Duclos-Jalbert-Araar decomposition of redistributive effect: implementation issues

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Abstract

The models decomposing the redistributive effect of fiscal systems into vertical and horizontal effects are extensively used by practitioners. The Duclos, Jalbert and Araar's (2003) model, despite its advantages, has not yet been extensively employed in empirical research, possibly due to certain difficulties emerging in its implementation. This paper addresses some of these problems and offers advice on how to solve them. Unfolding the estimation and calculation procedures it helps practitioners to properly apply the model. The procedures are first illustrated on small hypothetical population and then employed on the real data scenario for Croatian fiscal system. Connections with Kakwani's (1984) decomposition and the issue of vertical effect as a measure of potential redistributive effect are also thoroughly discussed.

Keywords: redistributive effect, vertical equity, horizontal inequity, pre-fiscal equals

JEL: D63, H22, H23

1. Introduction

Duclos, Jalbert and Araar (2003) (DJA) have designed a comprehensive model for measurement of vertical, classical horizontal inequity (CHI) and reranking effects of fiscal system. It is built into the framework of the Atkinson-Gini social welfare function (AGF), which first converts incomes into utilities employing Atkinson (1970) utility function, and then aggregates them using rank-dependent weights, which underlie the S-Gini coefficients proposed by Donaldson and Weymark (1980) and Yitzhaki (1983).[1]

The AGF framework gives the DJA model certain advantages over its competitors, the widely acknowledged Kakwani's (1984) (K84) and Aronson, Johnson and Lambert's (1994) (AJL) decompositions of RE, which are set up in the Gini environment. Namely, by incorporating the utility function, the DJA model enables a natural appraisal of CHI effect. On the other hand, to measure CHI effect the AJL model relies on somewhat arbitrary procedure of "groups of close equals" formation, while K84 does not even contain the separate CHI term.

Despite its great measurement potential, the DJA model has not yet become more widely employed by practitioners.[2] The model requires the estimation of expected post-fiscal incomes (EPI) and expected post-fiscal utilities (EPU) at different points of the pre-fiscal income distribution (PRFID), which calls for certain statistical expertise related to data smoothing and curve fitting methods. Inaccurate estimates of EPI and EPU result in unreliable values of indicators in the DJA model.

This paper carefully explains the procedures of data manipulations, calculations, and estimations needed to obtain the indices of the DJA model. Several hints are suggested that make the work of practitioners easier and their results more confident. The relationship between the DJA and other models is described. Two important issues emerged during the empirical research: the presence of large number of exact pre-fiscal equals and the non-increasing EPI are observed for the data employed. In that respect certain adjustments are made concerning calculation procedures and interpretation of results.

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[1] Araar and Duclos (2003, 2006) describe the properties of AGF based inequality indices: "Income inequality aversion is captured by decreasing marginal utilities, and aversion to rank inequality is captured by rank-dependent ethical weights, thus providing an ethically-flexible dual basis for the assessment of inequality and equity." Furthermore, it is shown that AGF is the only family of social evaluation functions "to obey a set of popular axioms in the income distribution literature" (the principle of transfers, the principle of population, scale invariance, etc.).

[2] One application can be found in Bilger (2008), who uses the DJA model to analyse redistributive features of the health system financing in Switzerland.

As a result, the paper can serve as a manual for practitioners applying the DJA methodology, accompanying the original Duclos, Jalbert and Araar's (2003) work. Furthermore, because of the relations of the DJA model with other models, such as K84, it can be useful to all practitioners employing standard methodologies in the field of income redistribution.

Section 2 briefly exposes the elements of the DJA model and its connections with the K84 decomposition. Section 3 extensively describes the procedures of data preparation, estimation and calculation of various elements of the DJA model. The methodology is first applied to small hypothetical population, and then, in section 4, to Croatian system of personal taxes and social benefits in 2008 and the results are thoroughly discussed. Section 5 concludes.

2. The DJA model

The change of income inequality induced by fiscal system is called the redistributive effect (RE). In measurement terms, we have that $\Delta = I(X) - I(N)$, where Δ represents RE, while $I(X)$ and $I(N)$ are indices of pre- and post-fiscal income inequality.[3]

In the DJA model, inequality indices $I(\cdot)$ are derived using Atkinson-Gini social welfare function, proposed by Araar and Duclos (2003, 2006):

$$W(X,\varepsilon,\nu) = \int_0^1 U(X(p),\varepsilon)\omega(p,\nu)dp \tag{1}$$

where ε is ethical parameter configuring the Atkinson's (1970) utility function, $U(X(p),\varepsilon) = (X(p))^{1-\varepsilon}/(1-\varepsilon)$ for $\varepsilon \neq 1$, with p denoting the quantiles of pre-fiscal income distribution, and $X(p)$ the income at p.[4] The term ν is another ethical parameter, characterizing the Donaldson and Weymark's (1980) and Yitzhaki's (1983) S-Gini rank-dependent weighting scheme, $\omega(p,\nu) = \nu(1-p)^{\nu-1}$. The equally distributed equivalent income (EDEI) is an inverse function of $W(\cdot)$ and is obtained as $\xi(X,\varepsilon,\nu) = ((1-\varepsilon)W(X,\varepsilon,\nu))^{1/(1-\varepsilon)}$ for $\varepsilon \neq 1$. Finally, the Atkinson-Gini inequality index is calculated as follows:

$$I(X) = 1 - \xi(X,\varepsilon,\nu)/\mu^X \tag{2}$$

where μ^X is the mean pre-fiscal income. $I(N)$ is obtained analogously, using the quantiles of post-fiscal income distribution.

[3] Generally, pre-fiscal (post-fiscal) income is income before (after) taxes and benefits.

[4] To simplify the presentation, the formulas referring to the case when $\varepsilon = 1$ are omitted.

The DJA model decomposes RE as follows:

$$\Delta = V - C - R = [I(X) - I(N^E)] - [I(U^P) - I(N^E)] - [I(N) - I(U^P)] \tag{3}$$

The vertical effect, $V = I(X) - I(N^E)$, represents the potential RE or the reduction of inequality that would be achieved by the counterfactual, horizontally equitable system.[5] The discrepancy between potential and actual RE is divided into CHI effect, $C = I(U^P) - I(N^E)$, and reranking effect, $R = I(N) - I(U^P)$, which measure two different manifestations of horizontal inequity (HI). The former effect (C) measures HI emerging from violation of the 'classical horizontal equity principle', which says that equals should be treated equally. The latter effect (R) evaluates HI arising from the infringement of the 'no-reranking principle', requiring that fiscal process does not change ranks of income units in transition from pre- to post-fiscal income.[6]

In equation (3), $N^E(p)$ represents expected post-fiscal incomes, obtained as $N^E(p) = \int_0^1 N(q \mid p)dq$, where $N(q \mid p)$ denotes a post-fiscal income at the *q*th quantile among all those income units belonging to the *p*th quantile of pre-fiscal income distribution. $U^P(p,\varepsilon)$ is the expected post-fiscal utility at the *p*th quantile of PRFID, obtained as $U^P(p,\varepsilon) = \int_0^1 U(N(q \mid p),\varepsilon)dq$. For $N^E(p)$ and $U^P(p,\varepsilon)$ we obtain respective social welfare functions $W(N^E,\varepsilon,\nu) = \int_0^1 U(N^E(p),\varepsilon)\omega(p,\nu)dp$ and $W(U^P,\varepsilon,\nu) = \int_0^1 U^P(p,\varepsilon)\omega(p,\nu)dp$, while corresponding inequality indices are $I(N^E,\varepsilon,\nu) = 1 - \xi(N^E,\varepsilon,\nu)/\mu^N$ and $I(U^P,\varepsilon,\nu) = = 1 - \xi(U^P,\varepsilon,\nu)/\mu^N$, where μ^N is the mean post-fiscal income.

When $\varepsilon = 0$, utilities are identical to incomes: $U(y,0) = y$. Therefore, we have that $U(N(q \mid p),0) = N(q \mid p)$ across all p and $N(q \mid p)$, and it follows that $W(N^E,0,\nu) = W(U^P,0,\nu)$ and $I(N^E,0,\nu) = I(U^P,0,\nu)$. The consequence for the DJA model is that CHI effect collapses to zero, and the decomposition (3) can be rewritten as:

$$\Delta(0,\nu) = V(0,\nu) - R(0,\nu) = (I(X,0,\nu) - I(N^E,0,\nu)) - (I(N,0,\nu) - I(N^E,0,\nu)) \tag{4}$$

[5] For more detailed interpretation of different effects, see discussion in section 4.4 below.

[6] For example, families A and B have pre-fiscal incomes of 10$, while C and D have 20$. Suppose that A, B, C and D end up with post-fiscal incomes of 8, 16, 12, 24, respectively. Among pre-fiscal equals (A and B; C and D) CHI has occurred, while between pre-fiscal unequals (B and C) reranking has taken place.

