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Abstract

This paper presents a new cooperative equilibrium for strategic form games, denoted Conjectural Cooperative Equilibrium (CCE). This concept is based on the expectation that joint deviations from any strategy profile are followed by an optimal and noncooperative reaction of non deviators. We show that CCE exist for all symmetric supermodular games. Furthermore, we discuss the existence of a CCE in specific submodular games employed in the literature on environmental agreements. *Keywords*: Strong Nash Equilibrium, Cooperative Games, Public Goods. JEL Classification: C7

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1 Introduction

Intuitively a *cooperative equilibrium* is a collective decision adopted by a group of individuals that can be viewed as *stable* (i.e., an equilibrium) against all feasible deviations by single individuals or by proper subgroups. While modelling the possibilities of cooperation may not pose the social scientist particular problems, at least once an appropriate economic or social situation is clearly outlined, the definition of *stability* may be a more demanding task for the modeler. This because the outcome, and the profitability, of players' deviations heavily depends on the *conjectures* they make over the reaction of other players. As an example, a neighborhood rule to keep a common garden clean possesses different stability properties whether the conjectured reactions in the event of shirking is, in turn, that the garden would be kept clean anyway or, say, that the common garden would be abandoned as a result. Similarly, countries participating to an international environmental agreement will possess different incentive to comply with the prescribed pollution abatements whether defecting countries expect the other partners to be inactive or to retaliate.

The main focus of the present paper are cooperative equilibria of games in strategic form. A cooperative equilibrium of a game in strategic form can be defined as a *strategy profile* such that no subgroup of players can "make effective" - by means of alternative strategy profiles - utility levels higher for its members than those obtained at the equilibrium. As expressed in the example above, the content of the equilibrium concept depends very much on the utility levels that each coalition can potentially make effective and this, in turn, depends on the *conjectures* over the reactions induced by deviations. In this paper we propose a cooperative equilibrium for games in strategic form, based on the assumption that players deviating from an arbitrary strategy profile have *non zero conjectures* on the reaction of the remaining players. More precisely, the *conjectural cooperative equilibrium* we propose assumes that the remaining players are expected to optimally and independently react according to their best response map.

1.1 Related literature

The problem of defining cooperative equilibrium concepts have been centered on the formulation of *conjectures* ever since the pioneering work of von Neumann and Morgenstern's (1944). The concepts of α and β core, formally studied by Aumann (1967), are based on their early proposal of representing the worth of a coalition as the aggregate payoff that it can guarantee its members in the game being played. Formally obtained as the minmax and maxmin payoff imputations for the coalition in the game played against its complement, the α and β characteristic functions express the behaviour of extremely risk averse coalitions, acting *as if* they expected their rivals to minimize their payoff. Although fulfilling a rationality requirement in zero sum games, α and β-assumptions do not seem justifiable in most economic settings. Moreover, the little profitability of coalitional objections usually yield very large set of solutions (e.g., large cores). Another important cooperative equilibrium proposed by Aumann (1959), denoted *Strong Nash Equilibrium*, extends the Nash Equilibrium assumption of "zero conjectures" to every coalitional deviation. Accordingly, a Strong Nash Equilibrium is defined as a strategy profile that no group of players can profitably object, given that remaining players are expected not to change their strategies. Strong Nash Equilibria are at the same time Pareto optima and Nash Equilibria; in addition they satisfy the Nash stability requirement for each possible coalition. As a consequence, the set of Strong Nash Equilibria is often empty, preventing the use of this otherwise appealing concept in most economic problems of strategic interaction.

Other approaches have looked at the choice of forming coalitions as a strategy in well defined games of coalition formation (see Bloch (1997) for a survey). Among others, the gamma and delta games in Hart and Kurz (1985) constitute a seminal contribution.[1] The gamma game, in particular, is related to the present analysis, since it predicts that if the grand coalition N is objected by a subcoalition S, the complementary set of players splits and act as a noncooperative fringe. On the same behavioural assumption is based the concept of γ core, introduced by Chander and Tulkens (1997) in the analysis of environmental agreements, where a characteristic function is obtained as the Nash equilibrium between the forming coalition and all individual players in its complement. As in the present approach, based on deviations in the underlying strategic form game, the γ core assumes that the forming coalition expects outside players to move along their (individual) reaction functions. Differently from our approach, however, there the forming coalition forms before choosing its Nash equilibrium strategy in the game against its rivals, while here deviating coalitions directly switch to new strategies in the underlying game, expecting their rivals to react in the same manner as followers in a Stackelberg game. In applying our concept to the analysis of stability of environmental coalitions, we may interpret these differences as the description of different structures in the process of deviation. While the γ core seems to describe settings in which the formation of a deviating coalition is publicly observed before the choice of strategies, our approach best fits situations in which deviating coalitions can implement their new strategies

[1] More precisely, Hart and Kurz (1983) present endogenous coalition formation games and look at the Strong Nash of these games. Other related papers (i.e., Chander and Tulkens (1998), Yi (1998)) look at the Nash equilibrium taking as given the gamma and delta rule of coalition formation.

before their formation is monitored, enjoying a positional advantage.

The *conjectural cooperative equilibrium* we propose in this paper, by assuming that remaining players are expected to optimally react according to their best response map, introduces a very natural rationality requirement in the equilibrium concept. Moreover, the coalitional incentives to object are considerably weakened with respect to the Strong Nash Equilibrium, thus ensuring the existence of a cooperative conjectural equilibrium in all symmetric games in which players' actions are strategic complements in the sense of Bulow et al. (1985), i.e., in all supermodular games (see Topkis (1998)).

1.2 An example of a conjectural cooperative equilibrium

Before formally defining the conjectural cooperative equilibrium, it is easy to introduce the mechanics at works for the existence of such an equilibrium by means of the following 3x3 bi-matrix game.

