Bertrand-Edgeworth equilibrium with a large number of firms

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Abstract

We examine a model of price competition with strictly convex costs where the firms simultaneously decide on both price and quantity, are free to supply less than the quantity demanded, and there is discrete pricing. If firms are symmetric then, for a large class of residual demand functions, there is a unique equilibrium in pure strategies whenever, for a fixed grid size, the number of firms is sufficiently large. Moreover, this equilibrium price is within a grid-unit of the competitive price. The results go through to a large extent when the firms are asymmetric, or they are symmetric but play a two stage game and the tie-breaking rule is 'weakly manipulable'.

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Key words: Bertrand equilibrium, Edgeworth paradox, tie-breaking rule, rationing rule, folk theorem of perfect competition.

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1 Introduction

Let us consider a Bertrand duopoly where the firms decide on both their price and output levels and the firms are free to supply less than the quantity demanded. Edgeworth (1897) argues that in such models equilibria in pure strategies may not exist (see Dixon (1987), or Friedman (1988) for formal statements of the problem). In the literature this is often referred to as the Edgeworth paradox. In this paper we seek to provide a resolution of this paradox.

We focus on the case where the firms make their price and output decisions simultaneously, though we also examine a variant where the firms first decide on their prices, and then on their output levels (we restrict attention to pure strategies). We assume that the price level varies over a grid, where the size of the grid can be arbitrarily small. There are generally two problems associated with the existence of pure strategy equilibrium under price competition. The first reason has to do with the well known open-set problem. The second one has to do with the fact that the profit function of a firm may not be quasi-concave in its own price. The grid assumption allows us to side-step the open set problem, and focus on the second one. This assumption can also be motivated by appealing to the practice of integer pricing, or to the fact that there are minimum currency denominations. Other papers that model such discrete pricing include Dixon (1993) and Roy Chowdhury (1999).

We examine two main classes of residual demand functions, one where the tie-breaking rule (henceforth TBR) is 'weakly manipulable' (and the rationing rule is satisfied by a parametric class of rationing rules, though not the proportional one),[1] and another where it is 'strongly manipulable'. Suppose that several firms are charging the same price. If the TBR is weakly manipulable, then, up to a level, the residual demand coming to such a firm is responsive to an increase in its own output level. Beyond this level, however, the residual demand may be insensitive to an increase in own output (this

[1]From now on, for ease of exposition, we shall often use the shorthand - weakly manipulable TBR - to refer to this *combination* of a weakly manipulable TBR along with the associated restriction on the rationing rule.

happens whenever the output levels of the other firms charging this price are 'reasonably large'). If, however, the TBR is strongly manipulable, then, irrespective of the output levels of the other firms charging this price, such a firm can increase the residual demand coming to it by increasing its own output level. (Later, in Remarks 2 and 6, we argue that there are very few papers in the literature that analyze the case where the TBR is effectively strongly manipulable). Further, both kinds of TBRs allow for spill-overs in the sense that, in the event of a tie in price, it cannot be that there is unmet demand at this price, while some of these firms have output that they cannot sell for lack of demand.

We first consider the case where firms are symmetric (we later analyze the asymmetric case also). To begin with we examine the setup where the firms simultaneously decide on both their prices and quantities. For both kinds of TBRs we demonstrate that if, *for any given grid-size*, the number of firms (n) is large enough, then there is a unique Nash equilibrium where the equilibrium price is within a grid-unit of the competitive price. Moreover, the output levels of individual firms become vanishingly small as n becomes very large.

The proofs of the existence results work as follows. Suppose all firms charge the lowest possible price in the grid that is greater than the marginal cost at zero. If the TBR is weakly manipulable and n is large, then the residual demand coming to every firm is small, so that it is residual demand rather than marginal cost which determines firm supply. In that case price would not equal marginal cost, and firms may have no incentive to increase their price levels. Next consider strongly manipulable TBRs. For n large, competition among firms may lead to excess production so that a firm that deviates and charges a higher price may have no demand at all.

We then turn to the aggregate output level. If the TBR is weakly manipulable, then, in equilibrium, aggregate output equals demand. If, however, it is strongly manipulable, then interestingly every firm produces more than what it sells, so that the equilibrium involves excess production. For this case, consider the limiting value of the aggregate output as n is taken to infinity. It turns out to be finite if the marginal cost at the origin is strictly positive. Otherwise, aggregate output diverges to infinity.

We then examine the case where, for a fixed n, the grid size is taken to zero. We find that, for all sufficiently small grid sizes, no single price equilibrium (i.e. equilibrium where all firms that supply a positive output charge the same price) exists. Whether, for small grid sizes, there exists equilibria involving different prices is an open question. In case they do, all such equilibria are bounded away from the competitive price if the grid size is sufficiently small.

We next examine the case where the firms are asymmetric. The results for the symmetric case generalize in a natural fashion when the marginal cost at zero is the same for all firms. Otherwise, the earlier results go through if it is the number of 'efficient' firms (a firm is said to be efficient if its marginal cost at zero is less than equal to that of any other firm) that is taken to infinity. Further, the results for the one-stage game 'go through' if the TBR is weakly manipulable and symmetric firms play a two stage game, where, in stage 1, they decide on their price, and in stage 2 on their output levels.

Next Section 2 describes the basic framework. Section 3 analyzes a one stage game with symmetric firms, while the asymmetric case is examined in Section 4. Section 5 analyzes the two stage game. Section 6 relates the paper to the literature and concludes. Finally, some proofs which are either too long, or of mainly technical interest, are in the Appendix.

2 The Framework

There are n identical firms, all producing the same homogeneous good. The market demand function is $q = d(p)$ and the common cost function of all the firms is $c(q)$.

A1. $d : [0, \infty) \rightarrow [0, \infty)$. The function $d(p)$ is continuous on $[0, \infty)$. Further $\exists p^{\max}$, $0 < p^{\max} < \infty$, such that $d(p) > 0$ if $0 \leq p < p^{\max}$, and $d(p) = 0$ if $p \geq p^{\max}$. Moreover, $\forall\, p', p''$, such that $p^{\max} \geq p'' > p' \geq 0$, it is the case that $d(p') > d(p'')$.

A2. $c : [0, \infty) \rightarrow [0, \infty)$. The function $c(q)$ is continuous, increasing and strictly convex on $[0, \infty)$ and twice differentiable on $(0, \infty)$. Moreover,

$c(0) = 0$ and $p^{\max} > \lim_{q \to 0+} c'(q) = c'(0)$.

We assume that prices vary over a grid. The set of feasible prices $F = \{\hat{p}_0, \hat{p}_1, \cdots\}$, where $\hat{p}_0 = 0$, and $\hat{p}_j = \hat{p}_{j-1} + \alpha$, $\forall j \in \{1, 2, \cdots\}$, where $\alpha > 0$. Let p_i (respectively q_i) denote the price charged (respectively quantity produced) by firm i, where $p_i \in F$ and q_i is a continuous variable ranging over $[0, \infty)$.

Let $P = (p_1, \ldots, p_n)$, $Q = (q_1, \ldots, q_n)$ and $P, Q = (p_1, \ldots, p_n, q_1, \ldots, q_n)$.

For any P, Q, let Q_{p_i} (respectively Q^{p_i}) denote the vector generated from Q by deleting all q_j, $j \in \{1, 2, \cdots, n\}$, such that $p_j \geq p_i$ (respectively $p_j \leq p_i$). Clearly, Q_{p_i} (respectively Q^{p_i}) denotes the output vector of the firms charging less (respectively more) than p_i. Similarly, let P^{p_i} denote the price vector of the firms charging more than p_i.

Let $R_i(P, Q)$, $R_i : [0, \infty)^{2n} \to [0, \infty)$, denote the residual demand facing firm i as a function of the price quantity vector in the market.

If $S(j)$ denotes some statement involving firm j, then $\sum_{k|S(k)} q_k$ denotes the sum of q_k over all k, $k \in \{1, 2, \cdots, n\}$, such that $S(k)$ holds.

A3. (i) $\forall p \geq 0$, if $\sum_{j|p_j = p} q_j \geq d(p)$, then $R_i(P, Q)|_{p_i > p} = 0$.

(ii) $\forall p \geq 0$, $\sum_{i|p_i = p} R_i(P, Q) \leq d(p)$.

(iii) For any P, Q, and $\forall i, j$ such that $i \neq j$, let $P^{ij}, Q^{ij} = (p_1^{ij}, \ldots, p_n^{ij}, q_1^{ij}, \ldots, q_n^{ij})$ satisfy $p_i^{ij} = p_j$, $p_j^{ij} = p_i$, $q_i^{ij} = q_j$, $q_j^{ij} = q_i$, and $\forall k \notin \{i, j\}$, $p_k^{ij} = p_k$ and $q_k^{ij} = q_k$. Then $R_i(P, Q) = R_j(P^{ij}, Q^{ij})$.

(iv) $\forall P, Q$, $R_i(P, Q)$ is independent of both Q^{p_i} and P^{p_i}.

(v) $\forall P, Q$, $R_i(P, Q)|_{\forall j \neq i,\, p_j \neq p_i}$ is continuous in q_i. Furthermore, $\max_{q_i} R_i(P, Q)|_{\forall j \neq i, \text{ either } p_j > p_i, \text{ or } p_j < p_i \text{ and } q_j = 0} = d(p_i)$.

(vi) Consider $P, Q = (p_1, \ldots, p_n, q_1, \ldots, q_n)$ and $P', Q' = (p'_1, \ldots, p'_m, q'_1, \ldots, q'_m)$ (where $m \geq n$) such that $p_i = p'_i$ for some $i \leq n$, no firm other than i charges p_i (respectively p'_i) in P, Q (respectively P', Q') and, $\forall p < p_i$, $\sum_{k|p_k = p} q_k = \sum_{m|p'_m = p} q'_m$. Then the residual demand of firm i is the same irrespective of whether it faces P, Q or P', Q'.

For any price p, A3(i) states that if the total output of all firms charging p is at least $d(p)$, then all firms who charge prices greater than p obtain no demand. A3(ii) states that the aggregate residual demand of all the firms charging some price p can be at most $d(p)$. A3(iii) is a symmetry assumption. A3(iv) states that the residual demand of firm i, say, is independent of the price and output levels of the firms who charge prices higher than p_i. Next, note that given A3(ii), $\max_{q_i \geq 0} R_i(P,Q)|_{\forall j \neq i,\, p_j \neq p_i}$ equals $\max_{q_i | 0 \leq q_i \leq d(p_i)} R_i(P,Q)|_{\forall j \neq i,\, p_j \neq p_i}$, which, given the continuity assumption in A3(v), is well defined. A3(v) states that if there is a single firm, say i, charging the effectively lowest price, then its maximal residual demand is $d(p_i)$. Finally, A3(vi) states that in case firm i is the only firm charging p_i, then the residual demand of firm i remains unchanged at any other price quantity vector (with possibly different number of firms) where (a) firm i alone charges p_i and, (b) $\forall p < p_i$, the aggregate output of firms charging p is the same as that under the original vector.

For any P, define $P(i) = (p_1(i), \ldots, p_n(i))$, such that $p_j(i)|_{j \neq i,\, p_j = p_i} = p^{\max}$, $p_i(i) = p_i$ and $p_k(i)|_{p_k \neq p_i} = p_k$.

Given A3(iii) and A3(v), $\forall p_i < p^{\max}$ we can define

$$R^{p_i}(P,Q) = \max_{q_i} R_i(P(i),Q).$$

Given any P, Q such that $p_i < p^{\max}$, $R^{p_i}(P,Q)$ denotes the maximal residual demand for firm i if all other firms charging p_i switch to charging $p^{\max}$, and, given this new price vector, firm i sets q_i appropriately.

Given A3(iii), $\forall i, j$ such that $i \neq j$, $R^{p_i}(P,Q) = \max_{q_i} R_i(P(i),Q) = \max_{q_j^{ij}} R_j(P^{ij}(j), Q^{ij}) = R^{p_j^{ij}}(P^{ij}, Q^{ij})$. Thus $R^{p_i}(P,Q)$ depends on the magnitude of p_i, but not on the identity of the firm charging p_i. Hence we can define

$$R^{\overline{p}}(P,Q) = R^{p_i}(P,Q)|_{p_i = \overline{p}}.$$

2.1 Weakly Manipulable TBR

In this sub-section we introduce the notion of weakly manipulable TBRs. Assumption 4(i) below is a restriction on the TBR, whereas 4(ii) is a restriction on both the rationing rule, as well as the TBR. For any set S, let $N(\text{S})$ denote the number of elements in S.

A4.(i) $\forall p$, such that $0 \leq p < p^{\max}$, define $M_p = \{i|\ p_i = p\}$ and $K_p = \{j|\ p_j = p,\ q_j = 0\}$. Further, let $N(M_p) = m_p$ and $N(K_p) = k_p$ ($\leq m_p$). Then $R_i(P,Q)|_{i\in M_p - K_p} = q_i$, if either $\sum_{j|_{j\in M_p-K_p}} q_j \leq R^p(P,Q)$, or $\sum_{j|_{j\in M_p-K_p}} q_j > R^p(P,Q)$ and $q_i \leq \frac{R^p(P,Q)}{m_p-k_p}$. Otherwise, $R_i(P,Q)|_{i\in M_p-K_p} \geq \frac{R^p(P,Q)}{m_p-k_p}$.

Next, $\forall p_i, p$ satisfying $p^{\max} > p_i \geq p > 0$, define

$$r_i(p_i,p,n) = \max_{q_i} R_i(P,Q)|_{\forall j\neq i,\ p_j=p \text{ and } q_j=\frac{d(p)}{n}, \text{ and } p_i\geq p}.$$[2]

Given A3(ii) and the first sentence in A3(v), $r_i(p_i,p,n)$ is well defined for $p_i > p$. Moreover, from A4(i), $r_i(p_i,p,n)|_{p_i=p}$ is well defined and, from A3(ii) and A4(i), equals $\frac{d(p)}{n}$.