It can be shown that $I(X,0,\nu)$, $I(N,0,\nu)$ and $I(N^E,0,\nu)$ are the S-Gini coefficient of pre-fiscal income, $G(X,\nu)$, the S-Gini coefficient of post-fiscal income, $G(N,\nu)$, and the S-Gini concentration coefficient of post-fiscal income, $D(N,\nu)$, respectively.[7]

Consequently, $V(0,\nu)$ is equal to the S-Gini Kakwani's (1984) index of vertical effect, $V^K(\nu)=G(X,\nu)-D(N,\nu)$, and $R(0,\nu)$ is the S-Gini Atkinson (1980) and Plotnick (1981) index of reranking, $R^{AP}(\nu)=G(N,\nu)-D(N,\nu)$.[8] The Kakwani's (1984) decomposition of RE into vertical and horizontal components can be rewritten in the S-Gini terms as:

$$\Delta(\nu)=V^K(\nu)-R^{AP}(0,\nu)=(G(X,\nu)-D(N,\nu))-(G(N,\nu)-D(N,\nu)) \tag{5}$$

On the other hand, when $\nu=1$, the weights $\omega(p,\nu)$ are all equal and reranking disappears: $R=I(N)-I(U^P)=0$. For $\varepsilon>0$, the vertical and CHI effect, $V(\varepsilon,1)$ and $C(\varepsilon,1)$, become the indices consistent with the Duclos and Lambert's (2001) model of HI measurement.[9]

3. Calculation of indices

3.1. Basic data preparation

A typical research uses the following data for a household or family k: (a) unequivalized pre- and post-fiscal incomes, $\dot{X}_k$ and $\dot{N}_k$, (b) survey frequency (or sampling) weights, f_k, and (c)

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[7] Independent proof of this relationship can be found in Yitzhaki and Olkin (1991) who derive the 'relative concentration curve' of post-fiscal income N with respect to pre-fiscal income X as $C^{N,x}(p)=(\mu^N)^{-1}\int_{-\infty}^{F_X^{-1}(p)} m(t)dF_X(t)$, where $m(x)=E\{N \mid X=x\}$ corresponds to $N^E(p)$. Duclos and Araar (2006) present the same concentration curve as $C^{N,x}(p)=(\mu^N)^{-1}\int_0^p N^E(r)dr$, from which the S-Gini concentration coefficient is obtained as $D(N,\nu)=\int_0^1 (p-C^{N,x}(p))\varpi(p,\nu)dp$, where $\varpi(p,\nu)=\nu(\nu-1)(1-p)^{\nu-2}$ are rank-dependent weights.

[8] Originally, all these indices are defined for $\nu=2$, where $I(X,0,2)$ is the standard Gini coefficient.

[9] These authors have derived their indices using the „cost of inequality" approach, compared to the "change of inequality" approach, used in this paper. Duclos, Jalbert and Araar (2003) employ both methods to derive the DJA model.

equivalence factor β_k.[10] Equivalized pre- and post-fiscal incomes are $X_k = \dot{X}_k / \beta_k$ and $N_k = \dot{N}_k / \beta_k$. The frequency weights are defined as $\phi_k = f_k \beta_k$.[11]

To obtain M_i^x, the matrix $M_k^0 = \{X_k, N_k, \phi_k\}$ is sorted lexicographically – first by increasing order of pre-fiscal income and then, within each group of pre-fiscal equals, in increasing order of post-fiscal income. Sorting M_k^0 by increasing order of post-fiscal income, the matrix M_i^n, is obtained. Now, from M_k^x and M_k^n we extract the relevant columns – income vectors X_i^x, N_i^x, N_i^n and the frequency weights ϕ_i^x and ϕ_i^n.

3.2. The estimates of quantiles

We turn to the estimation of the quantiles $\hat{p}_i^x$ and the weights $\hat{\varpi}_i^{x,\nu}$ needed for computation of social welfare in (1). For the sample with s units, they can be obtained in the following way:

$$\hat{p}_i^x = (2S)^{-1} \sum_{j=1}^{i} (\phi_j^x + \phi_{j-1}^x) \tag{6}$$

$$\hat{\omega}_i^{x,\nu} = (S)^{-1} \nu (1 - \hat{p}_i^x)^{\nu-1} \tag{7}$$

where $S = \sum_{j=1}^{s} \phi_j^x$ and $\phi_0^x = 0$. Analogously, $\hat{p}_k^n$ and $\hat{\omega}_k^{n,\nu}$ are obtained.

Assume a group of q exact pre-fiscal equals, whose pre-fiscal incomes are $X_{b+1}^x =,\ldots,= X_{b+q}^x$, where b is a value between 0 and $s-q$. Since the weights $\hat{\omega}_i^{x,\nu}$ are strictly decreasing in $\hat{p}_i^x$, we have that $\hat{\omega}_b^{x,\nu} > \ldots > \hat{\omega}_{b+q}^{x,\nu}$. However, this contradicts the notion of the indices based on rank-dependent weights: the units with same pre-fiscal income belong to the same quantile, and should have equal weights. Thus, we need the new set of weights, $\hat{\varpi}_r^{x,\nu}$. They are equal to $\hat{\omega}_r^{x,\nu}$ if the unit r is not a member of exact-equals groups. On the other hand, if the unit r does belong to the above mentioned group of exact pre-fiscal equals, they are calculated as:

[10] For an explanation of these items see the concrete example of Croatian data in section 3.2.

[11] It is considered here that a household k has β_k 'equivalent' members instead of some number u_k of 'real' individuals; thus, each equivalized income pair (X_k, N_k) will be counted ϕ_k and not $f_k u_k$ times.

$$\widehat{\omega}_r^{x,\nu} = (\sum\nolimits_{j=1}^{q} \phi_j^x)^{-1} \sum\nolimits_{c=b+1}^{b+q} \phi_c^x \cdot \hat{\omega}_c^{x,\nu} \tag{8}$$

Typically, exact pre-fiscal equals are quite rare in real-world samples. However, in the research on Croatian individual taxes and cash social benefits, one of the scenarios treated public pensions as social benefits and hence they were not part of pre-fiscal income. Therefore, the sample contained a large number of zero pre-fiscal income equals. Below we will see how the results would be affected if the transformation of $\hat{\omega}_i^{x,\nu}$ into $\widehat{\omega}_i^{x,\nu}$ was not made.

3.3. Utilities, social welfare and indices of inequality

The following equations show how to obtain utilities, Gini-Atkinson welfare index and the inequality index for pre-fiscal income X_i^x, when $\varepsilon \neq 1$:

$$U(X_i^x, \varepsilon) = (X_i^x)^{1-\varepsilon} / (1-\varepsilon) \tag{9}$$

$$\widehat{W}(X_i^x, \varepsilon, \nu; \widehat{\omega}_i^{x,\nu}) = \sum\nolimits_{j=1}^{s} U(X_j^x, \varepsilon) \cdot \phi_j^x \cdot \widehat{\omega}_j^{x,\nu} \tag{10}$$

$$\widehat{I}(X_i^x, \varepsilon, \nu; \widehat{\omega}_i^{x,\nu}) = 1 - [(1-\varepsilon)\widehat{W}(X_i^x, \varepsilon, \nu; \widehat{\omega}_i^{x,\nu})]^{\frac{1}{1-\varepsilon}} / \hat{\mu}(X_i^x) \tag{11}$$

where $\hat{\mu}(X_i^x) = (S)^{-1} \sum_i^s \varphi_i^x X_i^x$ is the mean pre-fiscal income. Analogously to (9), (10) and (11), other utilities, welfare and inequality indices are obtained, as shown in the Appendix.

In section 3.1 the adapted set of weights is derived to account for the case of pre-fiscal equals. The inequality indices obtained for N_i^x, as shown by formulas (23) and (25) in the Appendix, will be different: $\hat{I}(N_i^x, \varepsilon, \nu; \hat{\omega}_i^{x,\nu}) > \widehat{I}(N_i^x, \varepsilon, \nu; \widehat{\omega}_i^{x,\nu})$.[12]

[12] To see why, recall how the weights $\hat{\varpi}_i^{x,\nu}$ and $\widehat{\varpi}_i^{x,\nu}$ are constructed; also, remember that incomes N_i^x within each group of exact equals are sorted in ascending order. Simple example will demonstrate the algebraic effect. Three units all have the same pre-fiscal income of 20; their post-fiscal incomes are 10, 20 and 30. The respective weights under the scheme A are 3, 2 and 1, while the weights B are 2, 2, 2. The sum-product of post-fiscal incomes and weights A (weights B) is 100 (<120). Similarly, $\hat{W}(N_i^x, \varepsilon, \nu; \hat{\omega}_i^{x,\nu}) < \widehat{W}(N_i^x, \varepsilon, \nu; \widehat{\omega}_i^{x,\nu})$. On the other hand, the sum-product of *pre-fiscal* incomes and weights A (weights B) is 120 (=120). Hence, $\hat{W}(X_i^x, \varepsilon, \nu; \hat{\omega}_i^{x,\nu}) = \widehat{W}(X_i^x, \varepsilon, \nu; \widehat{\omega}_i^{x,\nu})$.