	A	**B**	**C**
A	x, x	d, h	a, c
B	h, d	b, b	e, f
C	c, a	f, e	y, y

Suppose, in the game above, that (b, b) is an efficient outcome, i.e, such to maximize the sum of players' payoff. To be a cooperative equilibrium, the outcome (b, b) has to be immune from either player switching her own strategy, given their expectation that the rival would optimally react to the switch. When players actions are strategic substitutes (and the game submodular), each player's reaction map is downward sloped, implying that any move from (b, b) by one player would generate a predicted outcome on the asymmetric diagonal of the matrix. If we let, in the example, $a > b > c > h$, and $b > \frac{a+c}{2}$, then the efficient outcome (b, b) will not certainly be a conjectural cooperative equilibrium, for player 1 will profitably deviate from it (from B to A), conjecturing that her rival's best reply will go in the opposite direction (from B to C), and getting a payoff of $a > b$. The same will happen if $c > b > a > e$, in which case player 2 deviates by switching from B to C. In contrast, suppose that the game above is supermodular, with the associated increasing reaction maps. In this case, the conjectured outcomes in case of deviations from outcome (b, b) are only (x, x) and (y, y). As a result, if either player finds it profitable to switch either to A or to C (with $x > b$ and $y > b$, respectively) then the assumption that (b, b) is an efficient outcome is contradicted. We can conclude that (b, b) is a conjectural cooperative equilibrium of the symmetric game described above whenever supermodularity holds. Note that in our example, if $d > b$, the efficient outcome (b, b) is a conjectural cooperative equilibrium although it is neither a Strong Nash Equilibrium nor a Nash Equilibrium.[2] The above example, although providing a clear insight of how both supermodularity and symmetry work in favour of the existence of an equilibrium, contains two substantial simplifications: the presence of only two players, ruling out existence problems related to the formation of coalitions, as well as the restriction to 3 strategies, thus forcing the increasing best replies to generate symmetric outcomes, from which, the fact that (B,B) is an equilibrium, directly follows. However, in the paper we are able to show that the existence result holds for any number of players and strategies, provided a symmetry assumption on the effect of players' own strategies on the payoff of rivals is fulfilled.

The paper is organized as follows. The next section introduce the conjectural cooperative equilibrium in the standard setup of strategic form games. Section 3 presents the main paper result: for a well defined class of games, symmetric supermodular games, a conjectural cooperative equilibrium always exists. Section 4 discusses in detail the meaning of this result and presents a descriptive example of an environmental economy whose cooperative conjectural equilibrium exists depending on individuals' preferences. Section 5 concludes.

2 Set Up

We consider a **game in strategic form** $G = \left(N, (X_i, u_i)_{i \in N}\right)$, in which $N = \{1, ..., i, ..., n\}$ is the set of players, X_i is the set of strategies for player i, with generic element x_i, and $u_i : X_1 \times ... \times X_n \to R_+$ is the payoff function of player i. We denote by $S \subset N$ any coalition of players, and by $\bar{S}$ its complement with respect to N. For each coalition S, we denote by $x_S \in X_S \equiv \prod_{i \in S} X_i$ a profile of strategies for the players in S, and use the notation $X = X_N$ and $x = x_N$. A **Pareto Optimum (PO)** for G is a strategy profile such that there exists no alternative profile which is preferred by all players to and strictly preferred by at least one player. The Pareto Optimum x^e is **efficient** if it maximizes the sum of the payoffs of all players in N. In the example discussed in the above introduction, letting outcomes be ordered as follows: $a > b > c > d > e > h > x > y$, and assuming that $b > \frac{a+c}{2}$, the profiles (a, c), (c, a) and (b, b) are all Pareto Optima, while the efficient profile is (b, b).

A **Nash Equilibrium (NE)** for G is defined as a strategy profile $\bar{x} \in X_N$ such that no player has an incentive to change his own strategy, i.e., such that there exists no $i \in N$ and

[2] Similarly, in a 2x2 Prisoner's Dilemma, although no Strong Nash Equilibria exist, the efficient strategy profile, that is not even a Nash equilibrium, turns out to be a CCE.

$x_i \in X_i$ such that

$$u_i(x_i, \bar{x}_{N \setminus i}) > u_i(\bar{x}).$$

Nash equilibria are stable with respect to individual deviations, given that the effect of such deviations is evaluated keeping the strategies played by the other players fixed at the equilibrium levels.

In trying to formulate equilibrium concepts that allow coalitions of players to coordinate in the choice of their strategies, a natural extension of the Nash equilibrium is given by the concept of **Strong Nash equilibrium (SNE)**, a strategy profile that no coalition of players can improve upon given that the effect of deviations is, again, evaluated keeping the strategies of other players fixed at the equilibrium levels. So, $\hat{x} \in X_N$ is a SNE for G if there exists no $S \subset N$ and $x_S \in X_S$ such that

$$\begin{aligned} u_i(x_S, \hat{x}_{\bar{S}}) &\geq u_i(\hat{x}) \;\; \forall i \in S; \\ u_h(x_S, \hat{x}_{\bar{S}}) &> u_h(\hat{x}) \;\; \text{for some } h \in S. \end{aligned}$$

Obviously, all SNE of G are both Nash Equilibria and Pareto Optima. As a result, SNE fails to exist in many economic problems, and in particular, whenever Nash Equilibria fail to be Optimal. Although the lack of existence of SNE can be interpreted as a poor specification of the game theoretic model, it precludes the use of this otherwise appealing concept of a cooperative equilibrium in many important applications.

In this paper we propose a concept of cooperative equilibrium for G based on the introduction of non-zero conjectures in the evaluation of the profitability of coalitional deviations. The concept we propose captures the idea that players outside a deviating coalition are expected to react by making optimal choices (contingent on the strategy profile played in the deviation) as independent and noncooperative players. In order to describe the conjectured optimizing reactions of players outside a deviating coalition S, let us define first the restricted game $G(x_S)$ obtained from G by considering the restricted set of players $\bar{S}$, and parameterizing payoffs by letting each j in $\bar{S}$ obtain the payoff $u_j(x_{\bar{S}}, x_S)$ out of the profile $x_{\bar{S}}$, for each $x_{\bar{S}} \in X_{\bar{S}}$. We denote by $R_{\bar{S}} : X_S \to X_{\bar{S}}$ the map associating with each joint strategy x_S of coalition S the set $R_{\bar{S}}(x_S)$ of Nash Equilibria of the restricted game $G(x_S)$. The set $R_{\bar{S}}(x_S)$ describes the conjecture of coalition S on the possible reactions of players in $\bar{S}$ to the choice of the joint strategy x_S.