A4.(ii) Consider P,Q such that $\exists p', p''$, $p^{\max} \geq p'' > p' > 0$, such that firm i ($\leq n$) is charging a price p_i, where $p'' \geq p_i \geq p'$, there are m' ($n-1 \geq m' \geq 1$) firms (other than i) charging p', and no other firms charge any price p, $p'' > p \geq p'$. Then $\max_{q_i} R_i(P,Q)$ is twice differentiable in p_i over (p',p''), $\max_{q_i} R_i(P,Q)$ is decreasing in p_i over $[p',p'')$, and $\frac{\partial[\max_{q_i} R_i(P,Q)]}{\partial p_i}$ and $\frac{\partial^2[\max_{q_i} R_i(P,Q)]}{\partial p_i^2}$ are both (weakly) decreasing in p_i over (p',p''). Further, define $P^k, Q^k = (p_1^k,\cdots,p_k^k,q_1^k,\cdots,q_k^k)$ such that k is some integer satisfying $k \geq n$, $P^n,Q^n = P,Q$ (where P,Q is as defined in A4(ii) earlier), and, $\forall l \geq n$, we have that $p_j^{l+1} = p_j^l$ $\forall j \leq l$, $p_{l+1}^{l+1} = p'$, $q_k^{l+1}|_{k\neq i, p_k^{l+1}\neq p'} = q_k^l$, $q_j^{l+1}|_{j\neq i, p_j^{l+1}=p'} = \frac{R^{p'}(P^{l+1},Q^{l+1})}{m'+l+2-n}$, and $q_i^{l+1} = \max_{q_i^{l+1}} R_i(P^{l+1},Q^{l+1})$. Let $R_i^k(P^k,Q^k)$ denote the residual demand of firm i when facing P^k,Q^k. Then $\lim_{p_i^k\to p'+} \frac{\partial[\max_{q_i^k} R_i^k(P^k,Q^k)]}{\partial p_i^k}$ is (weakly) increasing in k and $\lim_{k\to\infty}\lim_{p_i^k\to p'+} \frac{\partial[\max_{q_i^k} R_i^k(P^k,Q^k)]}{\partial p_i^k} < 0$. Finally, $\forall P,Q$ s.t. $r_i(p_i,p,n)$ is well defined, $r_i(p_i,p,n)$ is (weakly) concave in p_i.

Given A4(ii), note that $\lim_{p_i\to p'+} \frac{\partial[\max_{q_i} R_i(P,Q)]}{\partial p_i}$, $\lim_{p_i\to p'+} \frac{\partial^2[\max_{q_i} R_i(P,Q)]}{\partial p_i^2}$ and $\lim_{k\to\infty}\lim_{p_i^k\to p'+} \frac{\partial[\max_{q_i^k} R_i^k(P^k,Q^k)]}{\partial p_i^k}$ are well defined, $r_i(p_i,p,n)$ is decreasing in p_i and $\lim_{n\to\infty}\lim_{p_i\to p+}\ r_i'(p_i,p,n) < 0$.

[2]For ease of exposition we suppress the fact that $r_i(p_i,p,n)$ is a function of q_j, $j \neq i$.

We next relate Assumption 4 to the literature. We first consider A4(i). Note that any firm i, $i \in M_p - K_p$, can increase the residual demand coming to it by increasing its output level q_i till $\frac{R^p(P,Q)}{m_p - k_p}$ (in fact the residual demand equals q_i). Beyond this output level $\frac{R^p(P,Q)}{m_p - k_p}$, however, the residual demand of firm i may not respond to an increase in its output level. Suppose all other firms in M_p supply at least $\frac{R^p(P,Q)}{m_p - k_p}$. Then, from A4(i), all these firms have a residual demand of at least $\frac{R^p(P,Q)}{m_p - k_p}$. Thus, from A3(ii), the residual demand coming to firm i is at most $\frac{R^p(P,Q)}{m_p - k_p}$. This formalizes the notion that the TBR is weakly manipulable.

Further, note that A4(i) allows for the possibility that if some of the firms supply less than $\frac{R^p(P,Q)}{m_P - k_p}$, then the residual demand facing the other firms may be greater than $\frac{R^p(P,Q)}{m_P - k_p}$. Such spill-over of unmet residual demand is, in fact, allowed for by Davidson and Deneckere (1986), Deneckere and Kovenock (1996) and Kreps and Scheinkman (1983). (This TBR is also discussed in Vives (1999).) Thus the TBR formalized through A4(i) is in the spirit of the above literature.[3]

We then claim that the restrictions on $r_i(p_i, p, n)$ are satisfied by a parametric class of rationing rules (though not the proportional one). Using the combined rationing rule introduced by Tasnádi (1999b), suppose $r_i(p_i, p, n) = \max\{d(p_i) - \frac{n-1}{n} d(p)[(1-\lambda)\frac{d(p_i)}{d(p)} + \lambda],\ 0\}$, where $\lambda \in [0, 1]$. For $\lambda = 1$, this satisfies the efficient rationing rule, whereas for $\lambda = 0$, this satisfies the proportional rationing rule. For intermediate values of λ, other rationing rules emerge (see Tasnádi (1999b) for an interpretation). Clearly, if $d(p_i)$ is concave then $r_i(p_i, p, n)$ is decreasing and concave in p_i. Moreover, it is the case that $\lim_{n\to\infty} \lim_{p_i \to p+} r_i'(p_i, p, n) = \lambda d'(p)$. So $\forall \lambda > 0$, and $\forall p < p^{\max}$, $\lim_{n\to\infty} \lim_{p_i \to p+} r_i'(p_i, p, n) < 0$.

2.2 Strongly Manipulable TBR

We then define strongly manipulable TBRs.

[3]Papers which consider TBRs that do not allow for such spill-over, include Dixon (1984), Maskin (1986), Levitan and Shubik (1972) and Yoshida (2002).

A5. Consider any P, Q. Let $M_{\tilde{p}} = \{j|\ p_j = \tilde{p}\}$ and $N(M_{\tilde{p}}) = m_{\tilde{p}}$. Then

$$R_i(P,Q)|_{i\in M_{\tilde{p}}} = \begin{cases} q_i, & \text{if } \sum_{j|_{j\in M_{\tilde{p}}}} q_j \leq R^{\tilde{p}}(P,Q), \\ \gamma(q_i, \sum_{j|j\in M_{\tilde{p}}, j\neq i} q_j) R^{\tilde{p}}(P,Q), & \text{if } \sum_{j|_{j\in M_{\tilde{p}}}} q_j > R^{\tilde{p}}(P,Q), \end{cases} \tag{1}$$

where $\gamma : [0,\infty)\times[0,\infty)-\{(0,0)\} \to [0,1]$ and $\sum_{i\in M_{\tilde{p}}} \gamma(q_i, \sum_{j|j\in M_{\tilde{p}}, j\neq i} q_j) = 1$.

The first line of Eq. (1) captures the idea that the TBR allows for spillovers of unmet residual demand. We assume that $\gamma(q_i, \sum_{j\neq i} q_j)$ satisfies the following assumption.

A6. (i) $\gamma_1(q_i, \sum_{j\neq i} q_j)$, $\gamma_{11}(q_i, \sum_{j\neq i} q_j)$ and $\gamma_{12}(q_i, \sum_{j\neq i} q_j)$ are well defined on $(0,\infty)\times(0,\infty)$.

(ii) $\gamma_1(q_i, \sum_{j\neq i} q_j) > 0$, $\gamma_{11}(q_i, \sum_{j\neq i} q_j) < 0$ and $\gamma_{11}(q_i, \sum_{j\neq i} q_j) < \gamma_{12}(q_i, \sum_{j\neq i} q_j)$. Moreover, $\gamma_{12}(q_i, \sum_{j\neq i} q_j) < 0$ whenever $\sum_{j\neq i} q_j > q_i$.

(iii) $\gamma_1(x, (n-1)x)$ is decreasing in both x and n. Moreover, $\lim_{x\to 0} \gamma_1(x, (n-1)x) \to \infty$ and $\lim_{x\to\infty} \gamma_1(x, (n-1)x) = 0$.

(iv) If $\lim_{r\to\infty} a(r) = 0$ and $\lim_{r\to\infty} b(r) = L$, where $0 \leq L < \infty$, then $\lim_{r\to\infty} \gamma_1(a(r), b(r)) = \frac{1}{L}$ if $L > 0$, and $\lim_{r\to\infty} \gamma_1(a(r), b(r)) \to \infty$, if $L = 0$.

(v) If $\lim_{r\to\infty} a(r) = 0$ and $\lim_{r\to\infty} b(r) \to \infty$, then $\lim_{r\to\infty} \gamma_1(a(r), b(r)) = 0$.

(vi) If, for any strictly increasing sequence of natural numbers $< n_m >$, $\lim_{m\to\infty} x(n_m) = D > 0$, then $\lim_{m\to\infty} \gamma_1(x(n_m), (n_m - 1)x(n_m)) = 0$.

Consider any price p_i such that $R^{p_i}(P,Q) > 0$. Then, irrespective of the output levels of the other firms charging p_i, any firm charging p_i can increase the residual demand coming to it by increasing its own output level. This formalizes the idea that the TBR is strongly manipulable.

Papers in the literature that adopt a strongly manipulable TBR include, for example, Allen and Hellwig (1986, 1993), Osborne and Pitchik (1986), Maskin (1986) (the first example provided by him) and Tasnádi (1999b). Appropriately extending the TBRs in these papers to the present context, one can write that

$$\gamma(q_i, \sum_{j\neq i} q_j) = \frac{q_i}{\sum_{j=1}^{m} q_j}.$$

Observe that in this case $\gamma_1(q_i, \sum_{j\neq i} q_j) = \frac{\sum_{j\neq i} q_j}{(\sum_{j=1}^{m} q_j)^2}$ and $\gamma_1(x, (n-1)x) = \frac{(n-1)}{n^2 x}$ so that A6 goes through.

This paper covers the case where the rationing rule is efficient (and the TBR is either weakly, or strongly manipulable), as well as the case where the rationing rule is proportional and the TBR is strongly manipulable. What happens in case a proportional rationing rule is coupled with a weakly manipulable TBR is an open question.[4]

3 One-stage Game: The Symmetric Case

In this section we consider a one-stage game where the i-th firm's strategy consists of simultaneously choosing both a price $p_i \in F$ and an output $q_i \in [0, \infty)$. All firms move simultaneously. We solve for the set of pure strategy Nash equilibria of this game.

We follow Edgeworth (1897) in assuming that firms are free to supply less than the quantity demanded, rather than Chamberlin (1933), who assumes that firms meet the whole of the demand coming to them.

Next let p^* be the minimum $p \in F$ such that $p > c'(0)$. Thus p^* is the minimum price on the grid which is strictly greater than $c'(0)$. We assume that α is not too large in the sense that $p^* < p^{\max}$. Since $p^* \in F$, let $p^* = \hat{p}_j$ for some integer j.

Let $q^* = c'^{-1}(p^*)$[5] and let n^* be the smallest possible integer such that $\forall N \geq n^*$,

$$\frac{d(p^*)}{N} < c'^{-1}(p^*) = q^*.$$

Thus for all N greater than n^*, if a firm charges p^* and sells $\frac{d(p^*)}{N}$, then the price p^* is strictly greater than marginal costs.

[4]While Roy Chowdhury (1999) does consider a similar case, the cost function used is discontinuous at zero and the TBR does not allow for spill-overs.

[5]Since, $\forall q > 0$, $c'(q)$ is well defined and strictly increasing, $c'^{-1}(p)$ is well defined $\forall p > c'(0)$. For $p \leq c'(0)$, we define $c'^{-1}(p) = 0$.

3.1 Weakly Manipulable TBR

For this case we argue that for a given grid size, p^* can be sustained as the unique Nash equilibrium price of this game whenever n is sufficiently large.

Let $\hat{n}$ be the smallest possible integer such that $\forall N \geq \hat{n}$,

$$[p^* - c'(\frac{d(p^*)}{N})]r_i'(p_i, p^*, N)|_{p_i \to p^*+} + \frac{d(p^*)}{N} < 0.^6$$

Comparing the definitions of n^* and $\hat{n}$, we find that $\hat{n} \geq n^*$.

We next define $\tilde{\pi}$ to be the profit of a firm that charges p^* and sells $\frac{d(p^*)}{n^*}$. Thus $\tilde{\pi} = \frac{p^* d(p^*)}{n^*} - c(\frac{d(p^*)}{n^*})$. Since $\frac{d(p^*)}{n^*} < q^*$, it follows that $\tilde{\pi} > -c(0)$, where $-c(0)$ denotes the profit of a firm which does not produce at all.

Now consider some $\hat{p}_i \in F$, such that $\hat{p}_i > p^*$. Let $\hat{q}_i$ satisfy $\hat{p}_i = c'(\hat{q}_i)$. Next consider a firm that charges $\hat{p}_i$ and sells $\frac{d(\hat{p}_i)}{k}$. Clearly the profit of such a firm is $\hat{p}_i \frac{d(\hat{p}_i)}{k} - c(\frac{d(\hat{p}_i)}{k})$.

We then define n_i to be the smallest possible integer such that $\forall k \geq n_i$, $\frac{d(\hat{p}_i)}{k} < \hat{q}_i$ and

$$\frac{\hat{p}_i d(\hat{p}_i)}{k} - c(\frac{d(\hat{p}_i)}{k}) < \tilde{\pi}.^7$$

Suppose that in any equilibrium the number of firms charging $\hat{p}_i$, say $\tilde{m}$, is greater than or equal to n_i. Then at least one of these firms would have a residual demand that is less than or equal to $\frac{d(\hat{p}_i)}{\tilde{m}}$. Since $\frac{d(\hat{p}_i)}{\tilde{m}} < c'^{-1}(\hat{p}_i)$, this firm would sell at most $\frac{d(\hat{p}_i)}{\tilde{m}}$ and have a profit less than $\tilde{\pi}$.

