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Abstract

We consider collective choice from two alternatives. Ex ante, each agent is uncertain about which alternative she prefers, and may be uncertain about the intensity of her preferences. An environment is given by a probability distribution over utility vectors that is symmetric across agents and neutral across alternatives. In many environments, the majority voting rule maximizes agents' ex-ante expected utilities among all anonymous and dominant-strategy implementable choice rules. But in some environments where the agents' utilities are stochastically correlated, other dominant-strategy choice rules are better for all agents. If utilities are stochastically independent across agents, majority voting is ex-ante optimal among all anonymous and incentive-compatible rules. We also compare rules from an interim-viewpoint.

1 Introduction

A traditional problem in political science is what rule should be used by a society for choosing between two alternatives if each agent is privately informed about her preferences, and if preferences are potentially conflicting. A natural approach to this problem—pioneered by Rae (1969), and subsequently followed by many authors[1]—is to compare rules with respect to the agents' ex-ante expected utilities. We extend existing models in three ways. First, rather than focussing on a pre-fixed set of choice rules, we consider

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[1] See the literature review at the end of this section.

the entire set of dominant-strategy implementable (resp., incentive compatible) rules, including rules with lottery outcomes. Hence, our contribution may be viewed as an exercise in mechanism design. Second, we allow the agents' preferences to be correlated. This seems appropriate for many applications. Suppose, for example, that there is uncertainty about whether there will be a military threat; if the threat occurs, all agents may be more inclined to prefer an increase of defense expenditure than otherwise. Third, we allow agents to be ex-ante uncertain about the intensities of their future preferences (i.e., preference intensities are state-dependent). Again, this appears natural in many applications. For example, a healthy person may care less about a reduction of the government's health expenditures than a sick person.[2]

Each agent has a privately known von-Neumann-Morgenstern utility for each of the two alternatives. Normalizing utilities from one alternative to 0, a *state* is a utility vector that specifies each agent's utility from the other alternative. From an ex-ante viewpoint, the state is distributed according to some commonly known probability distribution that defines the environment. Ex-ante, no agent knows her future utility. At the interim stage, each agent learns her utility and the choice is made. It is important to distinguish an agent's ex-ante preferences (over state-dependent lotteries over alternatives) from her interim preferences (over lotteries over alternatives). At the interim stage, only two preference relations are possible, depending on the *sign* of the agent's utility (we exclude indifference). Ex-ante, however, an agent may also be uncertain about the *absolute value* of her utility, that is, the intensity of her future preferences. In other words, she may face a trade-off between a future self that cares a lot and a different future self that cares little. Formally, this situation fits the state-dependent expected utility model (cf. Fishburn, 1970, Karni, 1985).

We focus on *symmetric* environments, in the sense that the utility probability distribution is symmetric across agents, that is, no agent is special. Given the symmetry across agents, it is natural to focus on *anonymous* choice rules, that is, choice rules that treat all agents in the same manner. Due to symmetry, these choice rules are ex-ante Pareto-ranked, so that no interpersonal utility comparison is needed to find an (ex-ante) optimal rule.[3]

[2]The potential importance of preference intensities for institutional design was first emphasized by Buchanan and Tullock (1962). A recent model that allows for uncertainty about preference intensities is Casella (2005).

[3]Our analysis extends to non-anonymous rules if a utilitarian welfare criterion is used, that is, if the sum of the agents' expected utilities is maximized; cf. footnote 13. See Harsanyi (1955) for a discussion of the utilitarian welfare criterion.

To further simplify matters, we focus on *neutral* environments, in the sense that the utility probability distribution is symmetric around 0. That is, ex ante both alternatives have the same chance to be preferred with a particular intensity. This implies that there always exists an optimal choice rule that is also *neutral*, in the sense that both alternatives are treated in the same manner.

From the revelation principle, it is without loss of generality to restrict attention to choice rules that are *incentive-compatible* for the agents, that is, no agent must have an incentive to misrepresent her utility if all other agents announce their utilities truthfully. For the most part, we refer to a particularly robust class of incentive-compatible choice rules: *dominant-strategy* choice rules, where announcing her true utility is a dominant strategy for each agent. A dominant-strategy choice rule is robust to the presence of additional private or public signals that may allow agents to revise their beliefs about other agents' utilities.

The class of *weak majority rules* is defined by the property that each alternative is chosen with probability 1/2 unless a sufficiently large fraction of the population prefers one alternative, in which case this alternative is chosen.

Our first result is that in any environment, some weak majority rule is optimal among all dominant-strategy rules; other anonymous and neutral choice rules can only be optimal in non-generic environments. Secondly, we provide a precise characterization of the class of environments where any given weak majority rule is optimal.

An important implication of our characterization is that the *(standard) majority rule* is optimal in any environment where the distribution of the utility vector is affiliated. The concept of affiliation (see Milgrom and Weber, 1982, p. 1118) captures a certain type of positive statistical dependence: roughly, the higher an agent's utility, the (weakly) more likely she considers high utilities for the other agents. To paraphrase our earlier example: the stronger an agent's preference for an increase in defense spending, the more likely she considers other agents to have strong preferences in the same direction. A special case of affiliation is stochastic independence. To the extent that affiliation is an appropriate assumption in applications, our result justifies the prevalence of the standard majority rule in real-world institutions.

In some applications, it may be reasonable to assume that members of a minority are somewhat more strongly affected by the collective choice than the members of the majority. In our model, such an assumption corresponds to environments where, conditional on the event that a certain alternative is preferred by few agents, the expected utility of any of these "minority agents" is (in absolute value) larger than the expected utility of the "majority agents". Our characterization result shows that it is in such environments where a weak majority rule can Pareto-dominate the standard majority rule.

The optimality of a weak majority rule in some environments may be a matter of curiosity, but it may also point to a potential improvement of some real-world institutions where the standard majority rule is being used.

Can a higher ex-ante expected utility be achieved if the dominant-strategy requirement is given up, that is, if all incentive-compatible anonymous rules are considered? We show that the answer is "no" if the agents' utilities are stochastically independent—the majority rule is still optimal among all incentive-compatible anonymous rules—, but with stochastically dependent utilities the answer can be "yes". Without independence, we do not have a useful characterization of incentive-compatibility, and a characterization of optimal rules appears difficult. We provide an example which shows that the optimal incentive-compatible rule can lie outside the class of weak majority rules. In the example, there are 3 agents, and both the standard majority rule and the (only other) weak majority rule are outperformed by a rule that chooses each alternative if and only if it is preferred by an *odd* number of agents. Such a rule appears rather fragile, suggesting that a planner's decision for or against the dominant-strategy requirement involves, in general, a trade-off between robustness and expected utility.

We also compare rules from an interim point-of-view, when each agent knows her utility. An ideal world would be one where, at the interim stage, all agents agree which is the best rule among all anonymous and neutral incentive-compatible rules. We show that the majority rule indeed has this *interim-dominance* property if the agents' utilities are stochastically independent. Finally, dropping the stochastic-independence assumption, we consider a simple class of affiliated environments with three agents and restrict attention to dominant-strategy rules. Here, a rule that is interim-dominant among dominant-strategy rules exists if and only if the positive stochastic dependence between the agents' utilities (as measured by the probability that all agents prefer the same alternative) is not too strong. When no interim-dominant rule exists, many rules are interim-undominated, including, for example, the rule where each agent is a dictator with equal probability.

The approach of evaluating the majority rule (in environments with two alternatives) from an expected-utility point-of-view goes back to Rae (1969), whose results were generalized by Badger (1972), Curtis (1972), Schofield (1972), and Taylor (1969). They compare agents' ex-ante expected utilities across *qualified majority rules*, in (possibly asymmetric and non-neutral) environments without uncertainty about preference intensities, and with stochastically independent utility distributions. One alternative is designated as the "status quo." A qualified majority rule stipulates that the status quo is chosen if and only if it is supported by a certain fraction of the population. It is shown that a standard majority rule maximizes the utilitarian welfare among all qualified majority rules, even if agents differ ex ante with respect to the probability of supporting the status quo; the only required assumption is that the *support* of the utility distribution is symmetric around 0.[4] Furthermore, asymmetries of the support, which Rae calls "positional preferences", can render the standard majority rule non-optimal, and any qualified majority rule may be optimal.

Barberà and Jackson (2004) start from the observation that if agents differ with respect to the probability of supporting the status quo, then they have different ex-ante preferences over the qualified majority rules. Barberà and Jackson seek qualified majority rules that are *self-stable* in a given environment, in the sense that for no other rule is there a qualified majority of agents who would strictly prefer the other rule. They go on to study constitutions that specify one rule for regular issues, and another one for changing the rule; the self-stability of such constitutions is analyzed.[5]

Barberà and Jackson (2006) study utilitarian welfare maximization in the class of weighted majority rules over two alternatives, in environments that are, in general, asymmetric across agents and non-neutral across alternatives; in their model, each voter represents an entire population of individuals, so that the optimal weights assigned to the various voters depend on the degree to which each vote reflects the utilities of the represented population. The theoretical results are compared to actual voting weights in the Council of the European Union.

