Inflation persistence and optimal positive long-run inflation

Dario Pontiggia

17. May 2007

Inflation Persistence and Optimal Positive Long-run Inflation*

Dario Pontiggia†
University of Glasgow, University of Milan-Bicocca

May 17, 2007

Abstract

In this paper we prove that (I) inefficient *natural level of output* (Friedman (1968)), (II) central bank's desire to stabilize output around a level that is higher than the inefficient *natural level of output*, (III) long-run Phillips curve trade-off, and (IV) inflation persistence result in optimal positive long-run inflation. The combination of (I), (II), and (III) makes positive inflation forever in principles desirable as it would result in positive output gap forever. Optimal positive steady-state inflation obtains if and only if there is a long-run incentive for positive inflation. Inflation persistence, defined as costly, in terms of output, disinflation, generates a long-run incentive for positive inflation. Optimal positive steady-state inflation obtains in the *basic neo-Wicksellian model* (Woodford (2003)) with inflation persistence due to backward-looking rule-of-thumb behaviour by price setters. Optimal positive long-run inflation also obtains in what we refer to as the nonmicrofounded model. Prescinding from hyperinflation, the formula for steady-state inflation is capable of providing a positive theory of inflation.

JEL classification: E31, E52.

Keywords: Optimal monetary policy, inflation persistence.

*I am grateful to Campbell Leith, Anton Muscatelli, and Patrizio Tirelli for valuable suggestions and comments. I thank Iona Moldovan and Jon Steinsson.

†If there are errors, I am solely responsible. Special thanks to those who know for love and game. E-mail: dario.pontiggia@unimib.it, 0412182p@student.gla.ac.uk.

This result poses the challenge for future researchers of finding a theoretical explanation for the optimality of positive inflation targets. Schmitt-Grohé and Uribe (2005, p. 52)

1 Introduction

In this paper we prove that (I) inefficient *natural level of output* (Friedman (1968)), (II) central bank's desire to stabilize output around a level that is higher than the inefficient *natural level of output*, (III) long-run Phillips curve trade-off, and (IV) inflation persistence result in optimal positive long-run inflation[1].

The combination of (I) and (II) implies that discretionary conduct of monetary policy produces the well-known inflation bias stressed by Kidland and Prescott (1977) and Barro and Gordon (1983).

The focus of this paper lies in deriving the long-run inflation target under the optimal commitment policy. The problem of what constitutes optimal inflation in the long-run is not trivial as monetary policy cannot simultaneously eliminate equilibrium distortions and distortions resulting from staggered price setting. With this respect, fiscal policy should not be assumed to fully offset equilibrium distortions: fiscal policy can either partially offset or exacerbate equilibrium distortions.

Allowing for equilibrium distortions in an economy characterized by (I) central bank's desire to stabilize output around a level that is higher than the inefficient *natural level of output* (by the steady-state efficiency gap, x^*) and (II) long-run Phillips curve trade-off makes positive inflation forever in principles desirable as it would result in positive output gap forever. Optimal positive steady-state inflation obtains if and only if there is long-run incentive for positive inflation. Inflation persistence, defined as costly, in terms of output, disinflation, generates a long-run incentive for positive inflation.

This paper owes a lot to the landmark work by Woodford (2003) and it can be interpreted as a natural extension of that contribution to the case of inflation persistence due to backward-looking rule-of-thumb behaviour by price setters, specified either à la Galì and Gertler (1999) or à la Steinsson (2003).

Optimal positive steady-state inflation breaks the surprising robustness of the optimality of a monetary policy that aims at complete price stability as it obtains in what Woodford (2003) labels the *basic neo-Wicksellian model*.

[1] Long-run inflation and steady-state inflation are used interchangeably.

Optimal positive long-run inflation target also obtains in what we refer to as the nonmicrofounded model.

Indeed, the optimal positive steady-state inflation that obtains in the nonmicrofounded model nests the one in the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour a là Galì-Gertler (1999) by price setters. The intuition for this is neat: microfounding inflation persistence a là Galì-Gertler (1999) does not affect the steady-state inflation under the optimal commitment policy.

Additionally, the nonmicrofounded model has by assumption the appealing property that the hybrid Phillips curve coefficients on future expected inflation and lagged inflation add up to one. In the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour by price setters, specified either à la Galì and Gertler (1999) or à la Steinsson (2003), the hybrid Phillips curve coefficients on future expected inflation and lagged inflation add up to one only in the limiting cases of (I) absence of inflation persistence or (II) absence of long-run Phillips curve trade off; namely zero optimal steady-state inflation.

On one hand, the nonmicrofounded model is not a Barro-Gordon (1983) model[2]: it could alternatively be defined as the purely forward-looking *basic neo-Wicksellian model* with nonmicrofounded inflation persistence.

On the other hand, the nonmicrofounded model is a Barro-Gordon (1983) model where (I) the hybrid **N***ew* ***K****eynesian* ***P****hillips* ***C****urve* (Roberts (1995)) replaces the Lucas-type aggregate supply function[3] and (II) central bank desires to stabilize output around a level that is higher than the natural rate of output.

Optimal positive long-run inflation target does not obtain in the *basic neo-Wicksellian model* with inflation persistence due to backward-looking price indexation (pioneered by Christiano, Eichenbaum and Evans (2005)) because backward-looking price indexation does not introduce any inflation persistence. Disinflation under backward-looking price indexation is costless as in the purely forward-looking *basic neo-Wicksellian model*.

Prescinding from hyperinflation, which is a phenomenon that can arise when the central bank's policy instrument is the nominal quantity of money, the formula for steady-state inflation is capable of providing a positive theory of inflation.

Section 2 lays out the models. Section 3 studies the long-run inflation target under the optimal commitment policy. Section 4 concludes. Appendix A reports the proof of propositions (1) and (2). Appendix B shows that a linear approximation to the production function suffices for a correct second-order approximation to the period utility of the representative household.

[2] See Walsh (2003, Ch. 8) for an excellent survey of the literature based on the Barro-Gordon (1983) model.

[3] Aggregate-supply relation and hybrid Phillips curve are used interchangeably.

2 The Models

The New Keynesian model considered here is the *basic neo-Wicksellian model* in Woodford (2003). It shares Woodford's notation[4], assumptions, and general formalism. It integrates it with the derivation of the hybrid Phillips curve and the central bank's objective in the case of backward-looking rule-of-thumb behaviour by price setters, specified either à la Galì and Gertler (1999) or à la Steinsson (2003). We only present the hybrid Phillips curves and the central bank's objectives that obtain under rule-of-thumb behaviour by price setters[5]. We remind the reader of the details of the economy considered and of the assumptions made. The demand side of the economy (i.e. expectational IS equation) is not reported here, it is the same as in Woodford (2003, 1.12, p. 246).

Proposition 1 *Consider a cashless economy with no endogenous capital accumulation, flexible wages, and specific labour markets (i.e. yeoman farmers). The producer's profit function is linearly homogeneous in its first three arguments (i.e. good's price, industry's price, and aggregate price level) and, for any value of the industry price and the aggregate price level, sinlge-peaked for some positive value of the good's price. Suppose that a fraction* $0 < \alpha < 1$ *of industries' prices remain fixed each period, with each price having a constant probability of being reset at any given period, as in Calvo (*1983*). Suppose also that a fraction* $0 \leq \omega < 1$ *of the* $1 - \alpha$ *industries' prices that are revised at any given period are reset according to backward-looking rule-of-thumb behaviour, as in Steinsson (*2003*). Suppose furthermore that profits are discounted according to the stochastic discount factor that equals on average* β*, with* $0 \leq \beta \leq 1$*. Then the aggregate inflation rate,* π_t*, and the aggregate output gap,* x_t*, in any period* t *must satisfy an aggregate-supply relation of the form*

$$\pi_t = \chi_f \beta E_t \pi_{t+1} + \chi_b \pi_{t-1} + \kappa_1 x_t + \kappa_2 x_{t-1} \tag{1}$$

[4] This is precisely true for all variables and structural parameters but two. First, we denote with ω the degree of rule-of-thumb behaviour by price setters rather than the elasticity of real marginal cost with respect to own output, which we denote with ϖ. Second, to avoid confusion with the Lagrangian multiplier associated with the period t hybrid Phillips Curve, φ_t, we denote with ϱ the parameter vector that indexes aspects of policy (i.e. monetary policy) that determine steady-state values of inflation and output gap, $\overline{\pi}$ and $\overline{x}$.

[5] The hybrid Phillips curve and the central bank's objective in the case of rule-of-thumb behaviour a là Steinsson (2003) correct the ones reported in Steinsson (2003). The hybrid Phillips curve and the central bank's objective in the case of rule-of-thumb behaviour a là Galì and Gertler (1999) coincide (up to x^*) with the ones reported in Amato and Laubach (2003).

with

$$\begin{aligned} \phi &= \alpha+\omega-(1-\beta)\omega\alpha; \chi_f = \frac{\alpha}{\phi}; \chi_b = \frac{\omega}{\phi}; \kappa_2 = \frac{(1-\alpha)\omega\delta}{\phi} \\ \kappa_1 &= \frac{(1-\omega)\alpha\kappa-(1-\alpha)\alpha\beta\omega\delta}{\phi}; \kappa = \frac{(1-\alpha)(1-\alpha\beta)(\sigma^{-1}+\varpi)}{(1+\varpi\theta)\,\alpha} \end{aligned} \tag{2}$$

If $\omega = 0$, (1) and (2) collapse to Woodford (2003, 2.12 and 2.13, p. 187). If the fraction $0 \leq \omega < 1$ of the $1-\alpha$ industries' prices that are revised at any given period are reset according to backward-looking rule-of-thumb behaviour, as in Galì-Gertler (1999) (i.e. $\delta = 0$), (1), standing (2), collapses to

$$\pi_t = \chi_f \beta E_t \pi_{t+1} + \chi_b \pi_{t-1} + \kappa_1 x_t \tag{3}$$

Proposition 2 *Consider a cashless economy with flexible wages, Calvo pricing, and backward-looking rule-of-thumb behaviour a là Steinsson (2003) by price setters. The discounted sum of utility of the representative household can be approximated to second-order by*

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t U_t &= -\Omega \sum_{t=0}^{\infty} \beta^t \left[\pi_t^2 + \lambda_1 (x_t - x^*)^2 + \lambda_2 \left[\pi_t - (\pi_{t-1} + (1-\alpha)\delta x_{t-1})\right]^2\right] \\ &\quad + t.i.p + O\left(\left\|\Phi_y, \widetilde{\xi}, \varrho, \Delta_{-1}^{1/2}\right\|^3\right) \end{aligned} \tag{4}$$

The definition of κ in (2) holds. The constant Ω is given by $\Omega = \overline{Y}\widetilde{u}_c(\sigma^{-1}+\varpi)\theta/2\kappa$. The steady-state efficiency gap x^* is given by $x^* \equiv \log(\overline{Y}^*/\overline{Y}) = \Phi_y/(\varpi+\sigma^{-1})$. The relative weight on output fluctuations is given by $\lambda_1 = \kappa/\theta$. The relative weight on $[\pi_t - (\pi_{t-1} + (1-\alpha)\delta x_{t-1})]^2$ is given by $\lambda_2 = \omega/[(1-\omega)\alpha]$. If $\omega = 0$, (4) collapses to Woodford (2003, 2.21 and 2.22, p. 400). In the presence of backward-looking rule-of-thumb behaviour a là Galì-Gertler (1999) by price setters, (4) collapses to

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t U_t &= -\Omega \sum_{t=0}^{\infty} \beta^t \left[\pi_t^2 + \lambda_1 (x_t - x^*)^2 + \lambda_2 (\pi_t - \pi_{t-1})^2\right] \\ &\quad + t.i.p + O\left(\left\|\Phi_y, \widetilde{\xi}, \varrho, \Delta_{-1}^{1/2}\right\|^3\right) \end{aligned} \tag{5}$$