Let $\hat{p}_k$ be the largest price belonging to F such that $\hat{p}_k \leq p^{\max}$.

Definition. $N_1 = \sum_{i=j+1}^{k} n_i + n^* - 1.$[8]

For the case where the TBR is weakly manipulable, Proposition 1 below provides a resolution of the Edgeworth paradox.

[6]Notice that $\lim_{n\to\infty}[\{p^* - c'(\frac{d(p^*)}{n})\}r_i'(p_i, p^*, n)|_{p_i\to p^*+} + \frac{d(p^*)}{n}] = \lim_{n\to\infty}[(p^* - c'(0))r_i'(p_i, p^*, n)|_{p_i\to p^*+}]$. Since, $p^* > c'(0)$ and $\lim_{n\to\infty}[r_i'(p_i, p^*, n)|_{p_i\to p^*+}] < 0$ (A4(ii)), this term is negative.

[7]Clearly the left hand side of this inequality is decreasing in k. Moreover, as k goes to infinity, this term goes to $-c(0) \leq 0$. Thus n_i is well defined.

[8]Note that the assumption that the demand function intersects the price axis is required for this definition.

Proposition 1. *Suppose Assumptions 1, 2, 3 and 4 hold. If* $n \geq \max\{\hat{n}, N_1\}$*, then the unique equilibrium involves all the firms charging a price of* p^**, and producing* $\frac{d(p^*)}{n}$*.*

Proof. Existence. From the definition of p^*, undercutting is not profitable. We then argue that for the i-th firm, charging a higher price, p_i, is not profitable either. We first claim that $r_i(p^*, p^*, n) = \frac{d(p^*)}{n}$. From A3(v) and A4(i), $r_i(p^*, p^*, n) \geq \frac{d(p^*)}{n}$, and $\forall j \neq i$, $R_j(P, Q) \geq \frac{d(p^*)}{n}$. Thus $\sum_{k \leq n} R_k(P, Q) \geq d(p^*)$. The claim now follows from A3(ii).

Since $n \geq \hat{n} \geq n^*$, $\frac{d(p^*)}{n} < c'^{-1}(p^*)$. Hence for any $p_i \geq p^*$,

$$c'^{-1}(p_i) \geq c'^{-1}(p^*) > \frac{d(p^*)}{n} = r_i(p_i, p^*, n)|_{p_i \to p^*+} \geq r_i(p_i, p^*, n), \tag{2}$$

where the last inequality follows from A4(ii). Since $c'^{-1}(p_i) > r_i(p_i, p^*, n)$, for any $p_i \geq p^*$, the deviant firm supplies the whole of the residual demand coming to it. Hence the profit of a firm which charges a price p_i ($\geq p^*$)

$$\pi(p_i, r_i(p_i, p^*, n)) = p_i r_i(p_i, p^*, n) - c(r_i(p_i, p^*, n)). \tag{3}$$

Clearly

$$\frac{\partial \pi(p_i, r_i(p_i, p^*, n))}{\partial p_i} = r_i'(p_i, p^*, n)[p_i - c'(r_i(p_i, p^*, n))] + r_i(p_i, p^*, n). \tag{4}$$

Next from equation (1) it follows that $\forall p_i \geq p^*,\ p_i > c'(r_i(p_i, p^*, n))$. Hence from the concavity of $r_i(p_i, p^*, n)$ it follows that $\pi(p_i, r_i(p_i, p^*, n))$ is concave in p_i.[9] Moreover,

$$\frac{\partial \pi(p_i, r_i(p_i, p^*, n))}{\partial p_i}|_{p_i \to p^*+} = [p^* - c'(\frac{d(p^*)}{n})] r_i'(p_i, p^*, n)|_{p_i \to p^*+} + \frac{d(p^*)}{n}. \tag{5}$$

Since $n \geq \hat{n}$, we have that $\frac{\partial \pi(p_i, r_i(p_i, p^*, n))}{\partial p_i}|_{p_i \to p^*+} < 0$. Next, from the concavity of $\pi(p_i, r_i(p_i, p^*, n))$ it follows that $\forall p_i \geq p^*$, the profit of any deviant firm is decreasing in p_i.

[9]This follows since

$$\begin{aligned}\frac{\partial^2 \pi(p_i, r_i(p_i, p^*, n))}{\partial p_i^2} = r_i''(p_i, p^*, n)[p_i - c'(r_i(p_i, p^*, n))] &+ 2r_i'(p_i, p^*, n) \\ &- c''(r_i(p_i, p^*, n)) r_i'^2(p_i, p^*, n).\end{aligned}$$

Next, from A3(v) and A4(i), all firms, by producing $\frac{d(p^*)}{n}$, can have a residual demand of at least $\frac{d(p^*)}{n}$. Thus, given that $\frac{d(p^*)}{n} < c'^{-1}(q^*)$, all firms produce at least $\frac{d(p^*)}{n}$. Hence, from A3(ii), the residual demand facing all firms is exactly $\frac{d(p^*)}{n}$.

Uniqueness. Step 1. We first claim that there cannot be an equilibrium where the output level of some of the firms is zero. This follows since these firms can always charge p^* and obtain a residual demand of at least $\frac{d(p^*)}{n}$ (A3(v) and A4(i)). Since $p^* > c'(0)$, producing a small enough positive output would increase their profit from $-c(0)$.

Step 2. We then argue that there cannot be some $\hat{p}_i$ $(\in F) > p^*$, such that some of the firms charge $\hat{p}_i$ and supply a positive amount. Suppose to the contrary that such a price exists. This implies that the total number of firms charging p^*, say $\tilde{n}$, can be at most $n^* - 1$. Otherwise, given A3(v) and A4(i) and the fact that $\frac{d(p^*)}{\tilde{n}} < c'^{-1}(p^*)$, all firms must be supplying at least $\frac{d(p^*)}{\tilde{n}}$. From A3(ii), all firms have a residual demand of $\frac{d(p^*)}{\tilde{n}}$. Hence all firms would supply $\frac{d(p^*)}{\tilde{n}}$ and the residual demand at any higher price, $\hat{p}_i$, would be zero.

Now consider some $\hat{p}_i > p^*$. Clearly, the number of firms charging $\hat{p}_i$ is less than n_i. Otherwise, some of these firms would have a profit less than $\tilde{\pi}$. Hence such a firm would have an incentive to deviate to p^*, when it can supply at least $\frac{d(p^*)}{n^*}$ and earn $\tilde{\pi}$. Thus the total number of firms producing a strictly positive amount is less than N_1, thereby contradicting step 1.

Step 3. Finally, note that by charging p^* and, by supplying $\frac{d(p^*)}{n}$, all firms can earn a strictly positive profit. Hence, from step 2, all firms charge p^*. Moreover, for a firm charging p^*, its profit level is strictly increasing in the output level till $\frac{d(p^*)}{n}$. Thus, given A3(ii), A3(v) and A4(i), all firms supply exactly $\frac{d(p^*)}{n}$. ■

Note that the equilibrium price is within α of the competitive price. The idea behind the existence result is simple. Consider a market price of p^*. If n is large then the residual demand coming to every firm is very small, so that it is residual demand rather than marginal cost which determines firm supply. In that case price would not equal marginal cost, and, given A3 and A4, firms may no longer have an incentive to increase their price levels.

3.2 Strongly Manipulable TBR

To begin with define $q'(n-1)$ as satisfying the following equation:

$$p^* d(p^*)\gamma_1(q,(n-1)q) = c'(q). \tag{6}$$

Thus if the market price is p^* and all the firms produce $q'(n-1)$, then, for all firms, marginal revenue equals marginal cost. It is easy to see that $q'(n-1)$ is decreasing in n.[10]

We are going to argue that for n large, the outcome where all the firms charge p^* and produce $q'(n-1)$, can be sustained as a Nash equilibrium. We then introduce a series of lemmas that we require for our analysis.[11]

Lemma 1. $\lim_{n\to\infty} p^* d(p^*)\gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) > \lim_{n\to\infty} c'(\frac{d(p^*)}{n-1})$.

Given Lemma 1, define M_1 to be the smallest integer such that $\forall n \geq M_1$,

$$p^* d(p^*)\gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) > c'(\frac{d(p^*)}{n-1}).$$

Lemma 2. $\forall n \geq M_1,\ (n-1)q'(n-1) > d(p^*)$.

Consider an outcome such that all the firms charge p^* and produce $q'(n-1)$. Then Lemma 2 suggests that if $n \geq M_1$, then the residual demand facing any firm that deviates and charges a price greater than p^* would be zero. This follows since the total production by the other firms will be enough to meet $d(p^*)$. Moreover, Lemma 2 also implies that $\forall n \geq M_1$, $q'(n-1) > 0$.

Next define

$$\hat{\pi} = \max_q \begin{cases} p^* d(p^*)\gamma(q,(n^*-1)q^*) - c(q), & \text{if } q > d(p^*) - (n^*-1)q^*, \\ p^* q - c(q), & \text{otherwise.} \end{cases} \tag{7}$$

Suppose that n^* of the firms charge p^*, and all other firms charge a higher price. Moreover, out of the n^* firms, suppose (n^*-1) of the firms produce

[10]Notice that given A6(iii), $q'(n-1)$ is well defined. That $q'(n-1)$ is decreasing in n, follows from Eq. (6) and the fact that $\gamma_1(x, nx)$ is decreasing in x and n.

[11]The proofs of lemmas 1-4, as well as Proposition 4 later, are in the appendix. The proofs of other lemmas and propositions are available from the author.

q^* and the remaining firm produces q. Then $\hat{\pi}$ denotes the maximum profit that this firm can earn if it chooses its output level optimally.

Next consider some $\hat{p}_i$ $(\in F) > p^*$. Recall that $\hat{q}_i$ satisfies $\hat{p}_i = c'(\hat{q}_i)$. Let $\hat{n}_i$ be the minimum integer such that $\forall k \geq \hat{n}_i$, $\frac{d(\hat{p}_i)}{k} < \hat{q}_i$ and

$$\frac{\hat{p}_i d(\hat{p}_i)}{k} - c(\frac{d(\hat{p}_i)}{k}) < \hat{\pi}.$$

Lemma 3. *If the number of firms charging $\hat{p}_i$ is greater than or equal to $\hat{n}_i$, then the profit of some of these firms would be less than $\hat{\pi}$.*

Lemma 3 provides an interpretation of $\hat{n}_i$. We need a further definition.

Definition. $M_2 = \sum_{i=j+1}^{k} \hat{n}_i + n^* - 1$.

We then state and prove the next proposition.

Proposition 2. *Suppose Assumptions 1, 2, 3, 5 and 6 hold and, moreover, let $n \geq \max\{M_1, M_2\}$. Then the unique equilibrium involves all the firms charging p^*, producing $q'(n-1)$ and selling $\frac{d(p^*)}{n}$.*

Proof. Existence. Step 1. Since, from Lemma 2, $(n-1)q'(n-1) > d(p^*)$, it is not possible for any firm to increase its price and gain, as the deviating firm will have no residual demand. Of course, from the definition of p^* it follows that undercutting is not profitable either.

Step 2. We then argue that none of the firms can change its output level and gain. Suppose firm i produces q_i, while the other firms produce $q'(n-1)$. Then the profit of the i-th firm

$$\pi_i(q_i, q', p^*) = p^* d(p^*)\gamma(q_i, (n-1)q') - c(q_i). \tag{8}$$

Observe that the profit function is concave in q_i [12] and $\frac{\partial \pi_i(q_i,q',p^*)}{\partial q_i}|_{q_i=0} > 0$.[13]

[12]This follows since $\frac{\partial^2 \pi_i(q_i,q',p^*)}{\partial q_i^2} = p^* d(p^*)\gamma_{11}(q_i, (n-1)q') - c''(q_i) < 0$.

[13]Suppose not, i.e. let $p^* d(p^*)\gamma_1(0, (n-1)q') - c'(0) \leq 0$. Then,

$$\begin{aligned} c'(q'(n-1)) &= p^* d(p^*)\gamma_1(q'(n-1), (n-1)q'(n-1)) \\ &< p^* d(p^*)\gamma_1(0, (n-1)q'(n-1)) \ \text{(as } \gamma_{11} < 0) \ \leq \ c'(0), \end{aligned}$$

We then notice that

$$\frac{\partial \pi_i(q_i, q', p^*)}{\partial q_i}|_{q_i=q'} = p^* d(p^*)\gamma_1(q', (n-1)q') - c'(q') = 0, \tag{9}$$

where the last equality follows from Eq. (6). Thus none of the firms has an incentive to change their output levels. Finally, given that $\gamma(q_i, \sum_{j\neq i} q_j)$ is symmetric, all the firms must be selling an identical amount, i.e. $\frac{d(p^*)}{n}$.

Uniqueness. Step 1. We first argue that all the firms must be producing strictly positive amounts in equilibrium. Suppose to the contrary that firm i has an output level of zero.

(i) First consider the case where the total production by the firms charging p^* is less than $d(p^*)$. (Clearly, all firms charging prices less than p^* would have an output level of zero). Let the i-th firm charge p^*. Since $p^* > c'(0)$, the profit of firm i would increase if it produces a sufficiently small amount.

(ii) Next consider the case where the total production by the firms charging p^* is greater than $d(p^*)$. Without loss of generality let these firms be $1, \cdots, m$, where $m < i$, and let $q_1 > 0$. Note that

$$\begin{aligned} \frac{\partial \pi_i}{\partial q_i}|_{q_i=0} &= p^* d(p^*)\gamma_1(0, \sum_{j=1}^{m} q_j) - c'(0) \\ &> p^* d(p^*)\gamma_1(q_1, \sum_{j=2}^{m} q_j) - c'(q_1) \text{ (since } \gamma_{11} - \gamma_{12} < 0) = \frac{\partial \pi_1}{\partial q_1} = 0. \end{aligned}$$

But then firm i can increase its output slightly from zero and gain.

