Some observations in the high-frequency versions of a standard New-Keynesian model

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February 2010

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June 2011

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Abstract

In a small-scale New-Keynesian model with a hybrid Phillips curve and IS equation, the paper is concerned with an arbitrary frequency of the agents' synchronized decision making. It investigates the validity of a fundamental methodological precept according to which no substantive prediction or explanation of a well-defined macroeconomic period model should depend on the real time length of the period. While this principle is basically satisfied as the period goes to zero, the impulse-response functions of the high-frequency versions can qualitatively as well as quantitatively be fairly dissimilar from their quarterly counterpart. The result proves to be robust under variations of the degree of price stickiness. The main conclusion is that DSGE modelling may be more sensitive to its choice of the agents' decision interval.

JEL classification: C63; E31; E32; E52.

Keywords: Hybrid New-Keynesian model; high-frequency modelling; impulse-response functions; Foley's methodological precept.

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1. Introduction

This paper is concerned with an elementary methodological issue in the macroeconomic modelling of dynamic stochastic general equilibrium (DSGE) that is not only of academic interest but may also find some reflection in more practical work. These models are

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[1] We would like to thank Carl E. Walsh, Leopold von Thadden, Roland Winkler, Hans-Werner Wohltmann as well as two anonymous referees for many helpful comments. The Matlab/Dynare source code for the numerical experiments is available on request by sending an e-mail to `sacht@economics.uni-kiel.de`.

usually formulated in discrete time. As an approximation to reality, they assume that decisions are taken discontinuously and that all transactions of a certain class occur in the same, synchronized rhythm. There is, however, no argument that would allow one to identify a uniform "natural decision period". If the models, especially in numerical studies, invoke a definite period, which is mostly a quarter, the choice is entirely determined by convention or the frequency of the available data. The present paper, among other things, queries this practice and asks for a more careful justification.

Given the 'degree of freedom' regarding the decision frequency, a sound model will certainly be required to produce similar outcomes if the underlying period is changed. This is a fundamental methodological precept that goes (at least) back to May (1970). To quote from a more extensive discussion of this issue by Duncan Foley (1975, p. 310; his emphasis):

> *"No substantive prediction or explanation in a well-defined macroeconomic period model should depend on the real time length of the period."*

For simplicity, and because of the influence the paper had at its time, we may refer to this maxim as *Foley's precept.*

To clarify our perspective in this respect it may be mentioned already here that the questions sketched below should not be a great challenge to macrodynamic models with purely backward-looking expectations.[2] This is, however, less obvious for the rational expectations models in contemporary macroeconomics, and we will choose a small-scale and fairly standard micro-founded New-Keynesian model to illustrate this point. We will indeed reveal certain problems in this setting, where our aim can only be to draw attention to the possible occurrence of similar phenomena in other, perhaps less didactic models.

The general idea of Foley's precept can be expressed in several ways, with different degrees of detail. Most fundamentally, the requirement that the results of a model do not depend in any important way on the period is translated into the condition that it should be feasible to make the period arbitrarily short. Consequently, in the limit, one would arrive at a continuous-time respecification: "The method used to accomplish this is to retain the length of the period as an explicit variable in the mathematical formulation of a period model and to make sure that it is possible to find meaningful limiting forms of the equations as the period goes to zero" (Foley, 1975, p. 311). In brief, this criterion for consistent discrete-time macroeconomic modelling will henceforth be called Foley's Criterion *CT*, which is to be mnemonic for a well-defined continuous-time limit.

A reduction of the period length is tantamount to more frequent decisions and transactions within a given calendar time unit. A model version with a shorter period than

[2] For example, this may be inferred from the discussion of the relationship between a continuous-time and discrete-time version of such a model with stocks and flows in Chiarella et al. (2005, pp. 191ff).

in the original formulation is thus a high-frequency economy. If the period of the benchmark model serves as the time unit, which is fixed, and a high-frequency economy is constituted by a period of length $h < 1$, the latter will also be designated an h-economy.

To forestall a possible criticism of the significance or even counterfactuality of this concept if h is "too small", it should be pointed out as a basic problem that decisions in the macroeconomy are infrequently made by the single agents, but in short time intervals by different agents. While a synchronized quarterly decision period disregards this fact completely, shorter though still synchronized periods may be said to provide a slightly better approximation to it. At least, they are not *a priori* inferior to, or less reasonable than, the conventional assumption of the quarterly rhythm. [3]

While a study of Criterion CT in a standard DSGE model will be one subject of our paper, this is not to say that modelling should be done in continuous time. Nevertheless, if discrete-time modelling allows for different frequencies in the agents' decision-making, the question arises whether or not two versions of the same model with different periods h might have different dynamic properties. A positive finding in this respect would cast serious doubt on the conventional quarterly period and raise the problem of which frequency to choose in the end, or of how to justify proceeding with the quarter.

One approach to address this problem is that a calibration or estimation procedure takes explicit account of different decision intervals on the part of the economic agents and tries to let the data decide about an appropriate period. [4] As long as this is not systematically done, one may resort to the concept of a *robust* period length $h^\star$. Quite in line with the abovementioned maxim, we mean by this an upper-bound on h that is low enough to rule out any overly strong dissimilarities in the model's most fundamental properties across the high-frequency versions with $h \le h^\star$. For example, if in a quarterly model a dynamic property P in its monthly version, P(month), turns out to be too different from P(quarter) but not so much from P(week), the quarter as a suitable decision interval may better be abandoned and replaced with a month. The second subject of the paper is an investigation of the question whether, in the given model and if the choice of the period is not to be too arbitrary, there is a need for such a reconsideration.

The first and most elementary property of a DSGE model that one should check under different frequencies is determinacy. In the model we are examining, however, determinacy turns out to pose no problem. [5] Instead we direct the focus on the model's

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[3] An explicit modelling of overlapping infrequent decisions of (otherwise homogeneous) agents, perhaps with different decision frequencies of different groups of agents, would be a much more ambitious perspective for future work.

[4] We only know of two examples of such work in the literature, namely, Christiano (1985) and Aadland (2001).

[5] This is different in the high frequency versions of the Benhabib and Farmer (1994) real business cycle model. Hintermaier (2005) discusses a set of numerical parameters such that, for instance, the quarterly economy is determinate but not the weekly version, and Anagnostopoulos and Giannitsarou (2008, 2010) make a similar point; see, in particular, pp. 11ff in their 2010 paper for

impulse-response functions (IRFs) as a familiar device to summarize its basic dynamic properties. For our purpose it suffices to limit the interest to just one kind of shocks, namely, monetary policy shocks, which after all are a widespread subject in the literature.[6]

The similarity or dissimilarity of the time paths of two IRFs can be assessed along two dimensions. On the one hand, it will be required that they have a similar shape. We refer to this criterion of qualitative similarity as Criterion *QualS*. It would typically fail to apply if, after the impact effect, one IRF immediately turns around and converges in a monotonic way back to the steady state level, and the other initially moves further away from the equilibrium for a while until it starts to revert.

The second dimension is that of quantitative similarity, which may be referenced as Criterion *QuantS*. What exactly constitutes a numerical threshold below (above) which Criterion *QuantS* is said to be fulfilled (or violated) is a matter of convenience, which is no problem if the threshold is explicitly stated. This will become apparent in the course of the discussion of our results.

The present paper investigates the three criteria sketched above for the high-frequency versions of a quarterly fairly standard three-equations model of the New-Keynesian variety, which is subjected to a contractionary shock in the Taylor rule. Assuming full flexibility regarding the wage side of the economy, we generally admit backward-looking elements in the form of a hybrid Phillips curve for price inflation and a hybrid IS equation for the output gap. Four different variants are thus obtained, depending on which of the hybrid components are switched on or off. The next section specifies the transformation of the quarterly model into a high-frequency version, where for better comparability across different decision intervals the variables continue to be expressed as quarterly magnitudes. Section 3 derives some analytical results for the limiting process $h \to 0$. After setting up a benchmark parameter scenario, Section 4 turns to a numerical investigation of the IRFs in the four model variants that compares the outcome of a quarterly, monthly and daily frequency. It therefore addresses the basic and so far largely neglected question of whether the quarterly decision interval is a sufficiently robust base period. Section 5 presents a short robustness analysis. Concentrating on a comparison of the quarterly with the possibly still convenient monthly economies, it examines how the gaps between the two IRFs at selected points in time increase or decrease when the model's price stickiness parameter is varied over an empirically relevant range. Section 6 concludes, and an

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an economic discussion of why determinacy may depend on the length of the adjustment period (jointly with a tax rate on labour).

[6] It nevertheless would be interesting to investigate whether our results would carry over to models incorporating productivity shocks, although for this a more general model like Smets and Wouters (2007) seems more appropriate (a framework in which productivity shocks have been found to be more important for an explanation of the variation in US inflation than the policy shocks).

appendix contains some technical details and additional diagrams.