Definition 1 *A* ***Conjectural Cooperative Equilibrium (CCE)*** *is a strategy profile* $\widetilde{x}$

such that there exists no coalition S, strategy profiles $x_S \in X_S$ and $x_{\bar{S}} \in R_{\bar{S}}(x_S)$ such that:

$$\begin{aligned} u_i(x_S, x_{\bar{S}})) &\geq u_i(\widetilde{x}) \ \ \forall i \in S \\ u_h(x_S, x_{\bar{S}})) &> u_h(\widetilde{x}) \ \ \textit{for some } h \in S. \end{aligned}$$

So defined, a CCE satisfies very restrictive stability requirements. According to definition 1, any coalition S can look for improvements upon any proposed strategy profile by selecting among its feasible joint profiles $x_S \in X_S$ and, for each possible x_S it may choose, by selecting among all the Nash responses of players in $\bar{S}$ (formally, the set $R_{\bar{S}}$) the most profitable strategy $x_{\bar{S}}$. Definition 1 is indeed well defined both when the set $R_{\bar{S}}(x_S)$ may be empty for some (possibly all) $x_S \in X_S$, and when the set $R_{\bar{S}}(x_S)$ is multivalued for some (possibly all) $x_S \in X_S$. In this sense, it applies to all games in strategic form. This generality comes at the price of the arguably unreasonable assumption that a deviating coalition faces no constraint in selecting among the possibly non unique reactions of outside players. A more realistic approach would assume that a deviating coalition should form expectations about which equilibrium reaction would be played by outside players, and that these expectations should be based on some sort of rationality requirement on the behaviour of such outside players. We remark, however, that the present approach generates a smaller set of equilibria than would result from any arbitrary selection from the set of Nash responses of outside players. Our result of existence of a CCE in all supermodular games, contained in theorem 1 in section 3.3, would therefore extend to any equilibrium concept associated with the choice of such a selection. In addition, lemmas 7-10 show that the present definition generates the same set of equilibria that would result from the selection of the Pareto dominant element of the set $R_{\bar{S}}(x_S)$. Since the existence of such elements is not generally ensured, but always holds on the class of symmetric supermodular games for which our result is obtained (see section 3.1 for definitions), we have chosen to present definition 1 in its present, and more general, form.

3 Existence of a conjectural cooperative equilibrium in supermodular games

This section contains our main result, showing that if a strategic form game G is supermodular, and satisfies some symmetry requirements, then admits a conjectural cooperative equilibrium.

3.1 Supermodularity

We start by defining the concept of a supermodular function and by recording some results in the theory of supermodularity that will be used in the analysis of the next section. For a partially ordered set $A \subset R^n$ and any pair of elements x, y of A, we define the join element $(x \wedge y)$ and the meet element $(x \vee y)$ as follows:

$$\begin{aligned} (x \wedge y) &= (\min\{x_1, y_1\}, ..., \min\{x_n, y_n\}); \\ (x \vee y) &= (\max\{x_1, y_1\}, ..., \max\{x_n, y_n\}). \end{aligned}$$

Definition 2 *The set A is a sublattice of R^n if $(x \vee y) \in A$ and $(x \wedge y) \in A$ for all $x, y \in A$.*

Definition 3 *The function $f : A \to R$ is supermodular if for all $x, y \in A$:*

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

Definition 4 *Let X, Y be partially ordered sets. The function $f : X \times Y \to R$ has increasing differences in (x, y) on $X \times Y$ if the term $f(x, y'') - f(x, y')$ is increasing in x for all $y'' > y'$.*

Increasing differences describe a complementarity property of the function f, whose marginal increase with respect to y is increasing in x. If A is the Cartesian product of partially ordered sets, then the fact that f is supermodular on A implies that f has increasing difference in all pairs of sets among those whose product originates A (see Topkis (1998) for a formal statement and proof of this fact).

Definition 5 *The game in strategic form $G = \left(N, (X_i, u_i)_{i \in N}\right)$ is supermodular if the set X of feasible joint strategies for N is a sublattice of R^n, if the payoff functions $u_i(x_i, x_{-i})$ is supermodular in x_i and has increasing differences in (x_i, x_{-i}) on $X_i \times X_{-i}$.*

We will extensively exploit two properties of supermodular games, related to the existence of a Nash Equilibrium and to the behaviour of the set of Nash equilibria in response to changes in a fixed parameter on which these equilibria depend. We recall these properties below, and refer to Topkis (1998) for proofs.

Lemma 1 *Let $G = \left(N, (X_i, u_i)_{i \in N}\right)$ be a supermodular game, with X nonempty and compact and u_i upper hemicontinuous in x_i for all i. Then the set of Nash equilibria of G is nonempty and admits a greatest and least element.*

Lemma 2 *Let $G_t = \left(N, \left(X_i, u_i^t\right)_{i \in N}\right)_{t \in T}$ be a set of supermodular games, parameterized by t, with T being a partially ordered set. Let the assumptions of Lemma 1 hold. Then the greatest and least elements of the set of Nash equilibria of G are non decreasing in t on T.*

3.2 Assumptions and preliminary results

We impose the following lattice structure and continuity assumptions on our game in strategic form.

Assumption 1 *X_i is a compact sublattice of R, for all $i \in N$.*

Assumption 2 *u_i is continuous and supermodular in x_i on X_i for each $x_{-i} \in X_{-i}$, and exhibits increasing differences on $X_i \times X_{-i}$.*

Our requirement of continuity of u_i is unnecessarily strong for the establishment of existence and monotonicity of Nash equilibria in the next lemmas. However, we will need such assumption to ensure the existence of a strategy profile with certain properties in X as a step towards the proof of theorem 1 (see lemma 9). In addition to assumptions 1 and 2, we impose two symmetry requirements on G.