Börgers (2004) assumes that voters have not only private utilities for

[4]The logic behind this result is quite simple; see Barberà and Jackson (2004, discussion below Lemma 4, p. 1025) for a review. In fact, the argument by Barberà and Jackson extends immediately to any correlated utility distribution, as long as only utilities 1 and −1 are possible.

[5]Messner und Polborn (2004, Section 4) introduce this idea of self-stability in an overlapping-generations model where multiple regular issues come up over time, and where an agent's utility gain from upsetting the status-quo alternative in a given issue decreases with her age. Here, an agent's age determines her preferences over qualified majority rules to be used for regular issues. Specifically, the median voter among the agents who are currently alive prefers a qualified majority of more (less) than 3/4 of the electorate if an agent's utility gain from upsetting the status-quo in a regular issue decreases with age in a concave (convex) fashion (see Messner und Polborn, 2004, Proposition 1).

the alternatives, but also private costs of voting. Maintaining neutrality, symmetry, and stochastic independence across agents, he shows that the voluntary majority voting rule yields a higher utilitarian welfare than compulsory majority voting, but the first-best welfare level is not achieved with either voting mechanism.[6]

Earlier papers that consider the entire class of incentive compatible choice rules include Börgers and Postl (2009) and Jackson and Sonnenschein (2007). Börgers and Postl (2009) consider a class of environments with private utilities where two agents have to make a collective choice from three alternatives. They show that the first-best utilitarian welfare cannot be achieved with any incentive-compatible choice rule, and they provide insights into the nature of second-best rules. Jackson and Sonnenschein (2007) show, for a quite general class of Bayesian collective decision problems, that there exist incentive-compatible choice rules that approximate the first-best arbitrarily closely if sufficiently many independent identical decision problems are linked.

Taking a completely different approach, May (1952) characterizes the majority rule as the unique choice rule having three intuitive properties in a model where only ordinal preferences are specified. The first two properties, neutrality and anonymity, are analogous to our correspondingly named properties. The difference is that in our model these properties are not primitives, but are justified by expected-utility (resp., utilitarian welfare, cf. footnote 13) maximization. The third property, positive responsiveness, has no analogue in our model.

2 Model

Suppose that $n \geq 2$ agents must collectively choose one of two alternatives, A or B. Each agent has state-dependent expected-utility preferences over lotteries over A and B. Normalizing the utility from alternative A to 0, a state is a random vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, where u_i is agent i's utility from alternative B; if $u_i < 0$, then agent i prefers alternative A. We may say that the absolute value $|u_i|$ measures the intensity of the agent's preferences.

In the beginning, nature chooses a state according to some cumulative probability distribution F on $\mathbb{R}^n$. We focus on environments F that are (1) symmetric across agents and (2) neutral across the alternatives A and B,

[6] See Krasa and Polborn (2009) for qualifications and extensions of Börgers' results to non-neutral environments.

that is, if $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n) \sim F$,[7]

$$\begin{aligned} (\tilde{u}_{\pi(1)}, \ldots, \tilde{u}_{\pi(n)}) &\sim F \quad \text{for all permutations } \pi, & (1) \\ -\tilde{u} &\sim F. & (2) \end{aligned}$$

Our goal is to find an optimal rule for choosing an alternative.

We make two simplifying technical assumptions. First, first moments are finite, $E[|\tilde{u}_1|] < \infty$. This guarantees that all expectations occurring throughout the analysis are finite. Second, agents are not indifferent between alternatives,[8]

$$\Pr[\exists i : \ \tilde{u}_i = 0] \quad = \quad 0.$$

We define each agent's type space as $V = \text{supp}(\tilde{u}_i) \setminus \{0\}$. For example, if $\tilde{u}_i$ has the support $[-1, 1]$, then $V = [-1, 1] \setminus \{0\}$.

A *choice rule* is a (Borel measurable) function

$$\phi : \ V^n \to [0, 1],$$

where $\phi(u)$ is interpreted as the probability that alternative B is chosen if the realized utility profile is $u \in V^n$.

Given that the environment is symmetric across agents (1), it is natural to focus on *anonymous* choice rules, that is, for all $u \in V^n$,

$$\phi(u_{\pi(1)}, \ldots, u_{\pi(n)}) \quad = \quad \phi(u) \quad \text{for all permutations } \pi. \tag{3}$$

We will focus on anonymous rules throughout the paper (cf. footnote 13). Hence, all agents obtain the same expected utility

$$U(\phi) \quad = \quad E[\tilde{u}_1 \phi(\tilde{u})].$$

In particular, if one agent prefers one anonymous choice rule over another, then all agents do—no interpersonal utility comparisons are needed in order to compare different anonymous choice rules.

An anonymous choice rule ϕ is *incentive-compatible* if, for all $u_1, u_1' \in V$,

$$u_1 E[\phi(\tilde{u}) \mid \tilde{u}_1 = u_1] \quad \geq \quad u_1 E[\phi(u_1', \tilde{u}_{-1}) \mid \tilde{u}_1 = u_1]. \tag{4}$$

From the revelation principle and (1), the set of anonymous and incentive-compatible choice rules equals the set of symmetric Bayesian Nash equilibrium outcomes that can be obtained from any symmetric game that may

[7] Read: "The random vector $\tilde{u}$ is distributed according to F."

[8] This assumption is not essential for the results, but simplifies definitions and proofs.

be played by the agents. Hence, subject to the symmetry restriction, it is without loss of generality to restrict attention to incentive-compatible choice rules.

An anonymous choice rule ϕ is a *dominant-strategy* rule if, for all $u_1, u_1' \in V$ and $u_{-1} \in V^{n-1}$,

$$u_1\phi(u) \quad \geq \quad u_1\phi(u_1', u_{-1}). \tag{5}$$

From the revelation principle, the anonymous and dominant-strategy choice rules are precisely the possible symmetric dominant-strategy equilibrium outcomes obtained from any symmetric game that may be played by the agents. Dominant-strategy choice rules are "robust" in the sense that each agent has an incentive to reveal her type independently of her beliefs about others.[9] Most of our results refer to dominant-strategy choice rules.

Our analysis begins with a characterization of the dominant-strategy rules among the anonymous rules (Lemma 1). Some notation is required. For any $u \in \mathbb{R}^n$, let $[u]^+ = \{i \mid u_i > 0\}$ denote the set of agents who prefer alternative B. For any anonymous rule ϕ and all $k = 0, \ldots, n$, let

$$q_k(\phi) \quad = \quad E[\phi(\tilde{u}) \mid |[u]^+| = k]$$

denote the probability of choosing B conditional on the event that k agents prefer B. An anonymous rule is a dominant-strategy rule if and only if (i) the probability that alternative B is chosen is a weakly increasing function of the number of agents preferring alternative B, and (ii) the probability that B is chosen is independent of any preference intensities. In particular, any anonymous and dominant-strategy rule ϕ is fully described by the $n+1$ numbers $q_0(\phi), \ldots, q_n(\phi)$.

Lemma 1 *An anonymous rule ϕ is a dominant-strategy rule if and only if (i) $q_0(\phi) \leq \ldots \leq q_n(\phi)$ and, (ii) for all $u \in V^n$ and all $k = 0, \ldots, n$, if $|[u]^+| = k$ then $\phi(u) = q_k(\phi)$.*

The proof of the "only if"-part begins by showing (*Step 1*) that if two utility vectors differ only with respect to the absolute value (not the sign) of a single agent's utility, then alternative B is chosen with the same probability for both utility vectors. Next (*Step 2*), any two utility vectors that differ

[9]Vice versa, any choice rule that is incentive compatible with respect to arbitrary beliefs is a dominant-strategy choice rule. See Ledyard (1978) and Bergemann and Morris (2005) for detailed results, and Chung and Ely (2007) for related results in a setting with transferable utility.

with respect to the absolute value of any number of agents' utilities lead to the same choice probabilities; this follows from *Step 1* via a sequence of equalities. *Step 3* uses symmetry to show that the identities of the agents preferring each alternative are irrelevant for the choice—only their number is important; hence, condition (ii) holds. Because no agent with a negative utility can have an incentive to announce a positive utility, the $q_k(\phi)$-numbers must be weakly increasing in k, that is, condition (i) holds (*Step 4*). The proof of the "if"-part is straightforward: by mis-representing her utility, an agent can change the collective choice only if she mis-represents the sign of her utility, but such a deviation can only reduce the probability that her preferred alternative is chosen. The details of the proof can be found in the Appendix.