2. The concept of a high-frequency economy

The New-Keynesian model underlying our investigations combines two well-known specifications of a hybrid IS equation and of a hybrid Phillips curve derived from a Calvo setting. Thus, with respect to a period of given length, let y_t be the output gap in period t, π_t the rate of price inflation (being zero in the steady state), i_t the nominal interest rate, and v_t a serially correlated monetary policy shock. The latter will be the only exogenous force imposed on the system, which is set in motion by a one-time impulse ε^v in period $t=0$. The natural real rate of interest is treated as a constant, which coincides with the households' time preference rate ρ. Assuming a Taylor rule with reactions to the contemporaneous values of inflation and economic activity, the model reads as follows:

$$\begin{aligned}
\pi_t &= \gamma_f\, E_t[\pi_{t+1}] \;+\; \gamma_b\, \pi_{t-1} \;+\; \lambda\, y_t \\
y_t &= \frac{1}{1+\chi}\, E_t[y_{t+1}] \;+\; \frac{\chi}{1+\chi}\, y_{t-1} \;-\; \frac{1-\chi}{(1+\chi)\,\sigma}\,(i_t - E_t[\pi_{t+1}] - \rho) \\
i_t &= \rho \;+\; \phi_\pi\, \pi_t \;+\; \phi_y\, y_t \;+\; v_t \\
v_{t+1} &= \alpha\, v_t\ , \qquad \text{where} \;\; v_0 = \varepsilon^v
\end{aligned} \tag{1}$$

It goes without saying that E_t denotes the mathematical expectation operator conditional on information up to period t, and that all coefficients are constant and positive or nonnegative, as the case may be. The autocorrelation α is contained between zero and one, ϕ_π and ϕ_y are the central bank's policy parameters, σ is the households' intertemporal elasticity of substitution in consumption, and χ is the parameter that measures the degree of the households' external habit formation ($0 \le \chi < 1$), where we adopt the specification in Smets and Wouters (2003, p. 1134).[7] The coefficients in the Phillips curve are composed of σ and three further structural parameters:

$$\begin{aligned}
\gamma_f &= \frac{\beta\,\theta}{\theta \,+\, \omega\,[1-\theta(1-\beta)]}\ , \qquad \beta \;=\; 1/(1+\rho) \\
\gamma_b &= \frac{\omega}{\theta \,+\, \omega\,[1-\theta(1-\beta)\,]} \\
\lambda &= \frac{(\sigma+\eta)\;(1-\omega)\,(1-\theta)\,(1-\beta\theta)}{\theta \,+\, \omega\,[1-\theta(1-\beta)]}
\end{aligned} \tag{2}$$

Here, β is the discount factor; η the substitution elasticity of labour in the households' utility function; θ the degree of price stickiness, i.e., $(1-\theta)$ is the fraction of all firms

[7] See also Abel (1990, p. 38), who first introduced this concept of habit formation into the macroeconomic literature. $\chi = 1$ would disconnect output and inflation from the real interest rate, while $\chi > 1$ is inadmissible because it would imply a falling steady state utility in consumption (Fuhrer, 2000, p. 370).

that happen to reset their price in a given period ($0 < \theta < 1$); and ω is the fraction of the rule-of-thumb price setters ($0 \le \omega < 1$), i.e. when resetting the price, these firms are 'backward-looking' and simply extrapolate the recent history of aggregate prices, whereas a fraction $(1-\omega)$ of the firms reset their price optimally, given the constraint on the timing adjustments and using all information to forecast future marginal costs. The deviations mc_t of the latter from their steady state value feature as the driving force in the determination of inflation. On the supposition that mc_t is linked to the output gap by the factor $(\sigma + \eta)$, the hybrid Phillips curve in (1) is seen to be identical to the one put forward by Galí and Gertler (1999, pp. 209–211).[8]

Certainly, system (1) includes the equations of the New-Keynesian baseline model as special cases (but not the polar case of a purely backward-looking model). For easier reference below, the two degrees of rule-of-thumb pricing ω and habit formation χ may serve to distinguish the following four model variants:

NKB :	the New-Keynesian baseline case,	$\omega = 0,\ \chi = 0$
HPC :	the hybrid Phillips curve,	$\omega > 0,\ \chi = 0$
HIS :	the hybrid IS equation,	$\omega = 0,\ \chi > 0$
HMP :	the hybrid model proper,	$\omega > 0,\ \chi > 0$

It will be noted that the Phillips curve has a backward-looking element with a composite parameter $\gamma_b > 0$ if and only if $\omega > 0$.

So far, no particular presumption concerning the length of the period has been invoked; it could be a quarter, as usual, as well as a month, a week or a day. In order to compare two economies with different frequencies of the synchronized actions, we now fix the time unit as a quarter, say, and generally allow the agents to make their decisions and carry out the corresponding transactions every h quarters ($0 < h \le 1$). As mentioned above, such a high-frequency economy is called an h-economy.

For a direct comparison of two h-economies, the flow variables are uniformly expressed as 'quarterized' magnitudes. Accordingly, a nominal interest rate i_t of, for example, 1.50% per quarter means that, in an h-economy, hundred dollars earn $h \cdot 1.50$ dollars over the period $[t, t{+}h)$. The time preference rate ρ is to be interpreted analogously.[9] Denoting the log price prevailing over the same time interval by p_t, the quarterized inflation rate is given by

$$\pi_t \;=\; (p_t - p_{t-h})\,/\,h \tag{3}$$

[8] See also Galí et al. (2001, pp. 1246ff) and Henzel and Wollmershäuser (2008, p. 819).

[9] While this treatment introduces a compound interest effect, it is certainly not strong enough to explain the differences over the first few quarters that we will point out in the impulse-response functions below. Incidentally, the procedure can also be found in the modelling of search and matching processes; see, e.g., Rogerson et al. (2005, p. 963).

It is not necessary to spell out the details that lead from the microfoundations to the structural Phillips curve and IS equation in an h-economy.[10] All of the agents' optimization procedures and the subsequent mathematical operations go through unaltered if the following rules are obeyed: (i) the output gap y_t as a ratio of two flow magnitudes need not be transformed since it has no time dimension; (ii) the inflation, interest and discount rates π_t, i_t and ρ in (1) have to be replaced with $h\pi_t$, hi_t, $h\rho$, respectively; (iii) the frequency-dependent parameters in (1) and (2) have to be suitably adjusted.

Regarding the last point, let us first list the parameters that remain unaffected by a change in the length of the period. These are the two elasticities σ and η, the rule-of-thumb price setting parameter ω, and the habit-formation parameter χ.[11] The IS equation can then immediately be reformulated for an h-economy as $y_t = [1/(1+\chi)]\, E_t[y_{t+h}] + [\chi/(1+\chi)]\, y_{t-h} + [(1-\chi)/(1+\chi)\,\sigma]\,(hi_t - E_t[p_{t+h} - p_t] - h\rho)$, and with definition (3) the last term can be rewritten as $h\,[(1-\chi)/(1+\chi)\,\sigma]\,(i_t - E_t[\pi_{t+h}] - \rho)$.

The two policy parameters determining the interest rate pose no problems, either. The Taylor rule adjusted to the h-economy reads $hi_t = h\rho + \phi_\pi(p_t - p_{t-h}) + h\phi_y\, y_t + hv_t$. Dividing this equation by h and again using (3), the same formulation for the interest rate reactions as in (1) is obtained.

The values of two of the structural parameters, which enter eq. (2) and the Phillips curve, are dependent on the frequency of decision-making: the discount factor β, which in an h-economy becomes $\beta = \beta(h) = 1/(1 + h\rho)$, and the price stickiness θ. As in a period of length h the fraction of firms resetting their price will be $h(1-\theta)$, we have $\theta(h) = 1 - h(1-\theta)$ if the pure symbol θ is retained for the constituent stickiness parameter from the quarterly setting. The stickiness remains the same in the sense that on average a firm is allowed to reset the price every $1/[1 - \theta(h)]$ periods of length h, which, independently of h, means every $h/[1 - \theta(h)] = h/[1 - 1 + h(1-\theta)] = 1/(1-\theta)$ quarters.

With $\beta(h)$ as just defined, $(1-\beta)$ turns into $h\rho/(1+h\rho)$, and the expression $(1-\theta)\,(1-\beta\theta)$ in the specification of λ in (2) turns into $[1 - \theta(h)]\,[1 - \beta(h)\theta(h)] = h^2\,(1-\theta)\,(1+\rho-\theta)/(1 + h\rho)$. Treating the other frequency-dependent terms in (2) in a similar way, the h-economy counterparts of the composite coefficients in (2) result like:

[10] The Calvo setting is helpful in this respect, whereas markup pricing together with Taylor's staggered wage contracts would be more difficult to treat; see Christiano (1985).

[11] In the quarterly model, current consumption c_t^k of household k and past aggregate consumption C_{t-1} (scaled to the household's consumption level) enter the utility function as the difference $c_t^k - \chi C_{t-1}$. This expression requires χ to be dimensionless as in the h-economy it simply becomes $hc_t^k - \chi\, hC_{t-h}$, if the consumption variables are quarterized, too. Note furthermore that firms show their willingness or ability to reoptimize their price in a *point in time*. Hence the parameter ω is not frequency-dependent, either.

$$
\begin{aligned}
\beta(h) &= \frac{1}{1+h\rho} \\
\gamma_f(h) &= \frac{1\,-\,h(1-\theta)}{(1+h\rho)\;a(h)} \\
\gamma_b(h) &= \frac{\omega}{a(h)} \\
\lambda(h) &= h^2\;b(h) \\
a(h) &= \omega\;+\;\frac{[\,1\,+\,h(1-\omega)\rho\,]\,[\,1\,-\,h(1-\theta)\,]}{1+h\rho} \\
b(h) &= \frac{(\sigma+\eta)\;(1-\omega)\,(1-\theta)\,(1+\rho-\theta)}{a(h)}
\end{aligned}
\tag{4}
$$

It will be observed that these terms remain well-defined as the period becomes infinitesimally small. In particular, $a(h)$ tends to $(1+\omega)$ as $h \to 0$, so that the limits of $\gamma_f(h)$ and $\gamma_b(h)$ are $1/(1+\omega)$ and $\omega/(1+\omega)$, respectively. We also remark that for $h=1$, $b(1)$ coincides with the definition of λ in (2).