Assumption 3 *(Symmetric Players)**:** For all $x \in X$ and all pairwise permutations $p : N \to N$:*

$$u_{p(i)}\left(x_{p(1)}, ..., x_{p(n)}\right) = u_i\left(x_1, ..., x_n\right).$$

Assumption 4 *(Symmetric Externalities)**:** One of the following two cases must hold:*

1. *Positive externalities: $u_i(x)$ increasing in $x_{N \setminus i}$ for all i and all $x \in X_N$;*
2. *Negative externalities: $u_i(x)$ decreasing in $x_{N \setminus i}$ for all i and all $x \in X_N$.*

Assumption 3 requires that players payoffs are neutral to switches in the strategies played by other players, and that pairwise switches in strategies are mirrored by pairwise switches in payoffs. In other words, only strategies matter, and not who plays them. Assumption 4 requires that the effect of a change in other players' strategies on one's own payoff is monotonic, and its sign is the same for all players. Many well known games (including Cournot, Betrand and public good games) satisfy this symmetry assumption. The next results directly follow from an applications to our game G of the properties of supermodular games listed in lemmas 1 and 2.

Lemma 3 *Let assumptions 1 and 2 hold. For all $x_S \in X_S$, the set of Nash equilibria $R_{\bar{S}}(x_S)$ is nonempty and has a greatest and a least element.*

Proof 1 *Application of lemma 1.*

Let $r^g_{\bar{S}}$ and $r^l_{\bar{S}}$ the selections of the map $R_{\bar{S}}$ obtained by considering its greatest and least element, respectively.

Lemma 4 *Let assumptions 1 and 2 hold. The maps $r^u_{\bar{S}}$ and $r^l_{\bar{S}}$ are non decreasing in x_S.*

Proof 2 *Application of lemma 2.*

We finally make use of the symmetry assumptions 3 and 4 to show that the set $R_{\bar{S}}(x_S)$ is Pareto ranked.

Lemma 5 *Let assumptions 1-4 hold. If the payoff functions exhibit positive (negative) externalities, then for all x_S the element $r^g_{\bar{S}}(x_S)$ $(r^l_{\bar{S}}(x_S))$ Pareto dominates all other elements in $R_{\bar{S}}$ on the set of players $\bar{S}$.*

Proof 3 *Let $j \in \overline{S}$, $x_{\bar{S}} \in R_{\bar{S}}(x_S)$ and $x'_{\bar{S}} = r^g_{\bar{S}}(x_S)$ for some $x_S \in X_S$. Let externalities be positive. The following inequality follows:*

$$u_j(x_S, x'_{\bar{S}\setminus j}, x'_j) \geq u_j(x_S, x'_{\bar{S}\setminus j}, x_j) \geq u_j(x_S, x_{\bar{S}}).$$

The first inequality is due to the Nash equilibrium property of $x'_{\bar{S}}$ for the restricted game $G(x_S)$. The second inequality is due to positive externalities. Since the argument applies to all j in $\bar{S}$ and for all $x_{\bar{S}} \in R_{\bar{S}}(x_S)$, the result follows. The proof for the case of negative externalities is similar and is omitted.

3.3 Results

This section contains our main result: all games satisfying assumptions 1-4 admit a Conjectural Cooperative Equilibrium. The proof of theorem 1 is constructive: we show that every efficient symmetric strategy profile in X_N satisfies the conditions for being a CCE. Before proving this fact in theorem 1, we establish a few preliminary results. We first show that an efficiency symmetric strategy profile always exists under assumptions 1-4.

Lemma 6 *Let G satisfy assumption 1-4. Then there exists an efficient strategy profile $x^e \in X_N$ such that $x^e_i = x^e_j$ for all $i, j \in N$.*

Proof 4 *Compactness of each X_i implies compactness of X. Continuity of each u_i implies continuity of the social payoff function $u_N = \sum_{i \in N} u_i$. Existence of an efficient profile directly follows from Weiestrass theorem. To show that there exists a symmetric efficient profile, we need to exploit the supermodularity properties of payoff functions. Consider any arbitrary* *asymmetric profile x, with $x_i \neq x_j$ for some players i and j. By the symmetry assumption on payoff functions, we write*

$$u_N(x) = u_N(x_i, x_j, x_{N\setminus\{i,j\}}) = u_N(x_j, x_i, x_{N\setminus\{i,j\}}) \tag{1}$$

where we have used the convention of writing the strategies of players i and j as first and second elements of x, respectively. Since the sum of supermodular functions is itself supermodular, assumptions 1 and 2 imply:

$$2 \cdot u_N(x) \leq u_N\left(x_i, x_i, x_{N\setminus\{i\cup j\}}\right) + u_N\left(x_j, x_j, x_{N\setminus\{i\cup j\}}\right). \tag{2}$$

It follows that either

$$u_N(x) \leq u_N\left(x_i, x_i, x_{N\setminus\{i\cup j\}}\right) \tag{3}$$

or

$$u_N(x) \leq u_N\left(x_j, x_j, x_{N\setminus\{i\cup j\}}\right) \tag{4}$$

or both.

Suppose that (3) holds, and let $x' = \left(x_i, x_i, x_{N\setminus\{i\cup j\}}\right)$. This is without loss of generality for the ongoing argument. If $x_k = x_i$ for all $k \in N\setminus\{i \cup j\}$ our proof is complete. If not, then let $x_k \neq x_i$. In this case, again by supermodularity of payoff functions, we write

$$2 \cdot u_N(x') \leq u_N\left(x_i, x_i, x_i, x_{N\setminus\{i\cup j\cup k\}}\right) + u_N\left(x_i, x_k, x_k, x_{N\setminus\{i\cup j\cup k\}}\right). \tag{5}$$

Condition (5) implies, again, that either

$$u_N(x') \leq u_N\left(x_i, x_i, x_i, x_{N\setminus\{i\cup j\cup k\}}\right) \tag{6}$$

or

$$u_N(x') \leq u_N\left(x_i, x_k, x_k, x_{N\setminus\{i\cup j\cup k\}}\right) \tag{7}$$

or both. Suppose first that only (7) holds. Using the definition of x' we obtain

$$u_N\left(x_i, x_i, x_k, x_{N\setminus\{i\cup j\cup k\}}\right) \leq u_N\left(x_i, x_k, x_k, x_{N\setminus\{i\cup j\cup k\}}\right). \tag{8}$$