Interestingly, in the presence of rule-of thumb behaviour by price setters, the utility-based central bank's loss functions (5) and (4) can now be seen as penalizing variations in inflation as well as variations in the difference between general inflation and rule-of-thumb price increases. Following the theoretical literature on optimal monetary policy, we later assume that the central bank's policy instrument is the short-term nominal interest rate. The assumption reflects the actual practice of monetary policy by large central banks such as the European

Central Bank, the Federal Reserve, and the Bank of England. In the microfounded model, the combination of (I) cashless economy (i.e. there are no costs associated with varying the nominal interest rate) and (II) central's bank control of the nominal interest rate implies that the expectational IS equation imposes no real constraint on the central bank. Given the central bank's optimal choices for inflation and output gap, the expectational IS equation simply determines the path of nominal interest rate necessary to achieve the optimal path for the output gap[6]. As a consequence, it is more convenient to treat output gap as if it were the central bank's policy instrument. Accordingly, the nonmicrofounded model is given by the hybrid Phillips curve

$$\pi_t = (1-\varepsilon)\beta E_t \pi_{t+1} + \varepsilon \pi_{t-1} + k_n x_t \tag{6}$$

and the ad-hoc monetary policy objective

$$\sum_{t=0}^{\infty} \beta^t \left[\pi_t^2 + \lambda_n (x_t - x^*)^2\right] \tag{7}$$

In (6), ε is a measure of the degree of inflation persistence in the economy. (6) is the NKPC which, in order to capture the inflation persistence found in the data, has been augmented with lagged inflation. Accordingly, (7) closely resembles the monetary policy objective that would obtain in the purely forward-looking *basic neo-Wicksellian model*. Yet, (7) is not, as in Woodford (2003, p. 400), a second-order approximation to the discounted sum of utility of the representative household. In the nonmicrofounded model, both the hybrid Phillips curve and the central bank's objective are assumed rather than being derived from first principles as we want the parameters in (6) and (7) not to be microfounded.

A few observations are necessary before proceeding.

Output gap, x_t, is the deviation of output, Y_t, from what Friedman (1968) labels the *natural level of output*, Y_t^n.

In the *basic neo-Wicksellian model*, the *natural level of output* is the equilibrium level of output in the absence of nominal rigidity (i.e. sticky prices). The steady-state level of output is the equilibrium level of output that obtains in the absence of exogenous real shocks (i.e. exogenous shocks to technology, to government purchases, to household's impatience to consume, and to the household's willingness to supply labour). The natural steady-state level of output, $\overline{Y}$, is the equilibrium level of output that obtains (I) in the absence of nominal rigidity and (II) in the absence of exogenous real shocks. The *natural level of output* is inefficiently low

[6] Given positive inflation (hence positive output gap), the zero lower bound on nominal interest rate is not a matter of concern.

due to equilibrium distortion (i.e. monopolistic competition). On one hand, fluctuations in the natural rate of output (due to the exogenous real disturbances) equal fluctuations in the efficient level of output, Y_t^*. Hence, optimal monetary policy should not react to any of the exogenous real shocks. On the other hand, the efficient steady-state level of output, $\overline{Y}^*$, is higher than the natural steady-state level of output, $\overline{Y}$. The wedge between the efficient level of output and the *natural level of output* is thus constant over time. We refer to the constant over time wedge as the steady-state efficiency gap, $x^* \equiv \log(\overline{Y}^*/\overline{Y})$.

In the nonmicrofounded model, the *natural level of output* is the linear trend in output.

Moreover, the two models differ in terms of structural parameters. The structural parameters in the nonmicrofounded model are five: (I) the steady-state efficiency gap, x^*, (II) the discount factor, β, (III) the degree of inflation persistence, ε, (IV) the output gap coefficient, κ_n, and (V) the relative weight on output fluctuations, λ_n. In the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour by price setters a là Galì-Gertler (1999), the structural parameters are seven: (I) the parameter that summarizes the distortion in the natural steady-state level of output due to monopolistic competition, Φ_y, (II) the discount factor, β, (III) the degree of backward-looking rule-of-thumb behaviour by price setters, ω, (IV) the degree of nominal rigidity, α, (V) the elasticity of real marginal cost with respect to own output, ϖ, (VI) the intertemporal elasticity of substitution of aggregate expenditure, σ[7], and (VII) the elasticity of substitution between any two goods, θ. In the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour by price setters a là Steinsson (2003), the structural parameters are eight: Φ_y, β, ω, α, ϖ, σ, θ, and (VIII) the degree of indexation to past output gap by backward-looking rule-of-thumb price setters, δ.

We can now define inflation persistence.

Proposition 3 *Inflation persistence is costly, in terms of output, disinflation.*

The aggregate-supply relations (3), (1), and (6) exhibit inflation persistence.

3 The Optimal Long-Run Inflation

In this section, we prove that (I) inefficient *natural level of output* (Friedman (1968)), (II) central bank's desire to stabilize output around a level that is higher than the inefficient

[7] Alternatively, standing the definition of σ, the sixth structural parameter can be defined as the elastiticy of real marginal cost with respect to aggregate ouput, σ^{-1}.

natural level of output, (III) long-run Phillips curve trade-off, and (IV) inflation persistence result in positive long-run inflation target under the optimal commitment policy.

Optimal positive long-run inflation target obtains in both the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour by price setters and in the nonmicrofounded model.

Indeed, the optimal positive steady-state inflation that obtains in the nonmicrofounded model nests the one in the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour a là Galì-Gertler (1999) by price setters. The intuition for this is neat: microfounding inflation persistence a là Galì-Gertler (1999) does not affect the steady-state inflation under the optimal commitment policy.

Additionally, the nonmicrofounded model has by assumption the appealing property that the hybrid Phillips curve coefficients on future expected inflation and lagged inflation add up to one. In the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour by price setters, specified either à la Galì and Gertler (1999) or à la Steinsson (2003), the hybrid Phillips curve coefficients on future expected inflation and lagged inflation add up to one only in the limiting cases of (I) absence of inflation persistence (i.e. $\omega = 0$) or (II) absence of long-run Phillips curve trade off (i.e. $\beta = 1$); namely zero optimal steady-state inflation.

The analysis is conducted in a purely deterministic setting, certainty equivalence guarantees that the results obtained hold in the presence of random disturbances.

Under the optimal commitment policy, the central bank chooses paths for inflation and output gap to minimize the future discounted sum of losses from date 0 (i.e. when the policy is implemented) onward subject to the constraint that the paths must satisfy the aggregate supply relation each period.

In the nonmicrofounded model, we simply assume both the hybrid Phillips curve and the central bank's objective. In the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour by price setters, the hybrid Phillips curve (namely a log-linear approximation to the model structural equations) suffices for a correct linear approximation to the optimal commitment policy only in the case of small equilibrium distortions (i.e. x^* is small enough). Given the assumed deterministic setting, the solution for the optimal paths of inflation and output is accurate up to a residual that is only of second-order. This is enough for a characterization of the first-order consequences of allowing for the empirically realistic case of equilibrium distortions (i.e. for inefficiency of the natural rate of output).

Precisely, we analytically derive the unique long-run inflation targets that, following Woodford

(1999), are *optimal from a timeless perspective*, $\overline{\pi}$.

> *A constant inflation target* $\overline{\pi}$ *is optimal from a timeless perspective if the problem of minimising* the discounted sum of losses *subject to the constraint that the bounded sequences,* $\{\pi_t, x_t\}_{t=0}^{\infty}$*, satisfy the* aggregate supply curve *for each* $t \geq 0$*, and the additional constraint that* $\pi_0 = \overline{\pi}$*, has a solution in which* $\pi_t = \overline{\pi}$ *for all* t. Woodford (2003, p. 475).

The two commitment policies in the literature differ as the requirement that $\pi_0 = \overline{\pi}$ under timeless-perspective is replaced by the initial condition $\varphi_{-1} = 0$ (i.e. no fulfilment of expectations existing prior to the policy implementation) in the case of zero-optimal commitment policy. The two commitment policies in the literature thus share the same target[8]. Accordingly, we also assume that both inflation and output gap in the period before policy is implemented (i.e. date -1) are at their values of zero (i.e. the optimal paths for inflation and output gap are flat at their respective long-run optimal targets). As long as inflation at date -1 is nonzero (and/or output gap at date -1 is nonzero under Steinsson's rule-of-thumb), inflation persistence implies that the optimal commitment policy, either zero-optimal or timeless-perspective, involves transition paths for inflation and the output gap to their respective long-run targets.

Moreover, in the case of larger equilibrium distortion and in the context of a purely-forward looking cashless economy, Benigno and Woodford (2005) show that the central bank's welfare criterion must also include transitory quadratic terms. However, these extra terms do not affect the characterization of optimal policy from a timeless perspective. Insofar as this result extends to the case of an economy with inflation persistence (which needs to be verified), optimal long-run inflation would not be affected.

We consider first the case of inflation persistence due to backward-looking rule-of-thumb behaviour by price setters a là Steinsson (2003). A central bank acting under commitment faces the problem of choosing bounded deterministic paths for inflation and the output gap, $\{\pi_t, x_t\}_{t=0}^{\infty}$, to minimise (4) subject to the constraint that the sequences must satisfy (1) each period. We form the following Lagrangian.

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{1}{2}\pi_t^2 + \frac{\lambda_1}{2}(x_t - x^*)^2 + \frac{\lambda_2}{2}\left[\pi_t - (\pi_{t-1} + (1-\alpha)\delta x_{t-1})\right]^2 \\ +\varphi_t\left[\pi_t - \chi_f \beta \pi_{t+1} - \chi_b \pi_{t-1} - \kappa_1 x_t - \kappa_2 x_{t-1}\right] \end{array} \right\} \quad (8)$$

[8] There is a unique optimal long-run inflation target. Hence, we can refer to it as the optimal long-run inflation (i.e. optimal steady-state inflation).

where (and henceforth) φ_t is the Lagrangian multiplier associated with the period t hybrid Phillips Curve. Differentiating with respect to π_t and x_t, we get the two first-order conditions

$$\left\{ \begin{array}{c} \pi_t + \varphi_t + \lambda_2 \left[\pi_t - (\pi_{t-1} + (1-\alpha)\delta x_{t-1})\right] \\ -\chi_f \varphi_{t-1} - \beta\chi_b\varphi_{t+1} - \beta\lambda_2\left[\pi_{t+1} - (\pi_t + (1-\alpha)\delta x_t)\right] \end{array} \right\} = 0 \tag{9}$$

$$\lambda_1(x_t - x^*) - \beta\lambda_2(1-\alpha)\delta\left[\pi_{t+1} - (\pi_t + (1-\alpha)\delta x_t)\right] - \kappa_1\varphi_t - \beta\kappa_2\varphi_{t+1} = 0 \tag{10}$$

for each $t \geq 0$.

Proposition 4 *Consider a cashless economy with flexible wages, Calvo pricing, backward-looking rule-of-thumb behaviour a là Steinsson (*2003*) by price setters, and no real disturbances. Assume that the initial price dispersion of prices $\Delta_{-1} \equiv var\left\{\log_{-1}(I)\right\}$ is small, initial inflation is zero $\pi_{-1} = 0$, initial output gap is zero $x_{-1} = 0$, and equilibrium distortions (measured by Φ_y) are small as well, so that an an approximation to the welfare of the representative household of the form (4) is possible, with the steady-state efficiency gap, x^*, a small parameter ($x^* = O(\|\Phi_y\|)$). Then, at least among inflation paths in which inflation remains forever in a certain interval around zero, there is a unique policy that is optimal from a timeless perspective. Under this policy, the positive optimal long-run inflation is given by*

$$\begin{aligned} \overline{\pi} \quad = \quad & \left\{ \frac{(1-\alpha)(1-\beta)\kappa\theta^{-1}\omega\left[(1-\omega)\alpha\kappa + (1-\alpha)(1-\alpha\beta)\omega\delta\right]}{\left\{ \begin{array}{c} (1-\omega)(1-\alpha)(\theta^{-1}-\delta)(1-\beta)^2\alpha\omega\kappa + \\ \left[(1-\omega)\alpha\kappa + (1-\alpha)^2\beta\omega\delta\right]\left[(1-\omega)\alpha\kappa + (1-\alpha)(1-\alpha\beta)\omega\delta\right] \end{array} \right\}} x^* \right\} \\ & + O(\left\|\Delta_{-1}^{1/2}, \Phi_y\right\|^2) \end{aligned} \tag{c}$$

Optimal steady-state inflation is zero (I) in the absence of inflation persistence (i.e. $\omega = 0$) or (II) in the absence of long-run Phillips curve trade off (i.e. $\beta = 1$).