3.3. Estimation of expected post-fiscal incomes and utilities

$\hat{I}(X_i^x,\varepsilon,\nu;\hat{\omega}_i^{x,\nu})$ and $\hat{I}(N_i^n,\varepsilon,\nu;\hat{\omega}_i^{n,\nu})$ are sample estimates of pre- and post-fiscal income inequality, $I(X)$ and $I(N)$. As we can see from (3), the full application of the DJA model requires the estimates of another two indices, $I(N^E)$ and $I(U^P)$, which are derived from the income and utility variables, $N^E(p)$ and $U^P(p,\varepsilon)$. $N^E(p)$ are expected post-fiscal incomes (EPI), and $U^P(p,\varepsilon)$ are expected post-fiscal utilities (EPU), at each quantile of PRFID, p. Their sample representatives will be denoted as N_i^E and $U_{i,\varepsilon}^P$, respectively.

The counterfactual post-fiscal incomes N_i^E are the estimates of EPI for each value of pre-fiscal income X_i^x. To obtain them, we must smooth a dataset (X_i^x, N_i^x), i.e. approximate the mean response curve m^E in the regression relationship $N_i^x = m^E(X_i^x)+\delta_i$. The basic form of the curve m^E is chosen by the analyst from the great variety of possible choices, such as OLS polynomial regressions, kernel regressions, local polynomial regressions, Gini regressions, Fourier transformations, etc. Let $\tilde{m}^E(X_i^x)$ be the approximation of m^E; then, $N_i^E = \tilde{m}^E(X_i^x)$.

Following the same approach as for N_i^E, for some chosen value of ε, we can estimate the regression relationship $U(N_i^x,\varepsilon) = m^{P,\varepsilon}(X_i^x)+\delta_i$ to obtain the approximation $\tilde{m}^{P,\varepsilon}(X_i^x)$ and another vector of fitted values $U_{i,\varepsilon}^P = \tilde{m}^{P,\varepsilon}(X_i^x)$. It shows expected post-fiscal utilities at different points of the PRFID.

Estimation of N_i^E and $U_{i,\varepsilon}^P$ is relatively difficult task as it involves the use of statistical techniques which are still regarded as non-standard. On the other hand, relatively small inaccuracies can result in large biases in the final indices. Here, we explain two useful hints for users who empirically apply the DJA model.

The following identity says that the sample estimate of $I(U^P)$ can be obtained simply using post-fiscal incomes N_i^x, accompanied by proper weights, $\hat{\omega}_i^{x,\nu}$:

$$\hat{I}(N_i^x,\varepsilon,\nu;\hat{\omega}_i^{x,\nu}) = \hat{I}(U_{i,\varepsilon}^P,\varepsilon,\nu;\hat{\omega}_i^{x,\nu}) \tag{12}$$

Note that each different value of ε chosen by an analyst (for example, in sensitivity analysis) requires new estimation of $U_{i,\varepsilon}^{P}$. However, thanks to the property (12), $U_{i,\varepsilon}^{P}$ does not need to be estimated at all, but instead we can simply use $\widehat{I}(N_i^x,\varepsilon,\nu;\widehat{\omega}_i^{x,\nu})$.

When $\varepsilon=0$, it follows that $U(N_i^x,\varepsilon=0)=N_i^x$, and $U_{i,\varepsilon=0}^{P}=N_i^E$. From (12) we derive the following equality:

$$\widehat{I}(N_i^E,0,\nu;\widehat{\omega}_i^{x,\nu})=\widehat{I}(N_i^x,0,\nu;\widehat{\omega}_i^{x,\nu}) \tag{13}$$

Thus we arrive to the second hint: if the elements of N_i^E are appropriately estimated, the inequality indices $\widehat{I}(N_i^E,0,\nu;\widehat{\omega}_i^{x,\nu})$ for different values of parameter ν should be equal to the inequality indices obtained for $\widehat{I}(N_i^x,0,\nu;\widehat{\omega}_i^{x,\nu})$. If there are significant differences in these indices, based on N_i^E and N_i^x, we should suspect that the present approximation $\tilde{m}^E(X_i^x)$ is poor, and try some different configuration of the function.

3.4. The DJA and K84 decompositions – implementation formulas

Having defined all the needed indices, we can present RE and its decompositions in terms of sample estimates formulas. RE is obtained as $\widehat{\Delta}(\varepsilon,\nu)=\widehat{I}(X_i^x)-\hat{I}(N_i^n)$. According to the DJA model from (3), RE is decomposed as follows:

$$\begin{aligned}
&\widehat{\Delta}(\varepsilon,\nu)=\widehat{V}(\varepsilon,\nu)-\widehat{C}(\varepsilon,\nu)-\widehat{R}(\varepsilon,\nu)=\\
&=[\widehat{I}(X_i^x)-\widehat{I}(N_i^E)]-[\widehat{I}(U_{i,\varepsilon}^P)-\widehat{I}(N_i^E)]-[\widehat{I}(N_i^n)-\widehat{I}(U_{i,\varepsilon}^P)]=\\
&=[\widehat{I}(X_i^x)-\widehat{I}(N_i^E)]-[\widehat{I}(N_i^x)-\widehat{I}(N_i^E)]-[\widehat{I}(N_i^n)-\widehat{I}(N_i^x)]
\end{aligned} \tag{14}$$

where the last row manifests the property (12), by which $\widehat{I}(U_{i,\varepsilon}^P)=\widehat{I}(N_i^x)$. The differences in the brackets – $\widehat{V}(\varepsilon,\nu)=\widehat{I}(X_i^x)-\widehat{I}(N_i^E)$, $\widehat{C}(\varepsilon,\nu)=\widehat{I}(U_{i,\varepsilon}^P)-\widehat{I}(N_i^E)=\widehat{I}(N_i^x)-\widehat{I}(N_i^E)$ and $\widehat{R}(\varepsilon,\nu)=\widehat{I}(N_i^n)-\widehat{I}(U_{i,\varepsilon}^P)=\widehat{I}(N_i^n)-\widehat{I}(N_i^x)$ – are the vertical, CHI and reranking effects of the DJA model.

For analytical reasons, we can divide both $\widehat{R}(\varepsilon,\nu)$ and $\widehat{C}(\varepsilon,\nu)$ into two parts:

$$\begin{aligned}
\widehat{R}(\varepsilon,\nu)&=[\widehat{I}(N_i^n)-\hat{I}(N_i^x)]+[\hat{I}(N_i^x)-\widehat{I}(N_i^x)]\\
\widehat{C}(\varepsilon,\nu)&=[\hat{I}(N_i^x)-\widehat{I}(N_i^E)]-[\hat{I}(N_i^x)-\widehat{I}(N_i^x)]
\end{aligned} \tag{15}$$

The difference $\hat{\psi}(\varepsilon,\nu)=\hat{I}(N_i^x)-\widehat{I}(N_i^x)$ in (15) is positive and shows by how much the true value of reranking effect (CHI effect) is underestimated (overestimated), if $\hat{I}(N_i^x)$ is used instead of $\widehat{I}(N_i^x)$. The terms $\widehat{R}^{\psi}(\varepsilon,\nu)=\hat{I}(N_i^n)-\hat{I}(N_i^x)$ and $\widehat{C}^{\psi}(\varepsilon,\nu)=\hat{I}(N_i^x)-\widehat{I}(N_i^E)$ will be referred to as the 'underestimated reranking term' and 'overestimated CHI term'.