For this case, using again supermodularity, we write

$$2u_N\left(x_i, x_k, x_k, x_{N\setminus\{i\cup j\cup k\}}\right) \leq u_N\left(x_i, x_i, x_k, x_{N\setminus\{i\cup j\cup k\}}\right) + u_N\left(x_k, x_k, x_k, x_{N\setminus\{i\cup j\cup k\}}\right). \tag{9}$$

Using (8) and (9) we obtain that

$$u_N\left(x_i, x_k, x_k, x_{N\setminus\{i\cup j\cup k\}}\right) \leq u_N\left(x_k, x_k, x_k, x_{N\setminus\{i\cup j\cup k\}}\right). \tag{10}$$

Conditions (8) and (10) directly imply

$$u_N(x') \leq u_N\left(x_k, x_k, x_k, x_{N\setminus\{i\cup j\cup k\}}\right). \tag{11}$$

We have therefore shown that either (6) or (9) must hold. By iteration of the same operation for each additional player in $N\setminus\{i\cup j\cup k\}$*, we obtain the conclusion that there exists some symmetric profile* x^s *for which* $u_N(x^s) \geq u_N(x)$*. Since the starting profile* x *was arbitrary, and by the existence of an efficient profile proved in the first part of this proof, we conclude that a symmetric efficient profile* x^e *always exists under assumptions 1-4.*

We now consider the possible joint strategies that an arbitrary coalition S can use in order to improve upon an efficient profile x^e. In particular, we focus on the "best" strategies S can adopt, by this meaning the profiles $x^*(S) \in X_N$ satisfying the two following properties: *i)* $x^*_{\bar{S}} \in R_{\bar{S}}(x^*_S)$; *ii)* there exists no $x'_S \in X_S$ and $x'_{\bar{S}} \in R_{\bar{S}}(x'_S)$ such that $u_i\left(x'_S, x'_{\bar{S}}\right) \geq u_i(x^*)$ $\forall i \in S$ and $u_h(x'_S, r_{\bar{S}}(x'_S)) > u_h(x^*)$ for at least one $h \in S$. In words, $x^*(S)$ is a Pareto optimal profile for coalition S in the set $F(S)$ of all profiles that are consistent with the reaction map $R_{\bar{S}}$:

$$F(S) = \{x \in X_N : x_{\bar{S}} \in R_{\bar{S}}(x_S)\}.$$

Note that $F(S)$ is a compact set by the compactness of X_N and by the closedness of the Nash correspondence $R_{\bar{S}}(x_S)$.

Lemma 7 *Let* G *satisfy assumptions 1-4. Then for all* $x' \in F(S)$ *there exists some profile* $x^*(S) \in X_N$ *which is a best strategy for* S *in the sense of conditions i) and ii) above and such that* $u_i(x^*(S)) \geq u_i(x')$ *for all* $i \in S$.

Proof 5 *Let* $x' \in F(S)$*. If* $x' = x^*(S)$ *for some* $x^*(S)$ *then the lemma is proved for* x'*. If* $x' \neq x^*(S)$ *for all* $x^*(S)$*, then let the set*

$$P_i(x') = \left\{x_N \in F(S) : u_i(x) \succeq u_i(x')\right\}$$

define the set of strategy profiles that are weakly preferred by player i *to* x'*. The set* $P_i(x')$ *is nonempty by the fact that* $x' \neq x^*(S)$ *for all* $x^*(S)$*, and it is closed and bounded by continuity of* u_i *and by compactness of* $F(S)$*. Since this holds for all* $i \in S$*, it follows that the set* $P_S(x') = \cap_{i\in S} P_i(x')$ *is closed and bounded.*[3] *Moreover, it is non empty because* $x' \neq x^*(S)$*. We can therefore conclude that the problem*

$$\max_{x\in P_S(x)} \sum_{i\in S} \lambda_i u_i(x)$$

[3] We remind here that S is a finite set.

has a solution for all λ in the interior of the $\#S-1$ dimensional unitary simplex. Call $x(\lambda)$ such a solution. Clearly, $x(\lambda)$ satisfies conditions i) and ii) defining the profile $x^*(S)$. Also, clearly $x(\lambda)$ Pareto dominates x' on the set of players S, which concludes the proof.

By lemma 7, we can restrict our analysis to the "best" choices $x^*(S)$ of coalition S, since if S cannot profitably deviate by any such profiles, it cannot deviate by means of any profile in $F(S)$. We remark here that in the choice of a "best" profile $x^*(S)$, coalition S is assumed able to select among all the possible (equilibrium) reactions of $\bar{S}$, as specified by $R_{\bar{S}}$, in order to maximize its joint payoff. This is in line with our definition of a CCE, in which this ability of S was implicitly assumed. The next lemma shows that under assumptions 3 and 4 the best choice of S always selects strategies for $\bar{S}$ that are greater (least) elements of the set $R_{\bar{S}}(x_S)$, depending on the sign of the externality being positive or negative, respectively.

Lemma 8 *Let G satisfy positive (negative) externalities. Let $S \subset N$ and $x' \in F(S)$. Then, $u_i\left(x'_S, r^g_{\bar{S}}(x'_S)\right) \geq u_i(x')$ (respectively, $u_i\left(x'_S, r^l_{\bar{S}}(x'_S)\right) \geq u_i(x')$) for all $i \in S$.*

Proof 6 *We show only the case of positive externalities; the proof for negative externalities is symmetric and left to the reader. Since $r^g_{\bar{S}}(x'_S) \geq x_{\bar{S}}$ for all $x_{\bar{S}} \in R_{\bar{S}}(x'_S)$, and since $x'_{\bar{S}} \in R_{\bar{S}}(x'_S)$, positive externalities imply that $u_i\left(x_S, r^g_{\bar{S}}(x'_S)\right) \geq u_i\left(x_S, x'_{\bar{S}}\right)$ for all x_S..*

The implications of lemmas 7 and 8 are better illustrated by referring to the sets $F^g(S) \subseteq F(S)$ and $F^l \subseteq F(S)$, defined as follows:

$$\begin{aligned} F^g(S) &= \left\{x \in F(S) : x_{\bar{S}} = r^g_{\bar{S}}(x_S)\right\}; \\ F^l(S) &= \left\{x \in F(S) : x_{\bar{S}} = r^l_{\bar{S}}(x_S)\right\}. \end{aligned}$$