The following class of dominant-strategy choice rules is particularly important for our analysis. For any $n/2 < m \leq n$, the *weak majority rule* ϕ^m with threshold m is defined by

$$q_k(\phi^m) = \begin{cases} 0 & \text{if } k \leq n-m, \\ \frac{1}{2} & \text{if } n-m < k < m, \\ 1 & \text{if } k \geq m. \end{cases}$$

According to ϕ^m, if there is a "weak majority" of fewer than m agents for either A or B, then a random choice is made. Only if the majority is "strong" enough, A or B is chosen with certainty. The *(standard) majority rule* ϕ_{maj} is defined as the weak majority rule with minimum threshold; according to ϕ_{maj}, a random choice is made only if each alternative is preferred by $n/2$ agents.

Other dominant-strategy choice rules include the *random choice rule* ϕ_{random}, where each alternative is always chosen with probability $q_k(\phi_{\text{random}}) = 1/2$ $(k = 0, \ldots, n)$, the *random dictatorship rule* ϕ_{dictator}, where each agent makes a dictatorial choice with probability $1/n$, so that $q_k(\phi_{\text{dictator}}) = k/n$ for all $k = 0, \ldots, n$, and the *qualified majority rules* ϕ_q^m defined by

$$q_k(\phi_q^m) = \begin{cases} 0 & \text{if } k < m, \\ 1 & \text{if } k \geq m, \end{cases}$$

where alternative B is chosen if and only if it is supported by a "qualified majority" of at least m $(n/2 < m \leq n)$ agents.

An agent's expected utility from an anonymous dominant-strategy-implementable rule ϕ can be expressed as

$$U(\phi) = \sum_{k=0}^{n} p_k e_k q_k(\phi),$$

where we use the abbreviations

$$e_k = E[\tilde{u}_1 \mid |[\tilde{u}]^+| = k]$$

and

$$p_k = \Pr[|[\tilde{u}]^+| = k].$$

By Lemma 1, a choice rule ϕ is *optimal among anonymous dominant-strategy rules* if and only if $q_k = q_k(\phi)$ $(k = 0, \ldots, n)$ solves the following problem:

$$(*) \qquad \max_{q_0,\ldots,q_n} \sum_{k=0}^{n} p_k e_k q_k$$

$$\text{s.t.} \quad 0 \leq q_0 \leq \ldots \leq q_n \leq 1.$$

An immediate observation is that the random choice rule ϕ_{random} is never optimal: any solution of $(*)$ satisfies $q_0 = 0$ and $q_n = 1$ because $e_0 < 0$ and $e_n > 0$.

For later use, the following implications of (2) are useful:

$$e_k = -e_{n-k} \tag{6}$$

and, similarly,

$$p_k = p_{n-k}. \tag{7}$$

Observe that, for any list of numbers $e_0, \ldots, e_n$ satisfying $e_0 < 0$ and (6), and any probability distribution $(p_0, \ldots, p_n)$ satisfying (7), there exists an environment F with these parameters.[10]

3 Optimal dominant-strategy rules

In this section, we deal with the problem of characterizing optimal dominant-strategy choice rules. As a first step, we show that without loss of generality

[10] To see this, imagine a three-stage random procedure where first the number k of agents preferring alternative B is chosen according to the distribution $(p_0, \ldots, p_n)$; the identities of the B-preferrers are chosen according to a uniform distribution on the k-elementary sets of agents; the utility of each agent who supports B is chosen conditionally independently across agents with any expected utility $e_k^+ > 0$; the utility of each agent who supports A is chosen conditionally independently with expected utility $-e_{n-k}^+ < 0$ such that $ke_k^+ - (n-k)e_{n-k}^+ = ne_k$.

we may focus on *neutral rules*, that is, rules that are invariant with respect to the labelling of alternatives: if preferences are reversed, choice probabilities are reversed as well. Formally, a choice rule is *neutral* if,

$$\forall u \in V^n: \quad \phi(-u) \quad = \quad 1 - \phi(u).$$

The weak majority rules, the random choice rule, and the random-dictatorship rule are neutral. The qualified majority rules are not neutral. More generally, the following result is immediate from the definition of a neutral rule.[11]

Lemma 2 *An anonymous dominant-strategy rule ϕ is neutral if and only if $q_k(\phi) + q_{n-k}(\phi) = 1$ for all $k = 0, \ldots, n$.*

While neutrality of a rule may be a useful property more generally,[12] in neutral environments (2) the focus on neutral rules can be justified solely on utility grounds. Lemma 3 shows that for any choice rule there exists a neutral choice rule that yields the same expected utility. The neutral choice rule is constructed from the original choice rule by assigning probability 1/2 to both the original rule and a rule that reverses the original choice if all agents' preferences are reversed. As shown in the Appendix, this construction preserves all relevant properties of the original rule.[13]

Lemma 3 *Let ϕ be an anonymous and incentive-compatible (resp., dominant-strategy) choice rule. Then the choice rule ψ,*

$$\psi(u) \quad = \quad \frac{\phi(u) + 1 - \phi(-u)}{2}, \tag{8}$$

[11]Combining Lemma 1 and Lemma 2, one obtains a characterization of neutral, anonymous, and dominant-strategy rules, which is a straightforward extension of Barberà's (1979, Theorem 3) corresponding result in an ordinal-preference model. Barberà's analysis, however, is considerably more general in the sense that, building on Gibbard (1977), he also incorporates environments with more than two alternatives, in which case an additional property, "alternative-independence", becomes relevant.

[12]Observe that implementing a non-neutral rule requires that one alternative is unambiguously distinguished (say, as a "status-quo"). Otherwise agents may disagree at the interim-stage about the labelling of alternatives. For example, if the alternatives are two candidates who compete for an office, and an $m = (2/3)n$-qualified majority if required for the choice of alternative B, then everybody would like their favored candidate to be labelled "A".

[13]Our focus on anonymous rules can be justified in a manner parallel to Lemma 3: for any incentive-compatible (resp., dominant-strategy) choice rule, there exists an anonymous incentive-compatible (resp., dominant-strategy) choice rule that yields the same utilitarian welfare. The anonymous rule is constructed by taking the average of all the rules obtained by permutations of the player roles in the original rule.

is anonymous, incentive-compatible (resp., dominant-strategy), and neutral. Moreover, $U(\psi) = U(\phi)$.

Applying this result to a qualified majority rule $\phi = \phi_q^m$, we have $\psi = \phi^m$. Hence, for any qualified majority rule there is a weak majority rule that yields the same expected utility,

$$U(\phi_q^m) \quad = \quad U(\phi^m). \tag{9}$$

Our goal is to characterize optimal choice rules. Hereby we focus on neutral rules, which, by Lemma 3, is always possible. Let $\hat{n}$ denote the smallest integer strictly greater than $n/2$. Using problem $(*)$, (6), (7), and Lemma 2, one sees that the neutral rules among the optimal rules are the solutions of problem

$$(*n) \qquad \max_{q_{\hat{n}},\dots,q_n} \sum_{k=\hat{n}}^{n} p_k e_k (2q_k - 1)$$

$$\begin{aligned} \text{s.t.} \qquad & q_{\hat{n}} \geq 1/2, \\ & q_k - q_{k-1} \geq 0 \quad (k = \hat{n}+1,\dots,n), \\ & -q_n \geq -1, \\ & q_k = 1 - q_{n-k} \quad (k < \hat{n}). \end{aligned}$$

To solve $(*n)$, one first chooses $q_{\hat{n}},\dots,q_n$ to maximize the objective subject to the inequality constraints, and then defines $q_0,\dots,q_{\hat{n}-1}$ via the equality constraints.

Proposition 1 provides an essentially complete characterization of the optimal neutral rules. There exists a number m ($\hat{n} \leq m \leq n$) such that an agent's expected utility is non-positive if between k and $m-1$ agents prefer alternative B for all $k = \hat{n},\dots,m-1$ and such that an agent's expected utility is non-negative if between m and $k-1$ agents prefer alternative B for all $k = m+1,\dots,n+1$. Any weak majority rule ϕ^m with such a threshold m is optimal. In particular, any weak majority threshold is optimal in some environments.