Step 2. We then argue that there cannot be some $\widehat{p}_i$ $(\in F) > p^*$ such that some firms charge $\widehat{p}_i$ and supply a positive amount.

Suppose to the contrary that such a price exists. This implies that the total number of firms charging p^*, say $\tilde{n}$, can be at most $n^* - 1$. Suppose not, i.e. let the number of firms be n^* or more. Moreover, let the aggregate production by these firms be less than $d(p^*)$. Clearly, all $\tilde{n}$ firms must be producing q^*. (Since there is excess demand at this price, the residual demand constraint cannot bind, and the output level of all firms must be such that price equals marginal cost.) But, from the definition of n^*, this implies that total production is greater than $d(p^*)$, which is a contradiction.

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which is a contradiction.

Now consider some $\widehat{p}_i > p^*$. Clearly, the number of firms charging $\widehat{p}_i$ is less than $\hat{n}_i$. Since otherwise some of these firms would have a profit less than $\hat{\pi}$. But they can always ensure a profit of $\hat{\pi}$ by charging p^*. Thus the total number of firms producing a strictly positive amount is less than N_2, thus contradicting step 1. Hence all the firms must be charging p^*.

Step 3. Let $\tilde{q} = (\tilde{q}_1, \cdots, \tilde{q}_n)$, denote the equilibrium output vector. First note that it cannot be the case that $\sum_i \tilde{q}_i < d(p^*)$. Since $n \geq M_2 > n^* - 1$, for some j, $\tilde{q}_j < c'^{-1}(p^*)$, and this firm will have an incentive to increase its output.

We then establish that the equilibrium output vector must be symmetric. Suppose not, and without loss of generality let $\tilde{q}_2 > \tilde{q}_1 > 0$. Then,

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1}|_{\tilde{q}} &= p^* d(p^*)\gamma_1(\tilde{q}_1, \sum_{i \neq 1} \tilde{q}_i) - c'(\tilde{q}_1) \\ &> p^* d(p^*)\gamma_1(\tilde{q}_2, \sum_{i \neq 2} \tilde{q}_i) - c'(\tilde{q}_2) \text{ (as } \gamma_{11} - \gamma_{12} < 0) = \frac{\partial \pi_2}{\partial q_2}|_{\tilde{q}}. \end{aligned} \tag{10}$$

This, however, is a contradiction, since in equilibrium $\frac{\partial \pi_1}{\partial q_1}|_{\tilde{q}} = 0 = \frac{\partial \pi_2}{\partial q_2}|_{\tilde{q}}$.

Step 4. Finally, we argue that there cannot be another symmetric equilibrium where the (common) output level of the firms is different from $q'(n-1)$. Clearly, in any symmetric equilibrium, the production level of all the firms must satisfy Eq. (6) which has a unique solution. ■

The idea behind the existence result is as follows. If the number of firms is large enough, then competition will drive all the firms to excess production in an attempt to manipulate the residual demand. This excess production ensures that if any of the firms charges a price greater than p^*, then the residual demand facing this firm will be zero. Thus none of the firms have an incentive to charge a price which is greater than p^*.

We then turn to the limit properties of the equilibrium output levels as n becomes large.

Lemma 4. $\lim_{n \to \infty} q'(n-1) = 0$.

Lemma 4 demonstrates that the output level of each firm becomes vanishingly small as the number of firms becomes very large. Recall, however, that the equilibrium involves excess production. The next proposition examines whether in the limit aggregate production, $nq'(n-1)$, approaches the demand level, $d(p^*)$, or not.

Proposition 3. *Suppose Assumptions 1, 2, 3, 5 and 6 hold.*
(i) If $c'(0) = 0$, then $\lim_{n\to\infty} nq'(n-1) \to \infty$.
(ii) If $c'(0) > 0$, then $\lim_{n\to\infty} nq' = d(p^*)\frac{p^*}{c'(0)}$.

Therefore the limiting behavior of the aggregate production level, $nq'(n-1)$, depends on the value of $c'(0)$. If $c'(0) = 0$, then aggregate production increases without bounds. Whereas it converges to $d(p^*)\frac{p^*}{c'(0)}$ if $c'(0) > 0$. As α goes to zero this term goes to $d(c'(0))$.

The folk theorem of perfect competition suggests that the perfectly competitive outcome can be interpreted as the limit of some oligopolistic equilibrium as the number of firms becomes large. While this issue has been thoroughly investigated in the context of Cournot competition (e.g. Novshek (1980), Novshek and Sonnenschein (1983) and Ruffin (1971)), in the Bertrand framework it remains relatively unexplored.[14]

The analysis so far allow us to discuss if, in the present framework, the folk theorem holds or not. From Propositions 1 and 2 we know that, for a given grid size α, there is a unique equilibrium price that is within α of the competitive one whenever n is sufficiently large. Also, from Proposition 1 and Lemma 4, the output levels of the individual firms are close to zero whenever n is large. Furthermore, the aggregate output is well behaved if the TBR is weakly manipulable, and reasonably so if the TBR is strongly manipulable and $c'(0) > 0$.

Hence, for the case where the TBR is weakly manipulable, or it is strongly manipulable and the marginal cost at the origin is positive, our results, perhaps, provide a non-cooperative foundation for the theory of perfect competition, and hence for the folk theorem.

[14]There are notable exceptions though, e.g. Allen and Hellwig (1986) and Vives (1986).

3.3 Limit Results for a Fixed n and α Small

Note that, in Propositions 1-3, the analysis is carried out for a *given grid size* α, while n is taken to be large. In this sub-section, *for a given* n, we focus on examining if, for all sufficiently small α, a single price equilibrium (defined below) exists. The objective is to examine the sensitivity of the analysis to the nature of the limiting process.

Definition. A Nash equilibrium $P^N, Q^N = (p_1^N, \cdots, p_n^N, q_1^N, \cdots, p_n^N)$ is said to be a single price equilibrium (henceforth SPE) if, $\forall i \neq j$, $q_i^N > 0$ and $q_j^N > 0$ imply that $p_i^N = p_j^N$.

We require a few more assumptions and notations. A7(i) below imposes a lower bound on $R_i(P,Q)$, whereas A7(ii) states that $R_i(P,Q)$ is decreasing in the output of the other firms charging p_i or less.

A7. (i) $\forall P, Q$, $R_i(P,Q) \geq \max\{0, d(p_i) - \sum_{j|j\neq i,\, p_j \leq p_i} q_j\}$.
(ii) $\forall P, Q$ and j, such that $j \neq i$ and $p_j \leq p_i$, $R_i(P,Q)$ is weakly decreasing in q_j.

Assumption 8 is a continuity assumption.

A8. Consider P, Q such that all firms other than i charge p' and supply $\frac{d(p')}{n}$, and firm i charges $p_i \geq p'$ and supplies $\max_{q_i} R_i(P,Q)$. Then $\lim_{p_i \to p'+} \frac{\partial[\max_{q_i} R_i(P,Q)]}{\partial p_i}$ is continuous in p'.

Next we define $p_c(n)$ as solving $p = c'(\frac{d(p)}{n})$.
Given A1 and A2, it is easy to see that $p_c(n)$ is well defined, $p^{\max} > p_c(n) > c'(0)$, $p_c(n)$ is decreasing in n and $\lim_{n\to\infty} p_c(n) = c'(0)$.
Next, $\forall n \geq 2$, and $\forall p \geq 0$, let $\pi_n(p) = \frac{pd(p)}{n} - c(\frac{d(p)}{n})$.

A9. (i) $\forall n \geq 2$, and $\forall p$ s.t. $p^{\max} \geq p \geq 0$, $\pi_n(p)$ is concave in p.
(ii) $\forall p$ s.t. $c'(0) \leq p \leq p_c(n)$, $\lim_{q \to \frac{d(p)}{n}+} pd(p)\gamma_1(q, (n-1)\frac{d(p)}{n}) < c'(\frac{d(p)}{n})$.

A9(i) is a concavity assumption. A9(ii) is satisfied by the example of a strongly manipulable TBR that follows A6.

For any vector $A = (a_1, \cdots, a_m)$, let $\underline{A} = \{a_1, \cdots, a_m\}$. Proposition 4 below is the central result of this sub-section.

Proposition 4. *(i)* $\lim_{\alpha \to 0} p^*(\alpha) = c'(0)$.
(ii) Suppose Assumptions 1, 2, 3 and 4 hold.
(a) $\lim_{\alpha \to 0} N_1(\alpha) \to \infty$.
(b) Let Assumptions 7, 8 and 9(i) hold and let n *be fixed. (A)* $\exists \alpha' > 0$, *such that* $\forall \alpha,\ 0 < \alpha < \alpha'$, *no SPE exists. (B) Further,* $\exists \tilde{p}(n) > c'(0)$ *and* $\tilde{\alpha}(\tilde{p}(n)) > 0$, *such that* $\forall \alpha,\ 0 < \alpha < \tilde{\alpha}(\tilde{p}(n))$, *in any equilibrium* P, Q, $\min\{\underline{P}\} > \tilde{p}(n)$.
(iii) Suppose Assumptions 1, 2, 3, 5 and 6 hold.
(a) $\lim_{\alpha \to 0} M_2(\alpha) \to \infty$.
(b) Let Assumptions 7, 8 and 9 hold and let n *be fixed. (A)* $\exists \alpha'' > 0$ *such that* $\forall \alpha,\ 0 < \alpha < \alpha''$, *no SPE exists. (B) Further,* $\exists \widehat{p}(n) > c'(0)$ *and* $\widehat{\alpha}(\widehat{p}(n)) > 0$, *such that* $\forall \alpha,\ 0 < \alpha < \widehat{\alpha}(\widehat{p}(n))$, *in any equilibrium* P, Q, $\min\{\underline{P}\} > \widehat{p}(n)$.

Let us fix n. Proposition 4 shows that, for all sufficiently small α, no single price equilibrium exists,[15] and *all* equilibria (in case they exist) are bounded away from the competitive price. Thus Proposition 4 demonstrates that for the earlier results to go through, n needs to be increasing at a 'relatively' faster rate compared to the rate of decrease in α.

4 One-stage Game: The Asymmetric Case

Deneckere and Kovenock (1996) is one of the very few papers that examine Bertrand-Edgeworth competition in an asymmetric framework. In a price-setting duopoly where the firms differ in both their unit costs and capacities, they characterize the set of equilibria and then, as an application, re-examine the Kreps and Scheinkman (1983) model with asymmetric costs, demonstrating that the Cournot equilibrium capacity levels need not emerge

[15] Whether there can be equilibria that are not single price is an open question. Other papers to focus on SPE include Dixon (1993).

in equilibrium. In keeping with our approach, however, in this section we shall be interested in the case where the number of firms is large.

Let there be m types of firms with the cost function of the l-th type being $c_l(q)$. The number of type l firms is denoted by n^l, where $\sum_l n^l = n$.

4.1 Weakly Manipulable TBR

Let p_l^* denote the minimum $p \in F$ such that $p > c_l'(0)$. Let $R_{il}(P,Q)$ denote the residual demand function facing the i-th firm of type l. Moreover, let $r_{il}(p_i, p, n) = R_{il}(P,Q)$, if $p_i \geq p$, and, $\forall j \neq i$, $p_j = p$ and $q_j = \frac{d(p)}{n}$. The residual demand satisfies appropriately modified versions of A3 and A4.[16] Next define n_l^* and $\hat{n}_l$ in a manner analogous to that of n^* and $\hat{n}$ respectively, only taking care to use the cost function of the l-th type, $c_l(q)$, instead of $c(q)$ in the definitions. (We can argue, as before, that, $\forall l$, $\hat{n}_l \geq n_l^*$.)

Definition. $\hat{N} = \max\{\hat{n}_1, \cdots, \hat{n}_m\}$.

We require some further notations. Let

$$\tilde{\pi}_l \ = \ \frac{p_l^* d(p_l^*)}{\max_q \ n_q^*} - c_l(\frac{d(p_l^*)}{\max_q \ n_q^*}).$$

Next consider some $\widehat{p}_x \in F$, such that $\widehat{p}_x > p_l^*$. Let $\widehat{q}_{lx}$ satisfy $\widehat{p}_x = c_l'(\widehat{q}_{lx})$. Clearly if a type l firm charges $\widehat{p}_x$ and sells $\frac{d(\widehat{p}_x)}{r}$, then the profit of such a firm is $\widehat{p}_x \frac{d(\widehat{p}_x)}{r} - c_l(\frac{d(\widehat{p}_x)}{r})$.

We then define n_{lx} to be the smallest possible integer such that $\forall r \geq n_{lx}$, $\frac{d(\widehat{p}_x)}{r} < \widehat{q}_{lx}$ and

$$\widehat{p}_x \frac{d(\widehat{p}_x)}{r} - c_l(\frac{d(\widehat{p}_x)}{r}) < \tilde{\pi}_l.$$

Suppose that in any equilibrium the number of firms charging $\widehat{p}_x$, say $\tilde{m}$, is greater than or equal to $\max_q n_{qx}$. Then at least one of these firms, say of type l, would have a residual demand that is less than or equal to $\frac{d(\widehat{p}_x)}{\tilde{m}}$. Since $\frac{d(\widehat{p}_x)}{\tilde{m}} < c_l'^{-1}(\widehat{p}_x)$, this firm would supply at most $\frac{d(\widehat{p}_x)}{\tilde{m}}$ and have a profit less than $\tilde{\pi}_l$.

[16] A4(ii) should be modified so that, the restrictions are on $R_{il}(P,Q)$, rather than on $R_i(P,Q)$. The changes needed in A4(i) and A3 are equally obvious.