Setting $\varepsilon=0$ and following (5), we can calculate the S-Gini K84 decomposition:

$$
\begin{aligned}
&\widehat{\Delta}(\nu)=\widehat{V}^K(\nu)-\widehat{R}^{AP}(\nu)=\\
&=[\widehat{I}(X_i^x,0,\nu;\widehat{\omega}_i^{x,\nu})-\widehat{I}(N_i^x,0,\nu;\widehat{\omega}_i^{x,\nu})]-[\hat{I}(N_i^n,0,\nu;\hat{\omega}_i^{n,\nu})-\widehat{I}(N_i^x,0,\nu;\widehat{\omega}_i^{x,\nu})]
\end{aligned}
\tag{16}
$$

Analogously to the above procedure for the DJA model, we can decompose $\widehat{V}^K(\nu)$ and $\widehat{R}^{AP}(\nu)$ to show how the unadjusted weights $\hat{\omega}_i^{x,\nu}$ can produce the wrong estimates of these indices:

$$
\begin{aligned}
\widehat{V}^K(\nu)&=[\widehat{I}(X_i^x,0,\nu;\widehat{\omega}_i^{x,\nu})-\hat{I}(N_i^x,0,\nu;\hat{\omega}_i^{x,\nu})]+[\hat{I}(N_i^x,0,\nu;\hat{\omega}_i^{x,\nu})-\widehat{I}(N_i^x,0,\nu;\widehat{\omega}_i^{x,\nu})]\\
\widehat{R}^{AP}(\nu)&=[\hat{I}(N_i^n,0,\nu;\hat{\omega}_i^{n,\nu})-\hat{I}(N_i^x,0,\nu;\hat{\omega}_i^{x,\nu})]+[\hat{I}(N_i^x,0,\nu;\hat{\omega}_i^{x,\nu})-\widehat{I}(N_i^x,0,\nu;\widehat{\omega}_i^{x,\nu})]
\end{aligned}
\tag{17}
$$

Equation (17) shows that the use of unadjusted weights will underestimate both the Kakwani vertical and Atkinson-Plotnick reranking effect by the amount of $\hat{\psi}(0,\nu)=\hat{I}(N_i^x,0,\nu;\hat{\omega}_i^{x,\nu})-\widehat{I}(N_i^x,0,\nu;\widehat{\omega}_i^{x,\nu})$.[13]

3.5. Hypothetical data example

The first example employs a hypothetical population of twelve income units. Table 1 presents most of the vectors needed for computation of different indices. The bottom six units are pre-fiscal exact equals with zero pre-fiscal incomes X_i^x, while the remaining units have different pre-fiscal incomes. The two sets of weights are presented: the original ones, $\hat{\omega}_i^{x,\nu}$, obtained by (7), which assume that all units have different pre-fiscal incomes, and the weights $\widehat{\omega}_i^{x,\nu}$, which are same as the original ones for the units #7 to #12, while for the units #1 to #6, they are equal to $\widehat{\varpi}_i^{x,\nu}=\sum_{c=1}^{6}\widehat{\varpi}_c^{x,\nu}/6$, as the rule (8) requires.

[13] The K84 model and the relevant indices are usually calculated using formulas different from the present ones. Note that these other calculation procedures should also be adjusted to account for presence of exact equals.

TABLE 1

HYPOTHETICAL EXAMPLE (UTILITIES FOR $\varepsilon = 0.5$; WEIGHTS FOR $\nu = 2$)

#	$\hat{\varpi}_i^{x,\nu}$	$\widehat{\varpi}_i^{x,\nu}$	X_i^x	N_i^x	N_i^n	N_i^E	$U(X_i^x,\varepsilon)$	$U(N_i^x,\varepsilon)$	$U(N_i^n,\varepsilon)$	$U(N_i^E,\varepsilon)$	$U_{i,\varepsilon}^P$
1	0.160	0.125	0	10	10	52	0.00	6.32	6.32	14.43	13.70
2	0.146	0.125	0	20	20	52	0.00	8.94	8.94	14.43	13.70
3	0.132	0.125	0	30	30	52	0.00	10.95	10.95	14.43	13.70
4	0.118	0.125	0	50	50	52	0.00	14.14	14.14	14.43	13.70
5	0.104	0.125	0	80	75	52	0.00	17.89	17.32	14.43	13.70
6	0.090	0.125	0	110	80	52	0.00	20.98	17.89	14.43	13.70
7	0.076	0.076	50	100	100	76	14.14	20.00	20.00	17.44	16.15
8	0.063	0.063	100	75	110	100	20.00	17.32	20.98	20.00	18.61
9	0.049	0.049	150	150	125	124	24.49	24.49	22.36	22.27	21.06
10	0.035	0.035	200	125	150	148	28.28	22.36	24.49	24.32	23.52
11	0.021	0.021	300	250	200	196	34.64	31.62	28.28	27.99	28.43
12	0.007	0.007	400	200	250	244	40.00	28.28	31.62	31.22	33.34
	1	1	1200	1200	1200	1200	161.56	223.31	223.31	229.85	223.31

Post-fiscal incomes N_i^x of the first six units are sorted in ascending order, following the procedure from section 2.1. There is a large variation among incomes within this group of exact pre-fiscal equals. Furthermore, the unit #6 has larger post-fiscal income than the units #7 and #8, which is the evidence of reranking; other instances of reranking are between the units #5 and #8, #7 and #8, #9 and #10, #11 and #12.

Table 1 presents different utility vectors obtained for $\varepsilon = 0.5$. Two specific vectors, N_i^E and $U_{i,\varepsilon}^P$, are estimated in the following way. They are calculated as $N_i^E = \tilde{a}^E + \tilde{b}^E X_i^x$ and $U_{i,\varepsilon}^P = \tilde{a}^P + \tilde{b}^P X_i^x$, where $\tilde{a}^E$, $\tilde{b}^E$, $\tilde{a}^P$ and $\tilde{b}^P$ are coefficients obtained by Gini regressions in which X_i^x was an independent variable, while N_i^x and $U(N_i^x,\varepsilon)$ were respective dependent variables.[14]

[14] See Schechtman, Yitzhaki and Artsev (2008) for details about Gini regressions. The beta coefficient for the first regression is $\tilde{b}^E = COV(N_i^x, \widehat{\tau}_i^{x,\nu}) / COV(X_i^x, \widehat{\tau}_i^{x,\nu})$, where $\widehat{\tau}_i^{x,\nu} = s\widehat{\varpi}_i^{x,\nu} / \nu$. The alpha coefficient is $\tilde{a}^E = \hat{\mu}(N_i^x) - \widehat{b}^E \hat{\mu}(X_i^x)$.

FIGURE 1

EXPECTED POST-FISCAL INCOMES AND UTILITIES (UTILITIES FOR $\varepsilon = 0.5$; GINI REGRESSIONS FOR $\nu = 2$)

(a) (b)

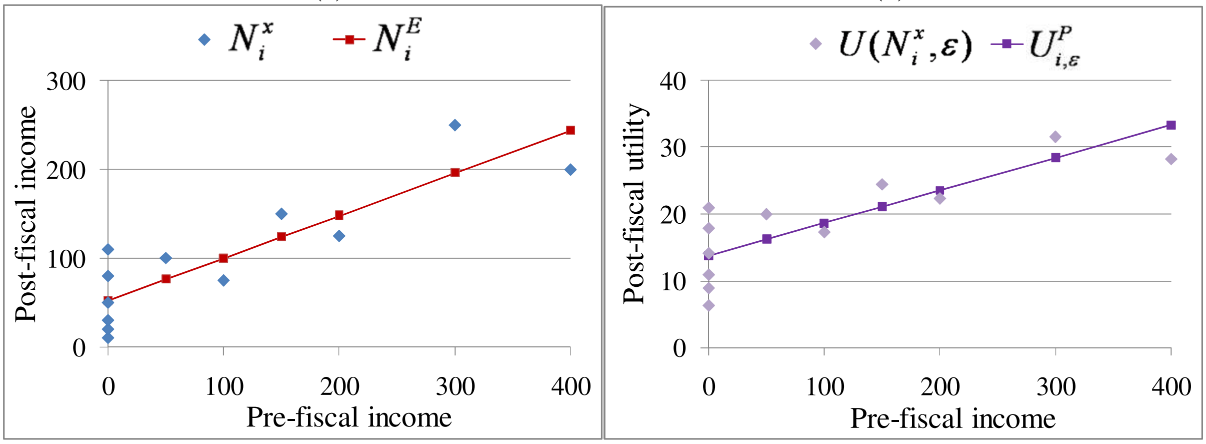


Figure 1a shows actual post-fiscal incomes N_i^x and expected post-fiscal incomes N_i^E, plotted against pre-fiscal incomes X_i^x. Figure 1b presents utilities of actual post-fiscal incomes $U(N_i^x, \varepsilon = 0.5)$ and their expected values $U_{i,\varepsilon}^P$.

The inequality indices are calculated for four combinations of parameters ε and ν, and presented in Table 2. The use of two sets of weights, $\hat{\varpi}_i^{x,\nu}$ and $\widehat{\varpi}_i^{x,\nu}$, enables us to make distinction between $\hat{I}(N_i^x)$ and $\widehat{I}(N_i^x)$, and to properly capture the reranking and CHI effects.

As the discussion concerning (15) reveals, in presence of pre-fiscal equals the measure $\widehat{R}^\psi(\varepsilon,\nu)$ would underestimate the true amount of reranking by $\hat{\psi}(\varepsilon,\nu)$, which is quite high in our hypothetical case for $\nu = 2$ and $\nu = 3$. At the same time, CHI effect is overestimated if $\widehat{C}^\psi(\varepsilon,\nu) = \hat{I}(N_i^x) - \widehat{I}(N_i^E)$ is used as a measure.