Proposition 1 *In any symmetric and neutral environment, some weak majority rule is optimal among anonymous dominant-strategy rules. The weak majority rule* ϕ^m *with threshold* m $(\hat{n} \leq m \leq n)$ *is optimal if and only if*

$$E[\tilde{u}_1 \mathbf{1}_{k \leq |[\tilde{u}^+]| \leq m-1}] \quad \leq \quad 0 \quad \textit{for all} \quad k = \hat{n},\dots,m-1, \tag{10}$$

$$E[\tilde{u}_1 \mathbf{1}_{m \leq |[\tilde{u}^+]| \leq k-1}] \quad \geq \quad 0 \quad \textit{for all} \quad k = m+1,\dots,n+1. \tag{11}$$

In a generic subset of environments, any neutral rule that is optimal among anonymous dominant-strategy rules is a weak majority rule.

The proof consists of an analysis of problem $(*n)$, which is linear and can be solved with standard Lagrange methods; details can be found in the Appendix. Geometrically speaking, the reason behind the (generically unique) optimality of weak majority rules is that these rules correspond to the corners of the constraint set of problem $(*n)$; by a fundamental lemma of linear programming, any linear problem has a corner solution. Now compare any two weak-majority rules ϕ^m and ϕ^k with $k < m$. The difference is that the probability of choosing B conditional on between k and $m-1$ agents being in favor of B is increased by 1/2 if one replaces rule ϕ^m with rule ϕ^k. Condition (10) states that the expected-utility gain from such a replacement is non-positive. Similarly, condition (11) states that rule ϕ^m is at least as good as any rule ϕ^k with $k > m$.

From (9), the weak majority rule ϕ^m is optimal if and only if the qualified majority rule ϕ_q^m is optimal. Hence, Proposition 1 also yields a characterization of optimal qualified majority rules. Qualified majority rules play an important role in many contexts where one alternative is naturally distinguished as the "status quo", so that there is no reason to focus on neutral rules (see, e.g., the examples cited by Messner and Polborn (2004), and Barberà and Jackson (2004, 2006)).[14]

The interpretation of conditions (10) and (11) is straightforward. If belonging to a "weak majority" of between $k \leq m-1$ and $m-1$ agents implies that the average agent's utility is non-positive, and if belonging to a "strong majority" of between m and $k' \geq m$ agents implies that the average agent's utility is non-negative, then a random choice should be made if fewer than m agents prefer B, and B should be chosen with certainty if at least m agents prefer B.

Corollary 1 *The majority rule ϕ_{maj} is optimal among the anonymous dominant-strategy rules if and only if*

$$E[\tilde{u}_1 \mathbf{1}_{\hat{n} \leq |[\tilde{u}^+]| \leq k'}] \quad \geq \quad 0 \quad \textit{for all} \quad k' = \hat{n}, \ldots, n. \tag{12}$$

The condition (12) for the optimality of the standard majority rule may appear stronger than expected. For example, the plausible condition "the

[14]Extending the proof of Proposition 1, one obtains a characterization of optimal rules in arbitrary symmetric (possibly non-neutral) environments. In a generic subset of the symmetric environments, any rule that is optimal among anonymous dominant-strategy rules is a qualified majority rule. Hence, weak-majority rules are most useful if either the environment is neutral, or attention is restricted to neutral rules.

event of belonging to a majority of between $n/2$ and n agents implies a non-negative expected utility" is *not* sufficient.

For many applications, it seems appropriate to assume a positive statistical dependence of the agents' utilities: if one agent assigns a high utility to an alternative, then the other agents are also more likely to assign high utilities than otherwise. The statistical concept of affiliation provides a convenient model for such a dependence. Specifically, any two agents' utilities are affiliated if, the higher one agent's utility, the (weakly) higher her belief about the other agent's utility, in the sense of likelihood-ratio dominance.[15] Affiliation also includes stochastic independence as a special case.

Corollary 2 shows that the standard majority rule is optimal in any environment where the agents' utilities are affiliated. This may be seen as an explanation for the prevalence of the standard majority rule in many real-world institutions.

Corollary 2 *Suppose that F is affiliated. Then the majority rule is optimal among all anonymous dominant-strategy rules.*

Proof. Using Milgrom and Weber (1982, Theorem 5), affiliation implies that $e_k \leq e_{k+1}$ for all $k < n$. Using this together with (6), $e_k > 0$ for all $k \geq \hat{n}$. Hence, condition (12) is satisfied. *QED*

Here is an example of an environment where the "weakest majority" rule ϕ^n is optimal.[16]

Example 1 *Suppose that there are $n = 3$ agents and the utility space is $V = \{-3, -1, 1, 3\}$. Let G be any distribution on $\mathbb{R}$ with supp$(G) = V$. Let P and N denote two events, each of which is realized with probability $1/2$. Define F such that conditional on event P, each agent's utility is independently realized according to G, and conditional on event N, the* negative *of each agent's utility is independently realized according to G. Let r_{-3}, r_{-1}, r_1, r_3 denote the probabilities of the utility levels $-3, -1, 1, 3$ according to G.*

[15] The distribution F is *affiliated* if (i) $W = V \cup \{0\}$ is an interval, (ii) F has a positive density f on W^n, and

$$f(u \vee v) f(u \wedge v) \leq f(u) f(v) \quad \text{for all } u, v \in W,$$

where $u \vee v$ denotes the component-wise maximum, and $u \wedge v$ denotes the component-wise minimum; see Milgrom and Weber (1982, p. 1098).

[16] Incidentally, in this example the random-dictatorship rule (though not optimal) is also better for all agents than the majority rule.

If r_{-1} is sufficiently close to 1 and r_3/r_1 is sufficiently large, then the weak majority rule ϕ^3 is optimal among all anonymous and dominant-strategy rules, and the standard majority rule ϕ^2 is not optimal.

The idea behind this construction is to define G such that, if an agent's utility is positive, then it tends to be large, while it tends to be small (in absolute value) if it is negative. Moreover, negative utility is much more likely under G than positive utility. Hence, negative utility is likely to occur conditional on event P, while positive utility is likely to occur conditional on event N. Thus, conditional on the event that only one agent has a positive utility while two agents have negative utilities, event P is much more likely than event N. Conditional on event P, it is very likely that the one agent's large positive utility outweighs the other two agents' small negative utilities, so that the average utility is positive. In other words, $e_1 > 0$. By neutrality, then, $e_2 < 0$, so that the weak majority rule ϕ^3 is optimal (Proposition 1) and the standard majority rule is not (Corollary 1); details of the proof are in the Appendix.

4 Optimal incentive-compatible rules

In this section, we give up the dominant-strategy requirement and deal with the problem of optimality among incentive-compatible choice rules. In another dimension, the results here are less general because we assume that utilities are stochastically independent across agents.

We begin with a characterization of incentive-compatibility (Lemma 4). Incentive compatibility is equivalent to the requirement that, from the viewpoint of any agent given her utility, the probability that alternative B is chosen is weakly larger if the agent prefers B compared to when she prefers A, and the probability is independent of the intensity of her preferences.

Lemma 4 *Suppose that F is stochastically independent across agents. Then an anonymous choice rule ϕ is incentive-compatible if and only if there exist numbers $q^+, q^- \in [0, 1]$ such that,*

$$q^+ \geq q^- \quad \textit{and} \quad \forall u_1 \in V: \; E[\phi(u_1, \tilde{u}_{-1})] = \begin{cases} q^+ & \textit{if } u_1 > 0, \\ q^- & \textit{if } u_1 < 0. \end{cases} \tag{13}$$

To prove "only if", observe first that the probability that a given agent assigns to the collective choice being B only depends on her utility announcement, and—by stochastic independence—does not depend on her actual utility. Hence, whenever an agent prefers alternative B, her announcement will maximize, among all possible announcements, the probability that the collective choice is B. Hence, incentive compatibility implies that this probability is some number, q^+, that is independent of the agent's preference intensity. Furthermore, because the agent must not have an incentive to misrepresent the sign of her utility, the probability of collectively choosing B when the agent prefers A, denoted q^-, cannot be larger than the probability q^+. The "if"-part is straightforward. Details of the proof can be found in the Appendix.

Example 2 below demonstrates a crucial difference between dominant-strategy implementability (Lemma 1) and incentive-compatibility (Lemma 4): the latter allows the collective choice to depend on preference intensities.

Example 2 *Suppose that $n = 2$, and F is the uniform distribution on the discrete set $(\{-2, -1, 1, 2\})^2$. The choice rule*

$$\phi(u_1, u_2) = \begin{cases} 1 & \text{if } u_1 > 0 \text{ and } u_2 > 0, \\ 1/2 & \text{if } u_1 + u_2 = 0, \\ 0 & \text{otherwise}, \end{cases}$$

is incentive-compatible.