We restrict attention to two cases, though we also briefly discuss the other cases.

Case (i). $c'_1(0) = c'_2(0) = \cdots = c'_m(0)$.

Note that if, at a given price, any firm finds it profitable to produce a strictly positive amount, then so will all other firms. For this case let us redefine $p^* = p^*_1 = \cdots = p^*_m$.

Definition. $\tilde{N}_1 = \sum_{x=j+1,\cdots,k} \max_l n_{lx} + \max_l n^*_l - 1$.

We can now state our next proposition.

Proposition 5. *Suppose Assumptions 1, 2, and appropriately modified versions of 3 and 4 hold, and let* $c'_1(0) = c'_2(0) = \cdots = c'_m(0)$*. If* $n \geq \max\{\hat{N}, \tilde{N}_1\}$*, then the unique equilibrium involves all the firms charging a price of* p^**, and producing* $\frac{d(p^*)}{n}$*.*

Case (ii). $c'_1(0) < c'_2(0) < \cdots < c'_m(0)$.

Consider any p such that $c'_1(0) < p < c'_2(0)$. While at this price producing a small enough positive level of output is profitable for type 1 firms, firms of other types will not find it profitable to supply a positive level of output. Hence type 1 firms are, in some sense, the most efficient.

Definition. Let $p^*_1 = \widehat{p}_h$ (say). $\hat{N}_2 = \sum_{x=h+1,\cdots,k} n_{1x} + n^*_1 - 1$.

Proposition 6 below solves for the case when n^1 is large.

Proposition 6. *Suppose Assumptions 1, 2, and appropriately modified versions of 3 and 4 hold, and let* $c'_1(0) < c'_2(0) < \cdots < c'_m(0)$*. If* $\alpha < c'_2(0) - c'_1(0)$ *and* $n^1 \geq \max\{N^1_1, \hat{N}_2\}$*, then there is an equilibrium that involves all firms of type 1 charging* p^*_1 *and producing* $\frac{d(p^*_1)}{n^1}$*, and firms of all other types charging* $p^{\max}$ *and having an output level of zero. Furthermore, any equilibrium involves all firms of type 1 charging* p^*_1 *and producing* $\frac{d(p^*_1)}{n^1}$*, and all other firms having an output level of zero.*

In any equilibrium, note that all firms producing a positive amount (i.e.

type 1 firms) have the same strategies. Further, interpreting $c_1'(0)$ as the perfectly competitive price, p_1^* is within α of the competitive price.

Next suppose that $c_1'(0) = c_2'(0) = \cdots = c_j'(0) < c_{j+1}'(0) \leq \cdots \leq c_m'(0)$. From Propositions 5 and 6, for a given α, if the number of firms of type 1 to j is large enough, then there is an equilibrium where all such firms charge p_1^*, and all other firms charge $p^{\max}$ and have an output of zero. Furthermore, any equilibrium involves all firms of type 1 charging p_1^*, and all other firms having an output level of zero.

Finally, consider the case when $n_1, \cdots, n_j$ are exogenously given. In this case one can construct examples where no equilibrium may exist even if the number of firms of type $j+1$ to m is very large.[17]

4.2 Strongly Manipulable TBR

Let A1, A2, an appropriately modified version of A3, A5, and A6 hold for this case.[18]

We then define q_l^*, $\hat{q}_{il}$, $q_l'(n^l - 1)$, $\hat{n}_i^l$, M_1^l and M_2^l in a manner similar to that of q^*, $\hat{q}_l$, $q'(n-1)$, $\hat{n}_i$, M_1 and M_2 respectively, only taking care to use the cost function of the l-th type, $c_l(q)$, instead of $c(q)$.

We restrict attention to two cases, though we briefly consider the other cases.

Case (i). $c_1'(0) = c_2'(0) = \cdots = c_m'(0) = c'(0)$ (say).

Let $(q_{11}', \cdots, q_{n^1 1}', \cdots, q_{1m}', \cdots, q_{n^m m}')$ solve

$$p^* d(p^*) \gamma_1(q_{il}, \sum_j \sum_k q_{jk} - q_{il}) - c_l'(q_{il}) = 0, \ \forall i, l, \tag{11}$$

where q_{il}' denote the output level of the i-th firm of type l.

If, $\forall i, l$, $\sum_a \sum_b q_{ab} - q_{il} > q_{il} > 0$, then we can use the Gale-Nikaido (1965) univalence theorem to show that Eq. (11) has a unique solution (the

[17] The proof is available from the author.

[18] Thus $\gamma(q_i, \sum_{j \neq i} q_j)$ is assumed to be independent of firm type. This is for notational reasons alone. Let $\gamma^l(q_i, \sum_{j \neq i} q_j)$, the gamma function for type l firms, satisfy appropriately modified versions of A5 and A6. Under the additional assumption that $\lim_{a \to 0} \gamma_1^l(a, X) = \lim_{a \to 0} \gamma_1^k(a, X)$, $X > 0$, it is simple to check that all our results in this sub-section go through.

proof is available from the author). Moreover, the solution is symmetric, i.e. $\forall l$, $q'_{1l} = \cdots = q'_{n^l l} = q'_l$. Thus $(q'_1, \cdots, q'_m)$ solves:

$$p^* d(p^*) \gamma_1(q_l, (n^l - 1)q_l + \sum_{j \neq l} n^j q_j) - c'_l(q_l) = 0, \ \forall l. \tag{12}$$

The proofs of these lemmas $1' - 4'$ below are very similar to that of the corresponding lemmas 1-4 earlier, and hence omitted.

Lemma $1'$**.** $\lim_{n\to\infty} p^* d(p^*) \gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) > \lim_{n\to\infty} c'_l(\frac{d(p^*)}{n-1}), \forall l$.

Given Lemma $1'$, we can define $\widehat{M}_1$ to be the smallest possible integer such that $\forall l$ and $\forall n \geq \widehat{M}_1$,

$$p^* d(p^*) \gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) > c'_l(\frac{d(p^*)}{n-1}).$$

Lemma $2'$**.** If $\sum_l n^l \geq \widehat{M}_1$, then, $\forall l$, $(n^l - 1)q'_l + \sum_{j \neq l} n^j q'_j > d(p^*)$.

We need some more notations.
$n^{**} = \max\ n^*_l$.
$\tilde{q} = \max_{n'_1, \cdots, n'_m} \sum_l n'_l q^*_l$ such that $\sum_l n'_l q^*_l < d(p^*)$.

$$\overline{\pi}_l = \max_q \begin{cases} p^* d(p^*) \gamma(q, \tilde{q}) - c_l(q), & \text{if } q > d(p^*) - \tilde{q}, \\ p^* q - c_l(q), & \text{otherwise.} \end{cases} \tag{13}$$

Note that $\overline{\pi}_l$ represents the least possible profit that an l type firm can obtain by charging p^* when the aggregate output level of the other firms charging p^* is $\tilde{q}$, or less. Since $\tilde{q} < d(p^*)$, $\overline{\pi}_l > -c_l(0)$. Moreover, let $\overline{\pi} = \min_l \overline{\pi}_l$.

Next consider some $\hat{p}_i \in F$, such that $\hat{p}_i > p^*$. Let $\hat{n}_{il}$ be the minimum integer such that $\forall k \geq \hat{n}_{il}$, $\frac{d(\hat{p}_i)}{k} < \hat{q}_{il}$ and

$$\frac{\hat{p}_i d(\hat{p}_i)}{k} - c_l(\frac{d(\hat{p}_i)}{k}) < \hat{\pi}.$$

Lemma $3'$ below provides an interpretation of $\hat{n}_{il}$.

Lemma $3'$**.** *If the number of type l firms charging $\hat{p}_i$ ($\in F$) $> p^*$ is greater than or equal to $\hat{n}_{il}$, then the profit of some of these firms would be less than $\overline{\pi}$.*

Definition. $\widehat{M}_2 = n^{**} - 1 + \sum_{i=j+1}^{k} \max_l \hat{n}_{il}$.

Proposition 7. *Suppose Assumptions 1, 2, an appropriately modified version of 3, 5 and 6 hold and* $c_1'(0) = c_2'(0) = \cdots = c_m'(0) = c'(0)$. *If,* $\sum_l n^l \geq \max\{\widehat{M}_1, \widehat{M}_2\}$, *then there is a unique Nash equilibrium where all firms of type* l *charge* p^* *and produce* q_l'. *Moreover, all firms of the same type sell the same amount.*

We then turn to the limit properties of the equilibrium outputs. We first impose some structure on the limiting process. Let us fix some vector $(n^1, \cdots, n^m)$. We then define an r-economy to be one where the number of type l firms is rn^l. Let $(q_1'(r), \cdots, q_m'(r))$ solve an appropriately modified version of Eq. (12) for the r-economy.

Lemma $4'$**.** $\lim_{r\to\infty} q_l'(r) = 0$, $\forall l$.

We then examine whether in the limit the aggregate production, $\sum_j rn^j q_j'(r)$, approaches the demand level $d(p^*)$ or not.

Proposition 8. *Suppose Assumptions 1, 2, an appropriately modified version of 3, 5 and 6 hold and* $c_1'(0) = c_2'(0) = \cdots = c_m'(0) = c'(0)$.
(i) If $c'(0) = 0$, *then* $\lim_{r\to\infty} \sum_j rn^j q_j'(r) \to \infty$.
(ii) If $c'(0) > 0$, *then* $\lim_{r\to\infty} \sum_j rn^j q_j'(r) = d(p^*)\frac{p^*}{c'(0)}$.

Therefore, as in the case with symmetric firms, if $c'(0) = 0$, then aggregate production increases without bounds, whereas if $c'(0) > 0$, then aggregate production converges to $d(p^*)\frac{p^*}{c'(0)}$.

It is natural to ask if Lemma $4'$ and Proposition 8 go through in case, say, n^l is taken to infinity, while $\forall j \neq l$, n^j is kept constant. We can mimic the proof of Lemma $4'$ to show that $\lim_{n^l\to\infty} q_l' = 0$. Moreover, we can mimic the proof of Proposition 8 to demonstrate that if $c'(0) = 0$, then $\lim_{n^l\to\infty} \sum_j n^j q_j'(n^l) \to \infty$, and, if $c'(0) > 0$, then $\lim_{n^l\to\infty} \sum_j n^j q_j'(n^l) = d(p^*)\frac{p^*}{c'(0)}$.[19] However, what happens to $q_j'(n^l)|_{j\neq l}$, as n^l is taken to infinity,

[19] This is true since, for a corresponding version of Proposition 8 to go through, it is sufficient that $\lim_{n^l\to\infty} q_l' = 0$.

is an open question.

Case (ii). $c_1'(0) < c_2'(0) < \cdots < c_m'(0)$.

Proposition 9 below examines the case when n^1 is large.

Proposition 9. *Let Assumptions 1, 2, an appropriately modified version of 3, 5 and 6 hold and $c_1'(0) < c_2'(0) < \cdots < c_m'(0)$. Moreover, suppose that $\alpha < c_2'(0) - c_1'(0)$ and $n^1 \geq \max\{M_1^1, M_2^1\}$. Then there is an equilibrium that involves all firms of type 1 charging p_1^*, producing $q_1'(n^1 - 1)$ and selling $\frac{d(p_1^*)}{n^1}$, and all other firms charging $p^{\max}$ and having an output of zero. Furthermore, any equilibrium involves all firms of type 1 charging p_1^*, producing $q_1'(n^1 - 1)$ and selling $\frac{d(p_1^*)}{n^1}$, and all other firms having an output of zero.*

Next suppose $c_1'(0) = c_2'(0) = \cdots = c_j'(0) < c_{j+1}'(0) \leq \cdots \leq c_m'(0)$. Combining Propositions 7 and 9, it is easy to see that if $\sum_1^j n^l$ is large enough, then there is an equilibrium where all firms of type 1 to j charge p_1^*, and all other firms have an output of zero. What happens in case there are a large number of firms of type $j+1$ to m, and firms of type 1 to j are relatively few in number, is an open question.

5 Two-stage Model

We then examine the case where the firms are symmetric and play a two stage game where, in stage 1, the firms simultaneously announce their prices, and in stage 2, they simultaneously decide on their output levels. Moreover, in stage 2, the price vector announced in stage 1 is common knowledge.

Fudenberg and Tirole (1987) and Tirole (1988) both employ such two-stage models to provide a game-theoretic foundation of contestability. In a two stage framework with continuous prices, convex costs, and costs of turning customers away, Dixon (1990) finds that if the industry is large enough, then the competitive price will be an equilibrium. Moreover, if costs of turning consumers away are small, then all equilibria will be close to the competitive one. Maskin (1986) shows that under a two-stage framework an equilibrium exists (for general TBRs). Whereas in a symmetric two-stage framework with strictly convex costs, efficient rationing, and the equalshares TBR, Yoshida (2002) characterizes the symmetric mixed strategy equilibrium.

5.1 Weakly Manipulable TBR

We then solve for the subgame perfect Nash equilibrium (spNe) of this game.

We need some further notations. Consider some P, Q. Given A3(iv), $\forall p_i < p^{\max}$ we can define $R^{p_i}(P, Q_{p_i}) = R^{p_i}(P, Q)$.

Next let $(\overline{p}_1, \ldots, \overline{p}_n, \overline{q}_1(P), \ldots, \overline{q}_n(P))$ denote a spNe, where $\overline{p}_i$ denotes firms i's stage 1 strategy and $\overline{q}_i(P)$ denotes its stage 2 strategy (as a function of the stage 1 price vector P).

Finally, if $\forall i$, firm i produces $\overline{q}_i(P)$, then let $\overline{Q}_{p_i}(P)$ denote the output vector of all firms charging less than p_i.