The Phillips curve is now easily derived as follows. In an h-economy, we have $p_t - p_{t-h} = \gamma_f(h)\,E_t[p_{t+h} - p_t] + \gamma_b(h)\,(p_{t-h} - p_{t-2h}) + \lambda(h)$. So we only have to divide this equation by h to formulate the relationship in terms of the quarterized inflation rates, which is no problems by virtue of the expression we got for $\lambda(h)$. Finally, writing the autoregressive process for the monetary policy shocks in (1) as $v_{t+1} = v_t - (1-\alpha)v_t$ and taking into account that in an h-economy the updating occurs $1/h$ times per quarter, the equation becomes $v_{t+h} = v_t - h(1-\alpha)v_t$. On the whole, therefore, the hybrid New-Keynesian model with a general period of length h is described by the following set of equations:[12]

$$
\begin{aligned}
\pi_t &= \gamma_f(h)\,E_t[\pi_{t+h}] \;+\; \gamma_b(h)\,\pi_{t-h} \;+\; h\,b(h)\,y_t \\
y_t &= \frac{1}{1+\chi}\,E_t[y_{t+h}] \;+\; \frac{\chi}{1+\chi}\,y_{t-h} \;-\; h\,\frac{1-\chi}{(1+\chi)\,\sigma}\,(i_t - E_t[\pi_{t+h}] - \rho) \\
i_t &= \rho \;+\; \phi_\pi\,\pi_t \;+\; \phi_y\,y_t \;+\; v_t \\
v_{t+h} &= [\,1\,-\,h(1-\alpha\,)\,]\,v_t\;, \qquad \text{where}\;\; v_0 = \varepsilon^v
\end{aligned}
\tag{5}
$$

Obviously, setting $h=1$ recovers the quarterly model set up in eqs (1) and (2). When studied in the framework of these h-economies, the cases NKB, HPC, etc. listed above may be referred to as NKB(h), HPC(h), etc.

[12] To be exact, the dynamic variables of a given h-economy may also be identified by the letter h. In eq. (5) we abstain from this since it would look too clumsy here, but we introduce the notation later for greater clarity when comparing different h-economies.

3. Analytical results

Our investigation of the h-economies begins with Foley's Criterion CT, which checks the limiting behaviour of system (4) and (5) as the length of the period h becomes infinitesimally small.[13] Before turning to numerical simulations, let us see what kind of results can be obtained from a general analytical treatment. In fact, this approach is perfectly suited for the New-Keynesian baseline model, where it allows us to deduce all of the information we want.

We first show that in this case the limit process $h \to 0$ transforms (5) into two well-defined differential equations for π and y. In a framework with a one-time exogenous shock to the interest rate, rational expectations amount to perfect foresight, so that $E_t[\pi_{t+h}] = \pi_{t+h}$ and $E_t[y_{t+h}] = y_{t+h}$. The Phillips curve with $\omega = 0$, from which we have $\gamma_f(h) = \beta(h)$ and $\gamma_b(h) = 0$, may then be rearranged as $\pi_{t+h} - \pi_t = [1 - \beta(h)]\, \pi_{t+h} - hb(h)\, y_t$. Substituting the Taylor rule in the IS equation gives us, with $\chi = 0$, $y_{t+h} - y_t = (h/\sigma)\, [(\phi_\pi - 1)\, \pi_t - (\pi_{t+h} - \pi_t) + \phi_y\, y_t + v_t]$. Putting $b^o = b(0) = (\sigma + \eta)\, (1 - \theta)\, (1 + \rho - \theta)$, dividing by h and going to the limit, $h \to 0$, leads to the continuous-time formulation,

$$\begin{bmatrix} \dot{\pi} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \rho & -b^o \\ (\phi_\pi - 1)/\sigma & \phi_y/\sigma \end{bmatrix} \begin{bmatrix} \pi \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1/\sigma \end{bmatrix} v \qquad (6)$$

Determinacy in this system requires both eigen-values of the matrix in (6) to have positive real parts. This is the case if, and only if, the matrix has a positive determinant and a positive trace. As the latter is always ensured, it is immediately seen that a necessary and sufficient condition for determinacy of the continuous-time counterpart of NKB is

$$b^o\, (\phi_\pi - 1) \; + \; \rho\, \phi_y \; > \; 0 \qquad (7)$$

Intuition says that, for small values of $h > 0$, the conditions for the h-economies are close to (7) (see Franke and Flaschel, 2009, for the precise details). Even the condition for the quarterly model is not very different: $b^o(\phi_\pi - 1) + \rho\, \phi_y/(1+\rho) > 0$ (Bullard and Mitra, 2002, pp. 1115, 1125f).

A well-defined continuous-time formulation is therefore not only a matter of consistency, but it may also be helpful for the analytical tractability of determinacy conditions. Although they are no problem here since results for the original model are readily available, it could be relevant for more general models giving rise to higher-order matrices. It may then happen that the original discrete-time formulations are too complicated to check the Blanchard-Kahn conditions analytically, while their continuous-time counter-

[13] A "practitioner" to whom this issue of internal consistency appears too academic might directly proceed to the next section.

parts still offer some scope for a mathematical analysis.[14]

Another pleasant property of the baseline model is that it permits explicit closed-form solutions. Certainly, also their limits are found to be well-defined if we write them down for the h-economies (the precise expressions are given in the Appendix). Hence, Criterion CT is satisfied in all pertinent aspects. We summarize this as our first observation, where for better distinction the variables in the h-economies are denoted by $y_t^{(h)}$, $\pi_t^{(h)}$, $v_t^{(h)}$.

Observation 1

In the New-Keynesian baseline case NKB, the limiting forms of the h-economies (4), (5) for $h \to 0$ are well-defined for all $t \geq 0$ and given by eq. (6). In addition, there exist two continuous and strictly positive functions $f_\pi = f_\pi(h)$ and $f_y = f_y(h)$ defined on the closed unit interval $[0,1]$ such that for each $h > 0$ the solution paths of (4), (5) are determined by

$$\pi_t^{(h)} \;=\; -f_\pi(h)\, v_t^{(h)} \;, \qquad y_t^{(h)} \;=\; -f_y(h)\, v_t^{(h)} \qquad (t = 0, h, 2h, \;\dots\;) \;.$$

Thus, the continuous-time limits $\pi_t^{(0)}$, $y_t^{(0)}$ exist for all $t \geq 0$. Moreover, $\pi_h^{(h)} \to \pi_0^{(0)}$ and $y_h^{(h)} \to y_0^{(0)}$ as h tends to zero.

The last continuity property is an obvious implication of the continuity in h of the functions f_π, f_y and the shocks $v_t^{(h)}$ in the solution formulae. We nevertheless include this feature in the summary since it becomes more noteworthy further below.

The issue of transforming the hybrid model variants into continuous time is more cumbersome. We therefore confine ourselves to directly stating the main result and relegate the finer details to the Appendix.

Observation 2

If around some positive point in time $t > 0$ two smoothness conditions for inflation and the output gap over time and across frequencies are satisfied, then in the limit, $h \to 0$, and at that time t the Phillips curve and the IS equation turn into the following differential equations system:

$$\begin{bmatrix} \dot{\pi} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \rho & -b^o\,(1{+}\omega)/(1{-}\omega) \\ c\,(\phi_\pi{-}1) & c\,\phi_y \end{bmatrix} \begin{bmatrix} \pi \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix} v \tag{8}$$

where $c := (1{+}\chi)\,/\,[\sigma\,(1{-}\chi)]$ and $\dot{\pi}$ and $\dot{y}$ have to be interpreted as backward derivatives.

On the other hand, a backward continuous-time limit does not exist at $t{=}0$ (and it will later be seen that the forward derivatives do not exist, either).

[14] This is demonstrated in Franke and Flaschel (2009) for the 4×4 matrices arising from the high-frequency versions of the New-Keynesian model by Erceg et al. (2000) with sticky wages and prices. It is, however, again purely forward-looking, $\omega = \chi = 0$.

As it should be, eq. (6) is obtained from (8) for the polar case $\omega = 0$ and $\chi = 0$. For the precise smoothness conditions, see eqs (11) and (13) in the Appendix.

We are thus facing the problem if, or when, the smoothness conditions in Observation 2 are satisfied. Logically prior to it is a confirmation of the intuition (or hope) that the response functions $\pi_t^{(h)}$, $y_t^{(h)}$ do tend to finite values as the periods become infinitesimally short. These issues, however, can no longer be treated with analytical mathematical methods. In the next section, the investigation is therefore extended to numerical simulations of the different versions of the model.

4. Numerical investigation

The numerical setting in our simulations of the model relies on parameter values well-known from the literature. They are collected in Table 1, where the frequency-dependent coefficients are based on the quarterly time unit. The real natural rate of interest of about 4% per annum, log utility ($\sigma = 1$), and a unitary Frisch elasticity of labour supply ($\eta = 1$) are standard. The policy coefficients are the classical Taylor values (the coefficient on the output gap adjusted to the quarterly interest rate), while the value for α is associated with moderately persistent monetary policy shocks. A price rigidity of $\theta = 2/3$ implies an average price duration of three quarters. These parameters are borrowed from Galí (2008, p. 52). The values for ω and χ are extracted from two estimation studies, the one for ω from Galí et al. (2001, p. 1257) and the one for χ from Smets and Wouters (2003, p. 1143). Finally, the initial shock ε^v corresponds to a sudden 1% increase in the annualized nominal interest rate, which of course is just a matter of scale.

ρ	Natural real rate of interest (households' discount rate)	0.010
σ	Intertemporal elasticity of substitution in consumption	1.000
η	Intertemporal elasticity of substitution of labour	1.000
θ	Calvo degree of price stickiness	2/3
ω	Rule-of-thumb parameter in price setting	0.300
χ	Habit-formation parameter in consumption	0.600
ϕ_π	Weight on inflation in the Taylor rule	1.500
ϕ_y	Weight on the output gap in the Taylor rule	0.125
α	Persistence in the shock process	0.500
ε^v	Impulse to the interest rate (in per cent)	0.250

Table 1: Numerical parameter scenario.