Proof. Condition (9) has a solution with inflation constant over time only if the Lagrange multiplier is also constant over time. Substituting a constant value for the Lagrange multiplier in (9) and (10), the two conditions can be simultaneously satisfied only if

$$\overline{\pi} = \frac{\left\{ \begin{array}{c} \{(\kappa_1 + \beta\kappa_2)\left[(1-\beta)(1-\alpha)\delta\lambda_2\right]\} \\ + \left\{(1-\chi_f - \beta\chi_b)\left[\lambda_1 - \beta\lambda_2(1-\alpha)^2\delta^2\right]\right\} \end{array} \right\}}{(\kappa_2 + \beta\kappa_3)} \overline{x} + \frac{(1-\chi_f-\beta\chi_b)\lambda_1}{(\kappa_2+\beta\kappa_3)} \tag{11}$$

The hybrid Phillips curve (1) implies an upward-sloping relation

$$\overline{x} = \frac{(1-\beta\chi_f - \chi_b)}{(\kappa_2 + \kappa_3)}\overline{\pi} \tag{12}$$

between long-run inflation and long-run output gap. Combining (18) and (19) yields the optimal steady-state inflation

$$\overline{\pi} = \frac{(1-\chi_f-\beta\chi_b)(\kappa_2+\kappa_3)\lambda_1}{(\kappa_2+\beta\kappa_3)(\kappa_2+\kappa_3) + (1-\chi_f\beta-\chi_b)\left\{ \begin{array}{c} \left\{(1-\chi_f-\beta\chi_b)\left[\lambda_1-\beta\lambda_2(1-\alpha)^2\delta^2\right]\right\} \\ -\left\{(\kappa_2+\beta\kappa_3)\left[(1-\beta)(1-\alpha)\delta\lambda_2\right]\right\} \end{array} \right\}} x^* \tag{13}$$

The sign of the relationship is more easily determined by substituting for all the parameters in (13) in terms of structural parameters (keeping κ implicit)

$$\overline{\pi} = \frac{(1-\alpha)(1-\beta)\kappa\theta^{-1}\omega\left[(1-\omega)\alpha\kappa+(1-\alpha)(1-\alpha\beta)\omega\delta\right]}{\left\{ \begin{array}{c} (1-\omega)(1-\alpha)(\theta^{-1}-\delta)(1-\beta)^2\alpha\omega\kappa+ \\ \left[(1-\omega)\alpha\kappa+(1-\alpha)^2\beta\omega\delta\right]\left[(1-\omega)\alpha\kappa+(1-\alpha)(1-\alpha\beta)\omega\delta\right] \end{array} \right\}} x^* \tag{14}$$

which is (c). Given (I) $k > 0$ and (II) the rigour of mathematics (i.e. $\alpha = 1$ is outside the range for α as it would imply dividing by zero in deriving (1)), optimal long-run inflation is always positive and collapses to zero in the case of (I) absence of inflation persistence (i.e. $\omega = 0$) or (II) absence of long-run Phillips curve trade off (i.e. $\beta = 1$). ■

We now turn to the case of inflation persistence due to backward-looking rule-of-thumb behaviour by price setters a là Galì and Gertler (1999). What constitutes optimal long-run inflation is implied by setting $\delta = 0$ in (c), here we prefer to derive it so to stress that microfounding inflation persistence a là Galì and Gertler (1999) does not affect the optimal steady-state inflation. A central bank acting under commitment faces the problem of choosing bounded deterministic paths for inflation and the output gap, $\{\pi_t, x_t\}_{t=0}^{\infty}$, to minimise (5) subject to the constraint that the sequences must satisfy (3) each period. We form the following Lagrangian.

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{1}{2}\left[\pi_t^2+\lambda_1(x_t-x^*)^2+\lambda_2(\Delta\pi_t)^2\right] \\ +\varphi_t\left[\pi_t-\chi_f\beta\pi_{t+1}-\chi_b\pi_{t-1}-\kappa_1 x_t\right] \end{array} \right\} \tag{15}$$

Differentiating with respect to π_t and x_t, we get the two first-order conditions

$$\pi_t+\lambda_2(\pi_t-\pi_{t-1})-\beta\lambda_2(\pi_{t+1}-\pi_t)+\varphi_t-\chi_f\varphi_{t-1}-\beta\chi_b\varphi_{t+1}=0 \tag{16}$$

$$\lambda_1(x_t-x^*)-\kappa_1\varphi_t=0 \tag{17}$$

for each $t \geq 0$.

Proposition 5 *Consider a cashless economy with flexible wages, Calvo pricing, backward-looking rule-of-thumb behaviour a là Galì and Gertler (*1999*) by price setters, and no real* *disturbances. Assume that the initial price dispersion of prices* $\Delta_{-1} \equiv var\left\{\log_{-1}(I)\right\}$ *is small, initial inflation is zero* $\pi_{-1} = 0$*, and equilibrium distortions (measured by* Φ_y*) are small as well, so that an an approximation to the welfare of the representative household of the form (5) is possible, with the steady-state efficiency gap,* x^**, a small parameter (*$x^* = O(\|\Phi_y\|)$*). Then, at least among inflation paths in which inflation remains forever in a certain interval around zero, there is a unique policy that is optimal from a timeless perspective. Under this policy, the positive optimal long-run inflation is given by*

$$\overline{\pi} = \frac{(1-\alpha)(1-\beta)\omega\kappa}{(1-\omega)\alpha\theta\kappa + (1-\alpha)(1-\beta)^2\omega}x^* + O(\left\|\Delta_{-1}^{1/2}, \Phi_y\right\|^2) \tag{b}$$

Optimal steady-state inflation is zero (I) in the absence of inflation persistence (i.e. $\omega = 0$*) or (II) in the absence of long-run Phillips curve trade off (i.e.* $\beta = 1$*).*

Proof. Condition (16) has a solution with inflation constant over time only if the Lagrange multiplier is also constant over time. Substituting a constant value for the Lagrange multiplier in (16) and (17), the two conditions can be simultaneously satisfied only if

$$\overline{\pi} = -\frac{(1-\chi_f-\beta\chi_b)\lambda_1}{\kappa_1}(\overline{x} - x^*) \tag{18}$$

The hybrid Phillips curve (3) implies an upward-sloping relation

$$\overline{x} = \frac{(1-\beta\chi_f-\chi_b)}{\kappa_1}\overline{\pi} \tag{19}$$

between long-run inflation and long-run output gap. Combining (18) and (19) yields the optimal long-run inflation target

$$\overline{\pi} = \frac{(1-\chi_f-\beta\chi_b)\lambda_1\kappa_1}{\kappa_1^2 + (1-\chi_f-\beta\chi_b)(1-\beta\chi_f-\chi_b)\lambda_1}x^* \tag{20}$$

The sign of the relationship is more easily determined by substituting for all the parameters in (20) in terms of structural parameters (keeping κ implicit). Here, rather than simply substituting, we can double-check the result obtained. Combining (17) and (16), optimal paths for inflation and output gap satisfy

$$\begin{bmatrix} \pi_t \\ +\frac{\omega}{\alpha(1-\omega)}(\pi_t - \pi_{t-1}) \\ -\frac{\beta\omega}{\alpha(1-\omega)}(\pi_{t+1} - \pi_t) \end{bmatrix} = \frac{1}{(1-\omega)\alpha\theta}\begin{bmatrix} \alpha(x_{t-1} - x^*) \\ +\omega\beta(x_{t+1} - x^*) \\ -\phi(x_t - x^*) \end{bmatrix} \tag{21}$$

Solving analytically for the optimal paths for inflation and output gap would require combining (21) with (3) and solve the resulting difference equation. Here we are content with deriving the optimal long-run inflation. The hybrid Phillips Curve(3) can be rewritten in terms of structural parameters as

$$x_t = \frac{1}{\kappa}(\pi_t - \beta\pi_{t+1}) - \frac{\omega\beta}{(1-\omega)\kappa}(\pi_{t+1} - \pi_t) + \frac{\omega}{(1-\omega)\alpha\kappa}(\pi_t - \pi_{t-1}) \tag{22}$$

where the equivalence $\pi_{t+1} \equiv \omega\pi_{t+1} - (1-\omega)\pi_{t+1}$ is used to obtain a term in the rate of inflation acceleration at date $t+1$. Combining (21) and (22) yields

$$\begin{bmatrix} \pi_t + \\ \lambda_2(\pi_t - \pi_{t-1}) - \\ \beta\lambda_2(\pi_{t+1} - \pi_t) \end{bmatrix} = \frac{1}{(1-\omega)\alpha\theta} \left\{ \begin{array}{c} \alpha \begin{bmatrix} \frac{\pi_{t-1}}{\kappa} - \frac{\beta}{\kappa}\pi_t \\ -\frac{\omega\beta}{(1-\omega)\kappa}(\pi_t - \pi_{t-1}) \\ +\frac{\omega}{(1-\omega)\alpha\kappa}(\pi_{t-1} - \pi_{t-2}) - x^*_{t-1} \end{bmatrix} \\ +\beta\omega \begin{bmatrix} \frac{\pi_{t+1}}{\kappa} - \frac{\beta}{\kappa}\pi_{t+2} \\ -\frac{\omega\beta}{(1-\omega)\kappa}(\pi_{t+2} - \pi_{t+1}) \\ +\frac{\omega}{(1-\omega)\alpha\kappa}(\pi_{t+1} - \pi_t) - x^*_{t+1} \end{bmatrix} \\ -\phi \begin{bmatrix} \frac{\pi_t}{\kappa} - \frac{\beta}{\kappa}\pi_{t+1} \\ -\frac{\omega\beta}{(1-\omega)\kappa}(\pi_{t+1} - \pi_t) \\ +\frac{\omega}{(1-\omega)\alpha\kappa}(\pi_t - \pi_{t-1}) - x^*_t \end{bmatrix} \end{array} \right\}$$

In the steady-state all the terms in the rate of inflation acceleration drop out. Optimal long-run inflation is then given by

$$\overline{\pi} = \frac{\omega(1-\alpha)(1-\beta)\kappa}{(1-\omega)\alpha\theta\kappa + \omega(1-\alpha)(1-\beta)^2} x^* \tag{23}$$

which is (b) (i.e. (20) in terms of structural parameters, (c) under $\delta = 0$). Given (I) $k > 0$ and (II) the rigour of mathematics (i.e. $\alpha = 1$ is outside the range for α as it would imply dividing by zero in deriving (3)), optimal long-run inflation is always positive and collapses to zero in the case of (I) absence of inflation persistence (i.e. $\omega = 0$) or (II) absence of long-run Phillips curve trade off (i.e. $\beta = 1$). ■

We now turn to the nonmicrofounded model. A central bank acting under commitment faces the problem of choosing bounded deterministic paths for inflation and the output gap, $\{\pi_t, x_t\}_{t=0}^{\infty}$, to minimise (7) subject to the constraint that the sequences must satisfy (6) each period. We form the following Lagrangian.