The next step towards finding an optimal rule is to represent an agent's ex-ante expected utility via the parameters q^+ and q^- from Lemma 4. Given any anonymous and incentive-compatible rule ϕ,

$$\begin{aligned} U(\phi) &= E\left[\tilde{u}_1 \phi(\tilde{u})\right] \\ &= E\left[\tilde{u}_1 E\left[\phi(\tilde{u}_1, \tilde{u}_{-1}) \mid \tilde{u}_1\right]\right] \\ &= E\left[\tilde{u}_1 (q^+ \mathbf{1}_{\tilde{u}_1 > 0} + q^- \mathbf{1}_{\tilde{u}_1 < 0})\right] \\ &= q^+ E\left[\tilde{u}_1 \mathbf{1}_{\tilde{u}_1 > 0}\right] + q^- E\left[\tilde{u}_1 \mathbf{1}_{\tilde{u}_1 < 0}\right] \\ &= (q^+ - q^-) E\left[\tilde{u}_1 \mathbf{1}_{\tilde{u}_1 > 0}\right] \end{aligned} \tag{14}$$

Formula (14) shows that the possible dependence of collective choice on preference intensities (Example 2) is eventually inconsequential: ex-ante expected utilities are fully determined by the averages q^+ and q^-.

Because an agent prefers each alternative with probability 1/2, the number $\frac{1}{2}(q^+ - q^- + 1)$ equals the probability $\frac{1}{2}q^+ + \frac{1}{2}(1 - q^-)$ that the collective choice is congruent with a given agent's preferences. Hence, by (14) an optimal rule is one that maximizes this probability.[17]

[17] This observation is the starting point of Rae's (1969) analysis. In our model, the observation is an implication of incentive-compatibility and anonymity.

Proposition 2 *Suppose that F is stochastically independent across agents. Then the majority rule is optimal among all anonymous and incentive-compatible choice rules.*

To see why majority voting is optimal, consider the event that k agents prefer B. Agent 1 gains from increasing the probability of choosing B if $k-1$ of the other agents prefer B (that is, if she is one of the k agents preferring B), and otherwise loses out. In balance, agent 1 would like to see a high probability of choosing B if the probability that $k-1$ of the other agents prefer B exceeds the probability that k of the other agents prefer B. This condition holds if and only if more than half of the agents prefer B (this follows from basic properties of binomial distributions). The majority rule chooses B with the highest possible probability if it is preferred by more than half of the agents, and otherwise chooses B with the lowest possible probability. Hence, the majority rule is optimal. Details of the proof can be found in the appendix.

The majority rule makes obviously no use of any information about the agents' preference intensities. Hence, Proposition 2 demonstrates that, given the stochastic-independence assumption, one does not need to collect any information about preference intensities to implement an optimal choice rule.

In the absence of the stochastic-independence assumption, we do not have a useful characterization of incentive-compatibility, and we do not know whether optimality requires collecting information about preference intensities. Even if one restricts attention to choices rules that make no use of information about preference intensities, the set of incentive-compatible rules is, in general, larger than the set of dominant-strategy rules, and there exist environments where no dominant-strategy rule is optimal among incentive-compatible rules; here is an example.

Example 3 *Consider the environments from Example 1.*

If r_{-1} is sufficiently close to 1 and r_3/r_1 is sufficiently large, then the choice rule ϕ_{odd} that chooses alternative B if and only if it is preferred by an odd *number of agents is incentive-compatible and yields a higher expected utility than the optimal weak majority rule ϕ^3.*

Using the rule ϕ_{odd} yields the expected utility $p_1e_1 + p_3e_3$, whereas the rule ϕ^3 yields the expected utility $(1/2)p_1e_1 + (1/2)p_2e_2 + p_3e_3$, which is smaller because $e_1 > 0$ and $e_2 < 0$. In the Appendix we verify that ϕ_{odd} is incentive-compatible (4).

5 Interim-Optimality

In previous sections we have focussed exclusively on comparing rules from an ex-ante point of view, before any agent has observed her own utility. Now we compare rules at the interim stage, when each agent knows her utility. Given an anonymous rule ϕ, the interim expected utility of an agent with utility $u_i \in V$ is denoted

$$U(\phi, u_i) \quad = \quad u_i E[\phi(\tilde{u}) \mid \tilde{u}_i = u_i]. \tag{15}$$

If one considers neutral rules only, then a rule ϕ that is better than (or equal to) another rule for any agent with some positive utility $u_i > 0$ is also better (or equal) for this agent if she has the utility $-u_i$, because

$$\begin{aligned} U(\phi, u_i) &\overset{(2)}{=} u_i E[\phi(-\tilde{u}) \mid -\tilde{u}_i = u_i] \\ &\overset{(8)}{=} u_i E[1 - \phi(\tilde{u}) \mid \tilde{u}_i = -u_i] \\ &= u_i + U(\phi, -u_i). \end{aligned}$$

Hence, for any interim-comparison of neutral rules it is sufficient to consider positive utilities; the result of the comparison then carries over to the negative utilities.

An ideal world would be one where a choice rule can be implemented through unanimous consent at the interim stage. Unanimous consent means that all agents prefer the rule to any other feasible rule. Unanimous consent can possibly be achieved only if attention is restricted to anonymous rules (otherwise every agent would like to be dictator) and neutral rules (otherwise every agent would want a rule that favors her preferred alternative). The formal requirement for such unanimous consent is captured by the following definition. A rule ϕ is *interim-dominant among dominant-strategy (resp., incentive-compatible) rules* if, for any anonymous and neutral dominant-strategy (resp., incentive-compatible) rule ψ,

$$U(\phi, u_i) \geq U(\psi, u_i) \quad \text{for all } u_i > 0. \tag{16}$$

First we consider environments with stochastically independent utilities. Here, any neutral rule that is ex-ante optimal is also interim-dominant among incentive-compatible rules. To see this, consider the parameters q^+ and q^- from Lemma 4, for any anonymous and neutral incentive-compatible rule ϕ. Then $q^- = 1 - q^+$ because ϕ is neutral. Hence, (14) implies

$$U(\phi) \quad = \quad (2q^+ - 1) E\left[\tilde{u}_1 \mathbf{1}_{\tilde{u}_1 > 0}\right].$$

Hence, ϕ is ex-ante optimal if and only if it maximizes q^+ among all anonymous and neutral incentive-compatible rules (the "if"-part of this statement relies on Lemma 3). Because the interim expected utility of an agent with utility $u_i > 0$ is $U(\phi, u_i) = u_i q^+$, ex-ante optimality carries over immediately to interim-dominance. In particular, Proposition 2 generalizes as stated below, confirming that stochastically independent environments indeed constitute an ideal world.

Proposition 3 *Suppose that F is stochastically independent across agents. Then the majority rule is interim-dominant among incentive-compatible rules.*

Now consider environments without stochastic independence. Here, an interim-dominant rule may fail to exist. As shown below, this holds even if attention is restricted to dominant-strategy rules and to affiliated environments. As a response to the possible non-existence of interim-dominant rules, we also consider a weaker optimality concept that only requires that a rule cannot get replaced by another rule at the interim stage through unanimous consent. An anonymous rule ϕ is *interim-dominated* by an anonymous rule ψ if $U(\phi, u_i) \leq U(\psi, u_i)$ for all $u_i > 0$, and "$<$" for some $u_i > 0$. A rule is *interim-undominated (among dominant-strategy rules)* if it is not interim-dominated by any anonymous and neutral dominant-strategy rule.[18]

We consider a simple class of affiliated environments where the agents' utilities are conditionally independent, with two possible conditioning events. Formally, we consider environments F with a density f of the form

$$f(u) = \frac{1}{2}\left(g(-u_1) \cdot \ldots \cdot g(-u_n) + g(u_1) \cdot \ldots \cdot g(u_n)\right) \qquad (u_i \in V = (-1,1) \setminus \{0\}), \tag{17}$$

where

$$g : [-1,1] \to \mathbb{R} \text{ is weakly increasing, differentiable, and strictly positive.} \tag{18}$$

[18] Observe that, in addition, an interim-undominated rule cannot be dominated by any non-neutral and anonymous dominant-strategy rule ϕ, because otherwise the neutral rule (8) would also be a dominating rule. Similarly, extending the definition of interim-dominance to non-anonymous rules, the symmetrization construction of footnote 13 can be used to show that an interim-undominated rule is not dominated by any dominant-strategy choice rule, including all non-neutral and non-anonymous rules. It still does *not* follow that an interim-undominated rule is incentive-efficient in the sense of Holmstrom-Myerson (1983), because the latter concept refers to the entire set of incentive-compatible rules, rather than just dominant-strategy rules.

Any such F is affiliated.[19] One can imagine F as arising from a procedure where with probability 1/2 each agent's utility is drawn independently according to the density g ("event P"), and with probability 1/2 the negative of each agent's utility is drawn independently according to the density g ("event N").