Proposition 10. *Suppose Assumptions 1, 2, 3, and 4 hold. If* $n \geq \max\{\hat{n}, n^* + 1, N_1\}$*, then, the unique spNe of the two stage game involves* $\overline{p}_j = p^*$ *and* $\overline{q}_j(P) = min\{\frac{R^{p_j}(P, \overline{Q}_{p_j}(P))}{m_j(P)},\ c'^{-1}(p_j)\}, \forall j$*, where* $m_j(P)$ *denotes the number of firms charging* p_j.

Note that along the equilibrium path, in stage 1 all firms charge p^*, and in stage 2, all firms produce $\frac{d(p^*)}{n}$.

5.2 Strongly Manipulable TBR

In this case we find that a subgame perfect equilibrium may not exist. The problem is as follows. In stage 1, suppose that there are m (> 1) firms charging the lowest (say) price $\overline{p}$. In case $c'^{-1}(\overline{p}) > \frac{d(\overline{p})}{m}$ then, given that the strategy space is not bounded, an equilibrium for the stage 2 game may not exist. Since m can be small, we cannot use the techniques used in Section 3.2 to resolve this problem.

Example. Suppose $n = 2$, $d(p) = a - p$, $\gamma(q_i, q_j) = \frac{q_i}{q_1+q_2}$ and $c(q) = \frac{q^2}{2}$. Suppose that both firms charge $\overline{p}$, where $a/2 > \overline{p} > a/3$. Let $q'(p, 1)$ solve $pd(p)\gamma_1(q, q) = c'(q)$. Observe that $2q'(\overline{p}, 1) = \sqrt{\overline{p}(a - \overline{p})} < a - \overline{p}$, so that both firms producing $q'(\overline{p}, 1)$ cannot be an equilibrium. Whereas since

$c'^{-1}(\overline{p}) = \overline{p} > \frac{a-\overline{p}}{2}$, both firms producing $c'^{-1}(\overline{p})$ cannot be an equilibrium either. Finally, suppose to the contrary there is an asymmetric equilibrium, where firm i produces $\overline{q}_i$. Clearly, $\overline{q}_1 + \overline{q}_2 > d(\overline{p})$. Thus, from the first order conditions, $\overline{p}d(\overline{p})\gamma_1(\overline{q}_1, \overline{q}_2) = c'(\overline{q}_1)$ and $\overline{p}d(\overline{p})\gamma_1(\overline{q}_2, \overline{q}_1) = c'(\overline{q}_2)$. But taken together, these equations imply that $\overline{q}_1 = \overline{q}_2$.

6 Conclusion

To begin with we discuss the relationship of this paper to the literature. For convenience the discussion is organized around a few remarks.

Remark 1. Our results are consistent with Shubik (1959, Chapter 5) who demonstrates that, under Bertrand-Edgeworth competition (both one stage and two stage) with continuous prices and strictly convex cost functions, any pure strategy equilibrium must involve the competitive price.

In general of course, no pure strategy equilibrium exists. Dixon (1987, 1993) solves the non-existence problem for one stage games by introducing various rigidities, e.g. menu costs (formalized through the notion of *epsilon*-equilibrium in Dixon (1987)), and integer pricing (Dixon (1993)). In a two stage game, Dixon (1990) introduces rigidities that take the form of costs involved in turning consumers away. These papers demonstrate that, in the presence of the appropriate rigidities, there are "equilibrium" prices which are arbitrarily close to the competitive price whenever the industry is large enough, results that are close in spirit to the present one.

Remark 2. Next consider price competition with linear and capacity constrained cost functions. In a model with the efficient rationing rule, Vives (1986, proposition 2(iii)) shows that, *for a given firm size*, one obtains the perfectly competitive price as the number of firms goes to infinity. For the parallel rationing rule Börgers (1992) shows that iterated elimination of dominated strategies yields prices close to the competitive price.

For the proportional rationing rule, however, Allen and Hellwig (1986) demonstrate that in general, there is no pure strategy equilibrium. Moreover, in the limit, the mixed strategy equilibrium does not converge in the support. Since the cost functions are linear and capacity constrained, firms produce till capacity if at all (provided there is demand), but not beyond that. Thus, given the nature of the cost functions, the TBR is *effectively* weakly manipulable. Since in this paper we do not consider the case where a proportional rationing rule is coupled with a weakly manipulable TBR, the results in Allen and Hellwig (1986) are not inconsistent with ours.

Remark 3. In the present paper the limiting procedure involves taking the number of firms to infinity, *while keeping the grid-size constant.*[20] Arguably there are other papers in the literature that follow a similar limiting procedure, e.g. Dixon (1987, 1990, 1993) and Roy Chowdhury (1999). All these papers use limiting procedures that involve taking the number of firms to infinity, *while keeping constant the size of some relevant rigidity in the model.* In Dixon (1993) and Roy Chowdhury (1999) this rigidity takes the form of grid-pricing. While prices are modeled continuously in Dixon (1987, 1990), both involve rigidities (described in Remark 1 above).

Remark 4. Next note that in Allen and Hellwig (1986), Dixon (1987, 1990, 1993) and Vives (1986), the limiting procedure not only involves taking the number of firms to infinity, but also involves taking "firm size", relative to market demand, to zero. In Allen and Hellwig (1986) and Vives (1986) this is done by taking the capacity level of the firms to zero, while in Dixon (1987, 1990, 1993) this is done by replicating the market demand function. Under our approach, however, relative firm size is kept unchanged.

The idea is as follows. Under our approach the number of firms is exogenous (so that there is no exit) and there are no setup costs, so that one can assume, without loss of generality, that $c(0) = 0$. Moreover, since the cost function is strictly convex, the efficient scale of production is zero.[21] Hence, given that the firms are already 'very small' compared to market demand, the limiting procedure only involves taking the number of firms to infinity. Other papers to employ a similar limiting procedure include, Tasnádi (1999a) and Roy Chowdhury (1999) (Bertrand-Edgeworth), Novshek and Roy Chowdhury (2003) (Bertrand-Chamberlin) and Ruffin (1971) (Cournot).

Dixon (1993) shows that if one replicates both demand and firms, then

[20] Of course, in Section 3.3 we also examine the effect on equilibrium outcomes if the grid-size is taken to zero.

[21] Alternatively, given that there is no exit, the appropriate measure of efficient scale in this paper is $argmin_q \frac{c(q)-c(0)}{q} = 0$.

in a one stage model with weakly manipulable TBRs, Nash equilibria are non-unique. This demonstrates the importance of the replication procedure.

Remark 5. In this paper the equilibrium profit levels of the firms depend on the TBR. Given that there is grid-pricing and there is a unique single price equilibrium such that the residual demand binds strictly, this dependence is to be expected. *In the context of grid-pricing*, such dependence is not new in the literature though, e.g. Harrington (1989) and Maskin and Tirole (1988, footnote 13). Of course, the TBR will not affect the profit levels if the equilibrium is in atomless mixed strategies (e.g. Vives (1986), and Allen and Hellwig (1993) for the symmetric case).[22]

Remark 6. Interestingly, for strongly manipulable TBRs the equilibrium involves excess production, which is inefficient. Given the nature of the TBR, this result is, perhaps, only to be expected. While Allen and Hellwig (1986, 1993), Osborne and Pitchik (1986), and Tasnádi (1999b), all have srongly manipulable TBRs, these papers assume that the cost functions are linear and capacity constrained. Hence these TBRs are, in effect, not strongly manipulable, so that a similar effect does not appear in these papers. Thus the present paper is one of the very few that deal with strongly manipulable TBRs, in particular the limit properties of equilibrium when the TBR is strongly manipulable.

In conclusion, in this paper we re-examine the non-existence problem associated with pure strategy Nash equilibrium under price competition (i.e. the Edgeworth paradox). We consider a model of Bertrand-Edgeworth price competition with strictly convex costs and discrete pricing. If firms are symmetric than, for a large class of residual demand functions there is a unique equilibrium in pure strategies whenever, for a fixed grid size, the number of firms is sufficiently large. Moreover, the equilibrium price is within a grid unit of the competitive price. Our analysis also has interesting implications for the folk theorem of perfect competition. To a large extent, the results go through when the firms are asymmetric, or they are symmetric but play a two stage game and the TBR is weakly manipulable.

[22] Deneckere and Kovenock (1996) show that, for the classical Bertrand-Edgeworth model, equilibrium profits are invariant with respect to the TBR.

7 Appendix

Proof of Lemma 1. Notice that

$$\begin{aligned}
\lim_{n\to\infty} p^* d(p^*)\gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) &= p^* d(p^*)\frac{1}{d(p^*)} \text{ (from A6(iv))} \\
&= p^* > c'(0) = \lim_{n\to\infty} c'(\frac{d(p^*)}{n-1}).
\end{aligned}$$

■

Proof of Lemma 2. Suppose not, i.e. let $q'(n-1) \leq \frac{d(p^*)}{n-1}$. Observe that

$$\begin{aligned}
& p^* d(p^*)\gamma_1(q'(n-1), (n-1)q'(n-1)) \\
\geq\ & p^* d(p^*)\gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) \text{ (since } \gamma_1(x, nx) \text{ is decreasing in } x) \\
>\ & c'(\frac{d(p^*)}{n-1}) \text{ (since } n \geq M_1) \\
\geq\ & c'(q'(n-1)) \text{ (since } q'(n-1) \leq \frac{d(p^*)}{n-1}).
\end{aligned}$$

This, however, violates Eq. (6). ■

Proof of Lemma 3. Let the number of firms charging $\hat{p}_i$ be k, where $k \geq \hat{n}_i$. First consider the case where none of the other firms charge prices that are less than $\hat{p}_i$. Clearly, if all the firms charging $\hat{p}_i$ produce identical amounts then the maximum profit of all such firms is $\frac{\hat{p}_i d(\hat{p}_i)}{k} - c(\frac{d(\hat{p}_i)}{k})$. Since $k \geq \hat{n}_i$, this is less than $\hat{\pi}$.

Now consider the case where the output level of the firms charging $\hat{p}_i$ are not the same. Clearly, if the aggregate production by all such firms are less than equal to $d(\hat{p}_i)$, then some of the firms would be producing and selling less than $\frac{d(\hat{p}_i)}{k}$, and consequently would have a profit less than $\frac{\hat{p}_i d(\hat{p}_i)}{k} - c(\frac{d(\hat{p}_i)}{k}) < \hat{\pi}$. Whereas, if the aggregate production of such firms is greater than $d(\hat{p}_i)$, then some firms would sell less than $\frac{d(\hat{p}_i)}{k}$, while their production would be larger. Again their profit would be less than $\frac{\hat{p}_i d(\hat{p}_i)}{k} - c(\frac{d(\hat{p}_i)}{k})$.

Finally, if some of the other firms charge less than $\hat{p}_i$, then the residual demand at $\hat{p}_i$ would be even less than $d(\hat{p}_i)$. We can now mimic the earlier argument to claim that some of the firms charging $\hat{p}_i$ would have a profit less than $\frac{\hat{p}_i d(\hat{p}_i)}{k} - c(\frac{d(\hat{p}_i)}{k})$. ■

Proof of Lemma 4. Suppose to the contrary that $\lim_{n\to\infty} q'(n-1) = D$, where $D > 0$. Then

$$\begin{aligned}\lim_{n\to\infty} p^* d(p^*)\gamma_1(q'(n-1),(n-1)q'(n-1)) &= 0 \ \text{(from A6(vi))} \\ &< c'(D) = \lim_{n\to\infty} c'(q'(n-1)).\end{aligned}$$

This, however, violates Eq. (6). ■

Proof of Proposition 4. (i) Follows since $c'(0) < p^*(\alpha) \le c'(0) + \alpha$.

(ii)(a) Define $n^{**}(\alpha) = \frac{d(p^*(\alpha))}{c'^{-1}(p^*(\alpha))}$. Since $n^{**}(\alpha) \le n^*(\alpha) \le N_1(\alpha)$, from Proposition 4(i) it is sufficient to observe that $\lim_{\alpha\to 0} n^{**}(\alpha) \to \infty$.

(ii)(b)(A). To begin with, consider a candidate SPE where all firms charge $p_c(n)$. Clearly, the optimal quantity decisions must involve all firms supplying $\frac{d(p_c(n))}{n}$. We argue that such an outcome cannot be sustained as an equilibrium. Consider P, Q such that all firms other than i charge $p_c(n)$ and supply $\frac{d(p_c(n))}{n}$, and firm i charges $p_i \ge p_c(n)$ and supplies $\max_{q_i} R_i(P,Q)$. Let $\pi_i(p_i, \max_{q_i} R_i(P,Q)) = p_i \max_{q_i} R_i(P,Q) - c(\max_{q_i} R_i(P,Q))$. Since, $p_c(n) = c'(\frac{d(p_c(n))}{n})$, it follows that $\frac{\partial \pi_i(p_i, \max_{q_i} R_i(P,Q))}{\partial p_i}|_{p_i \to p_c(n)+} = [p_c(n) - c'(\frac{d(p_c(n))}{n})]\frac{\partial[\max_{q_i} R_i(P,Q)]}{\partial p_i}|_{p_i\to p_c(n)+} + \frac{d(p_c(n))}{n} = \frac{d(p_c(n))}{n} > 0$. Thus, by increasing its price from $p_c(n)$ by a sufficiently small amount, firm i can increase its profits. Next consider a SPE where all firms charge p'. For $p' < p_c(n)$, the outcome must involve all firms producing $c'^{-1}(p')$, whereas for $p' > p_c(n)$, the outcome must involve all firms producing $\frac{d(p')}{n}$. Clearly, as p' converges to $p_c(n)$, these output levels converge to $\frac{d(p_c(n))}{n}$. Hence, from A8, there exist $\overrightarrow{\alpha} > 0$ and $\epsilon > 0$, such that $\forall 0 < \alpha < \overrightarrow{\alpha}$ and $\forall p \in [p_c(n) - \epsilon,\ p_c(n) + \epsilon]$ such that $p \in F(\alpha)$, in any candidate SPE where all firms charge p, firm i can deviate to $p + \alpha$ and gain.