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{1}{2}\left[\pi_t^2 + \lambda_n(x_t - x^*)^2\right] \\ +\varphi_t\left[\pi_t - (1-\varepsilon)\beta\pi_{t+1} - \varepsilon\pi_{t-1} - k_n x_t\right] \end{array} \right\} \tag{24}$$

Differentiating with respect to π_t and x_t, we get the two first-order conditions

$$\pi_t + \varphi_t - (1-\varepsilon)\varphi_{t-1} - \beta\varepsilon\varphi_{t+1} = 0 \tag{25}$$

$$\lambda_n(x_t - x^*) - \kappa_n \varphi_t = 0 \tag{26}$$

for each $t \geq 0$.

Proposition 6 *Consider the nonmicrofounded model (i.e. (6) and (7)).Then, at least among inflation paths in which inflation remains forever in a certain interval around zero, there is a unique policy that is optimal from a timeless perspective. Under this policy, the positive optimal long-run inflation is given by*

$$\overline{\pi} = \frac{(1-\beta)\varepsilon\lambda_n\kappa_n}{\kappa_n^2 + (1-\varepsilon)(1-\beta)^2\varepsilon\lambda_n} x^* \tag{a}$$

Optimal steady-state inflation is zero (I) in the absence of inflation persistence (i.e. $\varepsilon = 0$) or (II) in the absence of long-run Phillips curve trade off (i.e. $\beta = 1$).

Proof. Condition (25) has a solution with inflation constant over time only if the Lagrange multiplier is also constant over time. Substituting a constant value for the Lagrange multiplier in (25) and (26), the two conditions can be simultaneously satisfied only if

$$\overline{\pi} = -\frac{(1-\beta)\varepsilon\lambda_n}{\kappa_n}(\overline{x} - x^*) \tag{27}$$

The hybrid Phillips curve (6) implies an upward-sloping relation

$$\overline{x} = \frac{(1-\beta)(1-\varepsilon)}{\kappa_n}\overline{\pi} \tag{28}$$

between long-run inflation and long-run output gap. Combining (27) and (28) yields the optimal long-run inflation target

$$\overline{\pi} = \frac{(1-\beta)\varepsilon\lambda_n\kappa_n}{\kappa_n^2 + (1-\varepsilon)(1-\beta)^2\varepsilon\lambda_n} x^* \tag{29}$$

which is (a). We can double-check the result obtained. Combining (25) and (26), optimal paths for inflation and output gap satisfy

$$\pi_t = \frac{\lambda_n}{\kappa_n}\begin{bmatrix} (1-\varepsilon)(x_{t-1} - x^*) \\ +\beta\varepsilon(x_{t+1} - x^*) \\ -(x_t - x^*) \end{bmatrix} \tag{30}$$

Solving analytically for the optimal paths for inflation and output gap would require combining (30) with (6) and solve the resulting difference equation. Here we are content with deriving the optimal long-run inflation target. The hybrid Phillips Curve (6) can be rewritten in terms of structural parameters as

$$x_t = \frac{(1-\varepsilon)}{\kappa_n}(\pi_t - \beta\pi_{t+1}) + \frac{\varepsilon}{\kappa_n}(\pi_t - \pi_{t-1}) \tag{31}$$

where the equivalence $\pi_t \equiv \varepsilon\pi_t - (1-\varepsilon)\pi_t$ is used to obtain a term in the rate of inflation acceleration at date $t-1$. Combining (30) and (31) yields

$$\pi_t = \frac{\lambda_n}{\kappa_n}\left\{\begin{array}{c} (1-\varepsilon)\left[\frac{(1-\varepsilon)}{\kappa_n}(\pi_{t-1}-\beta\pi_t)+\frac{\varepsilon}{\kappa_n}(\pi_{t-1}-\pi_{t-2})-x^*\right] \\ +\beta\varepsilon\left[\frac{(1-\varepsilon)}{\kappa_n}(\pi_{t+1}-\beta\pi_{t+2})+\frac{\varepsilon}{\kappa_n}(\pi_{t+1}-\pi_t)-x^*\right] \\ -\left[\frac{(1-\varepsilon)}{\kappa_n}(\pi_t-\beta\pi_{t+1})+\frac{\varepsilon}{\kappa_n}(\pi_t-\pi_{t-1})-x^*\right] \end{array}\right\}$$

In the steady-state all the terms in the rate of inflation acceleration drop out. The optimal long-run inflation is then confirmed to be given by (a). Given (I) $\kappa_n > 0$ and (II) $\lambda_n > 0$[9], optimal long-run inflation is always positive and collapses to zero in the case of (I) absence of inflation persistence (i.e. $\varepsilon = 0$) or (II) absence of long-run Phillips curve trade off (i.e. $\beta = 1$). ■

Comparing the optimal plan first-order condition for output gap, we note that if (I) $\kappa_n = \kappa_1$ and (II) $\lambda_n = \lambda_1$, then (17) and (26) coincide. Comparing the optimal plan first-order condition for inflation, we note that if (I) $(1-\varepsilon) = \chi_f$ and (II) $\varepsilon = \chi_b$, then (16) and (25) coincide up to the terms in the rate of inflation acceleration. Comparing the hybrid Phillips curves, we note that if (I) $\kappa_n = \kappa_1$ and (II) $\varepsilon = 0$, then (22) and (31) coincide up to the terms in the rate of inflation acceleration.

The terms in the rate of inflation acceleration do not matter for the determination of optimal long-run inflation. The conditions that guarantee that the optimal positive steady-state inflation that obtains in the nonmicrofounded model nests the one in the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour by price setters a là Galì-Gertler (1999) are thus (I) $\kappa_n = \kappa_1$, (II) $\lambda_n = \lambda_1$, and (III) $(1-\varepsilon) = \chi_f$ and $\varepsilon = \chi_b \Longleftrightarrow \chi_f + \chi_b = 1$. Under these generic conditions (20) is easily seen to coincide with (a).

Condition (III) is then satisfied for $\omega\alpha(1-\beta) = 0$, namely the sum of the coefficients on future expected inflation and lagged inflation in the hybrid Phillips curve implied by backward-looking rule-of-thumb behaviour by price setters, specified either à la Galì and Gertler (1999) or à la Steinsson (2003), is generally greater than one. Given the rigour of mathematics (i.e. $\alpha = 0$ is outside the range for α), the coefficients on future expected inflation and lagged inflation add up to one only in the limiting cases of (I) absence of inflation persistence (i.e. $\omega = 0$) or (II) absence of long-run Phillips curve trade off (i.e. $\beta = 1$), namely zero

[9] Note that in the nonmicrofounded model the relative weight on output fluctuations, λ_n, is not a function of the output gap coefficient, κ_n. Conversely, in all the *basic neo-Wicksellian models*, the relative weight on output fluctuations is a positive function of the output gap coefficient that obtains in the purely forward-looking *basic neo-Wicksellian model*, κ. The optimality of positive long-run inflation in the nonmicrofounded model is thus conditional on the central bank caring about output fluctuations.

optimal steady-state inflation. Conversely, the nonmicrofounded model has by assumption the appealing property that the hybrid Phillips curve coefficients on future expected inflation and lagged inflation add up to one.

The combination of (I) inefficient *natural level of output*, (II) central bank's desire to stabilize output around a level that is higher than the inefficient *natural level of output*, and (III) long-run Phillips curve trade-off makes positive inflation forever in principles desirable as it would result in positive output gap forever.

Positive inflation forever obtains if and only if there is long-run incentive for positive inflation, namely the stimulative effect of inflation on output is not offset by the output cost of inflation. Inflation persistence, as defined above, brings about a long-run incentive for positive inflation. Indeed, optimal positive long-run inflation does not obtain in the *basic neo-Wicksellian model* with inflation persistence due to backward-looking price indexation (pioneered by Christiano, Eichenbaum and Evans (2005)) because backward-looking price indexation does not introduce any inflation persistence. Disinflation under backward-looking price indexation is costless as in the purely forward-looking *basic neo-Wicksellian model*.

The generality of inflation persistence and positive optimal long-run inflation can be corroborated by looking at the optimal plan first-order condition for inflation. Note that in all models considered the optimal plan first-order condition for output gap determines a positive relationship between the long-run value of the Lagrange multiplier, $\overline{\varphi}$, and the long-run value of the output gap, $\overline{x}$. Precisely, $\overline{\varphi}$ is found to be a positive function of the difference between long-run output gap and the steady-state efficiency gap, x^*. Analysing the absence/presence of long-run incentive for positive inflation thus amounts to consider whether there is a long-run relationship between inflation and the Lagrange multiplier. If the stimulative effect of higher inflation on output is greater than the output cost of higher inflation, $\overline{\pi}$ would then be negatively related to $\overline{\varphi}$. Hence, optimal long-run inflation would be found to be a positive function of the steady-state efficiency gap. In what follows, we are analysing the optimal plan first-order condition for inflation so to check whether the coefficients on the Lagrange multipliers add up to zero.

In the purely forward-looking *basic neo-Wicksellian model*, the optimal plan implies that inflation evolves according to

$$\pi_t + \varphi_t - \varphi_{t-1} = 0 \tag{32}$$

The increase in output in any period caused by higher inflation in the same period, φ_t, is thus offset by the cost of the reduction in output in the previous period as a result of expected higher inflation, φ_{t-1}. Hence, there is no long-run incentive for positive inflation, the optimal long-run inflation is zero.

The same conclusion holds in the *basic neo-Wicksellian model* with inflation persistence due to backward-looking price indexation. As in Woodford (2003), the conclusion can be reached directly from the result for the Calvo pricesetting. Alternatively, the optimal plan implies that inflation evolves according to

$$(\pi_t - \gamma\pi_{t-1}) - \beta\gamma(\pi_t - \gamma\pi_{t-1}) + \varphi_t - \varphi_{t-1} + \beta\gamma\varphi_t - \beta\gamma\varphi_{t+1} = 0 \tag{33}$$

As in the purely-forward looking *basic neo-Wicksellian model*, the increase in output in any period resulting from higher inflation in the same period, φ_t, is offset by the cost of the reduction in output in the previous period as a result of expected higher inflation, φ_{t-1}. Moreover, the additional increase in output in any period resulting from inflation in the same period, $\beta\gamma\varphi_t$, is also offset by the reduction in output in the subsequent period, $\beta\gamma\varphi_{t+1}$. Once again, there is no long-run incentive for positive inflation, the optimal long-run inflation is zero.

In the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour a là Galì-Gertler (2003), the optimal plan implies that inflation evolves according to (16). Substituting for χ_f and χ_b in terms of structural parameters yields

$$\pi_t + \lambda_2(\pi_t - \pi_{t-1}) - \beta\lambda_2(\pi_{t+1} - \pi_t) + \varphi_t - \frac{\alpha}{\phi}\varphi_{t-1} - \frac{\beta\omega}{\phi}\varphi_{t+1} = 0 \tag{34}$$

Higher inflation in any period results in the usual output increase in the same period, φ_t, and reduction in output in both the previous period as a result of expected higher inflation, $(\alpha/\phi)\varphi_{t-1}$, and the subsequent period, $(\beta\omega/\phi)\varphi_{t-1}$. Recalling that $\phi \equiv \alpha + \omega\left[1 - \alpha(1-\beta)\right]$, the absolute value of the overall output cost of higher inflation in any period is given by

$$\frac{\alpha + \beta\omega}{\alpha + \omega\left[1 - \alpha(1-\beta)\right]} \tag{35}$$

Checking the relationship between the stimulative effect of higher inflation on output and the output cost of higher inflation thus amounts to solve the inequality

$$1 \geq \frac{\alpha + \beta\omega}{\alpha + \omega\left[1 - \alpha(1-\beta)\right]} \tag{36}$$

The solution is given by

$$\omega(1-\beta)(1-\alpha) \geq 0 \tag{37}$$

Note that (37) equally applies to the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour a là Steinsson (2003) as the Lagrange multipliers enter the optimal plan first-order condition for inflation in the same way. The stimulative effect of higher inflation is thus generally greater than the output cost of higher inflation. Not surprisingly, the stimulative effect of higher inflation equals the output cost of higher inflation (I) in the absence of inflation persistence (i.e. $\omega = 0$) or (II) in the absence of long-run Phillips curve trade off (i.e. $\beta = 1$). Otherwise, there exists a long-run incentive for positive inflation and the optimal long-run inflation, $\overline{\pi}$, is then found to be a positive function of the steady-state efficiency gap, x^*.