In addition to the quarterly decision period, we present simulations with a monthly and a daily frequency. Even if the latter is accepted as an approximation to the daily arrival of the infrequent but not synchronized decisions of the single agents, as mentioned in the introduction, it might be argued that this argument does not carry over to the policy rule of the central bank. However, the interest rate is determined on the financial markets and does not stay put between two explicit decisions of the central bank. In the spirit of the rational expectations framework it therefore makes sense to assume that within this interval the financial markets bring about adjustments of the interest rate that are in line with the central bank's Taylor rule. There are certainly more problematic (though conventional) assumptions in the model than this one. Nevertheless, a reader not sharing this view may in the following still accept the month as a less counterfactual decision period than the quarter.

The outcome of a first set of simulations with the parameters from Table 1 is shown in Figure 1 (the output gap in the left column is given in per cent, the rate of inflation in the right column in per cent per quarter). The impulse-response functions (IRFs) in the upper two panels illustrate the analytical results in Observation 1 for the baseline case. At each of the three frequencies we deal with, the two variables $y_t^{(h)}$ and $\pi_t^{(h)}$ converge monotonically back to the steady state values at the speed given by the shock process $v_t^{(h)}$. This is a qualitative similarity across the frequencies—which holds not only here but in all of the scenarios to follow—where convergence is an immediate consequence of the universal feature of linearity and stability of the systems, and monotonicity is maintained since the variations in h are not of that kind, or not strong enough, to turn real into complex eigen-values. Convergence, however, is seen to take longer at the higher frequencies, although this effect is not very pronounced; see the dotted (blue) lines for the monthly and the thinner solid (red) lines for the daily economy *versus* the bold (green) lines for the usual quarterly setting. A discussion of the quantitative impact effects $y_0^{(h)}$ (but apparently not $\pi_0^{(h)}$) across different period lengths concerns Criterion *QuantS* and is postponed until later.

Another property—which likewise holds true in all other scenarios—is that the immediate response at $t=0$ to the shock becomes stronger as h decreases. A general intuition for that is that if the the future is one month or one day ahead it has more of an impact on today's choices than if it is one quarter away (cf. Anagnostopoulos and Giannitsarou, 2010, p. 12). However, since in a full rational expectations system everything depends on everything else, it must be admitted that these, so to speak, compound interest effects could only provide a partial, or first-round, argument. Some more remarks on this issue will be made in the next section.

It may also be mentioned that in Figure 1 as well as in Figure 2 below, the time paths from the weekly economies are fairly close to those from the daily economies (which is the reason why they have not been included in the diagrams). In the descriptions to follow, the daily decision intervals could therefore, for all practical reasons and if preferred, be identified with a weekly decision interval.

The results for the fully hybrid model in the lower two panels of Figure 1 are noteworthy for a number of reasons. What immediately leaps to the eye are the distinct troughs in

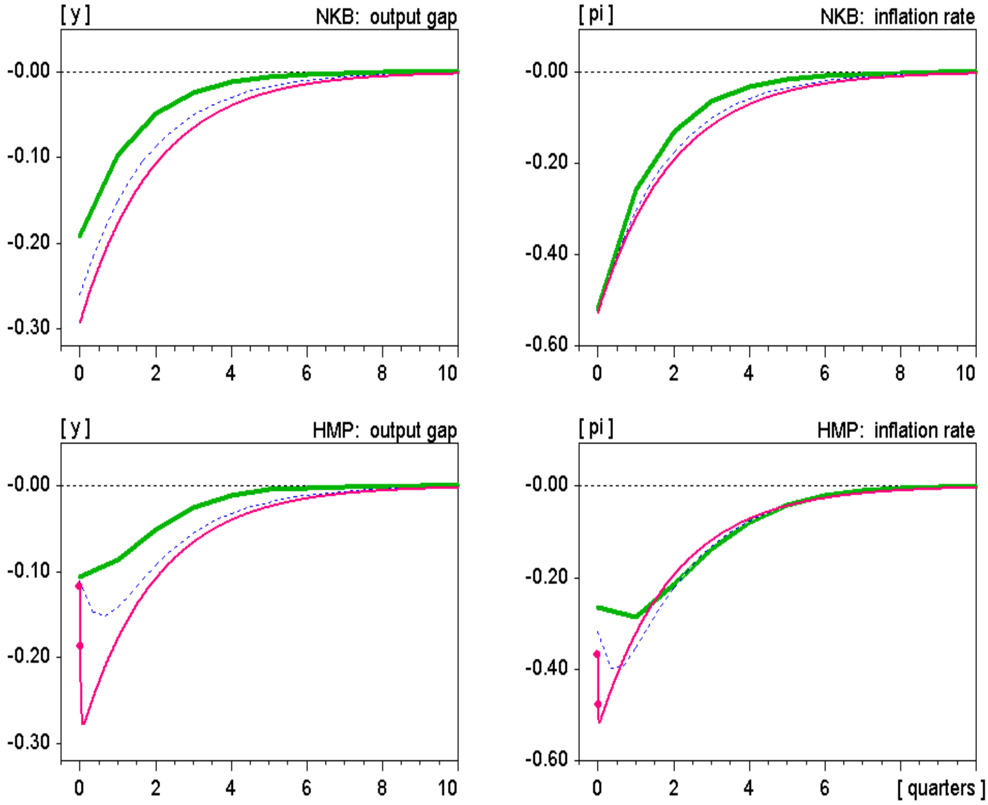


Figure 1: Impulse-response functions for NKB(h) and HMP(h).

Note: The bold (upper green) lines represent the quarterly model ($h = 1$), the dotted (blue) lines a monthly ($h = 1/3$) and the solid (red) lines a daily economy ($h = 1/90$). The two dots on the latter mark the first- and second-period values of the IRF in the hybrid case. This note equally applies to Figure 2.

the two impulse-response functions at the higher frequencies. That is, after the negative impact response to the shock, both the output gap and the inflation rate continue to decline for a short while. In the monthly economy the trough in the output gap is reached in the third month ($t = 2/3$), and in the daily economy the turnaround takes place after 9 days, where the further decline after the the initial period is by no means negligible ($y = -0.117$, -0.187, -0.279 on impact, at the second day, and in the trough at the 9th day, respectively). The inflation rate starts to return somewhat earlier to the steady state. Here the trough occurs in the second month for $h = 1/3$ and at the 5th day for $h = 1/90$ (with, in the latter case, $\pi = -0.478$ and -0.519 at the second and 5th day, respectively, which is less dramatic than for the output gap).

The fact that the daily economy produces deeper (or even considerably deeper) troughs than the monthly economy raises the question for the behaviour of these turning points when h is decreased further. Checking the limiting behaviour (and its monotonicity) up to an hourly frequency, $h = 1/(24 \cdot 90) = 1/2160$, we have every reason to believe that the troughs in HMP(h) are bounded from below (of course, for NKB(h) this is already known from Observation 1). Actually, the trough values of the daily economy are already fairly close to that of the hourly economy ($y = -0.287$ for the former *versus* $y = -0.293$ after 15 hours, and $\pi = -0.519$ *versus* $\pi = -0.528$ after 7 hours). Very similar results are found for the 'partial' model variants HPC(h) and HIS(h) as h is lowered to 1/2160. Hence at least for practical purposes, boundedness of the IRFs can be considered an established fact, which we take down in the next observation. In its formulation we adopt the convention that, for points in time t that are not an integer multiple of a given h, $y_t^{(h)}$ and $\pi_t^{(h)}$ are to be interpreted as the linearly interpolated values between the two values of the corresponding consecutive periods of the h-economy (between time kh and $(k+1)h$ if $kh < t < (k+1)h$ for some suitable $k \in \mathbb{N}$).

Observation 3

In all four model variants NKB(h), HPC(h), HIS(h), HMP(h) and for every $t \geq 0$ *it appears that the impulse-response functions* $\pi_t^{(h)}$, $y_t^{(h)}$ *converge towards finite values* $\pi_t^{(0)}$, $y_t^{(0)}$ *as the period length h becomes small. This has been numerically confirmed for* $h \to 1/(24 \cdot 90)$.

Taking the boundedness of the IRFs for granted, the main message of the lower two panels in Figure 1 is the contrast between the pronounced troughs of $y_t^{(h)}$ and $\pi_t^{(h)}$ in HMP(h) for all periods of length h equal to or shorter than a month, and the monotonic behaviour of the output gap or the only weakly nonmonotonic behaviour of inflation in the quarterly economy. To distinguish which of the two backward-looking features is responsible for which effect, Figure 2 plots the IRFs for, respectively, the model with the hybrid Phillips curve only (in the upper part) and with the hybrid IS equation only (in the lower part). In both variants the quarterly IRFs are monotonic, whereas HPC(h) causes the inflation rate $\pi_t^{(h)}$ and HIS(h) causes the output gap $y_t^{(h)}$ to be nonmonotonic as soon as $h \leq 1/3$.[15] Hence we have a number of cases where the precept of qualitative similarity of models with different period lengths is substantially violated.

These findings are stated as Observation 4, where we also add the obvious quantitative gaps between the quarterly and daily—and therefore weekly—economies (recall the remark above that the weekly and daily IRFs are quite close). This leads us to the conclusion of the quarterly model's lack of robustness with respect to its decision interval, in the sense in which this concept was discussed in the Introduction.

Observation 4

For $\pi_t^{(h)}$ *in HPC(h), and for* $y_t^{(h)}$ *in HIS(h) and HMP(h), Criterion* QualS *is not satisfied in that, after the impact effect, the impulse-response functions of the quarterly*

[15] The timing of the troughs is the same as before.