Note that $\frac{d\pi_n(p)}{dp}|_{p\to p_c(n)+} = \frac{\partial \pi_i(p_i, \max_{q_i} R_i(P,Q))}{\partial p_i}|_{p_i\to p_c(n)+} = \frac{d(p_c(n))}{n} > 0$. Let $p'(n)$ be a global maximizer of $\pi_n(p)$ over $[p_c(n), p^{\max}]$, such that $\pi_n(p)$ is strictly increasing for $p \in [p_c(n), p'(n))$. Hence $p_c(n) < p'(n) < p^{\max}$. Next, $\forall p \ge 0$, let $\pi(p) = p\min\{c'^{-1}(p), d(p)\} - c(\min\{c'^{-1}(p), d(p)\})$ (given A1 and A2, $\pi(p)$ and $\pi_n(p)$ are continuous in p). Note that (a) $\forall p$ such that $p^{\max} \geq p > p_c(n)$, $c'^{-1}(p) > \frac{d(p)}{n}$, so that $\pi(p) > \pi_n(p)$, and (b) $\pi(p_c(n)) = \pi_n(p_c(n))$ (since $c'^{-1}(p_c(n)) = \frac{d(p_c(n))}{n}$).

Next we consider a sequence $< p^n >$ such that $p^1 = p'(n)$ and $\forall i \geq 2$, p^i is the minimum p such that $\pi(p) = \pi_n(p^{i-1})$. From property (a) in the earlier paragraph, $< p^n >$ is a decreasing sequence. (Consider some p^i, $p'(n) \geq p^i > p_c(n)$, $i \geq 1$. Thus $\pi(p^i) > \pi_n(p^i) > \pi_n(p_c(n)) = \pi(p_c(n))$. Hence, from the intermediate value theorem, there exists some $p < p^i$, such that $\pi(p) = \pi_n(p^i)$. Clearly, $p'(n) \geq p^{i+1} > p_c(n)$.) Further, from property (b), $< p^n >$ is bounded below by $p_c(n)$, so that it converges to $p_c(n)$. (Suppose it converges to some p'', $p'(n) > p'' > p_c(n)$. Then $\pi_n(p'') = \pi(p'')$, which contradicts property (a).) Thus, whenever $p_c(n_{+}\epsilon < p'(n)$ (where $\epsilon > 0$ is as defined earlier in the first paragraph of this proof of Proposition 4(ii)(b)(A)), there exists $N > 1$ such that $p^{N-1} \geq p_c(n) + \epsilon > p^N$.

Next, since $p_c(n) - \epsilon < p_c(n)$, $\exists \overbrace{\alpha} > 0$ be such that $\forall 0 < \alpha < \overbrace{\alpha}$, $d(p_c(n) - \epsilon + \alpha) - (n-1)c'^{-1}(p_c(n) - \epsilon) > c'^{-1}(p_c(n) - \epsilon + \alpha)$. Consider $p < p_c(n) - \epsilon$. Then $d(p+\alpha) - c'^{-1}(p+\alpha) > d(p_c(n) - \epsilon + \alpha) - c'^{-1}(p_c(n) - \epsilon + \alpha) > (n-1)c'^{-1}(p_c(n) - \epsilon) > (n-1)c'^{-1}(p)$. Hence, $\forall \alpha < \overbrace{\alpha}$ and $p \leq p_c(n) - \epsilon$, $d(p+\alpha) - (n-1)c'^{-1}(p) > c'^{-1}(p+\alpha)$.

Define $\alpha' = \min\{p_N - p_{N-1}, \overbrace{\alpha}, \overrightarrow{\alpha}\}$ and consider $0 < \alpha < \alpha'$. Consider some candidate single price equilibrium where the active firms charge $p \in F(\alpha)$. Since firms can always charge $p^*(\alpha)$ and sell $\frac{d(p^*(\alpha))}{n}$, all firms must be active in this equilibrium.

First, we can rule out any candidate SPE p, such that $p < p_c(n) - \epsilon$. Since $\alpha < \overbrace{\alpha}$, we can show that a firm can deviate to $p+\alpha$ and make a gain. The output levels of the other firms who charge p, are $c'^{-1}(p)$. Given A7(i), the residual demand facing the deviating firm is at least $d(p+\alpha) - (n-1)c'^{-1}(p)$. Next note that since $\alpha < \overbrace{\alpha}$,

$$d(p+\alpha) - (n-1)c'^{-1}(p) > c'^{-1}(p+\alpha).$$

Thus, if a firm charging p deviates to $p+\alpha$, then it can supply till its marginal cost. Since $p + \alpha > p$, its profit will increase.

Next, consider a candidate SPE p, such that $p \in [p_c(n) - \epsilon, p_c(n) + \epsilon]$. Since $\alpha < \overrightarrow{\alpha}$, a firm can charge $p + \alpha$ and gain. Next consider p such that $p'(n) \geq p > p_c(n) + \epsilon$. Since $\alpha < p^N - p^{N-1}$, a firm can undercut by charging $p - \alpha$ and gain. (Of course, if $p_c(n) + \epsilon \geq p'(n)$, then this step is redundant.) Next, for $p_c(n) \leq p < p'(n)$, let $\mathrm{p}(p)$ be the maximum price such that $\pi_n(p) = \pi_n(\mathrm{p}(p))$ and $\mathrm{p}(p) > p$. For some candidate SPE where all firms charge p, where $p'(n) < p \leq \mathrm{p}(p_c(n) + \epsilon)$, a firm can deviate to $\mathrm{p}^{-1}(p) - \alpha$, and gain. Finally, for any candidate SPE where all firms charge p, where $p > \mathrm{p}(p_c(n) + \epsilon)$, a firm can deviate to p'_α, where $p'_\alpha \in F(\alpha)$ is the price on the grid which is closest to $p'(n)$, and gain.

(ii)(b)(B). Fix n. Let $\widetilde{p}(n)$ ($\widetilde{p}$ from now on) satisfy $c'^{-1}(\tilde{p}) < \frac{d(\tilde{p})}{2(n-1)}$. Such a $\tilde{p}$ exists since $(n-1)c'^{-1}(c'(0)) < \frac{d(c'(0))}{2}$. Let $\tilde{p}_\alpha$ denote the smallest $p \in F(\alpha)$ such that $p > \tilde{p}$. Next, let $\tilde{\alpha}(\tilde{p})$ be such that, $\forall \alpha < \tilde{\alpha}(\tilde{p})$, $(n-1)c'^{-1}(\tilde{p}_\alpha) < \frac{d(\tilde{p}_\alpha)}{2}$ (such an $\tilde{\alpha}(\tilde{p})$ exists since $\lim_{\alpha \to 0} \tilde{p}_\alpha = \tilde{p}$).

Suppose to the contrary there is an equilibrium where the lowest price charged $p' \leq \tilde{p}$. The profit of a firm charging p' is at most $\tilde{p}c'^{-1}(\tilde{p}) - c(c'^{-1}(\tilde{p}))$. Now suppose a firm charging this price deviates to $\tilde{p}_\alpha \leq \tilde{p} + \alpha$. The output levels of the other firms who charge $\tilde{p}_\alpha$ or less, are at most $c'^{-1}(\tilde{p}_\alpha)$. Given A7, the residual demand facing the deviating firm is at least $d(\tilde{p}_\alpha) - (n-1)c'^{-1}(\tilde{p}_\alpha)$. Next note that

$$d(\tilde{p}_\alpha) - (n-1)c'^{-1}(\tilde{p}_\alpha) > \frac{d(\tilde{p}_\alpha)}{2} > (n-1)c'^{-1}(\tilde{p}_\alpha) \geq c'^{-1}(\tilde{p}_\alpha).$$

Thus, if a firm charging p' deviates to $\tilde{p}_\alpha$, then it can supply till its marginal cost. Since $\tilde{p}_\alpha > p'$, its profit will increase.

(iii)(a) The proof mimics that of Proposition 4(ii)(a).

(iii)(b) The proof is similar to that of Proposition 4(ii)(b) and is available from the author. ■

8 Some Additional Proofs

Proof that $\hat{n} \geq n^*$. Suppose that $N \geq \hat{n}$. Then, from the definition of $\hat{n}$, it follows that $r_i'(p_i, p, n)(p^* - c'(\frac{d(p^*)}{N})) < 0$. Since $r_i'(p_i, p, n) < 0$, $p^* - c'(\frac{d(p^*)}{N}) > 0$, so that $N \geq n^*$. ■

Proof of Proposition 3. From Lemma 4, $\lim_{n\to\infty} q'(n-1) = 0$. Hence $\lim_{n\to\infty} nq'(n-1) = \lim_{n\to\infty}(n-1)q'(n-1)$. Moreover, from Eq. (6), A6(ii) and the fact that $q'(n-1)$ is decreasing in n, it follows that $(n-1)q'(n-1)$ is increasing in n.[23]

(i) Let $c'(0) = 0$, and suppose to the contrary that $\lim_{n\to\infty}(n-1)q'(n-1) = l$, where l is finite. Then

$$\begin{aligned} & \lim_{n\to\infty} p^* d(p^*)\gamma_1(q'(n-1), (n-1)q'(n-1)) \\ = & \frac{p^* d(p^*)}{l} \text{ (from A6(iv))} \\ > & 0 = c'(0) = \lim_{n\to\infty} c'(q'(n-1)), \end{aligned}$$

where the last equality follows from Lemma 4. But this contradicts Eq. (6).

(ii) Let $c'(0) > 0$ and suppose to the contrary that $\lim_{n\to\infty}(n-1)q'(n-1)$ diverges to infinity. In that case

$$\lim_{n\to\infty} p^* d(p^*)\gamma_1(q'(n-1), (n-1)q'(n-1)) = \lim_{n\to\infty} c'(q'(n-1)),$$

which, from A6(v) and Lemma 4, implies that $c'(0) = 0$. But this is a contradiction. Hence let $\lim_{n\to\infty}(n-1)q'(n-1) = L$, where L is finite. We then mimic the earlier argument to show that $L = d(p^*)\frac{p^*}{c'(0)}$. ■

[23] Suppose the number of firms increase from n to $n+1$, so that $q'(n) < q'(n-1)$. Now suppose to the contrary that $(n-1)q'(n-1) \geq nq'(n)$. Then

$$\begin{aligned} p^* d(p^*)\gamma_1(q'(n), nq'(n)) &> p^* d(p^*)\gamma_1(q'(n-1), nq'(n-1)) \text{ (since } \gamma_1(x, nx) \text{ is decreasing in } x) \\ &> p^* d(p^*)\gamma_1(q'(n-1), (n-1)q'(n-1)) \text{ (since } \gamma_1(x, nx) \text{ is decreasing in } n) \\ &= c'(q'(n-1)) \text{ (from Eq. (6))} > c'(q'(n)), \end{aligned}$$

which contradicts Eq. (6).

Properties of $p_c(n)$. (i) Note that $c'(d(0)/n) > 0$, $c'(d(p^{\max})/n) = c'(0) < p^{\max}$, and $c'(q)$ is continuous and increasing in q. Hence $p_c(n)$ is well defined.

(ii) Suppose $p_c(n) \geq p^{\max}$. Then $p^{\max} \leq p_c(n) = c'(d(p^{\max})/n) = c'(0)$, which contradicts A2. Next suppose $p_c(n) = c'(0)$. Then, $c'(0) = c'(d(c'(0))/n)$, so that $d(c'(0)) = 0$, which is a contradiction.

(iii) Since $d(p)$ is decreasing in p, and $c(q)$ is convex, $p_c(n)$ is decreasing in p.

(iv) Since $p_c(n)$ is decreasing in n, and bounded below by $c'(0)$, $\lim_{n\to\infty} p_c(n) = \overline{p}$ is well defined. Thus $\overline{p} = \lim_{n\to\infty} c'(d(\overline{p})/n) = c'(0)$. ■

Proof of Proposition 4(iii)(b)(A). Consider a P, Q where all firms charge $p_c(n)$. We then argue that, given P, the optimal quantity decisions must involve all firms supplying $\frac{d(p_c(n))}{n}$.

Suppose not. Then the aggregate output must be greater than demand at $p_c(n)$ (otherwise one of the firms will have an incentive to change its output). Hence the quantity decisions must be symmetric (we can mimic the argument in step 3 of Proposition 2 to show this), so that all firms supply $q'(p_c(n), n-1)$. Thus $q'(p_c(n), n-1) > \frac{d(p_c(n))}{n}$. Hence, from Eq. (6), $p_c(n)d(p_c(n))\gamma_1(\frac{d(p_c(n))}{n}, (n-1)\frac{d(p_c(n))}{n}) > p_c(n)d(p_c(n))\gamma_1(q'(p_c(n), n-1), (n-1)q'(p_c(n), n-1)) = c'(q'(p_c(n), n-1)) > c'(\frac{d(p_c(n))}{n})$. Note, however, that this contradicts A9(ii).

Now suppose all firms other than i charge $p_c(n)$ and supply $c'^{-1}(p_c(n))$, and firm i charges $p_i \geq p_c(n)$ and supplies $\max_{q_i} R_i(P, Q)$. Next suppose $\pi_i(p_i, \max_{q_i} R_i(P, Q)) = p_i \max_{q_i} R_i(P, Q) - c(\max_{q_i} R_i(P, Q))$. Since, at $p_c(n)$, price equals marginal cost, $\frac{\partial \pi_i(p_i, \max_{q_i} R_i(P,Q))}{\partial p_i}|_{p_i \to p_c(n)+} = \frac{d(p_c(n))}{n} > 0$.