Lemmas 8 implies that, under positive externalities, the same strategy profile $x^*(S)$, maximizing (by lemma 7) the aggregate payoff of S over the set $F(S)$ for some vector of weights λ, also maximizes the same aggregate payoff over the set $F^g(S)$. The same conclusion can be drawn, with respect to the set $F^l(S)$, for the case of negative externalities. This result is important for two reasons. First, it endows the somewhat problematic assumption that S can select among Nash reactions of players in $\bar{S}$ - which, as we said, is implicit in the definition of a CCE and of the set $F(S)$ above - with the more appealing interpretation that the Pareto dominant Nash equilibrium will be played by members of $\overline{S}$. This interpretation is supported by the result of Lemma 5, by which the greater and least elements of $R_{\bar{S}}(x'_S)$ are the best choices for $\bar{S}$ under positive and negative externalities, respectively. Second, the result of lemma 8 allows us to exploit the properties of the maps $r^g_{\bar{S}}(x_S)$ and $r^l_{\bar{S}}(x_S)$ in supermodular games. This is done in the next lemma, in which these properties are shown to imply that at $x^*(S)$ the strategies played by members of S and of $\bar{S}$ are ordered according to the sign of the externality: under positive externalities, players in S play "greater" strategies than those in $\bar{S}$, while the opposite is true under negative externalities.

Lemma 9 *Let $i \in S$ and $j \in \overline{S}$, and denote by $x \in X$ and $y \in X$ the strategies of player $i \in S$ and player $j \in \overline{S}$, respectively, at $x^*(S)$. Then:*
i) positive externalities imply $x \geq y$;
ii) negative externalities imply $y \geq x$.

Proof 7 *For simplicity of notation, let x^* denote the profile $x^*(S)$. Let $U_i(x,y) \equiv u_i\left(x^*_{S\backslash i}, x, x^*_{N\backslash S\backslash j}, y\right)$, and similarly let $U_j(x,y) = u_j\left(x^*_{S\backslash i}, x, x^*_{N\backslash S\backslash j}, y\right)$. We use supermodularity of u_i to write:*

$$U_i(y,y) + U_i(x,x) \geq U_i(x,y) + U_i(y,x). \tag{12}$$

By the properties of x^*,

$$U_j(x,y) \geq U_j(x,x), \tag{13}$$

implying by symmetry that

$$U_i(y,x) \geq U_i(x,x). \tag{14}$$

Using (12) and (14) we obtain

$$U_i(y,y) \geq U_i(x,y) = u_i(x^*). \tag{15}$$

Now suppose that $y > x$ and assume that the game has positive externalities. By lemma 4, the equilibrium best response map has non decreasing greatest element, so that

$$y > x \Rightarrow r^g_{\bar{S}}(x^*_{S\backslash i}, y) \geq r^g_{\bar{S}}(x^*_S) = x^*_{\bar{S}}. \tag{16}$$

By positive externalities

$$u_i(x^*_{S\backslash i}, y, r^g_{\bar{S}}(x^*_{S\backslash i}, y)) > u_i(x^*_{S\backslash i}, y, r^g_{\bar{S}}(x^*_S)) = U_i(y,y). \tag{17}$$

Equations (15) and (17) imply

$$u_i\left(x^*_{S\backslash i}, y, r^g_{\bar{S}}(x^*_{S\backslash i}, y)\right) > u_i(x^*). \tag{18}$$

Finally, since $y > x$, positive externalities also imply that for every player $k \in S\backslash i$:

$$u_k\left(x^*_{S\backslash i}, y, r^g_{\bar{S}}(x^*_{S\backslash i}, y)\right) \geq u_k(x^*). \tag{19}$$

Both 18 and 19 contradict the assumption that x^* is a Pareto Optimum. Suppose now that $y < x$ and assume that the game has negative externalities. Supermodularity of u_i and u_j imply

$$y < x \Rightarrow r^l_{\bar{S}}(x^*_{S\backslash i}, y) \leq r^l_{\bar{S}}(x^*_S) = x^*_{\bar{S}}. \tag{20}$$

By negative externalities

$$u_i(x^*_{S\backslash i}, y, r^l_{\bar{S}}(x^*_{S\backslash i}, y)) \geq u_i(x^*_{S\backslash i}, y, r^l_{\bar{S}}(x^*_S)) = U_i(y, y). \tag{21}$$

Again, equation (22) imply

$$u_i(x^*_{S\backslash i}, y, r^l_{\bar{S}}(x^*_{S\backslash i}, y)) > u_i(x^*). \tag{22}$$

and, by negative externalities,

$$u_k\left(x^*_{S\backslash i}, y, r^g_{\bar{S}}(x^*_{S\backslash i}, y)\right) > u_k(x^*) \tag{23}$$

for every $k \in S\backslash i$, a contradiction.

Since by lemma 7 we can restrict our attention to the profiles $x^*(S)$, we will use the above result as a characterizing of the strategies played in the only relevant profiles that may be used in any deviation from an efficiency profile x^e. The next result makes use of this characterization to prove that at any profile $x^*(S)$, the members of S cannot be better off than the members of $\overline{S}$. This result generalizes to the present setting of coalitional actions a well known property of the subgame perfect equilibrium in two player symmetric supermodular games, in which the "leader" is weakly worse off than the "follower".

Lemma 10 *Let $i \in S$ and $j \in \bar{S}$. Then $u_j(x^*(S)) \geq u_i(x^*(S))$.*

Proof 8 *For simplicity, let again x^* denote the profile $x^*(S)$. The following inequalities hold:*

$$u_j\left(x^*_S, x^*_{\bar{S}}\right) \geq u_j\left(x^*_S, x^*_{\bar{S}\backslash j}, x^*_i\right) \geq u_j\left(x^*_{S\backslash i}, x^*_j, x^*_{\bar{S}\backslash j}, x^*_i\right). \tag{24}$$

The first part is implied by the conditions defining the profile x^*; the second part follows from lemma 9 and assumption 4. By assumption 3, we also have

$$u_j\left(x^*_{S\backslash i}, x^*_j, x^*_{\bar{S}\backslash j}, x^*_i\right) = u_i\left(x^*_S, x^*_{\bar{S}}\right). \tag{25}$$

Inequalities (24) and (25) imply

$$u_j\left(x^*\right) \geq u_i\left(x^*\right),$$

which proves the result.