On the other hand, these problems do not occur in the last scenario, because when $\nu = 1$ the weights $\hat{\varpi}_i^{x,\nu}$ for all units are identical, reranking collapses to zero.

TABLE 2

INDICES OBTAINED FOR HYPOTHETICAL POPULATION

	$\varepsilon = 0$, $\nu = 2$	$\varepsilon = 0.5$, $\nu = 2$	$\varepsilon = 0.5$, $\nu = 3$	$\varepsilon = 0.5$, $\nu = 1$
$\widehat{I}(X_i^x)$	0.666667	0.924304	0.984997	0.546834
$\hat{I}(N_i^n)$	0.388889	0.479165	0.619086	0.134222
$\hat{I}(N_i^x)$	0.368056	0.465696	0.607441	0.134222

$\widehat{I}(N_i^x)$	0.319444	0.411984	0.500583	0.134222
$\widehat{I}(U_{i,\varepsilon}^P)$	0.319444	0.411984	0.500583	0.134222
$\widehat{I}(N_i^E)$	0.319444	0.349193	0.437733	0.072948
$\widehat{\Delta}(\varepsilon,\nu)$	0.277778	0.445139	0.365911	0.412612
$\widehat{V}(\varepsilon,\nu)$	0.347222	0.575111	0.547264	0.473886
$\widehat{C}(\varepsilon,\nu)$	0.000000	0.062791	0.062850	0.061274
$\widehat{R}(\varepsilon,\nu)$	0.069444	0.067181	0.118503	0.000000
$\hat{\psi}(\varepsilon,\nu)$	0.048611	0.053711	0.106858	0.000000
$\widehat{R}^{\psi}(\varepsilon,\nu)$	0.020833	0.013469	0.011645	0.000000
$\widehat{C}^{\psi}(\varepsilon,\nu)$	0.048611	0.116503	0.169708	0.061274

4. Real data example: Croatian personal taxes and cash social benefits

4.1. Data

The fiscal subsystem analyzed here consists of social security contributions (SSC; for the pension, health and unemployment insurance funds), personal income tax and surtax (PITS), public pensions and six types of cash social benefits.[15]

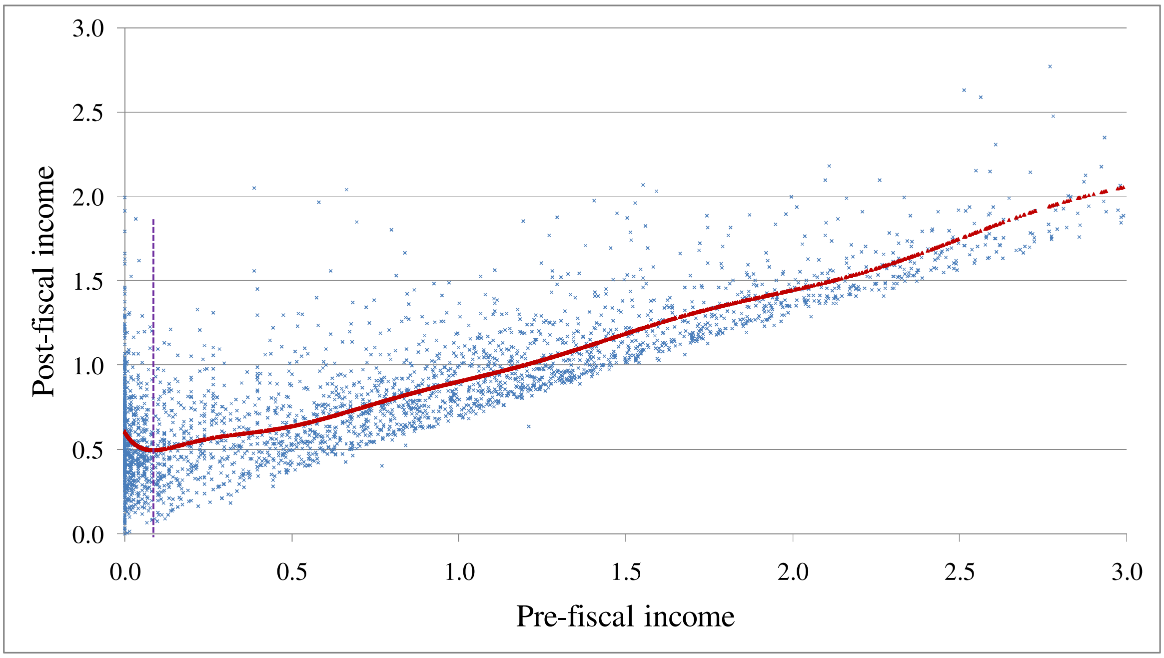
The data on incomes arrive from the Croatian household budget survey (Anketa o potrošnji kućanstava; APK). Since APK registers only net incomes of household members, it was a pre-requisite to build a microsimulation model to obtain amounts of pre-fiscal income, PITS and SSC. APK is available for years 2001 to 2008, and samples contain around 3,000 households. The analysis here is based on the 2008 APK sample, consisting of 3,108 households.

Pre- and post-fiscal incomes are obtained in the following way. Denote with $\dot{X}_k$, $\dot{T}_k$, $\dot{B}_k$ and $\dot{N}_k = \dot{X}_k - \dot{T}_k + \dot{B}_k$, the pre-fiscal income, the sum of all taxes paid, the sum of all benefits received and the post-fiscal income of household k. Incomes are deflated by the equivalence factor obtained using the 'modified OECD scale', $\beta_k = 1 + 0.5(a_k - 1) + 0.3c_k$, where a_k and c_k are number of adults and children in household *k*.

[15] Basic support allowances, unemployment benefit, child allowance, sick-leave benefit, maternity and layette supplement, and supplement for the injured and support for rehabilitation and employment of people with disabilities.

Income units are shown in Figure 2, with incomes expressed as ratios to the mean pre-fiscal income. Let us first concentrate on the units to the left of the dotted vertical line. They account for about 22% of all analyzed units. One half of them are units with zero pre-fiscal income; for the other half, the pre-fiscal income is greater than zero, but quite low – below 1/10 of the mean pre-fiscal income. The mean post-fiscal income of these two groups is 62% and 52% of mean pre-fiscal income, respectively; thus, the zero pre-fiscal income units have somewhat higher post-fiscal incomes.[16]

FIGURE 2
SCATTERGRAM OF PRE- AND POST-FISCAL INCOMES



EPI vector N_i^E is obtained using *Curve Fitting Toolbox 1.2* (henceforth CFT), an interactive tool for graphical data exploration working within Matlab R2007b. CFT enables the use of a dozen pre-programmed parametric and non-parametric models, as well as purely user-defined models. The model employed in this research is based on Fourier series – a sum of sine and cosine functions describing a periodic signal.[17]

[16] Pensioners prevail in this group, and there is a large variety in the level of pension income, as can be seen in Figure 2 if we look at the dots lying on the vertical axis.

[17] The number of 'terms' or 'harmonics' is set to 8; the Levenberg-Marquardt algorithm is employed and the robust fitting options are not used. The top twelve pre-fiscal income units are excluded from the fitting process and their values N_i^E are set to be equal to N_i^x. The same set-up was used in estimation of $U_{i,\varepsilon}^P$.

The advantage of this model is that it can provide us with the spoon-shape curve, which describes well the specific feature of the current data, where EPI initially falls. The EPI starting point (i.e. when pre-fiscal income is zero) is approximately equal to the mean post-fiscal income for the group of zero pre-fiscal income equals. Another two other desirable features of this particular method are: (a) the difference $\widehat{I}(N_i^E,0,\nu;\widehat{\omega}_i^{x,\nu})-\widehat{I}(N_i^x,0,\nu;\widehat{\omega}_i^{x,\nu})$ is very close to zero: for $\nu=1.2$ $(\nu=3)$ it is equal to 0.02% (0.14%) of RE; (b) the difference between the means of N_i^E and N_i^x is only 0.0001%. Thus, the estimate convincingly passes the tests proposed in section 3.3. above and the Appendix.

The results for the DJA decomposition for the Croatian fiscal system are shown in Table 3. The underestimation of the reranking term, $\hat{\psi}(\varepsilon,\nu)$, is small relative to RE (less than 2% of $\widehat{\Delta}$), but when compared to total HI, measured by $\widehat{C}(\varepsilon,\nu)+\widehat{R}(\varepsilon,\nu)$, it ranges from 3.3% to 5.6% for different combinations of ε and ν.