The following result presents a characterization of interim-dominance in environments of the form (17, 18) when there are three agents. We use the shortcut $G(0) = \int_{-1}^{0} g(x)\mathrm{d}x$ for the probability that an agent's utility is negative conditional on event P. The result shows that the existence of an interim-dominant rule among dominant-strategy rules depends on how informative the event P (and, hence, N) is about the sign of an agent's utility; an interim-dominant rule exists if and only if the event P is not too informative. Put differently, an interim-dominant rule exists if and only if the probability that all agents prefer the same alternative is not too large.

Proposition 4 *Let $n = 3$, and let F have a density of the form (17, 18). If $G(0) \geq 1/2 - 1/\sqrt{12}$, then the majority rule is interim-dominant among dominant-strategy rules. If $G(0) < 1/2 - 1/\sqrt{12}$, then there exists no rule that is interim-dominant among dominant-strategy rules, and* every *anonymous and neutral dominant-strategy rule ϕ with $q_3(\phi) = 1$ is interim-undominated.*

To understand this result, it is useful to represent interim-expected utilities by conditioning on the number of agents who prefer B. The probability that k ($0 \leq k \leq n$) agents prefer alternative B, conditional on the event that a given agent has any utility $u_i \in V$, is denoted

$$r(k, u_i) \quad = \quad \Pr[|[\tilde{u}]_+| = k \mid \tilde{u}_i = u_i].$$

For any anonymous and neutral dominant-strategy rule ϕ,

$$\begin{aligned} U(\phi, u_i) \quad &= \quad \sum_{k=0}^{n} r(k, u_i) q_k(\phi) \\ &\stackrel{\text{Lemma 2}}{=} \quad u_i \left(\sum_{k=\hat{n}}^{n} r(k, u_i) q_k(\phi) + \sum_{k=0}^{\hat{n}-1} r(k, u_i)(1 - q_{n-k}(\phi)) \right) \end{aligned}$$

[19] To see this, it is sufficient to verify that $\ln f$ is supermodular, that is, $\partial^2 \ln f / (\partial u_i \partial u_j) \geq 0$ for all $i \neq j$ (see Milgrom and Weber, 1982, Theorem 1).

Affiliation can fail if g is not weakly increasing as, for example, in a smoothed version of the environments used in Example 1 and Example 3.

$$
\begin{aligned}
= \quad & u_i \left(\sum_{k=\hat{n}}^{n} r(k,u_i) q_k(\phi) + \sum_{k=\hat{n}}^{n} r(n-k,u_i)(-q_k(\phi)) \right. \\
& \left. + \sum_{k=0}^{\hat{n}-1} r(k,u_i) \right) \\
= \quad & u_i \left(\sum_{k=\hat{n}}^{n} (r(k,u_i) - r(n-k,u_i)) q_k(\phi) + \sum_{k=0}^{\hat{n}-1} r(k,u_i) \right).
\end{aligned} \tag{19}
$$

Hence, an agent's interim preferences over rules depend on the expressions $r(k,u_i) - r(n-k,u_i)$ for $k = \hat{n}, \ldots, n$. Let $u_i > 0$. Because $r(n,u_i) - r(0,u_i) = r(n,u_i) > 0$, the agent always wants $q_n(\phi)$ to be as large as possible, that is, $q_n(\phi) = 1$. In environments with $n = 3$ agents, the only other expression to consider is for $k = \hat{n} = 2$. An agent wants $q_2(\phi)$ to be as small as possible if

$$
r(2,u_i) \quad < \quad r(1,u_i), \tag{20}
$$

and wants $q_2(\phi)$ to be as large as possible if

$$
r(2,u_i) \quad \geq \quad r(1,u_i). \tag{21}
$$

In particular, if (21) holds for all $u_i > 0$, then the majority rule is interim-dominant; if each direction of the inequality (20, 21) occurs with strict inequality for some $u_i > 0$, then no interim-dominant rule exists, and every anonymous and neutral dominant-strategy rule ϕ with $q_3(\phi) = 1$ is interim-undominated.

Let us explain here why no interim-dominant rule exists if $G(0)$ is small. Details of the omitted arguments towards the proof of Proposition 4 are in the Appendix. Consider an agent with a small utility $u_i > 0$. By continuity of g, we have $g(u_i) \approx g(-u_i)$. Hence, the utility u_i is about as likely to occur conditional on P as conditional on N, implying that the agent learns little about whether P or N occurs. Conditional on P, the other agents' utilities will most likely both be positive (because $G(0)$ is small), while conditional on N they will most likely both be negative. Hence, the agent believes it is much more likely that both other agents' utilities are negative than that one is negative and the other is positive, implying (20). Given that (20) holds for some u_i, condition (21) must hold with strict inequality for some other u_i; otherwise, by (19), the weak majority rule ϕ^3 would yield a higher ex-ante expected utility than the standard majority rule ϕ^2, contradicting Corollary 2.

6 Appendix

Proof of Lemma 1. "only if"

Step 1. For all $i \in N$ and $w, x \in V^n$, if $[w]^+ = [x]^+$ and $w_{-i} = x_{-i}$, then $\phi(w) = \phi(x)$.

To see this, let $i = 1$ (by symmetry, the argument extends to all $i \in N$). Suppose that $w_i > 0$ (if $w_i < 0$, the proof is analogous). Applying (5) with $u = w$ and $u' = x$ yields

$$w_1\phi(w) \geq w_1\phi(x_1, w_{-1}) = w_1\phi(x),$$

hence $\phi(w) \geq \phi(x)$. Applying (5) with $u = x$ and $u' = w$ yields $\phi(x) \geq \phi(w)$. Thus, $\phi(x) = \phi(w)$.

Step 2. For all $u, v \in V^n$, if $[u]^+ = [v]^+$, then $\phi(u) = \phi(v)$.

To see this, let $w^0 = u$, $w^1 = (v_1, u_{-1})$, $w^2 = (v_1, v_2, u_3, \ldots, u_n)$, $\ldots$, $w^n = v$. Using *Step 1*,

$$\phi(u) = \phi(w^0) = \phi(w^1) = \ldots = \phi(w^n) = \phi(v).$$

Step 3. For all $u, v \in V^n$, if $|[u]^+| = |[v]^+|$, then $\phi(u) = \phi(v)$. Thus, condition (ii) holds.

To see this, observe that there exists a permutation π such that $[u_\pi]^+ = [v]^+$. By *Step 2*, $\phi(u_\pi) = \phi(v)$. Hence, using symmetry (3), $\phi(u) = \phi(v)$.

Step 4. $q_{k-1}(\phi) \leq q_k(\phi)$ for all $k = 1, \ldots, n$. Hence, condition (i) holds.

To see this, observe that there exists $u \in V^n$ such that $|[u]^+| = k - 1$ and $u_1 < 0$. Define $u' \in V^n$ such that $u'_1 = -u_1$ and $u'_{-1} = u_{-1}$. Applying (5) yields

$$u_1 q_{k-1}(\phi) = u_1\phi(u) \geq u_1\phi(u') = u_1 q_k(\phi).$$

Dividing by u_1 implies $q_{k-1}(\phi) \leq q_k(\phi)$.

"if": Describing a rule ϕ by the $n + 1$ numbers $q_0(\phi), \ldots, q_n(\phi)$, it is straightforward to verify condition (5). *QED*

Proof of Lemma 3. Clearly, ψ is anonymous and neutral. To see incentive compatibility of ψ, observe that, for all $u_1, u'_1 \in V$,

$$\begin{aligned} -u_1 E[\phi(-u'_1, -\tilde{u}_{-1}) \mid \tilde{u}_1 = u_1] &\overset{(2)}{=} -u_1 E[\phi(-u'_1, \tilde{u}_{-1}) \mid -\tilde{u}_1 = u_1] \\ &= -u_1 E[\phi(-u'_1, \tilde{u}_{-1}) \mid \tilde{u}_1 = -u_1] \\ &\overset{(4)}{\leq} -u_1 E[\phi(-u_1, \tilde{u}_{-1}) \mid \tilde{u}_1 = -u_1] \\ &\overset{(2)}{=} -u_1 E[\phi(-u_1, -\tilde{u}_{-1}) \mid \tilde{u}_1 = u_1]. \end{aligned} \tag{22}$$

Hence,

$$\begin{aligned}
& u_1 E[\psi(u_1', \tilde{u}_{-1}) \mid \tilde{u}_1 = u_1] \\
= \;& \frac{1}{2}\left(u_1 E[\phi(u_1', \tilde{u}_{-1}) \mid \tilde{u}_1 = u_1] + u_1 - u_1 E[\phi(-u_1', -\tilde{u}_{-1}) \mid \tilde{u}_1 = u_1]\right) \\
\overset{(4),(22)}{\leq} \;& \frac{1}{2}\left(u_1 E[\phi(u_1, \tilde{u}_{-1}) \mid \tilde{u}_1 = u_1] + u_1 - u_1 E[\phi(-u_1, -\tilde{u}_{-1}) \mid \tilde{u}_1 = u_1]\right) \\
= \;& u_1 E[\psi(u_1, \tilde{u}_{-1}) \mid \tilde{u}_1 = u_1].
\end{aligned}$$

Analogously, one sees that ψ is a dominant-strategy rule if ϕ is a dominant-strategy rule.