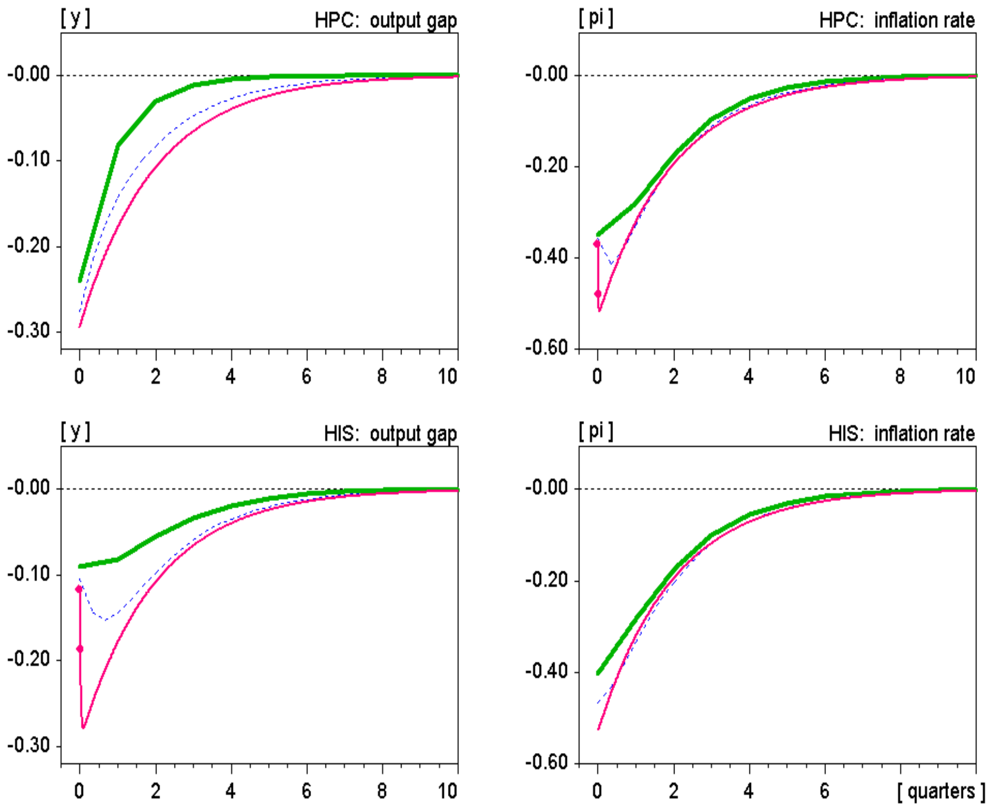


Figure 2: Impulse-response functions for HPC(h) and HIS(h).

economy return monotonically to zero, whereas for the h-economies with $h \leq 1/3$ they exhibit a markedly nonmonotonic behaviour.

Furthermore, the model's quarterly decision interval is not a robust period. That is, at least for weekly decision intervals and at least for the response of the output gap, Criterion QuantS *is clearly violated.*

At the end of this section we may once more return to the properties of the hybrid model when the period shrinks to zero. The implications from the results so far together with some formal arguments, which are given in the Appendix, can be summarized by another observation.

Observation 5

The forward derivatives of π at $t=0$ in HPC and HMP, and of y in HIS and HMP, do not exist. That is, the difference quotients $(\pi_h^{(h)} - \pi_0^{(h)})/h$ and $(y_h^{(h)} - y_0^{(h)})/h$ diverge as the period length h shrinks to zero. In addition, the corresponding second-period effects in the h-economies, $\pi_h^{(h)}$ and $y_h^{(h)}$, converge to finite values $\tilde{y}$ and $\tilde{\pi}$, respectively, where $\tilde{\pi} < \pi_0^{(0)}$ and $\tilde{y} < y_0^{(0)}$ with respect to the limits from Observation 3.

Owing to the nonexistence of the (backward and forward) time derivatives at $t=0$ and the discontinuity that the sequences of the second-period values converge to different values from the limits of the impact effects, Criterion CT for a transition to a continuous-time formulation is not perfectly satisfied. However, responsible for this is not necessarily the model as such but the sudden jump of the shock variable at $t=0$. A smoother shock scenario in the h-economies, where, for example, the impulse is distributed over the entire first quarter, may well be able to avoid this kind of deficiency.[16]

In particular, there might then also be good prospects for the assumption on the limits of the difference quotients in Observation 2 to be fulfilled, which would give rise to the differential equations (12) and (14). As a matter of fact, even in the present shock scenario the consecutive difference quotients become quite close after a short while if h is small enough. In the hourly economy, for example, the order of magnitude of $(\pi_{t+h}^{(h)} - \pi_t^{(h)})/h$ is 0.26 for some time after $t = 0.015$ quarters, while the absolute differences between two consecutive quotients are already as small as 0.000086 and lower; similarly so for the output gap, where $(y_{t+h}^{(h)} - y_t^{(h)})/h \approx 0.14$ at that time and the differences are less than 0.000065.

We can conclude from these remarks and the previous observations that Criterion CT may not be seriously incriminated. What nevertheless remains is Observation 4, that is, the failure of the model to meet the qualitative and partly also the quantitative consistency criterion, if the original quarterly decision interval is taken as the base period. The conclusion from this observation is that, at least for the present parameter scenario, the model and its generic dynamic properties should rather be discussed on the basis of a *daily or weekly decision interval.* Alternatively, as alluded to in the Introduction, the length of the decision interval might be an integral part of attempts to calibrate or estimate the model.

5. A comparison of quarterly and monthly economies

It was apparent from the diagrams in Section 4 that at least for the variables with a corresponding hybrid model component, the deviations of the daily (weekly) from the quarterly economies are by no means negligible. Both Criterion *QualS* and Criterion *QuantS* are clearly violated. We summarized that therefore, for the present parameter scenario, the quarter is not a robust period and tentatively indicated that a day or a week, say, would be much more appropriate in this respect.

The present section initiates a more extensive examination of Criterion *QuantS*, which would be necessary for more definite statements. Its perspective will, however, be more conservative in that we confine our interest to the quantitative differences between the impulse-response functions from the quarterly and monthly economies. If these are found

[16] In general, on the other hand, it depends on the specific kind of shock(s) considered whether such a supposition would make economic sense.

to be displeasingly large across a certain range of relevant parameters, we have a firm basis to conclude that a quarterly decision interval of the agents yields misleading results in some of the IRFs.

More specifically, a month would at least be shown to be a better candidate for a robust period than a quarter. It could then, in a next step, be decided whether a base period of a month, which is still in the range of (a subset of) available data, would also be preferred to a weekly decision interval—which possibly would better serve as a robust period but might be regarded as "implausibly" low. In this situation, a monthly decision interval could represent some sort of compromise.[17]

For a better evaluation of the quantitative results across different model versions or across different parameters, the deviations of the monthly from the quarterly economy are measured in percentage terms. Accordingly, we compute the expressions

$$d_t \;:=\; 100 \cdot |\, (x_t^{(h)} - x_t^{(1)}) \,/\, x_t^{(1)} \,| \,, \qquad h = 1/3\,, \quad x = \pi, y \tag{9}$$

For the non-hybrid cases, the deviations d_t on impact, at $t=0$, and at quarter $t=1$ are studied. For the variables associated with a hybrid Phillips curve or IS equation, which imply a nonmonotonic IRF, we have an additional look at the deviations in the second month, at $t = 1/3$. At least for the output gap this is a conservative statistic since, as seen from Figures 1 and 2, it takes another month until $y_t^{(1/3)}$ reaches its trough (while the quarterly IRF is already on the rise). Concerning the quarterly series at $t = 1/3$, the corresponding convex combination is used to enter (9), which is given by $x_t^{(1)} := (1-t)\, x_0^{(1)} + t\, x_1^{(1)}$.

	NKB		HMP		HPC		HIS	
	y	π	y	π	y	π	y	π
at $t = 0$	34	2	4	19	15	2	15	16
at $t = 1/3$	—	—	48	46	—	26	65	—
at $t = 1$	**55**	17	**63**	23	**76**	17	**75**	18

Table 2: Absolute percentage deviations d_t of monthly from quarterly IRFs in the benchmark parameter scenario.

To begin with the benchmark parameter scenario, Table 2 reports the deviations (9) that result from the IRFs in Figures 1 and 2, i.e., they quantify the gap between the bold and dotted lines. The table points out that not all of these gaps are negligible. There are in fact several instances where they are wider than, say, 25%. The deviations of more

[17] Nevertheless, Aadland (2001) shows the superiority of a weekly over a monthly decision interval in his calibration of a standard real business cycle model. Hence a week may not be dismissed altogether.

than 50% are in bold type because we think they could no longer be disregarded as being "not too serious". These notable cases are obtained for the output gap after a quarter, while the gaps in the inflation rate are less dramatic (except perhaps for the intermediate period in the fully hybrid model). It is also remarkable that the serious deviations of the output gap occur in all four model variants; whether the Phillips curve or the IS equation is hybrid or not plays no essential role for this.