TABLE 3
INDICES OBTAINED FOR THE REAL FISCAL SUBSYSTEM

	$\varepsilon=0$ $\nu=2$	$\varepsilon=0.5$ $\nu=2$	$\varepsilon=0.5$ $\nu=3$	$\varepsilon=0.5$ $\nu=1$
$\widehat{I}(X_i^x)$	0.514079	0.702777	0.845047	0.284834
$\hat{I}(N_i^n)$	0.299155	0.340488	0.448383	0.074615
$\hat{I}(N_i^x)$	0.254675	0.299256	0.374014	0.074615
$\widehat{I}(N_i^x)$	0.252521	0.296900	0.367694	0.074615
$\widehat{I}(U_{i,\varepsilon}^P)$	0.252653	0.297038	0.368091	0.074616
$\widehat{I}(N_i^E)$	0.252653	0.269239	0.336353	0.053050
$\widehat{\Delta}(\varepsilon,\nu)$	0.214924	0.362289	0.396663	0.210219
$\widehat{V}(\varepsilon,\nu)$	0.261426	0.433538	0.508693	0.231783
$\widehat{C}(\varepsilon,\nu)$	-0.000132	0.027661	0.031341	0.021564
$\widehat{R}(\varepsilon,\nu)$	0.046634	0.043588	0.080689	0.000000
$\hat{\psi}(\varepsilon,\nu)$	0.002153	0.002356	0.006320	0.000000
$\widehat{R}^{\psi}(\varepsilon,\nu)$	0.044480	0.041232	0.074369	0.000000
$\widehat{C}^{\psi}(\varepsilon,\nu)$	0.002022	0.030017	0.037661	0.021564
$\hat{\psi}(\varepsilon,\nu)$ $(\%\widehat{\Delta})$	1.00	0.65	1.59	0.00
$\hat{\psi}(\varepsilon,\nu)$ (%HI)	4.6	3.3	5.6	0.00
$\widehat{I}(U_{i,\varepsilon}^P)-\widehat{I}(N_i^x)$	-0.000132	-0.000138	-0.000396	-0.000002

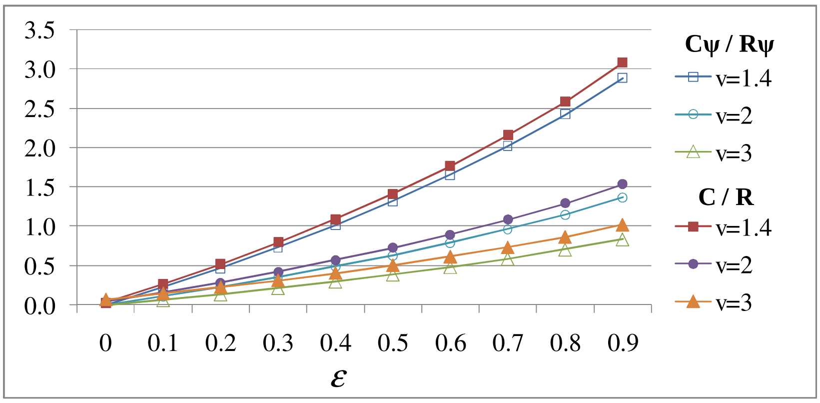
4.2. Reranking effect underestimated

As already noted, in presence of pre-fiscal equals, if the non-adapted weights $\hat{\varpi}_i^{x,\nu}$ are used, the reranking (CHI) effect will be underestimated (overestimated) by the amount $\hat{\psi}(\varepsilon,\nu)$. This fault may not be evident for $\varepsilon > 0$, but when $\varepsilon = 0$, it becomes quite obvious. Namely, the CHI effect does not exist in the DJA model by construction, i.e. $\hat{C}^{\psi}(0,\nu) = 0$, but in the presence of large number of exact pre-fiscal equals, it will be positive. This can be quite confusing to practitioners who are testing whether their EPI vector estimate is appropriate, following the recipe from section 3.3. To prevent this, they should not forget to use the adapted weights $\hat{\omega}_i^{x,\nu}$, in the beginning of the analysis.

It is interesting to see how the use of the wrong weights would affect the results. Duclos, Jalbert and Araar (2003) were analysing the ratio between the CHI and reranking terms, which indicates the relative importance of CHI versus reranking in the analysed fiscal system. Figure 3 shows these ratios for the Croatian fiscal system, for different values of ν and ε. Two different sets of ratios are presented: (a) the incorrect ones, $\hat{C}^{\psi}(\varepsilon,\nu) / \hat{R}^{\psi}(\varepsilon,\nu)$, and (b) the correct ones, $\hat{C}(\varepsilon,\nu) / \hat{R}(\varepsilon,\nu)$. They both increase in ε and $1/\nu$. The incorrect ratio, $\hat{C}^{\psi}(\varepsilon,\nu) / \hat{R}^{\psi}(\varepsilon,\nu)$, significantly overestimates the correct one, $\hat{C}(\varepsilon,\nu) / \hat{R}(\varepsilon,\nu)$, by up to 20 percentage points.

FIGURE 3

CHI / RERANKING RATIO

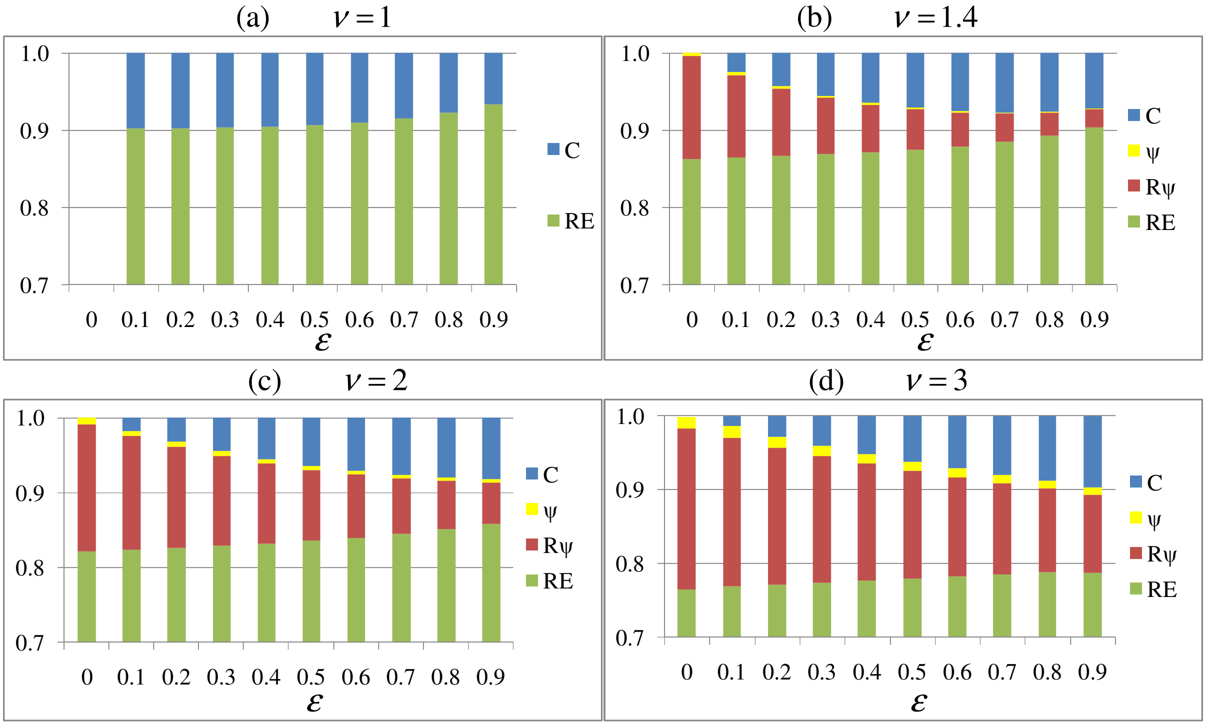


Legend: C / R = $\hat{C}(\varepsilon,\nu) / \hat{R}(\varepsilon,\nu)$, Cψ / Rψ = $\hat{C}^{\psi}(\varepsilon,\nu) / \hat{R}^{\psi}(\varepsilon,\nu)$

Let us now look at the structure of the vertical effect, $\hat{V}(\varepsilon,\nu)$, presented in Figure 4, where all components are separately identified as shares in $\hat{V}(\varepsilon,\nu)$. Note that $\hat{R}(\varepsilon,\nu) = \hat{R}^{\psi}(\varepsilon,\nu) + \hat{\psi}(\varepsilon,\nu)$. When $\nu = 1$, there can be no reranking and CHI makes about 10% of the vertical effect. If $\nu = 1$ and $\varepsilon = 0$, there is no inequality at all, and consequently no RE, CHI, and other effects.