In the nonmicrofounded model, the optimal plan implies that inflation evolves according to (25) (reported here for convenience)

$$\pi_t + \varphi_t - (1-\varepsilon)\varphi_{t-1} - \beta\varepsilon\varphi_{t+1} = 0$$

Higher inflation in any period results in the usual output increase in the same period, φ_t, and reduction in output in both the previous period as a result of expected higher inflation, $(1-\varepsilon)\varphi_{t-1}$, and the subsequent period, $\beta\varepsilon\varphi_{t+1}$. The absolute value of the overall output cost of higher inflation in any period is given by

$$1 - \varepsilon + \beta\varepsilon \tag{38}$$

Checking the relationship between the stimulative effect of higher inflation on output and the output cost of higher inflation thus amounts to solve the inequality

$$1 \geq 1 - \varepsilon + \beta\varepsilon \tag{39}$$

The solution is given by

$$\varepsilon(1-\beta) \geq 0 \tag{40}$$

The stimulative effect of higher inflation is thus generally greater than the output cost of higher inflation. Not surprisingly, the stimulative effect of higher inflation equals the output cost of higher inflation (I) in the absence of inflation persistence (i.e. $\varepsilon = 0$) or (II) in the absence of long-run Phillips curve trade off (i.e. $\beta = 1$). Otherwise, there exists a long-run incentive for positive inflation and the optimal long-run inflation, $\overline{\pi}$, is then found to be a positive function of the steady-state efficiency gap, x^*.

Without loss of generality, we consider the positive optimal long-run inflation that obtains in the nonmicrofounded model

$$\overline{\pi} = \frac{(1-\beta)\varepsilon\lambda_n\kappa_n}{\kappa_n^2 + (1-\varepsilon)(1-\beta)^2\varepsilon\lambda_n} x^* \equiv \left[(1-\beta)\varepsilon\lambda_n\kappa_n^{-1} + (1-\varepsilon)^{-1}(1-\beta)^{-1}\kappa_n\right] x^* \tag{41}$$

Optimal steady-state inflation is a function of (I) the steady-state efficiency gap, x^*, (II) the discount factor, β, (III) the degree of inflation persistence, ε, (IV) the output gap coefficient, κ_n, and (V) the relative weight on output fluctuations, λ_n. Long-run output gap is increasing in long-run inflation.

Proposition 7 *Optimal steady-state inflation is:*
(I) increasing in x^**,* $\partial\overline{\pi}/\partial x^* = (1-\beta)\varepsilon\lambda_n\kappa_n^{-1} + (1-\varepsilon)^{-1}(1-\beta)^{-1}\kappa_n$
(II) increasing in ε*,* $\partial\overline{\pi}/\partial\varepsilon = \left[(1-\beta)\lambda_n\kappa_n^{-1} + (1-\varepsilon)^{-2}(1-\beta)^{-1}\kappa_n\right]x^*$
(III) increasing in λ_n*,* $\partial\overline{\pi}/\partial x^* = (1-\beta)\varepsilon\kappa_n^{-1}$
if and only if $\kappa_n^2 < (1-\beta)^2(1-\varepsilon)\varepsilon\lambda_n$
(IV) decreasing in κ_n*,* $\partial\overline{\pi}/\partial\varepsilon = \left[-(1-\beta)\varepsilon\lambda_n\kappa_n^{-2} + (1-\varepsilon)^{-1}(1-\beta)^{-1}\right]x^*$
(V) decreasing in β*,* $\partial\overline{\pi}/\partial\beta = \left[-\varepsilon\lambda_n\kappa_n^{-1} + (1-\varepsilon)^{-1}(1-\beta)^{-2}\kappa_n\right]x^*$

The discount factor, β, is set equal to 0.99, appropriate for interpreting $t-(t-1) = 1$ *quarter*. The steady-state efficiency gap, x^*, is set equal to 0.2 as in Woodford (2003): 0.2 is the value implied by $x^* = \Phi_y/(\varpi + \sigma^{-1})$, under the assumption that (I) $\theta = 7.88$, (II) $\sigma^{-1} = 0.16$, (III) $\varpi = 0.473$, and (IV) equilibrium distortions are only due to monopolistic competition (i.e. there are no distorting taxes). The degree of inflation persistence, ε, varies between 0.1 and 0.9. The output gap coefficient, κ_n, takes 5 values $(0.01, 0.025, 0.05, 0.075, 0.1)$.The relative weight on output fluctuations, λ_n, takes 5 values $(0.01, 0.05, 0.1, 0.25, 0.5, 1)$.
Table 1 reports the annualized percentage inflation rate[10]: it ranges between 0.0799% and 66.055%. Note that, given κ_n and λ_n, the annualized percentage inflation rate is an arithmetic progression.

4 Discussion and Conclusion

The features that deliver an endogenously optimal positive long-run inflation are (I) inefficient *natural level of output*, (II) central bank's desire to stabilize output around a level that is higher than the inefficient *natural level of output*, (III) long-run Phillips curve trade-off, and (IV) inflation persistence. Inflation persistence (i.e. costly, in terms of output, disinflation) is what brings about a long-run incentive for positive inflation.
The result dissipates doubts about the application of existing New Keynesian models to policy advice and to empirical analysis thus providing a major input to our understanding of how central banks and governments interact in the macroeconomic policy arena, using their own policy instruments. Optimal positive steady-state inflation obtains in both the

[10] $(4 * \overline{\pi}) * 100$

nonmicrofounded model and in the *basic neo-Wicksellian model* with inflation persistence due to backward-looking rule-of-thumb behaviour by price setters. Optimal positive long-run inflation does not obtain in the *basic neo-Wicksellian model* with inflation persistence due to backward-looking price indexation because backward-looking price indexation does not introduce any inflation persistence.

The nonmicrofounded model has by assumption the appealing property that the hybrid Phillips curve coefficients on future expected inflation and lagged inflation add up to one. Indeed, this paper highlights the trickiness of microfounding inflation persistence. Given that the justification for introducing lagged inflation in the NKPC is mainly empirical, the result questions the worthiness of microfounding inflation persistence. Overall, what we prove here is that inflation persistence (i.e. costly, in terms of output, disinflation) brings the short-run in line with the long-run. Given a long-run Phillips curve trade-off, the dichotomy short-run and long-run is at least weakened.

Prescinding from hyperinflation, which is a phenomenon that can arise when the central bank's policy instrument is the nominal quantity of money, (41) is capable of providing a positive theory of inflation. Altissimo, Ehrmann and Smets (2006) combined with (41) is capable of providing a monetary history of the Euro Area. Changes in the degree of inflation persistence can be explained along the line of Frankel and Froot (1990)[11]. Negative inflation is compatible with either a negative output gap coefficient, κ_n, or a negative relative weight on output fluctuations, λ_n[12].

This paper relies on analytics only: positive (negative) inflation is positive (negative) output gap, namely output above (below) the *natural level of output*. In the first half of the twentieth century, the annual inflation rate in developed economies averaged only slightly above zero, in the past 50 years inflation has been the norm. 50 years of inflation (i.e. 50 years of output in excess of the *natural level of output*) can be argued to have substantially contributed towards climate change. With this respect, Ascari and Ropele (2007) show that an ECB-like stability oriented monetary policy (i.e. 2% target inflation rate in the medium term) brings about a substantial percentage loss in welfare with respect to a zero inflation target policy. It suffices for the central bank not to care about output fluctuations to reach zero inflation. Arguably, the most compelling task for central banks is to fix a time horizon (i.e. $t-(t-1)$): optimal monetary policy is such that inflation between t and $t-1$ averages zero.

Everything has been thought before, but the problem is to think of it again.
Goethe

[11] I thank Patrizio Tirelli for bringing this paper to my attention.

[12] Note that also a discount factor greater than one, $\beta > 1$, delivers a negative steady-state inflation.

5 References

Altissimo, F., M. Ehrmann, and F. Smets (2006). Inflation persistence and price-setting behaviour in the Euro Area: a summary of the IPN evidence. *European Central Bank Occasional Paper Series*. No. 46.

Ascari, G. and Ropele T. (2007). Optimal monetary policy under low trend inflation. *Journal of Monetary Economics*, forthcoming.

Amato, J. and T. Laubach (2003). Rule-of-thumb behaviour and monetary policy. *European Economic Review 47*, 791 – 831.

Barro, R. and D. Gordon (1983). A positive theory of monetary policy in a natural-rate model. *Journal of Political Economy 91*, 589 – 610.

Benigno, P. and M. Woodford (2005). Inflation stabilization and welfare: the case of a distorted steady state. *Journal of the European Economics Association 3*, 1185 – 1236.

Calvo, G. (1983). Staggered prices in a utility-maximising framework. *Journal of Monetary Economics 12*, 383 – 398.

Christiano, L., M. Eichenbaum, and C. Evans (2005). Nominal rigidities and the dynamic effects of a shock to monetary policy. *Journal of Political Economy 113*, 1 – 45.

Dixit, A. and Stiglitz J. (1977). Monopolistic competition and optimum product diversity. *American Economic Review 67*, 297 – 308.

Frankel, J. and Froot K. (1990). Chartists, fundamentalists, and trading in the foreign exchange market. *American Economic Review 80*, 181 – 185.

Friedman , M. (1968). The role of monetary policy. *American Economic Review 58*, 1 – 17.

Galì, J. and M. Gertler (1999). Inflation dynamics: a structural econometric analysis. *Journal of Monetary Economics 44*, 195 – 222.

Kidland, F. and E. Prescott (1977). Rules rather than discretion: the inconsistency of optimal plans. *Journal of Political Economy 85*, 473 – 491.

Roberts, J. (1995). New Keynesian economics and the Phillips curve. *Journal of Money, Credit and Banking 27*, 975 – 984.

Schmitt-Grohé, S. and M. Uribe (2005). Optimal inflation stabilization in a medium-scale macroeconomic model. *NBER Working Paper*, No. 11854.

Steinsson, J. (2003). Optimal monetary policy in an economy with inflation persistence. *Journal of Monetary Economics 50*, 1425 – 1456.

Walsh, C. (2003). *Monetary Theory and Policy*. MIT Press.

Woodford, M. (1999). Commentary: how should monetary policy be conducted in an era of price stability? *New Challenges for Monetary Policy*. Federal Reserve Bank of Kansas City.

Woodford, M. (2003). *Interest and Prices: Foundations of a Theory of Monetary Policy.* Princeton University Press.