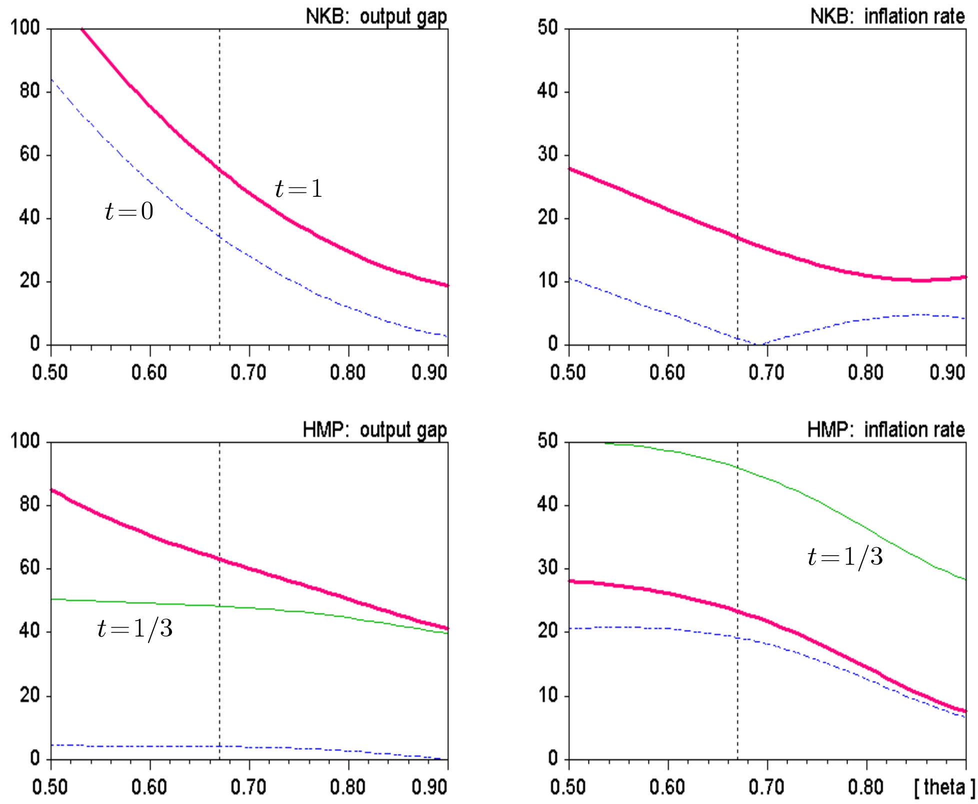


Figure 3: Absolute percentage deviations d_t of monthly from quarterly IRFs under variations of the price stickiness parameter θ; in NKB and HMP.

To get some understanding of the differences in the impact responses of the monthly economy, consider the baseline case NKB and recall the above intuition that today's reactions of the households and firms become stronger as the length of the decision period decreases, i.e., as the 'future' comes closer. Then, note that the Euler condition from the households' optimization problem gives rise to an annualized coefficient on the real interest rate, $(1-\chi)\,/\,[(1+\chi)\sigma]$, in the IS equation in (5) that is independent of h. By contrast, the annualized version of the coefficient on the output gap in the Phillips curve decreases as h decreases.[18] It may therefore be said that the first-round effects

[18] As detailed in (4), for some constant c this coefficient is given by $b(h) = c\,a(h)$, where $a(h)$ is easily seen to be increasing in h. Thus, for example, our parameters yield $1/a(h) = 0.60$ for $h=1$

just mentioned are weakened in, so to speak, a second round. In NKB the first-round effect is indeed almost completely undone. [19]

Taking this for granted, the reactions to the real interest rate in the IS equation can be viewed as reactions to the output gap itself. In the (hypothetical) process of forming self-fulfilling expectations we thus have some sort of a multiplier effect. In Table 2 this mechanism is seen to become even more important after a quarter at $t=1$, where the deviations of $y_1^{(h)}$ from $y_1^{(1)}$ are stronger than the deviations of $\pi_1^{(h)}$ from $\pi_1^{(1)}$ ($h=1/3$).

The quantitative effects of different frequencies summarized in Table 2 are based on a given numerical parameter scenario. This raises the question of the robustness of these results. Here we content ourselves with concentrating on the Calvo price stickiness as one central parameter. While $\theta \approx 2/3$ is a generally accepted benchmark that is supported by several empirical studies (Galí et al., 2001, p. 1255; Christiano et al., 2005, p. 18; or Álvarez et al., 2006, p. 578), one can also find lower and higher degrees of stickiness in the literature. Fabiani et al. (2007, p. 41) analyze data from surveys of 11000 firms which were conducted by the national banks of 9 European countries. The median number of price changes per year in these different countries is equal to one, which implies $\theta \approx 3/4$. By estimating a DSGE model with sticky prices and wages for the euro area with Bayesian techniques, Smets and Wouters (2003, p. 1144) obtain a much longer average duration of price contracts of two and a half years, or $\theta \approx 9/10$. In contrast, from the data of the U.S. Bureau of Labor Statistics, Bils and Klenow (2004, p. 953) derive evidence for firms changing their price every two quarters or even less, according to which θ can become as low as 1/2 and less.

This material gives us an idea of the range of price stickiness that a robustness check should cover, that is, $0.50 \leq \theta \leq 0.90$. Increasing the parameter in steps of 0.01 from the lower to the upper end of the interval, we compute for each of these values, and for $t = 0, 1/3, 1$, the same deviation statistics d_t as in Table 2. Doing this for all of the four model variants and plotting d_t against θ, the graphs presented in Figure 3 and 4 are obtained. The dotted vertical lines indicate the benchmark $\theta = 2/3$, which has been investigated above. [20]

The first result to observe is the monotonic relationship between θ and d_t. Except for $t=0$ for the inflation rate in NKB, the deviations diminish as the prices become stickier. For the IRFs of the output gap at $t=1$, on the other hand, the deviations increase well above 50% or even 100% when the prices become more and more flexible. The effects on

and $1/a(h) = 0.53$ for $h=1/3$.

[19] The presence of $b(h)$ in the coefficient on the driving variable in the Phillips curve and the absence of a similar term in the IS equation even carries over to the final closed-form solutions for $\pi_t^{(h)}$ and $y_t^{(h)}$, respectively; see eq. (10) in the Appendix. However, observe also that this type of reasoning is insufficient for HMP and HIS, where for the impact response we obviously have strong counteracting effects from the backward-looking element in the IS equation.

[20] Additional results covering d_t for $0 \leq t \leq 5$ are given in the Appendix.

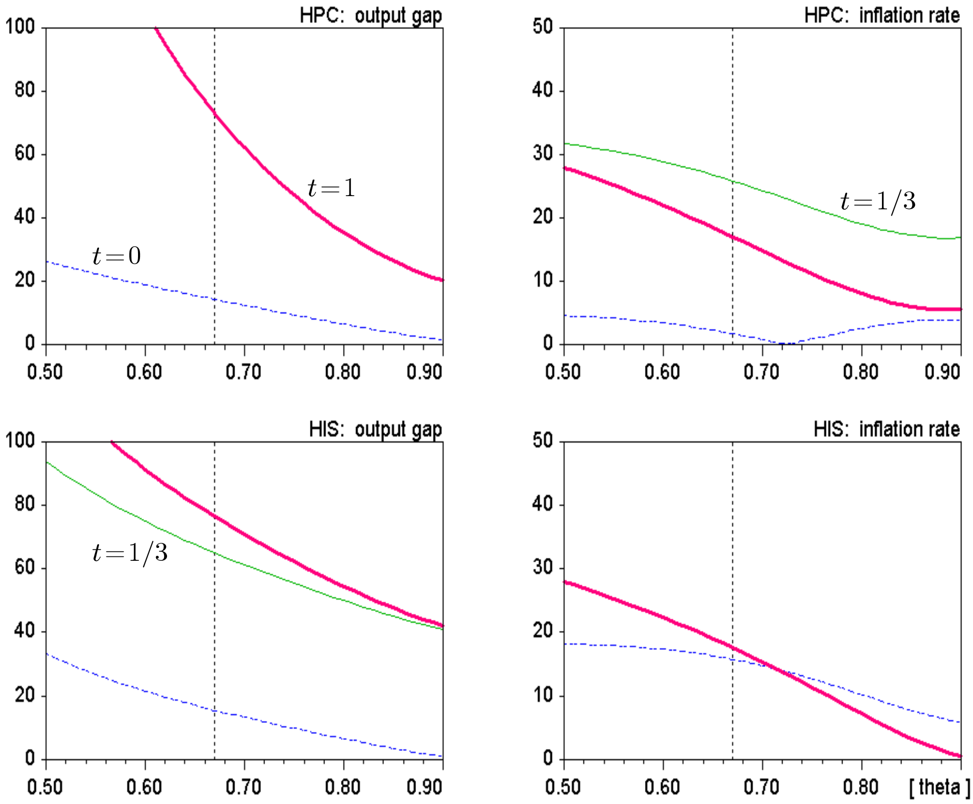


Figure 4: Absolute percentage deviations d_t of monthly from quarterly IRFs under variations of the price stickiness parameter θ; in HPC and HIS.

the inflation rate are similar in kind but more limited in size.[21]

A second remarkable outcome is that, if we focus on the bold (red) lines for the deviations at $t\!=\!1$, they are not very different across the variants NKB, HMP, HPC and HIS for either the output gap and the inflation rate. A side result is that the deviations in the impact effects are rather small (except for the lower price stickiness in the purely forward-looking case, when we look at the output gap). The intermediate effects at $t\!=\!1/3$ can be stronger than d_t for $t\!=\!1$, which is the case for the inflation rate, or they can be slightly weaker, which is seen in the output gap diagrams.

The minimum conclusion that can be drawn from this numerical robustness analysis is that, between the output gap IRFs at $t=1$ of the quarterly and monthly economies with a hybrid IS equation (and with or without a hybrid Phillips curve), there are substantial differences of more than 50%. This holds over the entire range of price stickiness

[21] The deviations of the output gap (but not inflation) can become extreme when the real business cycle core of the model is approached, i.e., when $\theta \to 0$. In HMP, for example, they tend to 1800% in this case. The reason for this phenomenon is that in the quarterly economy y_t is already close to zero at $t = 1$, whereas in the monthly economy convergence towards zero takes considerably longer; cf. eq. (9).

considered, and the effects are severely aggravated as the prices become more flexible. If in addition the different shapes of these IRFs are taken into account, the results of the quarterly economies concerning the output gap must be evaluated as somewhat special, or nonrepresentative.