When $\varepsilon = 0$ and $\nu > 1$, CHI in the DJA model does not exist. For $\nu > 1$ and $\varepsilon > 0$, the ratio $\widehat{C}(\varepsilon,\nu)/\widehat{V}(\varepsilon,\nu)$ increases with ε, while it is opposite for $\widehat{R}^{\psi}(\varepsilon,\nu)/\widehat{V}(\varepsilon,\nu)$. On the other hand, keeping ε constant and by increasing ν, the ratio $\widehat{C}(\varepsilon,\nu)/\widehat{V}(\varepsilon,\nu)$ increases only slightly, while the ratio $\widehat{R}^{\psi}(\varepsilon,\nu)/\widehat{V}(\varepsilon,\nu)$ increases significantly with ν.[18]

FIGURE 4
COMPOSITION OF VERTICAL EFFECT



Legend: C = $\widehat{C}(\varepsilon,\nu)$, ψ = $\hat{\psi}(\varepsilon,\nu)$, Rψ = $\widehat{R}^{\psi}(\varepsilon,\nu)$, RE = $\widehat{\Delta}$

4.4. The reranking effect of the counterfactual system

The models like DJA, K84 and AJL have achieved their popularity, among other things, due to apparently simple interpretation for policy purposes. The explanation usually goes like

[18] Notice that the ratio $\hat{\psi}(\varepsilon,\nu)/\widehat{V}(\varepsilon,\nu)$ has the same pattern of change as $\widehat{R}^{\psi}(\varepsilon,\nu)/\widehat{V}(\varepsilon,\nu)$, which is expected as we know that $\hat{\psi}(\varepsilon,\nu)$ is a component of overall reranking, $\widehat{R}(\varepsilon,\nu)$. The latter would be wrongly perceived as part of CHI if the weights $\hat{\omega}_i^{x,\nu}$ were used.

this: “If HI is somehow eliminated, RE would be enhanced by the amount equal to the HI effect(s)”. Thus, the vertical effect serves as a yardstick of ‘potential’ RE, namely, the higher level of RE that would be achieved by the fiscal system in the absence of HI.

The scholars usually disclaim that they are suggesting policy makers to eliminate HI in practice. However, to better understand the model and its results, we can ask at least hypothetically: how HI can eliminated?

Dardanoni and Lambert (2001:p.808) describe the curious relationship between two varieties of HI, concluding that CHI and reranking are interconnected concepts: “If the density function $h(x, y)$ is jointly continuous in x and y, as one finds in very large samples (...), then the one form of HI occurs if and only if the other does.”[19] They continue: “In particular, in the continuous case, both approaches define the absence of HI by the existence of a deterministic [post-fiscal income] function [$y = f(x)$]”. In fact, the values of this function are represented by the EPI curve $N^E(p)$ and estimated through $m^E(X_i^x)$, as in section 3.3.

We can conclude that the EPI curve $N^E(p)$ (and its empirical estimate N_i^E) eliminate both CHI and reranking. However, there is one important qualification, acknowledged by Dardanoni and Lambert (2001:p.808): the function $m^E(X_i^x)$ must be strictly increasing. Is this the case in reality? Not necessarily, as the following example shows.

Notice that the EPI curve in Figure 2 is decreasing on the interval [0,0.1] (the upper boundary being marked by the vertical dotted line). It means that the counterfactual system (CS) defined by EPI eradicates CHI, but is not free of reranking. Namely, in this CS the units in interval [0,0.1] have higher expected post-fiscal incomes than some units outside this interval: the latter ones are reranked by the former ones. Therefore, certain amount of HI will exist in the form of reranking albeit CHI is absent.

But, how large is this reranking effect caused by CS? We can compute the DJA model indicators for the CS as follows. Pre-fiscal incomes and frequency weights are equal to the original ones: $X_i^{x,cs} = X_i^x$ and $\varphi_i^{x,cs} = \varphi_i^x$. Post-fiscal incomes N_i^x are replaced by expected post-

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[19] A proof follows: “If $h(x_0, y_0) > 0$, then $h(x, y)$ will be strictly positive in some neighbourhood of (x_0, y_0), and one would necessarily find positive density at four points: $(x_0, y_0 + e)$, $(x_0, y_0 - e)$, $(x_0 - e, y_0 + e)$ and $(x_0 + e, y_0 - e)$ for some small $e > 0$. That is, CHI exists (consider the first two points) and so does reranking (the last two).”

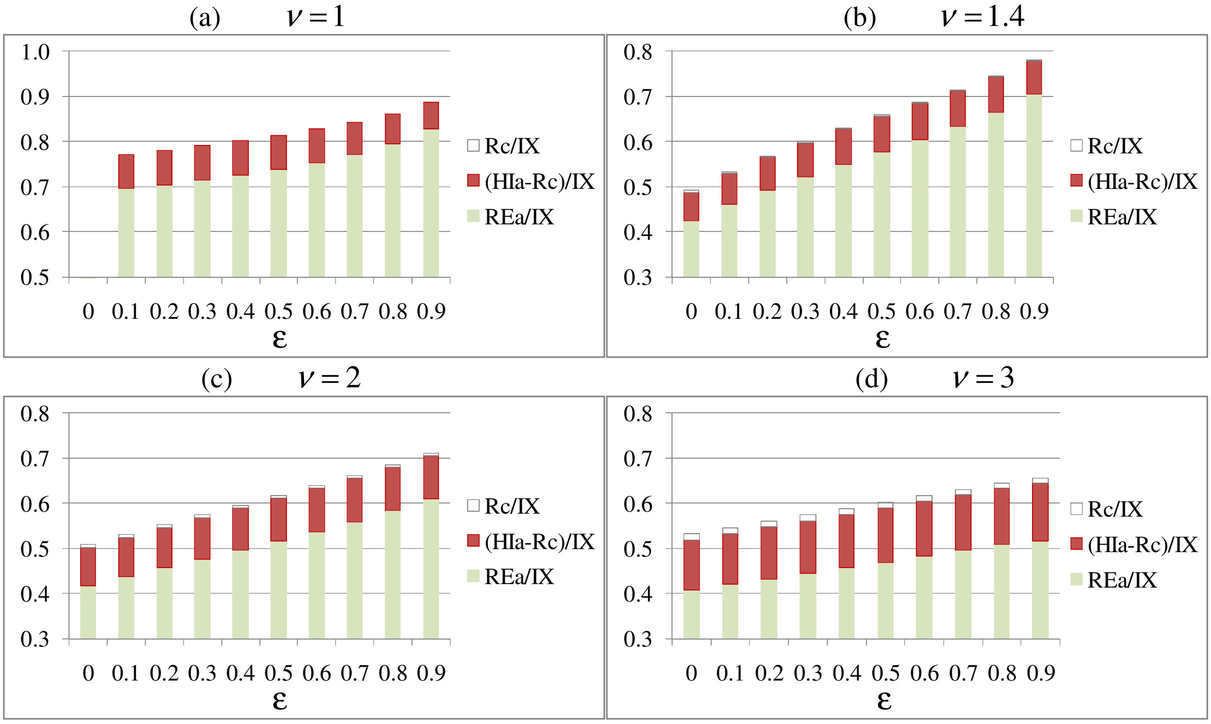
fiscal incomes N_i^E of the actual system: $N_i^{x,cs} = N_i^E$. Expected post-fiscal incomes of CS are equal to expected post-fiscal incomes N_i^E of the actual system; $N_i^{E,cs} = N_i^E = N_i^{x,cs}$. Post-fiscal incomes N_i^n and frequency weights φ_i^n are sorted together in increasing order to obtain $N_i^{n,cs}$ and $\varphi_i^{n,cs}$.

We know from (14) that the CHI effect of CS is zero for all ε, simply because $N_i^{E,cs} = N_i^{x,cs}$. The vertical effect is equal to the one achieved by the actual system, because $X_i^{x,cs} = X_i^x$ and $N_i^{E,cs} = N_i^E$. If the vertical effect of actual system really represents the *potential RE* with properties defined as above, the RE of CS should be equal to $\widehat{V}(\varepsilon,\nu)$. However, in our example, this is not the case, because CS did not eliminate all HI. Namely, the reranking effect of CS will be greater than zero, $\hat{I}(N_i^{n,cs}) - \widehat{I}(N_i^{E,cs}) > 0$, and RE of CS will be smaller than the vertical effect of the actual system: $\widehat{\Delta}^{cs}(\varepsilon,\nu) < \widehat{V}(\varepsilon,\nu)$.

Figure 5 shows different effects that 'constitute' the vertical effect, all of them expressed as shares in pre-fiscal income inequality: (a) redistributive effect, $\widehat{\Delta}$; (b) HI of the actual system reduced by the reranking of CS, $\widehat{H} + \widehat{R} - \widehat{R}^{cs}$; (c) reranking of CS itself, $\widehat{R}^{cs}$. The three components together make the vertical effect of actual system, $\widehat{V} = \widehat{\Delta} + \widehat{H} + \widehat{R}$, which supposedly represents the RE that would be achieved if HI would be eliminated. However, in our example this is not true: attainable RE is lowered because EPI induces reranking in amount measured by $\widehat{R}^{cs}$.