6 Tables

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\lambda_n = 0.01; \kappa_n = 0.01$	0.0799	0.1597	0.2395	0.3192	0.399	0.4789	0.5588	0.639	0.7194
$\lambda_n = 0.01; \kappa_n = 0.025$	0.032	0.064	0.096	0.128	0.1599	0.1919	0.2239	0.2559	0.288
$\lambda_n = 0.01; \kappa_n = 0.05$	0.016	0.032	0.048	0.064	0.08	0.096	0.112	0.128	0.144
$\lambda_n = 0.01; \kappa_n = 0.075$	0.0107	0.0213	0.032	0.0427	0.0533	0.064	0.0747	0.0853	0.096
$\lambda_n = 0.01; \kappa_n = 0.1$	0.008	0.016	0.024	0.032	0.04	0.048	0.056	0.064	0.072
$\lambda_n = 0.05; \kappa_n = 0.01$	0.3982	0.7937	1.1875	1.581	1.9753	2.3715	2.7709	3.1746	3.5839
$\lambda_n = 0.05; \kappa_n = 0.025$	0.1599	0.3196	0.4792	0.6388	0.7984	0.9582	1.1181	1.2784	1.439
$\lambda_n = 0.05; \kappa_n = 0.05$	0.08	0.1599	0.2399	0.3198	0.3998	0.4798	0.5598	0.6398	0.7199
$\lambda_n = 0.05; \kappa_n = 0.075$	0.0533	0.1067	0.16	0.2133	0.2666	0.3199	0.3733	0.4266	0.48
$\lambda_n = 0.05; \kappa_n = 0.1$	0.04	0.08	0.12	0.16	0.2	0.24	0.28	0.32	0.36
$\lambda_n = 0.1; \kappa_n = 0.01$	0.7929	1.5748	2.3506	3.125	3.9024	4.6875	5.4848	6.2992	7.1358
$\lambda_n = 0.1; \kappa_n = 0.025$	0.3195	0.6384	0.9568	1.2751	1.5936	1.9127	2.2325	2.5535	2.8759
$\lambda_n = 0.1; \kappa_n = 0.05$	0.1599	0.3198	0.4796	0.6394	0.7992	0.9591	1.1191	1.2792	1.4395
$\lambda_n = 0.1; \kappa_n = 0.075$	0.1066	0.2133	0.3199	0.4265	0.5331	0.6397	0.7464	0.8531	0.9598
$\lambda_n = 0.1; \kappa_n = 0.1$	0.08	0.16	0.2399	0.3199	0.3999	0.4799	0.5599	0.6399	0.7199
$\lambda_n = 0.25; \kappa_n = 0.01$	1.956	3.8462	5.7007	7.5472	9.4118	11.3208	13.3017	15.3846	17.6039
$\lambda_n = 0.25; \kappa_n = 0.025$	0.7971	1.5898	2.38	3.1696	3.9604	4.7544	5.5534	6.3593	7.1742
$\lambda_n = 0.25; \kappa_n = 0.05$	0.3996	0.7987	1.1975	1.5962	1.995	2.3943	2.7941	3.1949	3.5968
$\lambda_n = 0.25; \kappa_n = 0.075$	0.2666	0.533	0.7993	1.0655	1.3319	1.5983	1.8649	2.1318	2.399
$\lambda_n = 0.25; \kappa_n = 0.1$	0.2	0.3998	0.5997	0.7995	0.9994	1.1993	1.3993	1.5994	1.7996
$\lambda_n = 0.5; \kappa_n = 0.01$	3.8278	7.4074	10.8597	14.2857	17.7778	21.4286	25.3394	29.6296	34.4498
$\lambda_n = 0.5; \kappa_n = 0.025$	1.5886	3.1596	4.7207	6.2794	7.8431	9.4192	11.0149	12.6382	14.2971
$\lambda_n = 0.5; \kappa_n = 0.05$	0.7986	1.5949	2.39	3.1847	3.9801	4.7771	5.5766	6.3796	7.1871
$\lambda_n = 0.5; \kappa_n = 0.075$	0.5329	1.0652	1.597	2.1288	2.6608	3.1932	3.7264	4.2606	4.7962
$\lambda_n = 0.5; \kappa_n = 0.1$	0.3998	0.7994	1.1987	1.5981	1.9975	2.3971	2.7971	3.1974	3.5984
$\lambda_n = 1; \kappa_n = 0.01$	7.3394	13.7931	19.8347	25.8065	32	38.7097	46.2810	55.1724	66.055
$\lambda_n = 1; \kappa_n = 0.025$	3.1546	6.2402	9.2879	12.3267	15.3846	18.4900	21.6718	24.9610	28.3912
$\lambda_n = 1; \kappa_n = 0.05$	1.5943	3.1797	4.76	6.3391	7.9208	9.5087	11.1067	12.7186	14.3483
$\lambda_n = 1; \kappa_n = 0.075$	1.065	2.1273	3.1881	4.2485	5.3097	6.3728	7.4389	8.5091	9.5847
$\lambda_n = 1; \kappa_n = 0.1$	0.7993	1.5974	2.395	3.1923	3.99	4.7885	5.5883	6.3898	7.1935

Table 1: The Annualized Percentage Inflation Rate

7 Appendix A

The structural parameters are defined in Section 2.

Proof of Proposition 1. We denote with p_t^f and p_t^b the forward-looking reset price and the rule-of-thumb backward-looking reset price respectively. The aggregate price level, P_t, evolves according to

$$P_t = \left\{(1-\alpha)(p_t^*)^{1-\theta} + \alpha P_{t-1}^{1-\theta}\right\}^{\frac{1}{1-\theta}} \tag{42}$$

where $p_t^* = (1-\omega)p_t^f + \omega p_t^b$ denotes the overall reset price at time t. Henceforth, we log-linearise the structural equations around the natural steady-state level of output, $\overline{Y}$. The natural steady-state level of output is the equilibrium level of output that obtains (I) in the absence of nominal rigidity (i.e. sticky prices) and (II) in the absence of exogenous real shocks (i.e. $\widetilde{\xi}_t = 0$ where the vector $\widetilde{\xi}_t$ includes exogenous shocks to technology, to government purchases, to household's impatience to consume, and to the household's willingness to supply labour). If $\widetilde{\xi}_t = 0$ and $Y_t = \overline{Y}$ at all times, (42) has a solution with zero inflation at all times (i.e. $P_t = p_t^* = p_t^f = p_t^b = P_{t-1} = \overline{P}$ at all times). In the case of small enough fluctuations in $\widetilde{\xi}_t$ and Y_t around 0 and $\overline{Y}$ respectively, the solution to the log-linear approximate model is one in which any variable's log-deviation from its natural steady-state value (for instance, $\widehat{P}_t \equiv \log(P_t/\overline{P}\,)$) remains always close to 0[13]. (42) can be log-linearised as

$$\widehat{P}_t = (1-\alpha)\widehat{p}_t^* + \alpha\widehat{P}_{t-1} \tag{43}$$

with

$$\widehat{p}_t^* = (1-\omega)\widehat{p}_t^f + \omega\widehat{p}_t^b \tag{44}$$

Firms allowed to change their price at time t choose p_t^f so to maximise expected future profits subject to the demand they face. The producer's objective is given by

$$E_t \sum_{s=0}^{\infty} \alpha^s R_{t,t+s} \Pi(p_t(i), p_{t+s}^I, P_{t+s}, Y_{t+s}, \widetilde{\xi}_{t+s}) \tag{45}$$

The producer's nominal profit function, Π, is linearly homogeneous in its first three arguments (i.e. good's price, $p_t(i)$, industry's price, p_t^I, aggregate price level) and, for any value of the industry price and the aggregate price level, single-peaked for some positive value of the good's

[13]Henceforth, a variable's log-deviation from its natural steady-state value, which is denoted with a bar, is denoted with a hat.

price[14]. Under the assumption that all firms in a given industry change their prices at the same time, the common forward-looking reset price, p_t^f, is implicitly defined by the relation

$$E_t \sum_{s=0}^{\infty} \alpha^s R_{t,t+s} \Pi_1(p_t^f, p_t^f, P_{t+s}, Y_{t+s}, \widetilde{\xi}_{t+s}) = 0 \tag{46}$$

The first-order condition for optimal pricing (by all the supplier of good i, which belongs to industry I), $\Pi_1(p_t^f, p_t^f, P_{t+s}, Y_{t+s}, \widetilde{\xi}_{t+s}) = 0$, implicitly defines what Woodford (2003, p. 162) labels the *notional **S**hort-**R**un **A**ggregate **S**upply curve*. A log-linearisation to the SRAS is given by

$$\log(p_t^f / P_t) = \zeta x_t \tag{47}$$

where ζ is the elasticity of the notional SRAS curve and x_t is the aggregate output gap (i.e. $x_t \equiv \log(Y_t/Y_t^n) \equiv \widehat{Y}_t - \widehat{Y}_t^n$). Under the assumption of specific labour markets, ζ is given by

$$\zeta = \frac{(\sigma^{-1} + \varpi)}{(1 + \varpi\theta)} > 0 \tag{48}$$

Substituting (47) in (46) yields

$$E_t \sum_{s=0}^{\infty} \alpha^s R_{t,t+s} \left[\widehat{p}_t^f - \widehat{P}_{t+s} - \zeta x_{t+s}\right] = 0 \tag{49}$$

Supposing that profits are discounted using a discount factor that equals on average β gives

$$\widehat{p}_t^f = (1 - \alpha\beta) E_t \sum_{s=0}^{\infty} (\alpha\beta)^s \left[\widehat{P}_{t+s} + \zeta x_{t+s}\right] \tag{50}$$

Quasi-differencing (50) yields

$$\widehat{p}_t^* = (1 - \alpha\beta)\zeta x_t + (1 - \alpha\beta)\widehat{P}_t + \alpha\beta E_t \widehat{p}_{t+1}^* \tag{51}$$

Steinsson (2003) rule-of-thumb backward-looking reset price is given by

$$p_t^b = p_{t-1}^* \frac{P_{t-1}}{P_{t-2}} \left(\frac{Y_{t-1}}{Y_{t-1}^n}\right)^{\delta} \tag{52}$$

Log-linearising yields

$$\widehat{p}_t^b = \widehat{p}_{t-1}^* + \pi_{t-1} + \delta x_{t-1} \tag{53}$$

[14] Given a constant returns to scale production function (i.e. $y_t(i) = A_t h_t(i)$), the nominal profit function is given by

$$\Pi(p_t(i), p_t^I, P_t, Y_t, \widetilde{\xi}_t) \equiv p_t(i) y_t(i) - w_t^I h_t(i) \equiv p_t(i) \left(\frac{p_t(i)}{P_t}\right)^{-\theta} Y_t - \frac{v_h((p_t^I/P_t)^{-\theta} Y_t/A_t); \xi_t)}{u_c(C_t; \xi_t)} P_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta} \frac{Y_t}{A_t}$$

Rewriting (43) in terms of the aggregate inflation rate, $\pi_t \equiv \log(P_t/P_{t-1}) \equiv \widehat{P}_t - \widehat{P}_{t-1}$, gives

$$\pi_t = \frac{1-\alpha}{\alpha}(\widehat{p}_t^* - \widehat{P}_t) \tag{54}$$

Substituting (44) in (54) yields

$$\pi_t = \frac{1-\alpha}{\alpha}\left[(1-\omega)(\widehat{p}_t^f - \widehat{P}_t) + \omega(\widehat{p}_t^b - \widehat{P}_t)\right] \tag{55}$$

Using (43) to substitute for $\widehat{p}_{t-1}^*$ in (53) and subtracting $\widehat{P}_t$ from both sides, $\widehat{p}_t^b - \widehat{P}_t$ is given by

$$\widehat{p}_t^b - \widehat{P}_t = \frac{1}{1-\alpha}\pi_{t-1} - \pi_t + \delta x_{t-1} \tag{56}$$

Rewriting (51) in terms of $\widehat{p}_t^f - \widehat{P}_t$ yields

$$\widehat{p}_t^f - \widehat{P}_t = (1-\alpha\beta)\zeta x_t + \alpha\beta E_t(\widehat{p}_{t+1}^f - \widehat{P}_t) \tag{57}$$

Using (53) at *t+1* to substitute for $\widehat{p}_{t+1}^b$ in (44) and subtracting $\widehat{P}_t$ from both sides, gives

$$\widehat{p}_{t+1}^* - \widehat{P}_t = (1-\omega)(\widehat{p}_{t+1}^f - \widehat{P}_t) + \omega(\widehat{p}_t^* - \widehat{P}_{t-1} + \delta x_{t-1}) \tag{58}$$

(43) implies

$$\widehat{p}_t^* - \widehat{P}_{t-1} = \frac{1}{1-\alpha}\pi_t \tag{59}$$

Substituting (59) in (58), taking the expected value at t and solving for $E_t(\widehat{p}_{t+1}^f - \widehat{P}_t)$ yields

$$E_t(\widehat{p}_{t+1}^f - \widehat{P}_t) = \frac{1}{(1-\alpha)(1-\omega)}E_t(\pi_{t+1} - \omega\pi_t) \tag{60}$$