6. Conclusion

Referring to a standard New-Keynesian model, the Phillips curve and IS equation of which allow for a purely forward-looking and a hybrid version, this paper asked for the possible implications if one varies the length of the period, i.e., the agents' decision interval. The question can be seen as reviving Foley's (1975) long neglected methodological precept that the basic properties of a discrete-time model without a "natural period" should not depend on the length of the period.

Our interest concentrated on the impulse-response functions (IRFs) of a monetary policy shock. On the one hand, it was found that, if we abstract from the discontinuities arising from the impulse in the hybrid versions, the limit of the high-frequency economies as their period shrinks to zero is well-defined, thus satisfying Foley's most fundamental requirement. At the lower frequencies, on the other hand, we provided evidence of marked qualitative differences. In the hybrid variants of the model with a monthly or shorter decision interval, the IRFs initially reinforce the impact reaction for a short while and only then return to the steady state values. This produces a sharp spike in the solution path, which is absent in the quarterly model. Furthermore, even when limiting the investigation to a comparison of the quarterly and monthly economies, the quantitative gaps in the initial phase of the two IRFs are not negligible. In particular, as confirmed by a robustness analysis of *ceteris paribus* variations of the Calvo price stickiness parameter, stronger reactions of the monthly output gap of 50 per cent and more after the first quarter are by no means exceptional.

These effects are stronger than what might be expected from the, as we have called them in Section 4, pure compound interest effects. They may provide a first-round argument in an economic reasoning, but apart from the problem of the 'second-round' arguments it must be realized that they only contribute to an explanation of the impact responses and are not sufficient to understand the differences across the frequencies at higher lags of the IRFs. In the end we are afraid that we have to fall back on the agnostic view that the coefficients in the reduced-form solution of the rational expectations economy are highly nonlinear functions of the structural parameters, inclusive as it turns out of the frequency of decision-making (cf. the analytical expressions in the Appendix for the purely forward-looking model as the simplest case). Therefore, at least presently, we have to accept the numerical outcome of the IRFs as it is.

The results of our numerical exercises resuscitate the question, "which interval best represents agents' decision-making process?" (Aadland, 2001, p. 291). One answer is more practically oriented: use calibration or estimation methods to find out if variations of the period length can improve the matching of certain empirical moments or the value of an objective function in general. However, research in this direction seems to be rare.[22]

A theoretical answer returns to Foley's principle and requires robustness of the period on which one decides to settle down. Accordingly, a period may be called robust if it is an upper-bound on the length of the decision intervals with, in our case, essentially similar IRFs. From this point of view, the main conclusion of the paper is that, in the present modelling framework, the conventional quarter cannot be claimed to be a robust period length. The more general message is that DSGE modelling may be more sensitive to the possible "pitfalls of timing misspecification" (to use the words of Christiano, 1985, p. 397).[23]

Appendix

Solution of NKB(h)

For easier reference, we first restate the subcase NKB(h) of system (4), (5) where the contemporaneous Taylor rule and, regarding the compatibility with (1), the observation $b(1) = \lambda$ for $h=1$ are understood:

$$\begin{aligned}
\pi_t &= \beta(h)\, E_t[\pi_{t+h}] \; + \; h\, b(h)\, y_t \\
y_t &= E_t[y_{t+h}] \; - \; (h/\sigma)\,(i_t - E_t[\pi_{t+h}] - \rho) \\
v_{t+h} &= \alpha(h)\, v_t \;\; := \;\; [1 - h(1-\alpha)]\, v_t
\end{aligned}$$

The solution formula can directly be taken over from Galí (2008, p. 51). Adjusting the notation correspondingly and recalling $\alpha = \alpha(1)$, $\beta = \beta(1)$, the solution for the quarterly economy reads here,

$$\begin{aligned}
\pi_t &= -\beta(1)\,\Lambda\, v_t \\
y_t &= -(1 - \beta\alpha)\,\Lambda\, v_t \\
C &:= \sigma\,(1-\alpha)\,(1-\beta\alpha) \; - \; \alpha\, b(1) \\
\Lambda &:= 1\,/\,[\,C \, + \, b(1)\,\phi_\pi \, + \, (1-\beta\alpha)\,\phi_y\,]
\end{aligned}$$

Using $1 - \beta(h)\alpha(h) \; = \; h\,(1 + \rho - \alpha)/(1 + h\rho)$, the counterparts for C and Λ in the high-frequency economies become

[22] Christiano (1985) and Aadland (2001) may be recalled as the only two references regarding this question that we know. More modestly, Ahrens and Sacht (2011) estimate a high-frequency New-Keynesian Phillips curve on daily inflation data for Argentina. In particular, the results for the Calvo price stickiness parameter are quite in line with microeconomic evidence. Future estimations might also consider a differentiation of the decision intervals of households and firms.

[23] In particular, it would be an important question how seriously the structure of optimal monetary policy rules might be affected by changes in the length of the period.

$$
\begin{aligned}
C &= (\sigma/h)\,[1-\alpha(h)]\,[1-\beta(h)\alpha(h)] \;-\; \alpha(h)\,h\,b(h) \\
&= h\,\{\,\sigma\,(1-\alpha)\,(1+\rho-\alpha)\,/\,(1+h\rho) \;-\; [1-h(1-\alpha)]\,b(h)\,\} \\
&=:\; h\;c(h) \\
1/\Lambda(h) &= h\,[\,c(h) \;+\; b(h)\,\phi_\pi \;+\; (1+\rho-\alpha)\,\phi_y\,/\,(1+h\rho)\,] \\
&=:\; h\;d(h)
\end{aligned}
$$

The solution of NKB(h) can thus be written as

$$
\begin{aligned}
\pi_t^{(h)} &= -h\,b(h)\,\Lambda(h)\,v_t^{(h)} &&= -[\,b(h)\,/\,d(h)\,]\,v_t^{(h)} \\
y_t^{(h)} &= -[1-\beta(h)\alpha(h)]\,\Lambda(h)\,v_t^{(h)} &&= -\{\,(1+\rho-\alpha)\,/\,[(1+h\rho)\,d(h)]\,\}\,v_t^{(h)}
\end{aligned}
\qquad (10)
$$

Clearly, the expressions $b(h)$ and $c(h)$ tend to finite values as $h \to 0$. Limiting our interest to $\phi_\pi \geq 1$, it is also immediately seen that the sum $c(h) + b(h)\,\phi_\pi$ in $1/\Lambda(h)$ exceeds $-[1-(1-\alpha)]\,b(h) + b(h)$, which in turn is not less than zero for $h \geq 0$. This ensures that $d(h)$ is positively bounded away from zero as h becomes infinitesimally small.[24] Hence the functions $f_\pi = f_\pi(h)$, $f_y = f_y(h)$ have the properties stated in Observation 1.

Arguments underlying Observation 2

To address the issue of transforming the hybrid model variants into continuous time, consider a positive point in time $t > 0$ and periods of length $h = t/k$ for natural numbers $k \in \mathbb{N}$. If, beginning with the Phillips curve, we suitably rearrange the inflation rates, divide through by h, and use the parametric relationship

$$
\frac{1-\gamma_f(h)-\gamma_b(h)}{h} \;=\; c(h) \;:=\; \frac{(1-\omega)\,\rho\,[1-h(1-\theta)]}{(1+h\rho)\,a(h)}
$$

we obtain

$$
\frac{\pi_{t+h}-\pi_t}{h} \;-\; \frac{\pi_t-\pi_{t-h}}{h} \;=\; \frac{-[2\gamma_f(h)-1]}{\gamma_f(h)}\,\frac{\pi_t-\pi_{t-h}}{h} \;+\; \frac{c(h)\,\pi_{t-h} \;-\; b(h)\,y_t}{\gamma_f(h)} \qquad (11)
$$

Should the solution paths of the h-economies happen to exhibit sufficient smoothness over time and across frequencies, such that the difference quotients on the left-hand side of (11) tend to become equal as $h \to 0$, the time derivative of π can be obtained by setting the limit of the right-hand side equal to zero and solving it for $\dot{\pi}$. With $b^o = b(0)$ and the simplifications in the other frequency-dependent expressions for $h = 0$, this yields

$$
\dot{\pi} \;=\; \rho\,\pi \;-\; \frac{1+\omega}{1-\omega}\,b^o\,y \qquad \text{(evaluated at } t > 0\text{)} \qquad (12)
$$

In a similar way, a high-frequency hybrid IS equation can be rearranged as

[24] Showing this for the case $\phi_\pi < 1$, given that (7) is still satisfied, requires a little bit more effort.

$$\frac{y_{t+h}-y_t}{h} - \frac{y_t-y_{t-h}}{h} = \frac{-[2\delta_f-1]}{\delta_f}\frac{y_t-y_{t-h}}{h} + (1/\sigma\delta_f)\,[\,(\phi_\pi-1)\,\pi_t - (\pi_{t+h}-\pi_t)+v_t\,] \tag{13}$$

where $\delta_f := 1/(1+\chi)$. Applying the same assumption and reasoning as before leads to the following differential equation for the output gap,

$$\dot{y} = \frac{1+\chi}{(1-\chi)\,\sigma}\Big[(\phi_\pi-1)\,\pi + \phi_y\,y + v\Big] \qquad \text{(evaluated at } t>0) \tag{14}$$

In the way they were derived, the differential equations (12) and (14) cannot possibly be defined at $t=0$. From (11) and (13) it is clear that $\dot{\pi}$ and $\dot{y}$ must be interpreted as backward derivatives. Since π_{t-h}, y_{t-h} are predetermined at their steady state values and the monetary policy impulse causes an immediate jump of the h-economies out of this state, the difference quotients $(\pi_t-\pi_{t-h})/h$ and $(y_t-y_{t-h})/h$ become unbounded as h tends to zero—at least if it can be presumed that the impact responses do not become negligibly weak for $h\to 0$.