For $\nu = 1$ there can be no reranking and $\widehat{R}^{cs} = 0$. For $\nu > 1$ it increases with ν, from modest 0.3% of $\widehat{I}(X_i^x)$ for $\nu = 1.4$ and $\varepsilon = 0.5$, to 1.4% of $\widehat{I}(X_i^x)$ for $\nu = 3$ and $\varepsilon = 0$, when the share of $\widehat{R}^{cs}$ in HI of the actual system reaches 11.4%.

FIGURE 5
COMPOSITION OF VERTICAL EFFECT



Legend: Rc/IX = $\hat{R}^{cs} / \hat{I}(X_i^x)$, (HIa-Rc)/IX = $[\hat{H} + \hat{R} - \hat{R}^{cs}] / \hat{I}(X_i^x)$, REa/IX = $\hat{\Delta} / \hat{I}(X_i^x)$

Figure 5 shows the levels of actual and potential RE. For all combinations of parameters both potential and actual RE are quite high for Croatian system of personal taxes, public pensions and cash social benefits. For example, when $\nu = 1.5$ and $\varepsilon = 0.4$,[20] pre-fiscal inequality is reduced by no less than 57.6%, whereas the potential reduction (adjusted by $\hat{R}^{cs}$) is 65.6% .

5. Conclusion

The models decomposing redistributive effect of fiscal systems into vertical and horizontal effects are extensively used by practitioners. Duclos, Jalbert and Araar's (2003) model, despite its advantages over some other models, such as Kakwani's (1984) and Aronson, Johnson and Lambert's (1994) decompositions of RE, has not yet been extensively employed in empirical research, possibly due to certain difficulties emerging in its implementation.

This paper carefully explained the procedures of data manipulations, estimations and calculations needed to obtain the indices of the DJA model. Several hints are suggested that make the job of practitioners much easier and the results more confident. Firstly, it was shown

[20] The combination of parameters preferred by Duclos, Jalbert and Araar (2003).

that some indices can be calculated in more simple way: the estimation of EPU is unnecessary because the relevant index of inequality can be calculated simply using sample post-fiscal incomes. Secondly, a straightforward test is presented which indicates whether EPI is appropriately estimated: when the ethical parameter of the utility function is zero, the concentration coefficients obtained for EPI and sample post-fiscal incomes should be equal.

Furthermore, during the application of the model using the data on Croatian individual taxes and cash social benefits, a new problem emerged. One of the research scenarios treated public pensions as social benefits and hence not as a part of pre-fiscal income. Therefore, the sample contained a large number of zero pre-fiscal income equals. One of the common formulas for computation of the concentration coefficient of post-fiscal incomes produced flawed results, because it disregarded the fact that pre-fiscal equals should be assigned equal and not different weights. We explained which adjustments must be made in order to obtain the correct estimates.

Another peculiarity is observed during the study of Croatian fiscal system. Namely, the estimated EPI curve is not increasing in pre-fiscal income across the whole distribution of pre-fiscal incomes. This implies that the counterfactual fiscal system defined by EPI does not fully eliminates HI, as a certain amount is present in the form of reranking.

Besides serving as a 'cookbook' for implementation of the DJA model, the paper has also analysed its connections with the K84 decomposition, and the issue of vertical effect as a measure of potential redistributive effect. The findings of the analysis can be interesting to all researchers in the field of income redistribution measurement.

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Appendix:

Indices for N_i^n:

$$U(N_i^n,\varepsilon) = (N_i^n)^{1-\varepsilon} / (1-\varepsilon) \tag{18}$$

$$\hat{W}(N_i^n,\varepsilon,\nu;\hat{\omega}_i^{n,\nu}) = \sum_{j=1}^{s} U(N_j^n,\varepsilon)\cdot\phi_j^n\cdot\hat{\omega}_j^{n,\nu} \tag{19}$$

$$\hat{I}(N_i^n,\varepsilon,\nu;\hat{\omega}_i^{n,\nu}) = 1-[(1-\varepsilon)\hat{W}(N_i^n,\varepsilon,\nu;\hat{\omega}_i^{n,\nu})]^{\frac{1}{1-\varepsilon}} / \hat{\mu}(N_i^n) \tag{20}$$

Indices for N_i^x:

$$U(N_i^x,\varepsilon) = (N_i^x)^{1-\varepsilon} / (1-\varepsilon) \tag{21}$$

$$\hat{W}(N_i^x,\varepsilon,\nu;\hat{\omega}_i^{x,\nu}) = \sum_{j=1}^{s} U(N_j^x,\varepsilon)\cdot\phi_j^x\cdot\hat{\omega}_i^{x,\nu} \tag{22}$$

$$\hat{I}(N_i^x,\varepsilon,\nu;\hat{\omega}_i^{x,\nu}) = 1-[(1-\varepsilon)\hat{W}(N_i^x,\varepsilon,\nu;\hat{\omega}_i^{x,\nu})]^{\frac{1}{1-\varepsilon}} / \hat{\mu}(N_i^x) \tag{23}$$

$$\widehat{W}(N_i^x,\varepsilon,\nu;\widehat{\omega}_i^{x,\nu}) = \sum\nolimits_{j=1}^{s} U(N_j^x,\varepsilon)\cdot\phi_j^x\cdot\widehat{\omega}_i^{x,\nu} \tag{24}$$

$$\widehat{I}(N_i^x,\varepsilon,\nu;\widehat{\omega}_i^{x,\nu}) = 1-[(1-\varepsilon)\widehat{W}(N_i^x,\varepsilon,\nu;\widehat{\omega}_i^{x,\nu})]^{\frac{1}{1-\varepsilon}} / \hat{\mu}(N_i^x) \tag{25}$$

Indices for N_i^E:

$$U(N_i^E,\varepsilon) = (N_i^E)^{1-\varepsilon}/(1-\varepsilon) \tag{26}$$

$$\widehat{W}(N_i^E,\varepsilon,\nu;\widehat{\omega}_i^{x,\nu}) = \sum\nolimits_{j=1}^{s} U(N_j^E,\varepsilon)\cdot\phi_j^x\cdot\widehat{\omega}_j^{x,\nu} \tag{27}$$

$$\widehat{I}(N_i^E,\varepsilon,\nu;\widehat{\omega}_i^{x,\nu}) = 1-[(1-\varepsilon)\widehat{W}(N_i^E,\varepsilon,\nu;\widehat{\omega}_i^{x,\nu})]^{\frac{1}{1-\varepsilon}} / \hat{\mu}(N_i^E) \tag{28}$$

Indices for U_i^P:

$$\widehat{W}(U_{i,\varepsilon}^P,\varepsilon,\nu;\widehat{\omega}_i^{x,\nu}) = \sum\nolimits_{j=1}^{s} U_{i,\varepsilon}^P\cdot\phi_j^x\cdot\widehat{\omega}_j^{x,\nu} \tag{29}$$

$$\widehat{I}(U_{i,\varepsilon}^P,\varepsilon,\nu;\widehat{\omega}_i^{x,\nu}) = 1-[(1-\varepsilon)\widehat{W}(U_{i,\varepsilon}^P,\varepsilon,\nu;\widehat{\omega}_i^{x,\nu})]^{\frac{1}{1-\varepsilon}} / \hat{\mu}(N_i^x) \tag{30}$$

where $\hat{\mu}(N_i^n)$, $\hat{\mu}(N_i^x)$ and $\hat{\mu}(N_i^E)$ are means of post-fiscal income variables, equal to $\hat{\mu}(N_i^n) = (S)^{-1}\sum\nolimits_i^s \phi_i^n N_i^n$, $\hat{\mu}(N_i^x) = (S)^{-1}\sum\nolimits_i^s \phi_i^x N_i^x$ and $\hat{\mu}(N_i^E) = (S)^{-1}\sum\nolimits_i^s \phi_i^x N_i^E$, respectively. It is clear that $\hat{\mu}(N_i^x) = \hat{\mu}(N_i^n)$, because both N_i^n and N_i^x contain the same sample values, only differently sorted. However, we should also have that $\hat{\mu}(N_i^E) \approx \hat{\mu}(N_i^x)$, saying that the mean value of the estimated vector is very close to the sample-based value. This is another indicator showing the appropriateness of EPI estimation.

Yet another (approximate) equality that should exist if all calculations are done properly is the one between total (or mean) utilities:

$$\sum\nolimits_i^s \phi_i^x U(N_i^x,\varepsilon) = \sum\nolimits_i^s \phi_i^n U(N_i^n,\varepsilon) \approx \sum\nolimits_i^s \phi_i^x U_{i,\varepsilon}^P \approx \sum\nolimits_i^s \phi_i^x U(N_i^E,\varepsilon) \tag{31}$$