We are now ready to show that an efficient strategy profile x^e satisfies the requirements of a Conjectural Cooperative Equilibrium.

Theorem 1 *Let the game G satisfy assumption 1-4. Then, G admits a conjectural cooperative equilibrium.*

Proof 9 *Let x^e be a symmetric efficient strategy profile for G, that is, a symmetric strategy profile that maximizes the aggregate payoff of N. Let $u(x^e)$ denote the payoff of each agent at x^e. Suppose, by contradiction, that there exists a coalition $S \subset N$ such that for all $i \in S$:*

$$u_i(x^*(S)) \geq u(x^e) \tag{26}$$

with strict inequality for at least one $h \in S$. Note that by lemma 10, it must be that

$$\frac{\sum_{i \in S} u_i(x^*(S))}{s} \leq \frac{\sum_{j \in \bar{S}} u_j\left(x^*(S)\right)}{n-s}, \tag{27}$$

otherwise there would exist $i \in S$ and $j \in \bar{S}$ for which

$$u_i\left(x^*(S)\right) > u_j\left(x^*(S)\right).$$

By condition (27) we obtain the following implication:

$$\frac{\sum_{i \in S} u_i(x^*(S))}{s} > u(x^e) \Rightarrow \frac{\sum_{j \in \bar{S}} u_j\left(x^*(S)\right)}{n-s} > u(x^e). \tag{28}$$

We conclude that if $u_i(x^*(S)) \geq u(x^e)$ for all $i \in S$, with strict inequality for at least one $h \in S$, then using (26) and (28), we obtain

$$s\frac{\sum_{i \in S} u_i(x^*(S))}{s} + (n-s)\frac{\sum_{j \in \bar{S}} u_j\left(x^*(S)\right)}{n-s} > s\ u(x^e) + (n-s)\, u(x^e) \tag{29}$$

or,

$$\sum_{i \in N} u_i(x^*(S)) > n\ u(x^e) \tag{30}$$

which contradicts the efficiency of x^e.

4 On the Existence of Equilibria in Submodular Games

4.1 The Role of the Slope of the Reaction Map

Theorem 1 establishes sufficient conditions for the existence of a conjectural cooperative equilibrium of the game G. The crucial condition, strategic complementarity in the sense of Bulow et al. (1985), generates non decreasing best replies; in particular, the supermodularity of payoff functions implies that the Nash responses of players outside a deviating coalition are a non decreasing function of the strategies of coalitional members. This feature ensures that each players outside S is better off than each coalitional member of S when deviating. Deviations by proper subcoalitions of players are therefore little profitable, while the grand coalition, not affected by this "deviator's curse", produces a sufficiently big aggregate payoff for a stable cooperative outcomes to exist. In this section we show how the same mechanics responsible for our existence result on the class of supermodular games, provide useful insight for the analysis of games with strategic substitutes, as, for instance, environmental and public goods games. We will use as an illustration an environmental Cobb-Douglas economy to show that *as long as best replies are not "too" decreasing* (thereby providing deviating coalitions with a not "too" big positional advantage), stable cooperative outcomes exist.

4.2 An illustration using a Cobb-Douglas environmental economy

We consider an economy with set of agents $N = \{1, ..i, .., n\}$, in which $z \geq 0$ is the environmental quality enjoyed by agents, $x_i \geq 0$ is a private good, $p_i \geq 0$ is a polluting emission originated as a by-product of the production of x_i. We assume that for each i in N preferences are represented by the Cobb-Douglas utility function

$$u_i(z, x_i) = z^{\alpha} x^{\beta},$$

technology is described by the production function

$$x_i = p_i^{\gamma},$$

and emissions accumulate according to the additive law

$$z(p) = A - \sum_{i \in N} p_i \tag{31}$$

where A is a constant expressing the quality of a pollution-free environment. We will assume that γ, α and β are all positive and $\gamma \leq 1$. We associate with this economy the game G_e with players set N, strategy space $\left[0, p_i^0\right]$ for each i, with $\sum_{i \in N} p_i^0 < A$, and payoffs

$U_i(p_1, ..., p_n) = z^\alpha p_i^\delta$, where $\delta = \beta\gamma$. Using this (symmetric) setup, we can express the maximal per-capita payoff of each coalition S in the event of a deviation from an arbitrary strategy profile in G as follows:

$$u_i(S) = s^{-\delta} A^{\alpha+\delta} \alpha^{2\alpha} (\alpha + \delta)^{-\alpha-\delta} (\alpha + \delta (n - s))^{-\alpha} \delta^\delta. \tag{32}$$

This simple setup of an environmental economy can be used to illustrate how CCE exist when best replies are not *too* decreasing or, in other terms, when strategies are not too substitute. This in turn requires that players' utilities does not decrease *too much* with other players' choice, a property mainly depending on the level of log-concavity of the term $z(p)^\alpha$. We prove this analytically for the case $\delta = 1$, while we rely on numerical simulation for the general case. Note that $z(p)^\alpha$ is log-concave (and the game is not log-supermodular) for $\alpha > 0$, and best replies are decreasing. The environmental game admits a unique Nash equilibrium $\bar{p}$ with $\overline{\overline{p}}_i = \frac{A}{\alpha+n}$ for every $i \in N$, and a unique efficient profile p^e (by efficient we mean "aggregate welfare maximizer"). Simple algebra yields the following expression:

$$u_i(S) = s^{-1} A\alpha^{\alpha+1} \alpha^{2\alpha} (\alpha + 1)^{-\alpha-1} (\alpha + n - s)^{-\alpha}.$$

The profitability of individual deviation from the efficient strategy profile p^e is evaluated as follows:

$$u_i(p^e) - u_i(S) = \alpha^\alpha (\alpha + n - 1)^{-\alpha} n - 1 < 0 \Leftrightarrow \alpha < 1.$$