To prove the "Moreover"-part, observe that $E[\tilde{u}_1] = 0$ by (2). Hence,

$$\begin{aligned}
U(\psi) &= \frac{1}{2}\left(E[\tilde{u}_1 \phi(\tilde{u})] + 0 - E[\tilde{u}_1 \phi(-\tilde{u})]\right) \\
&\overset{(2)}{=} \frac{1}{2}\left(E[\tilde{u}_1 \phi(\tilde{u})] - E[-\tilde{u}_1 \phi(\tilde{u})]\right) \\
&= E[\tilde{u}_1 \phi(\tilde{u})] \;=\; U(\phi).
\end{aligned}$$

QED

Proof of Proposition 1. Problem ($*n$) has a solution because the constraint set is compact and the objective is continuous. Label the inequality constraints, in the order of appearance in ($*n$), by the numbers $l = \hat{n}, \hat{n}+1, \ldots, n, n+1$. Lagrange multipliers for these inequality constraints are denoted by

$$\lambda_l \leq 0. \tag{23}$$

The Slater condition is satisfied because there exists a point $(q_{\hat{n}}, \ldots, q_n) = (1/2+\epsilon, 1/2+2\epsilon, \ldots, 1/2+(n-\hat{n}+1)\epsilon)$ that satisfies all inequality constraints with strict inequality for any small $\epsilon > 0$. The first-order conditions for ($*n$) are (after dividing the objective by 2):

$$p_k e_k \;=\; \lambda_k - \lambda_{k+1} \qquad (k = \hat{n}, \ldots, n). \tag{24}$$

Suppose problem ($*n$) has a solution $(q_{\hat{n}}, \ldots, q_n)$ that is *not* a weak majority rule. Then there exists $l \geq \hat{n}$ such that $1/2 < q_l < 1$; let $l = \underline{l}$ be minimal with this property and $l = \bar{l}$ be maximal with this property. Then the constraints $\underline{l}$ and $\bar{l}+1$ are not binding. Hence, $\lambda_{\underline{l}} = 0$ and $\lambda_{\bar{l}+1} = 0$. Hence,

$$0 = \lambda_{\underline{l}} - \lambda_{\bar{l}+1} = \sum_{k=\underline{l}}^{\bar{l}} (\lambda_k - \lambda_{k+1}) \overset{(24)}{=} \sum_{k=\underline{l}}^{\bar{l}} p_k e_k$$

For generic environments, this equation is violated, so that only a weak majority rule can be optimal.

The weak majority rule ϕ^m with threshold m ($\hat{n} \leq m \leq n$) is optimal if and only if the first-order conditions together with the slackness conditions are satisfied at $(q_{\hat{n}}, \ldots, q_n)$, where

$$q_k = \begin{cases} \frac{1}{2} & \text{if } k < m, \\ 1 & \text{if } k \geq m. \end{cases}$$

The slackness conditions imply $\lambda_m = 0$. Using this, the equation system (24) can be solved by adding up equalities. One finds that, for all $k = \hat{n}, \ldots, n+1$,

$$\lambda_k = \begin{cases} \sum_{l=k}^{m-1} p_l e_l, & \text{if } k \leq m-1, \\ 0 & \text{if } k = m, \\ -\sum_{l=m}^{k-1} p_l e_l, & \text{if } k \geq m+1. \end{cases}$$

Together with (23) we obtain that ϕ^m is optimal if and only if

$$\begin{aligned} \sum_{l=k}^{m-1} p_l e_l &\leq 0 \quad \text{for all} \quad k = \hat{n}, \ldots, m-1, \\ \sum_{l=m}^{k-1} p_l e_l &\geq 0 \quad \text{for all} \quad k = m+1, \ldots, n+1, \end{aligned}$$

which is equivalent to the conditions (10) and (11). *QED*

Proof of claims in Example 1. Let $\hat{p}$ denote the probability of event P, conditional on the event that exactly one agent prefers B. Thus, using Bayes' rule,

$$\begin{aligned} \hat{p} &= \frac{\Pr[|[\tilde{u}]_+| = 1 \mid P] \cdot \Pr[P]}{\Pr[|[\tilde{u}]_+| = 1 \mid P] \cdot \Pr[P] + \Pr[|[\tilde{u}]_+| = 1 \mid N] \cdot \Pr[N]} \\ &= \frac{\Pr[\tilde{u}_1 > 0, \tilde{u}_2 < 0, \tilde{u}_3 < 0 \mid P]}{\Pr[\tilde{u}_1 > 0, \tilde{u}_2 < 0, \tilde{u}_3 < 0 \mid P] + \Pr[\tilde{u}_1 > 0, \tilde{u}_2 < 0, \tilde{u}_3 < 0 \mid N]} \\ &= \frac{(r_1 + r_3)(r_{-3} + r_{-1})^2}{(r_1 + r_3)(r_{-3} + r_{-1})^2 + (r_{-3} + r_{-1})(r_1 + r_3)^2} \\ &= \frac{r_{-3} + r_{-1}}{r_{-3} + r_{-1} + r_1 + r_3} \\ &= r_{-3} + r_{-1}. \end{aligned}$$

By symmetry, conditional on the event that exactly one agent prefers B and P occurs, each agent has the same probability of preferring B,

$$\Pr[[\tilde{u}]_+ = \{i\} \mid P, \ |[\tilde{u}]_+| = 1] = \frac{1}{3} \quad (i = 1, 2, 3). \tag{25}$$

We compute e_1 by conditioning separately on events P and N. Given any of these conditions, utilities are stochastically independent across agents. We also condition on the identity of the agent with the positive utility, yielding

$$\begin{aligned} e_1 \;=\;& \hat{p}(E[\tilde{u}_1 \mid [\tilde{u}]_+ = \{1\}, P] \cdot \Pr[[\tilde{u}]_+ = \{1\} \mid P,\ |[\tilde{u}]_+| = 1] \\ &+E[\tilde{u}_1 \mid [\tilde{u}]_+ = \{2\}, P] \cdot \Pr[[\tilde{u}]_+ = \{2\} \mid P,\ |[\tilde{u}]_+| = 1] \\ &+E[\tilde{u}_1 \mid [\tilde{u}]_+ = \{3\}, P] \cdot \Pr[[\tilde{u}]_+ = \{3\} \mid P,\ |[\tilde{u}]_+| = 1]) \\ &+(1-\hat{p})\underbrace{E[\tilde{u}_1 \mid |[\tilde{u}]_+| = 1, N]}_{\geq -3}, \end{aligned} \tag{26}$$

because the smallest possible utility, -3, is a lower bound for any conditional expectation.

Using (26) together with (25) and the fact that, conditional on event P, utilities are stochastically independent across agents,

$$e_1 \;\geq\; \hat{p}\left(E[\tilde{u}_1 \mid \tilde{u}_1 > 0, P] + 2E[\tilde{u}_1 \mid \tilde{u}_1 < 0, P]\right)\frac{1}{3} + (1-\hat{p})(-3).$$

Hence, using the definition of G,

$$e_1 \;\geq\; (r_{-3} + r_{-1})\left(\frac{r_1 + 3r_3}{r_1 + r_3} - 2\frac{3r_{-3} + r_{-1}}{r_{-3} + r_{-1}}\right)\frac{1}{3} + (r_1 + r_3)(-3).$$

For any small $\epsilon > 0$, let

$$r_{-3} = \epsilon,\ r_{-1} = 1 - 2\epsilon - \epsilon^2,\ r_1 = \epsilon^2,\ r_3 = \epsilon. \tag{27}$$

Then

$$\begin{aligned} e_1 \;\geq\;& (1-\epsilon-\epsilon^2)\left(\frac{\epsilon^2 + 3\epsilon}{\epsilon^2 + \epsilon} - 2\frac{3\epsilon + (1-2\epsilon-\epsilon^2)}{\epsilon + (1-2\epsilon-\epsilon^2)}\right)\frac{1}{3} + (\epsilon^2+\epsilon)(-3) \\ &\to \frac{1}{3} \quad \text{as} \quad \epsilon \to 0. \end{aligned}$$