Arguments underlying Observation 5

The reduced-form solutions of the h-economies are given by

$$z_t^{(h)} = \Omega^{(h)}\,z_{t-h}^{(h)} + \Gamma^{(h)}\,v_t^{(h)} \tag{15}$$

where $z^{(h)}=(\pi_t^{(h)},y_t^{(h)})'$, $\Omega^{(h)}\in\mathrm{I\!R}^{2\times2}$, $\Gamma^{(h)}\in\mathrm{I\!R}^{2\times1}$. As can also be inferred from Figures 1 and 2, the two matrices appear to converge towards finite and nonzero matrices Ω^o, Γ^o as $h\to 0$ (with the exception of $\Omega^{(h)}\equiv 0$ in the baseline model). Incidentally, for $h=1/(24\cdot 90)$ the transition matrix $\Omega^{(h)}$ is close to a diagonal matrix (with entries of approximately 0.30 and 0.60 on the main diagonal). This decoupling of prices and quantities might be an interesting result in itself.

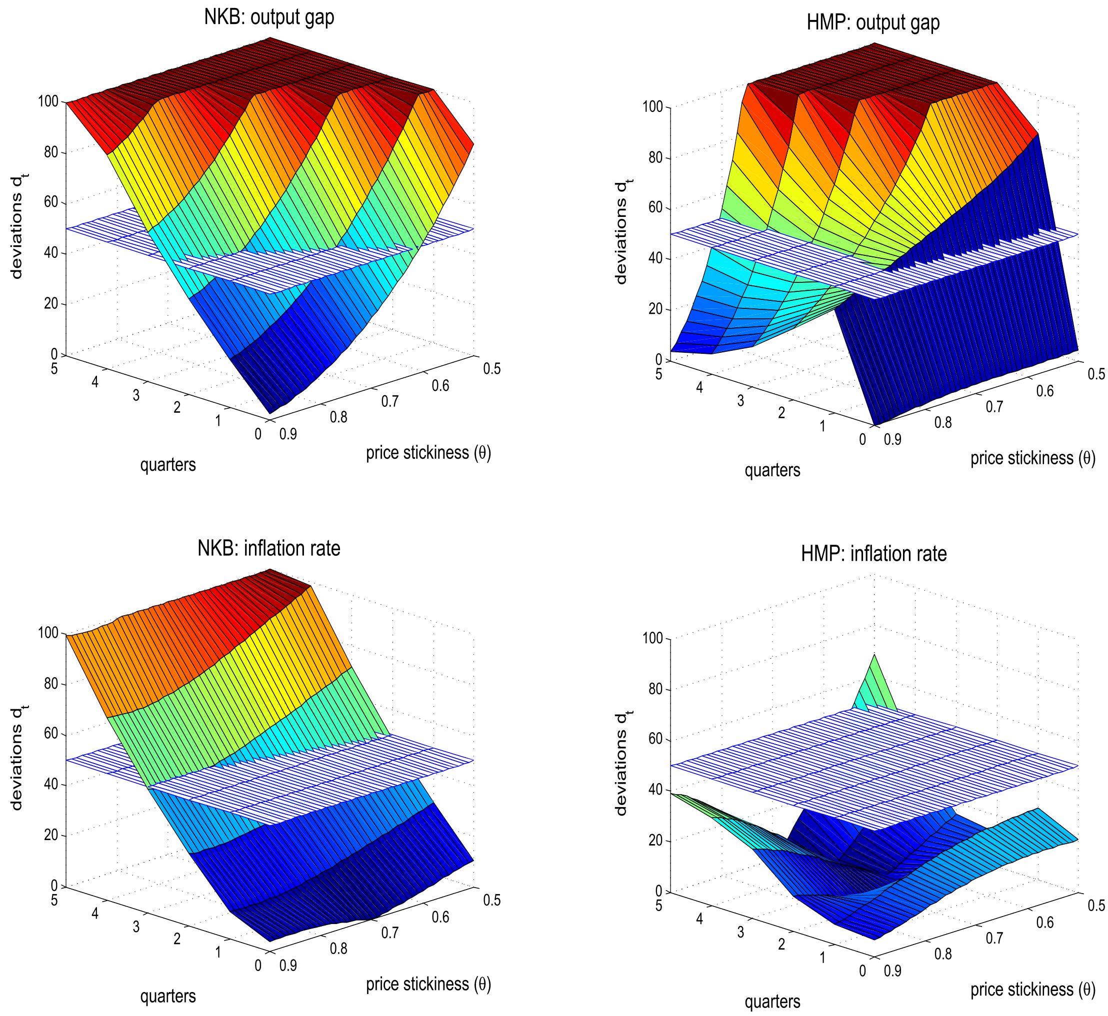
Equation (15) can be used to compute the forward difference quotients of inflation and the output gap. Leading it by one period, subtracting one equation from the other and dividing the result by h yields

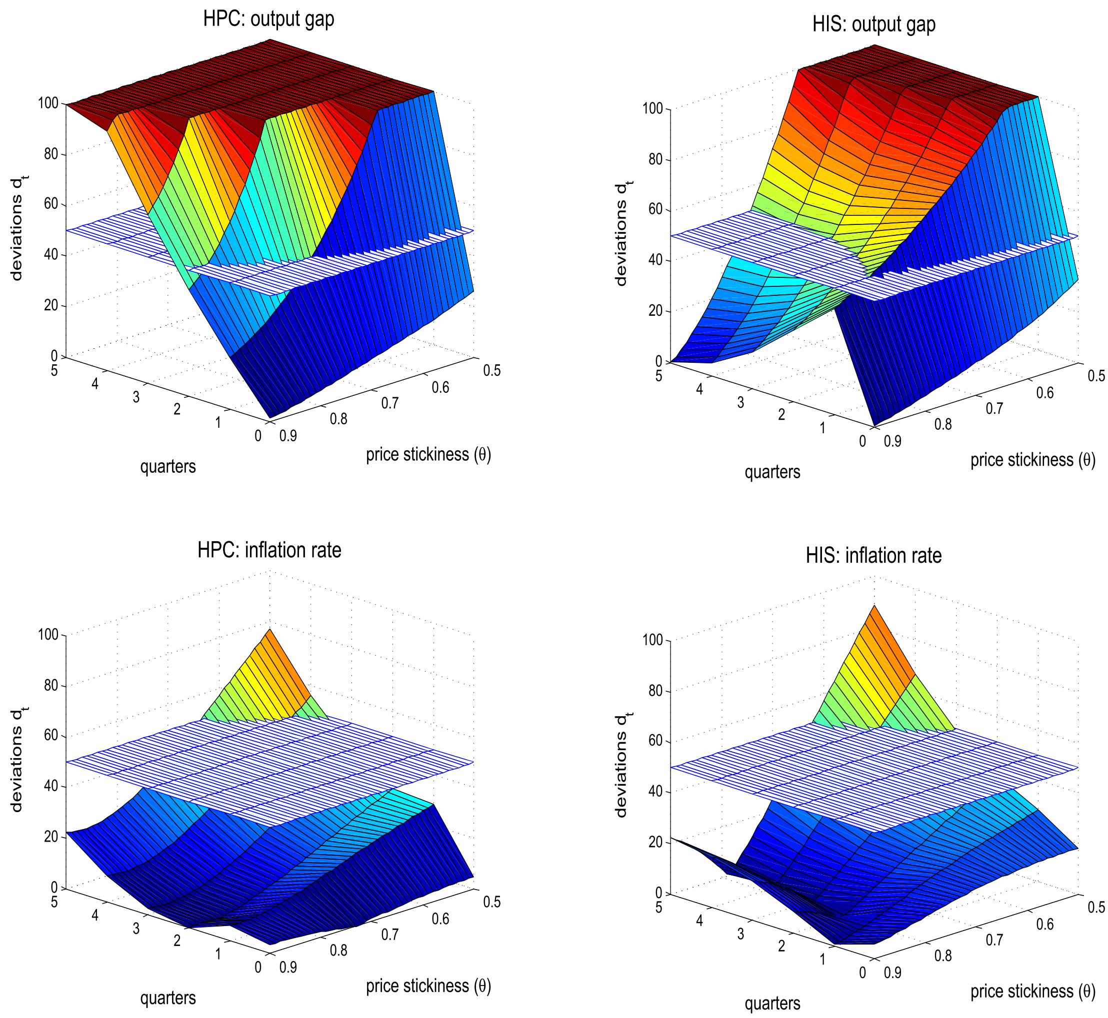
$$\frac{z_{t+h}^{(h)}-z_t^{(h)}}{h} = \Omega^{(h)}\,\frac{z_t^{(h)}-z_{t-h}^{(h)}}{h} + \Gamma^{(h)}\,\frac{v_{t+h}^{(h)}-v_t^{(h)}}{h} \tag{16}$$

Owing to $z_{-h}^{(h)}=0$, $v_{-h}^{(h)}=0$ and $z_0^{(h)}=\Gamma^{(h)}\,\varepsilon^v\to\Gamma^o\,\varepsilon^v\neq 0$, the first difference quotient on the right-hand side of (16), when evaluated at $t=0$, diverges as h becomes small. Since $(v_{t+h}^{(h)}-v_t^{(h)})/h=-(1-\alpha)\,v_t^{(h)}$ is bounded, the limit of the left-hand side of this equation is not well-defined, either. By the same token, the second-period effects $z_h^{(h)}$ remain bounded away from the impact effects $z_0^{(h)}$. These features, then, allow us to formulate Observation 5.

Deviations d_t from eq. (9) across quarters t and price stickiness θ

The surface of d_t above the plane $0 \leq t \leq 5$ and $0.50 \leq \theta \leq 0.90$ is truncated at $d = 100\%$. A plateau at $d = 50\%$ indicates an upper-bound on deviations that might still be regarded as acceptable.





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