Substituting (60) in (57) gives

$$\widehat{p}_t^f - \widehat{P}_t = (1-\alpha\beta)\zeta x_t + \frac{\alpha\beta}{(1-\alpha)(1-\omega)}E_t(\pi_{t+1} - \omega\pi_t) - \frac{\alpha\beta\omega\delta}{(1-\omega)}x_t \tag{61}$$

Substituting (56) and (61) in (55) yields

$$\pi_t = \frac{1-\alpha}{\alpha}\left\{\omega\begin{pmatrix} \frac{1}{1-\alpha}\pi_{t-1} \\ -\pi_t \\ +\delta x_{t-1}\end{pmatrix} + (1-\omega)\begin{bmatrix} \left[(1-\alpha\beta)\zeta - \frac{\alpha\beta\omega\delta}{(1-\omega)}\right]x_t \\ -\frac{\omega\alpha\beta}{(1-\alpha)(1-\omega)}\pi_t \\ +\frac{\alpha\beta}{(1-\alpha)(1-\omega)}E_t\pi_{t+1}\end{bmatrix}\right\} \tag{62}$$

Solving for inflation, π_t

$$\frac{\alpha + \alpha\beta\omega + (1-\alpha)\omega}{\alpha}\pi_t = \left\{\begin{array}{c} \frac{\alpha\beta}{\alpha}E_t\pi_{t+1} + \frac{\omega}{\alpha}\pi_{t-1} + \frac{(1-\alpha)\omega\delta}{\alpha}x_{t-1} \\ +\left[\frac{(1-\alpha)(1-\omega)(1-\alpha\beta)\zeta - (1-\alpha)\alpha\beta\omega\delta}{\alpha}\right]x_t \end{array}\right\} \tag{63}$$

delivers the hybrid Phillips curve for price inflation

$$\pi_t = \chi_f \beta E_t \pi_{t+1} + \chi_b \pi_{t-1} + \kappa_1 x_t + \kappa_2 x_{t-1} \tag{64}$$

where, given (48), the parameters are

$$\left\{ \begin{array}{c} \phi = \alpha + \omega - (1-\beta)\omega\alpha; \chi_f = \frac{\alpha}{\phi}; \chi_b = \frac{\omega}{\phi}; \kappa_2 = \frac{(1-\alpha)\omega\delta}{\phi} \\ \kappa_1 = \frac{(1-\omega)\alpha\kappa - (1-\alpha)\alpha\beta\omega\delta}{\phi}; \kappa = \frac{(1-\alpha)(1-\alpha\beta)(\sigma^{-1}+\varpi)}{(1+\varpi\theta)\alpha} \end{array} \right\} \tag{65}$$

■

In the case of small enough fluctuations in the production of each good, $\widehat{y}_t(i) \equiv \log\left(y_t(i)/\overline{Y}\right)$, around the natural steady-state level of output, $\overline{Y}$, small enough exogenous real shocks, and small enough equilibrium distortions (i.e. a small enough value of Φ_y), the period-utility of the representative household can be approximated to second-order as in Woodford (2003, 2.13, p. 396)

$$U_t = -\frac{\overline{Y}\widetilde{u}_c}{2}\left[(\sigma^{-1}+\varpi)(x_t - x^*)^2 + (1+\varpi\theta)\theta var_i \log p_t(i)\right] + t.i.p + O\left(\left\|\Phi_y, \widetilde{\xi}, \varrho\right\|^3\right) \tag{66}$$

$\widetilde{u}_c$ is the marginal utility of aggregate expenditure evaluated at the natural steady-state level of output (i.e. $\widetilde{u}_c \equiv \widetilde{u}_c(\overline{Y}, 0)$). The steady-state efficiency gap, $x^* \equiv \log\left(\overline{Y}^*/\overline{Y}\right)$, is given by $x^* \equiv \log(\overline{Y}^*/\overline{Y}) = \Phi_y/(\varpi + \sigma^{-1})$. $var_i \log p_t(i)$ is a measure of the degree of price dispersion across industries (i.e. goods). $t.i.p$ collect **t**erms that are **i**ndependent of monetary **p**olicy (i.e. irrelevant to the welfare ranking of alternative equilibria). The third-order residual is function of (I) the parameter that summarizes the distortion in the natural steady-state level of output due to monopolistic competition, Φ_y, (II) the vector that includes all the real exogenous disturbances, $\widetilde{\xi}$, and (III) the parameter vector that indexes aspects of policy (i.e. monetary policy) that determine the long-run equilibrium values of the endogenous variables (i.e. $\overline{\pi}$ and $\overline{x}$) in the absence of real exogenous disturbances, ϱ.

Proof of Proposition 2. Under Calvo (1983) staggered price setting and Steinsson (2003) backward-looking rule-of-thumb behaviour, the distribution of prices at any period, $\{p_t(i)\}$, consists of (I) α times the distribution of prices in the previous period, $\{p_{t-1}(i)\}$, (II) an atom of size $(1-\alpha)(1-\omega)$ at the forward-looking reset price, p_t^f, and (III) an atom of size $(1-\alpha)\omega$ at the rule-of-thumb backward-looking reset price, p_t^b

$$\{p_t(i)\} = \alpha\{p_{t-1}(i)\} + (1-\alpha)(1-\omega)p_t^f + (1-\alpha)\omega p_t^b \tag{67}$$

Let (I) $\Delta_t \equiv var_i \log p_t(i)$ denote the degree of price dispersion and (II) $\overline{P}_t \equiv E_i\{\log p_t(i)\}$. Hence, $\overline{P}_t - \overline{P}_{t-1} = E_i\left[\log\{p_t(i)\} - \overline{P}_{t-1}\right]$. Recalling $\log p_t^* = (1-\omega)\log p_t^f + \omega \log p_t^b$ and using (67), $\overline{P}_t - \overline{P}_{t-1}$ can be rewritten as

$$\begin{aligned}
\overline{P}_t - \overline{P}_{t-1} &= E_i\left[\log\{p_t(i)\} - \overline{P}_{t-1}\right] \\
&= E_i\left[\alpha\{\log p_{t-1}(i)\} + (1-\alpha)(1-\omega)\log p_t^f + (1-\alpha)\omega\log p_t^b - \overline{P}_{t-1}\right] \\
&= E_i\left[\begin{array}{c}\alpha\left[\{\log p_{t-1}(i)\} - \overline{P}_{t-1}\right] + (1-\alpha)(1-\omega)(\log p_t^f - \overline{P}_{t-1}) \\ +(1-\alpha)\omega(\log p_t^b - \overline{P}_{t-1})\end{array}\right] \\
&= \left[\begin{array}{c}\overbrace{\alpha E_i\left[\{\log p_{t-1}(i)\} - \overline{P}_{t-1}\right]}^{0} + (1-\alpha)(1-\omega)(\log p_t^f - \overline{P}_{t-1}) \\ +(1-\alpha)\omega(\log p_t^b - \overline{P}_{t-1})\end{array}\right] \\
&= (1-\alpha)(\log p_t^* - \overline{P}_{t-1}) \qquad (68)
\end{aligned}$$

Similarly, Δ_t can be rewritten as

$$\begin{aligned}
\Delta_t &= var_i\left[\log\{p_t(i)\} - \overline{P}_{t-1}\right] \\
&= E_i\left\{\left[\log\{p_t(i)\} - \overline{P}_{t-1}\right]^2\right\} - \left[E_i\log\{p_t(i)\} - \overline{P}_{t-1}\right]^2 \\
&= \left[\begin{array}{c}\alpha E_i\left\{\left[\log\{p_{t-1}(i)\} - \overline{P}_{t-1}\right]^2\right\} + (1-\alpha)(1-\omega)(\log p_t^f - \overline{P}_{t-1})^2 \\ +(1-\alpha)\omega(\log p_t^b - \overline{P}_{t-1})^2 - (\overline{P}_t - \overline{P}_{t-1})^2\end{array}\right] \qquad (69)
\end{aligned}$$

$\overline{P}_t$ is related to the **C**onstant **E**lasticity of **S**ubstitution Dixit-Stiglitz (1967) price index through the log-linear approximation

$$\overline{P}_t = \log P_t + O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^2\right) \qquad (70)$$

the second-order residual follows from the fact that the equilibrium inflation process (as the equilibrium output process) satisfies a bound of second order $O(\left\|\widetilde{\xi}, \varrho\right\|^2)$ together with a second-order bound on the initial (i.e. date -1, policy is implemented at date 0) degree of price dispersion, Δ_{-1}. Note that, as in Woodford (2003), Δ_{-1} is assumed to be of second order (that is why it enters the second-order residual in (70) to the power of 1/2). It then follows that this measure of price dispersion continues to be only of second order in the case of first-order deviations of inflation from zero. Recalling $\log p_t^b = \log p_{t-1}^* + \pi_{t-1} + \delta x_{t-1}$ and using (70), $\log p_t^b - \overline{P}_{t-1}$ is given by

$$\begin{aligned}
\log p_t^b - \overline{P}_{t-1} &= \log p_{t-1}^* + \pi_{t-1} + \delta x_{t-1} - \overline{P}_{t-1} \\
&= \log p_{t-1}^* - \overline{P}_{t-2} - (\overline{P}_{t-1} - \overline{P}_{t-2}) + \pi_{t-1} + \delta x_{t-1} \\
&= \log p_{t-1}^* - \overline{P}_{t-2} - \pi_{t-1} + \pi_{t-1} + \delta x_{t-1} + O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^2\right) \\
&= \log p_{t-1}^* - \overline{P}_{t-2} + \delta x_{t-1} + O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^2\right) \qquad (71)
\end{aligned}$$

Recalling $\log p_t^* = (1-\omega)\log p_t^f + \omega \log p_t^b$, $\log p_t^b = \log p_{t-1}^* + \pi_{t-1}$, and using (70), $\log p_t^f - \overline{P}_{t-1}$ is given by

$$\begin{aligned}
\log p_t^f - \overline{P}_{t-1} &= \frac{1}{1-\omega}\log p_t^* - \frac{\omega}{1-\omega}(\log p_{t-1}^* + \pi_{t-1} + \delta x_{t-1}) - \overline{P}_{t-1} \\
&= \frac{1}{1-\omega}(\log p_t^* - \overline{P}_{t-1}) - \frac{\omega}{1-\omega}(\log p_{t-1}^* + \pi_{t-1} - \overline{P}_{t-1} + \delta x_{t-1}) \\
&= \left[\begin{array}{c} \frac{1}{1-\omega}(\log p_t^* - \overline{P}_{t-1}) - \frac{\omega}{1-\omega}(\log p_{t-1}^* - \overline{P}_{t-2}) \\ -\frac{\omega\delta}{1-\omega}x_{t-1} + O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^2\right) \end{array}\right] \qquad (72)
\end{aligned}$$

Using (70), (68) becomes

$$\pi_t = (1-\alpha)(\log p_t^* - \overline{P}_{t-1}) + O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^2\right) \qquad (73)$$

Accordingly, (71) and (72) become respectively

$$\log p_t^b - \overline{P}_{t-1} = \frac{1}{1-\alpha}\pi_{t-1} + \delta x_{t-1} + O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^2\right) \qquad (74)$$

$$\log p_t^f - \overline{P}_{t-1} = \left[\begin{array}{c} \frac{1}{(1-\omega)(1-\alpha)}\pi_t - \frac{\omega}{(1-\omega)(1-\alpha)}\pi_{t-1} \\ -\frac{\omega\delta}{(1-\omega)}x_{t-1} + O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^2\right) \end{array}\right] \qquad (75)$$