Hence, $e_1 > 0$ if we choose ϵ sufficiently close to 0. From (6), then $e_2 = -e_1 < 0$. Thus, by Proposition 1, the weak majority rule ϕ^3 is optimal among all anonymous and dominant-strategy rules, and, by Corollary 1, the standard majority rule ϕ^2 is not optimal. *QED*

Proof of Lemma 4. Because of stochastic independence, (4) is equivalent to

$$\forall u_1, u_1' \in V: \quad u_1 E[\phi(u_1, \tilde{u}_{-1})] \;\;\geq\;\; u_1 E[\phi(u_1', \tilde{u}_{-1})].$$

This is equivalent to

$$\forall u_1, u_1' \in V: \quad E[\phi(u_1, \tilde{u}_{-1})] - E[\phi(u_1', \tilde{u}_{-1})] \begin{cases} \geq 0, & \text{if } u_1 > 0, \\ \leq 0, & \text{if } u_1 < 0. \end{cases} \tag{28}$$

Clearly, (13) implies (28). Vice versa, suppose that (28) is satisfied. Consider any $u_1 > 0$ and $u_1' > 0$. Switching the roles of u_1 and u_1' in (28) yields $E[\phi(u_1, \tilde{u}_{-1})] - E[\phi(u_1', \tilde{u}_{-1})] \leq 0$. Hence, $E[\phi(u_1, \tilde{u}_{-1})] = E[\phi(u_1', \tilde{u}_{-1})] =: q^+$. Similarly, one defines $q^- := E[\phi(u_1, \tilde{u}_{-1})]$ using any $u_1 < 0$. Then (28) implies $q^+ \geq q^-$. *QED*

Proof of Proposition 2. We start with any unanimous and incentive-compatible ϕ and express q^+ and q^- in terms of the conditional expectations $q_k(\phi)$. The probability that $k = 0, \ldots, n-1$ many out of the agents other than 1 prefer alternative B is

$$p_k^{n-1} = \frac{1}{2^{n-1}} \binom{n-1}{k},$$

by standard formulas for binomial distributions. Hence,

$$\begin{aligned} q^+ &= E[\phi(\tilde{u}) \mid \tilde{u}_1 > 0] \\ &= \sum_{k=0}^{n-1} p_k^{n-1} q_{k+1}(\phi), \end{aligned}$$

where the last index is $k+1$ because agent 1 prefers B. Similarly,

$$q^- = \sum_{k=0}^{n-1} p_k^{n-1} q_k(\phi).$$

Thus,

$$q^+ - q^- = q_n(\phi) - q_0(\phi) + \sum_{k=1}^{n-1} (p_{k-1}^{n-1} - p_k^{n-1}) q_k(\phi).$$

Because $p_{k-1}^{n-1} > p_k^{n-1}$ if and only if $k > n/2$, and "=" if and only if $k = n/2$, the majority rule ϕ_{maj} maximizes $q^+ - q^-$ and, hence, maximizes $U(\phi)$ by (14). *QED*

Proof that in Example 3, ϕ_{odd} is incentive-compatible. The probability assigned by an agent with utility 1 to the event P is

$$\Pr[P|1] = \frac{\frac{1}{2}r_1}{\frac{1}{2}r_1 + \frac{1}{2}r_{-1}} = \frac{r_1}{r_1 + r_{-1}}.$$

Hence, the probability assigned by an agent with utility 1 to the event that the other two agents have positive utilities is

$$\Pr[++|1] = (r_1 + r_3)^2 \Pr[P|1] + (r_{-1} + r_{-3})^2(1 - \Pr[P|1]),$$

the probability she assigns to the event that the other two agents have negative utilities is

$$\Pr[--|1] = (r_{-1} + r_{-3})^2 \Pr[P|1] + (r_1 + r_3)^2(1 - \Pr[P|1]),$$

and the probability assigned to the event that a certain other agent has a negative utility while the third agent has a positive utility is $\Pr[-+|1] = \Pr[+-|1] = (1 - \Pr[++|1] - \Pr[--|1])/2$.

Suppose the rule ϕ_{odd} is used. The expected utility of an agent with utility $u_1 = 1$ is

$$\Pr[++|1] + \Pr[--|1]. \tag{29}$$

If she deviates and announces a negative utility $u_1' < 0$, then alternative B is chosen if and only if exactly one other agent prefers B; hence, her expected utility from the deviation is

$$\Pr[+-|1] + \Pr[-+|1] = 1 - \Pr[++|1] - \Pr[--|1]. \tag{30}$$

Hence, incentive-compatibility for type $u_1 = 1$ holds if and only if

$$\Pr[++|1] + \Pr[--|1] \geq \frac{1}{2},$$

which is equivalent to the condition

$$(r_{-1} + r_{-3})^2 + (r_1 + r_3)^2 \geq \frac{1}{2}. \tag{31}$$

The same condition is equivalent to the incentive-compatibility for any other type. Condition (31) holds because $r_{-1} + r_{-3} + r_1 + r_3 = 1$ (recall the basic algebraic identity $(1 - x)^2 + x^2 \geq 1/2$ for all $x \in \mathbb{R}$). *QED*

Proof that in the environments of Proposition 4, (21) holds for all $u_i > 0$ if $G(0) \geq 1/3$, and (20) holds for all u_i close to 0 if $G(0) < 1/3$.

W.l.o.g., consider $i = 1$. For any $u_1 > 0$, Bayes rule yields that the probability that an agent with utility u_1 assigns to the event P is

$$\Pr[P|\tilde{u}_1 = u_1] \;=\; \frac{g(u_1)\Pr[P]}{g(u_1)\Pr[P] + g(-u_1)\Pr[N]} \;=\; \frac{g(u_1)}{g(u_1) + g(-u_1)}.$$

It is useful to express $r(2, u_1)$ by conditioning separately on events P and N,

$$\begin{aligned}
r(2, u_1) \;&=\; 2\Pr[\tilde{u}_2 > 0, \tilde{u}_3 < 0 \mid \tilde{u}_1 = u_1] \\
&=\; 2\Pr[\tilde{u}_2 > 0, \tilde{u}_3 < 0 \mid P, \tilde{u}_1 = u_1]\Pr[P|\tilde{u}_1 = u_1] \\
&\qquad +2\Pr[\tilde{u}_2 > 0, \tilde{u}_3 < 0 \mid N, \tilde{u}_1 = u_1](1 - \Pr[P|\tilde{u}_1 = u_1]) \\
&=\; 2\Pr[\tilde{u}_2 > 0, \tilde{u}_3 < 0 \mid P]\Pr[P|\tilde{u}_1 = u_1] \\
&\qquad +2\Pr[\tilde{u}_2 > 0, \tilde{u}_3 < 0 \mid N](1 - \Pr[P|\tilde{u}_1 = u_1]) \\
&=\; 2(1 - G(0))G(0). \qquad (32)
\end{aligned}$$

Similarly,

$$\begin{aligned}
r(1, u_1) \;&=\; \Pr[\tilde{u}_2 < 0, \tilde{u}_3 < 0 \mid P]\Pr[P|\tilde{u}_1 = u_1] \\
&\qquad + \Pr[\tilde{u}_2 < 0, \tilde{u}_3 < 0 \mid N](1 - \Pr[P|\tilde{u}_1 = u_1]) \\
&=\; G(0)^2 \frac{g(u_1)}{g(u_1) + g(-u_1)} + (1 - G(0))^2 \frac{g(-u_1)}{g(u_1) + g(-u_1)} \\
&=\; G(0)^2 + (1 - 2G(0)) \frac{1}{1 + \frac{g(u_1)}{g(-u_1)}}. \qquad (33)
\end{aligned}$$

Observe that $G(0) \leq 1/2$ because g is increasing. Hence,

$$1 - 2G(0) \;\geq\; 0. \qquad (34)$$

First consider cases $G(0) \geq 1/2 - 1/\sqrt{12}$. Observe that (33) together with (34) implies

$$r(1, u_1) \;\leq\; G(0)^2 + (1 - 2G(0))\frac{1}{2}$$

because $g(u_1)/g(-u_1)$ is weakly increasing in u_1. Hence, using (32) and elementary algebra,

$$r(2, u_1) - r(1, u_1) \geq 3G(0)(1 - G(0)) - \frac{1}{2} \geq 0.$$

That is, (21) holds for all $u_1 > 0$.

Second, consider cases $G(0) < 1/2 - 1/\sqrt{12}$. Then

$$\lim_{u_1 \to 0,\ u_1 > 0} r(1, u_1) \quad = \quad G(0)^2 + (1 - 2G(0))\frac{1}{2},$$

implying

$$\lim_{u_1 \to 0,\ u_1 > 0} r(2, u_1) - r(1, u_1) = 3G(0)(1 - G(0)) - \frac{1}{2} < 0.$$

That is, (20) holds for all u_1 sufficiently close to 0.

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