It follows that when the function $z(p)^\alpha$ is strictly concave ($\alpha < 1$), then no CCE exists. However, when $\alpha = 1$, the CCE is unique, and equal to p^e. It is also easy to show that for $\alpha > 1$ ($z(.)^\alpha$ convex) the strategy profile p^e is still a CCE. We conclude that the existence of a CCE only requires a not too strong log-concavity of $z(.)^\alpha$. This ensures that the marginal utility of each consumer does not decrease too much with the rivals' private consumption and hence, a deviating coalition, by expanding its pollution (and private consumption) does not exploit too much its advantage against complementary players. When this is the case, although the environmental game is a natural "strategic substitute" game, the CCE exists. It is interesting to relate the existence of a stable cooperative (and efficient) solution with the relative magnitude of the parameters α, β and γ, expressing the intensity of preferences for the environment and for private consumption, and the characteristics of technology. It turns out that in order for an agreement on emissions to be reached, agents must put enough weight on the environment in their preferences (high enough α), and emissions must not be too "productive" according to the available technology. In other words, this conclusion rephrases the common intuition that a clean environment is sustainable only if agents care enough for ambient quality. As we said, the analysis of existence of a CCE for the general case (that is, removing the assumption $\delta = 1$) is not possible in analytical terms. In what follows we show by means of computations that the set of CCEa of the game Γ_e can be characterized with respect to three possible configurations of the parameter α, β and γ of the economy: the case $\alpha = \beta\gamma$, in which the CCE is unique and assigning to each player the payoff $u_i(p^e)$ (for this case we provide an analytical proof); the case $\alpha > \beta\gamma$, in which the set of CCEa strictly includes the profile p^e; the case $\alpha < \beta\gamma$, in which the set of CCE is empty.

Proposition 1 *If $\alpha = \beta\gamma$ the unique CCE is the efficient profile p^e.*

Proof 10 *We first show that no profile $p \neq p^e$ can be a CCE. By 32 we obtain*

$$u_i(p^e) - u_i(\{i\}) = \frac{\alpha^\alpha A^{\alpha+\delta} (\alpha+\delta)^{-\alpha-\delta} \delta^\delta \left[(\alpha + \delta(n-1))^\alpha - \alpha^\alpha n^\delta\right]}{n^\delta (\alpha + \delta(n-1))^\alpha}$$

from which

$$u_i(p^e) - u_i(\{i\}) = 0 \Longleftrightarrow \left[(\alpha + \delta(n-1))^\alpha - \alpha^\alpha n^\delta\right] = 0;$$

Using the fact that $\delta = \beta\gamma$ we get

$$\left[(\alpha + \delta(n-1))^\alpha - \alpha^\alpha n^\delta\right] = [\alpha + \alpha(n-1)]^\alpha - (\alpha n)^\alpha = 0$$

from which

$$u_i(p^e) = u_i(\{i\}).$$

To show that p^e is a CCE, it suffices to show that $u_i(S) \leq u_i(p^e)$ for all coalitions S such that $s > 1$. Using 32 we obtain

$$u_i(p^e) - u_i(S) \geq 0 \Longleftrightarrow \left[s^\delta (\alpha + \delta(n-s))^\alpha - \alpha^\alpha n^\delta\right] \geq 0$$

which, using again the fact that $\delta = \beta\gamma$ reduces to

$$u_i(p^e) - u_i(S) \geq 0 \Longleftrightarrow [s(\alpha + \alpha(n-s))]^\alpha \geq (\alpha n)^\alpha.$$

The last condition can be rewritten as

$$u_i(p^e) - u_i(S) \geq 0 \Longleftrightarrow s + (n-s)s + s^2 \geq n + s^2$$

which is always satisfied since $s \geq 1$.

Proposition 2 *If $\alpha > \beta\gamma$ then p^e is a CCE.*

Proof 11 *We proceed by numerical simulations. Our aim is to show that whenever* $\alpha > \beta\gamma$ *the difference* $u_i(p^e) - u_i(S)$ *is positive for every* s*. We first consider the case* $s = 1$*. We plot the graph of*

$$f_i(\alpha, n) \equiv \max\{(u_i(p^e) - u_i(\{i\})), 0\}$$

for the fixed value of $\delta = 0.5$*. The domains are taken to be* $(1, 10000)$ *for* n *and* $(0, 1)$ *for* α*.From Figure 1 it is evident that* $u_i(p^e) > u_i(\{i\})$ *whenever* $\alpha > 0.5 = \delta$*. Similar graph are obtained for other values of* δ *in the range* $(0, 1)$*. We perform the same exercise for coalition of size* $s > 1$*. We plot the function*

$$f(\alpha, s) \equiv \max\{(u_i(p^e) - u_i(\{S\})), 0\}$$

for fix values of n *and* δ*. The domains are taken to be* $(\delta, 1)$ *for* α *and* $(1, n]$ *for* s*. For the case* $n = 1000$ *and* $\delta = 0.2$ *we obtain the following graph:In Figure 2 the graph of* $f(\alpha, s)$ *all lies above the zero plane for all values of* $s \in (1, n]$ *and of* $\alpha \in (\delta, 1)$*. Summing up, whenever* $\alpha > \delta$ *we found that* $u_i(p^e) > u_i\{i\}$ *for* $s \geq 1$*; we thus conclude that whenever* $\alpha > \delta$ *then* p^e *is a CCE.*

Proposition 3 *If* $\alpha > \beta\gamma$ *there exists no CCE.*

Proof 12 *We again proceed by numerical simulations and evaluate the function*

$$\hat{f}_i(\alpha, n) \equiv \min\{(u_i(p^e) - u_i(\{i\})), 0\}$$

for an arbitrary player $i \in N$ *and a fixed value of* δ*. The domains are taken to be* $(0, 1)$ *for* α *and* $[1, 10000]$ *for* n*. Figure 3 depicts the graph of* $\hat{f}_i(\alpha, n)$ *for the case* $\delta = 0.5$*.It is evident from Figure 3 (and from numerical evaluations around the point* $\alpha = 0.5$*) that for any value of* n *in the selected range,* $u_i(p^e) < u_i(\{i\})$ *for the whole range of values of* $\alpha < \delta$*. We thus conclude that for such values there is no CCE.*

The above results can be usefully summarized by plotting the value of the difference $[u_i(p^e) - u_i\{i\}]$ as a function of the parameter α for fixed values of δ, n and for $s = 1$.

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