Substituting (70), (74), and (75) in (69) yields

$$\begin{aligned}
\Delta_t &= \left[\begin{array}{c} \alpha E_i\left\{\left[\log\{p_{t-1}(i)\} - \overline{P}_{t-1}\right]^2\right\} + (1-\alpha)\omega(\frac{1}{1-\alpha}\pi_{t-1} + \delta x_{t-1})^2 - \pi_t^2 \\ +(1-\alpha)(1-\omega)\left[\frac{1}{(1-\omega)(1-\alpha)}\pi_t - \frac{\omega}{(1-\omega)(1-\alpha)}\pi_{t-1} - \frac{\omega\delta}{(1-\omega)}x_{t-1}\right]^2 \\ +O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^3\right) \end{array}\right] \\
&= \left[\begin{array}{c} \alpha E_i\left\{\left[\log\{p_{t-1}(i)\} - \overline{P}_{t-1}\right]^2\right\} + \frac{\omega}{1-\alpha}\pi_{t-1}^2 + (1-\alpha)\omega\delta^2 x_{t-1}^2 + 2\omega\delta\pi_{t-1}x_{t-1} \\ -\pi_t^2 + \frac{1}{(1-\omega)(1-\alpha)}\pi_t^2 + \frac{\omega^2}{(1-\omega)(1-\alpha)}\pi_{t-1}^2 + \frac{(1-\alpha)\omega^2\delta^2}{(1-\omega)}x_{t-1}^2 \\ -\frac{2\omega}{(1-\omega)(1-\alpha)}\pi_t\pi_{t-1} - \frac{2\omega\delta}{(1-\omega)}\pi_t x_{t-1} + \frac{2\omega^2\delta}{(1-\omega)}\pi_{t-1}x_{t-1} + O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^3\right) \end{array}\right] \\
&= \left[\begin{array}{c} \frac{\omega+(1-\omega)\alpha}{(1-\omega)(1-\alpha)}\pi_t^2 + \frac{\omega}{(1-\omega)(1-\alpha)}\pi_{t-1}^2 - \frac{2\omega}{(1-\omega)(1-\alpha)}\pi_t\pi_{t-1} + \frac{(1-\alpha)\omega\delta^2}{(1-\omega)}x_{t-1}^2 - \frac{2\omega\delta}{(1-\omega)}\pi_t x_{t-1} \\ +\frac{2\omega\delta}{(1-\omega)}\pi_{t-1}x_{t-1} + \alpha E_i\left\{\left[\log\{p_{t-1}(i)\} - \overline{P}_{t-1}\right]^2\right\} + O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^3\right) \end{array}\right] \\
&= \left[\begin{array}{c} \frac{\alpha}{(1-\alpha)}\pi_t^2 + \frac{\omega}{(1-\omega)(1-\alpha)}(\pi_t^2 + \pi_{t-1}^2 - 2\pi_t\pi_{t-1}) + \frac{(1-\alpha)\omega\delta^2}{(1-\omega)}x_{t-1}^2 \\ -\frac{2\omega\delta}{(1-\omega)}(\pi_t x_{t-1} - \pi_{t-1}x_{t-1}) + \overbrace{\alpha E_i\left\{\left[\log\{p_{t-1}(i)\} - \overline{P}_{t-1}\right]^2\right\}}^{\Delta_{t-1}} + O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^3\right) \end{array}\right] \\
&= \alpha\Delta_{t-1} + \frac{\alpha}{(1-\alpha)}\pi_t^2 + \frac{\omega}{(1-\omega)(1-\alpha)}\left[\pi_t - \pi_{t-1} - (1-\alpha)\delta x_{t-1}\right]^2 + O\left(\left\|\Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho\right\|^3\right) \qquad (76)
\end{aligned}$$

Integrating forward (76), starting from any small initial degree of price dispersion, Δ_{-1}, the degree of price dispersion in any period $t \geq 0$ is given by

$$\Delta_t = \left\{ \begin{array}{c} \sum_{s=0}^{\infty} \alpha^{t-s} \left[\frac{\alpha}{(1-\alpha)} \pi_t^2 + \frac{\omega}{(1-\omega)(1-\alpha)} \left[\pi_t - \pi_{t-1} - (1-\alpha)\delta x_{t-1} \right]^2 \right] \\ \alpha^{t-1}\Delta_{-1} + O\left(\left\| \Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho \right\|^3 \right) \end{array} \right\} \tag{77}$$

The term $\alpha^{t-1}\Delta_{-1}$ is **i**ndependent of monetary **p**olicy. Taking the discounted value of (77) over all periods $t \geq 0$ gives

$$\sum_{t=0}^{\infty} \beta^t \Delta_t = \left\{ \begin{array}{c} \frac{1}{1-\alpha\beta} \sum_{t=0}^{\infty} \beta^t \left[\frac{\alpha}{(1-\alpha)} \pi_t^2 + \frac{\omega}{(1-\omega)(1-\alpha)} \left[\pi_t - \pi_{t-1} - (1-\alpha)\delta x_{t-1} \right]^2 \right] \\ +t.i.p + O\left(\left\| \Delta_{-1}^{1/2}, \widetilde{\xi}, \varrho \right\|^3 \right) \end{array} \right\} \tag{78}$$

Taking the discounted value of (66) over all periods $t \geq 0$ yields

$$\sum_{t=0}^{\infty} \beta^t U_t = \left\{ \begin{array}{c} -\frac{\overline{Y}\widetilde{u}_c}{2} \left[(\sigma^{-1} + \varpi) \sum_{t=0}^{\infty} \beta^t (x_t - x^*)^2 + (1 + \varpi\theta)\theta \sum_{t=0}^{\infty} \beta^t \Delta_t \right] \\ +t.i.p + O\left(\left\| \Phi_y, \widetilde{\xi}, \varrho \right\|^3 \right) \end{array} \right\} \tag{79}$$

Substituting (78) in (79) and normalizing on inflation, the discounted sum of utility of the representative household can be approximated to second-order by

$$\sum_{t=0}^{\infty} \beta^t U_t = \left\{ \begin{array}{c} -\frac{\overline{Y}\widetilde{u}_c(\sigma^{-1}+\varpi)\theta}{2\kappa} \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} \pi_t^2 + \frac{\kappa}{\theta}(x_t - x^*)^2 \\ + \frac{\omega}{(1-\omega)\alpha} \left[\pi_t - \pi_{t-1} - (1-\alpha)\delta x_{t-1} \right] \end{array} \right]^2 \\ +t.i.p + O\left(\left\| \Phi_y, \widetilde{\xi}, \varrho, \Delta_{-1}^{1/2} \right\|^3 \right) \end{array} \right\} \tag{80}$$

where κ is defined as in (65). ■

8 Appendix B

We pick on the "choice of variables" issue (Woodford (2003), p. 388). The scenario is the one of small equilibrium distortions, namely

$$U_c(\overline{Y}, 0) = O\left(\|\Phi_y\| \right) \tag{81}$$

What we show here is that, when (81) holds, a linear approximation to the production function is indeed accurate for the purpose of policy analysis. Considering a first-order and not a second order approximation to the production function does not alter the approximate welfare measure (still given by (66)) but proves that in the presence of a not too inefficient natural steady-state level of output a linear approximation, when substituted into a second-order approximation to expected utility, yields a correct approximate welfare measure.

The period utility of the representative household, as a function solely of all $y_t(i)$, is given by Woodford (2003, 2.3, p. 392)[15]

$$U_t = \widetilde{u}(Y_t; \widetilde{\xi}_t) - \int_0^1 \widetilde{v}(y_t(i); \widetilde{\xi}_t) di \tag{82}$$

The first term in (82) can be approximated to second order (i.e. second-order Taylor expansion taken around $(\overline{Y}, 0)$) by

$$\widetilde{u}(Y_t; \widetilde{\xi}_t) = \overline{u} + \widetilde{u}_c \widetilde{Y}_t + \widetilde{u}_\xi \widetilde{\xi}_t + \frac{1}{2} \widetilde{u}_{cc} \widetilde{Y}_t^2 + \widetilde{u}_{c\xi} \widetilde{Y}_t \widetilde{\xi}_t + \frac{1}{2} \widetilde{\xi}_t' \widetilde{u}_{\xi\xi} \widetilde{\xi}_t + O\left(\left\| \Phi_y, \widetilde{\xi}, \varrho \right\|^3\right) \tag{83}$$

Substituting $\widetilde{Y}_t = \overline{Y} \widehat{Y}_t$ and dropping the terms that are higher than second order yields

$$\widetilde{u}(Y_t; \widetilde{\xi}_t) = \overline{u} + \overline{Y} \widetilde{u}_c \widehat{Y}_t + \widetilde{u}_\xi \widetilde{\xi}_t + \frac{1}{2} \overline{Y}^2 \widetilde{u}_{cc} \widehat{Y}_t^2 + \overline{Y} \widetilde{u}_{c\xi} \widetilde{\xi}_t \widehat{Y}_t + \frac{1}{2} \widetilde{\xi}_t' \widetilde{u}_{\xi\xi} \widetilde{\xi}_t + O\left(\left\| \Phi_y, \widetilde{\xi}, \varrho \right\|^3\right) \tag{84}$$

Taking all the steps as in Woodford (2003, Appendix E.1), yields

$$\widetilde{u}(Y_t; \widetilde{\xi}_t) = \overline{Y} \widetilde{u}_c \left\{ \widehat{Y}_t - \frac{1}{2} \sigma^{-1} \widehat{Y}_t^2 + \sigma^{-1} g_t \widehat{Y}_t \right\} + t.i.p + O\left(\left\| \Phi_y, \widetilde{\xi}, \varrho \right\|^3\right) \tag{85}$$

Using $\widetilde{Y}_t = \overline{Y} \widehat{Y}_t$, the second term in (82) can be approximated to second order by

$$\begin{aligned} \widetilde{v}(y_t(i); \widetilde{\xi}_t) \quad = \quad & \overline{v} + \overline{Y} \widetilde{u}_c (1 - \Phi_y) \widehat{y}_t(i) + \widetilde{v}_\xi \widetilde{\xi}_t + \frac{1}{2} \overline{Y}^2 \widetilde{v}_{yy} \widehat{y}_t(i)^2 + \overline{Y} \widetilde{v}_{y\xi} \widetilde{\xi}_t \widehat{y}_t(i) \\ & + \frac{1}{2} \widetilde{\xi}_t' \widetilde{u}_{\xi\xi} \widetilde{\xi}_t + O\left(\left\| \Phi_y, \widetilde{\xi}, \varrho \right\|^3\right) \end{aligned} \tag{86}$$

(86) delivers

$$\begin{aligned} \int_0^1 \widetilde{v}(y_t(i); \widetilde{\xi}_t) di \quad = \quad & \overline{Y} \widetilde{u}_c \left\{ (1 - \Phi_y) \widehat{Y}_t + \frac{1}{2} \varpi \widehat{Y}_t^2 - \varpi q_t \widehat{Y}_t + \frac{1}{2} (\theta^{-1} + \varpi) var_i \widehat{y}_t(i) \right\} \\ & + t.i.p + O\left(\left\| \Phi_y, \widetilde{\xi}, \varrho \right\|^3\right) \end{aligned} \tag{87}$$

Combining (85) and (87) yields

[15] We do not digress, see Woodford (2003) for more details. Note that in what follows we maintain Woodford's notation. ξ subscript denotes partial derivatives of $\widetilde{v}$ with respect to all exogenous disturbances in vector $\widetilde{\xi}_t$. Similarly, c subscript denotes partial derivatives of $\widetilde{u}$ with respect to the aggregate level of production, Y_t (i.e. $\partial \widetilde{u} / \partial Y_t = \partial u / \partial C_t$). All partial derivatives are evaluated at $(\overline{Y}, 0)$.

$$U_t = \overline{Y}\widetilde{u}_c \left\{ \Phi_y \widehat{Y}_t - \frac{1}{2}(\sigma^{-1} + \varpi)\widehat{Y}_t^2 + (\varpi q_t + \sigma^{-1} g_t)\widehat{Y}_t - \frac{1}{2}(\theta^{-1} + \varpi) var_i \widehat{y}_t(i) \right\} \quad (88)$$
$$+ t.i.p + O\left(\left\| \Phi_y, \widetilde{\xi}, \varrho \right\|^3 \right)$$

which then delivers (66).