Next, from A9(ii), $p_c(n)d(p_c(n))\gamma_1(\frac{d(p_c(n))}{n}, (n-1)\frac{d(p_c(n))}{n}) < c'(\frac{d(p_c(n))}{n})$. Hence, from continuity, for p' close enough to $p_c(n)$, $p'd(p')\gamma_1(\frac{d(p')}{n}, (n-1)\frac{d(p')}{n}) < c'(\frac{d(p')}{n})$. Consider some SPE where all firms charge such a p' close to $p_c(n)$, $p' \neq p_c(n)$, but the output vector do not involve every firm producing $\min\{c'^{-1}(p'), \frac{d(p')}{n}\}$. Then the output vector must involve excess supply, and hence, mimicing step 3 of Proposition 2, be symmetric. Thus all firms supply $q'(p', n-1) > \frac{d(p')}{n}$. This is a contradiction since, from A6(iii) and A9(ii), $p'd(p')\gamma_1(q'(p', n-1), (n-1)q'(p', n-1)) < c'(q'(p', n-1))$, which violates Eq.(6). Hence, for all such $p' > p_c(n)$, the only possible equilibrium involves the firms supplying $d(p')/n$. Whereas, for $p' < p_c(n)$, the outcome must involve all firms producing less than equal to $\frac{d(p')}{n}$. Since there is excess demand, all firms must supply $c'^{-1}(p')$. Then, from continuity (A8), $\exists \overleftarrow{\alpha} > 0$ and $\epsilon' > 0$, such that $\forall 0 < \alpha \leq \overleftarrow{\alpha}$ and $\forall p \in [p_c(n) - \epsilon',\ p_c(n) + \epsilon']$ such that $p \in F(\alpha)$, in any candidate single price equilibrium where all firms charge p, firm i can deviate to $p+\alpha$ and gain. Next define $p_N - p_{N-1}$ as in Proposition 4(ii)(b)(A). (We can adopt the same notation since we can assume, w.l.o.g., that $\epsilon = \epsilon'$, where ϵ is as defined in Proposition 4(ii)(b)(A)).

Next consider a candidate SPE P, Q such that all firms charge $p < p_c(n) - \epsilon'$. We then argue that such a candidate SPE must involve all firms supplying $c'^{-1}(p)$. (Suppose not. Then the aggregate output must be greater than demand at p (otherwise one of the firms will have an incentive to change its output). Hence the quantity decisions must be symmetric (we can mimic the argument in step 3 of Proposition 2 to show this), so that all firms supply $q'(p, n-1)$. Thus $q'(p, n-1) > \frac{d(p)}{n}$. Hence, from Eq. (6), $pd(p)\gamma_1(q'(p, n-1), (n-1)q'(p, n-1)) = c'(q'(p, n-1))$. Since $p < c'(\frac{d(p)}{n})$, from A9(ii) it follows that $pd(p)\gamma_1(\frac{d(p)}{n}, (n-1)\frac{d(p)}{n}) < c'(\frac{d(p)}{n})$. Since $q'(p, n-1) > \frac{d(p)}{n}$, it must be that $pd(p)\gamma_1(q'(p, n-1), (n-1)q'(p, n-1)) < c'(q'(p, n-1))$, which is a contradiction.) Next, since $p_c(n) - \epsilon' < p_c(n)$, $\exists \underbrace{\alpha} > 0$ such that $\forall 0 < \alpha < \underbrace{\alpha}$, $d(p_c(n) - \epsilon' + \alpha) - (n-1)c'^{-1}(p_c(n) - \epsilon') > c'^{-1}(p_c(n) - \epsilon' + \alpha)$. Consider $p < p_c(n) - \epsilon'$. Then $d(p+\alpha) - c'^{-1}(p+\alpha) > d(p_c(n) - \epsilon' + \alpha) - c'^{-1}(p_c(n) - \epsilon' + \alpha) > (n-1)c'^{-1}(p_c(n) - \epsilon') > (n-1)c'^{-1}(p)$. Hence, $\forall \alpha < \underbrace{\alpha}$ and $p \leq p_c(n) - \epsilon'$, $d(p+\alpha) - (n-1)c'^{-1}(p) > c'^{-1}(p+\alpha)$.

Next define $\alpha'' = \min\{p^N - p^{N-1}, \overleftarrow{\alpha}, \underbrace{\alpha}\}$, and let $\alpha < \alpha''$.

To begin with since $\alpha < \underbrace{\alpha}$, we can rule out SPE where firms charge $p < p_c(n) - \epsilon'$. This follows since in this case a firm can deviate to $p + \alpha$ and gain. This follows since such a firm can sell $c'^{-1}(p+\alpha)$, and charge $p + \alpha > p$. Whereas for any $p \in [p_c(n) - \epsilon',\ p_c(n) + \epsilon']$, one of the firms can increase its price to $p + \alpha$ and gain (since $\alpha < \overleftarrow{\alpha}$). Finally, for any candidate SPE where the firms charge p, such that $p \in [p_c(n) + \epsilon',\ p^{\max}]$, we can mimic the argument in Proposition 4(ii)(b)(A) to show that a firm can undercut and gain. This follows since in any such equilibrium the profit of the firms are bounded above by $\pi_n(p)$ and $\alpha < p^N - p^{N-1}$.

4(iii)(b)(B). Fix n. As in Proposition 4(ii)(b)(B), let $\widetilde{p}(n)$ ($\widetilde{p}$ from now on) satisfy $(n-1)c'^{-1}(\tilde{p}) < \frac{d(\tilde{p})}{2}$.

Next let $q'(p, n-1)$ solve $pd(p)\gamma_1(q,(n-1)q) = c'(q)$. Let $\overline{p}(n) > c'(0)$ ($\overline{p}$ from now on) be such that, $\forall c'(0) \leq p \leq \overline{p}$, $(n-1)q'(p,1) < d(p) - c'^{-1}(p)$. Such a $\overline{p}$ exists since, from A6(iii) and A9(ii), $c'(0)d(c'(0))\gamma_1(\frac{d(c'(0))}{n-1},(n-1)\frac{d(c'(0))}{n-1}) < c'(\frac{d(c'(0))}{n-1})$. Thus, $\frac{d(c'(0))}{n-1} > q'(c'(0),1)$. (Such a $q'(c'(0),1)$ exists since, from A6(iii), $\lim_{q\to 0} c'(0)d(c'(0))\gamma_1(q,(n-1)q) \to \infty > c'(0)$.) Hence the claim.

Define $\hat{p}(n) = \min\{\tilde{p}(n)/\beta, \overline{p}(n)/\beta\}$, where $\beta > 1$ ($\widehat{p}$ from now on).

Let $\hat{p}_\alpha$ denote the smallest $p \in F(\alpha)$ such that $p > \hat{p}$. Next, let $\hat{\alpha}(\hat{p})$ be such that, $\forall \alpha < \hat{\alpha}(\hat{p})$, $(n-1)c'^{-1}(\hat{p}_\alpha) < \frac{d(\hat{p}_\alpha)}{2}$, and $(n-1)q'(\hat{p}_\alpha, 1) < d(\hat{p}_\alpha) - c'^{-1}(\hat{p}_\alpha)$.

Suppose to the contrary there is an equilibrium where the lowest price charged $p' \leq \hat{p}$. The profit of a firm charging p' is at most $\hat{p}c'^{-1}(\hat{p}) - c(c'^{-1}(\hat{p}))$. Now suppose a firm charging this price deviates to $\hat{p}_\alpha \leq \hat{p} + \alpha$, where $\hat{p}_\alpha$ is smallest price on the grid that is greater that $\hat{p}$. The output levels of the other firms who charge less than $\hat{p}_\alpha$, are at most $\max\{c'^{-1}(\hat{p}_\alpha), q'(\hat{p}_\alpha, 1)\}$.

The argument is as follows. Consider some other firm j. If it produces less than or equal to $c'^{-1}(p_j)$, then there is nothing to prove. Hence suppose it is producing more that $c'^{-1}(p_j)$. Then there must be other firms charging this price, and the total output of all such firms must exceed the residual demand at p_j. Since this residual demand is bounded above by $d(p_j)$, we assume, without loss of generality, that the residual demand equals $d(p_j)$. We can mimic the argument in step 3 of Proposition 2 to show that the output level of all such firms are symmetric. Clearly, given A6(iii), $q'(p, n-1)$ is decreasing in n, and for $p \leq \overline{p}$, increasing in p. Hence the output of such a firm is bounded above by $q'(\hat{p}_\alpha, 1)$.

Given A7, the residual demand facing this firm is at least $d(\hat{p}_\alpha) - (n-1)\max\{c'^{-1}(\hat{p}_\alpha), q'(\hat{p}_\alpha, 1)\}$. Since $\alpha < \widehat{\alpha}(\widehat{p})$, note that

$$d(\hat{p}_\alpha) - (n-1)\max\{c'^{-1}(\hat{p}_\alpha), q'(\hat{p}_\alpha, 1)\} > c'^{-1}(\hat{p}_\alpha).$$

Thus, if a firm charging p' deviates to $\hat{p}_\alpha$, then it can supply till its marginal cost. Since $\hat{p}_\alpha > p'$, its profit will increase. ■

Proof of Proposition 5. Existence. Undercutting p^* is clearly not profitable. We then argue that for the i-th firm of type l, charging a higher price, p_i, is not profitable either.

As in Proposition 1, $\frac{d(p^*)}{n} = r_{il}(p^*, p^*, n)$. Notice that since $n \geq n_l^*$, $\frac{d(p^*)}{n} < c_l'^{-1}(p^*)$. Hence for any $p_i \geq p^*$,

$$c_l'^{-1}(p_i) \geq c_l'^{-1}(p^*) \geq r_{il}(p_i, p^*, n). \tag{14}$$

Since $c_l'^{-1}(p_i) > r_{il}(p_i, p^*, n)$, the deviant firm supplies the whole of the residual demand coming to it. Hence for a firm charging p_i ($\geq p^*$)

$$\pi_l(p_i, r_{il}(p_i, p^*, n)) = p_i r_{il}(p_i, p^*, n) - c_l(r_{il}(p_i, p^*, n)). \tag{15}$$

$$\frac{\partial \pi_l(p_i, r_{il}(p_i, p^*, n))}{\partial p_i} = r_{il}'(p_i, p^*, n)[p_i - c_l'(r_{il}(p_i, p^*, n))] + r_{il}(p_i, p^*, n). \tag{16}$$

Next from equation (5) it follows that $\forall p_i \geq p^*, \quad p_i > c_l'(r_{il}(p_i, p^*, n))$. Hence from the concavity of $r_{il}(p_i, p^*, n)$ it follows that $\pi_l(p_i, r_{il}(p_i, p^*, n))$ is concave in p_i. Moreover,

$$\frac{\partial \pi_l(p_i, r_{il}(p_i, p^*, n))}{\partial p_i}|_{p_i \to p^*+} = r_{il}'(p_i, p^*, n)|_{p_i \to p^*+}[p^* - c_l'(\frac{d(p^*)}{n})] + \frac{d(p^*)}{n}. \tag{17}$$

Since $n \geq \hat{n}_l$, we have that $\frac{\partial \pi_l(p_i, r_{il}(p_i, p^*, n))}{\partial p_i}|_{p_i \to p^*+} < 0$. Next, from the concavity of $\pi_l(p_i, r_{il}(p_i, p^*, n))$ it follows that $\forall p_i \geq p^*$, the profit of any deviant firm is decreasing in p_i.

Next, from Assumption 4(i), all firms have a residual demand of at least $\frac{d(p^*)}{n}$. Thus, given $\frac{d(p^*)}{n} < c_l'^{-1}(q^*)$, all firms produce at least $\frac{d(p^*)}{n}$. Hence, from A3(ii), A3(v) and A4(i), the residual demand of all firms is $\frac{d(p^*)}{n}$.

Uniqueness. Step 1. We can first mimic the proof of Proposition 1 to argue that there cannot be an equilibrium where the output level of some of the firms is zero.

Step 2. We then demonstrate that there cannot be some $\widehat{p}_y$ $(\in F) > p^*$, such that some of the firms charge $\widehat{p}_y$ and supply a positive amount.

Suppose to the contrary that such a price exists. This implies that the total number of firms charging p^*, say $\tilde{n}$, can be at most $\max_q n_q^* - 1$. Otherwise, $\tilde{n} \geq \max_q n_q^*$ and the residual demand facing all these firms would be exactly $\frac{d(p^*)}{\tilde{n}}$.[24] Since $\frac{d(p^*)}{\tilde{n}} < c_l'^{-1}(p^*)$, $\forall l$, all such firms would supply $\frac{d(p^*)}{\tilde{n}}$ and the residual demand at any higher price would be zero.

Now consider some $\widehat{p}_y > p^*$. Clearly, the number of firms charging $\widehat{p}_y$ is less than $\max_q n_{qy}$. Otherwise, some of these firms, say of type l, would have a profit less than $\tilde{\pi}_l$. Hence such a firm would have an incentive to deviate to p^*, when it can supply at least $\frac{d(p^*)}{\max_q n_q^*}$ and earn $\tilde{\pi}_l$. Thus the total number of firms producing a strictly positive amount is less than $\tilde{N}_1$, thereby contradicting step 1.

Step 3. We can finally mimic step 3 of Proposition 1 to argue that all firms have an output level of $\frac{d(p^*)}{n}$. ■

Proof of Proposition 6. Step A. Existence. Notice that since $\alpha < c_2'(0) - c_1'(0)$, it follows that $\forall i \geq 2$, $p_1^* < c_i'(0)$. Thus no firm of type i, where $i \geq 2$ can profitably charge a price of p_1^* and produce a strictly positive output level. For type 1 firms we can simply mimic the proof in Proposition 1 to claim that they cannot have a profitable deviation.

Step B. We first argue that in equilibrium all firms of type 1 charge p^* and produce $\frac{d(p_1^*)}{n^1}$, and all other firms have zero output.

First note that there cannot be an equilibrium where the output level of some of the type 1 firms is zero.

We then argue that there cannot be some $\widehat{p}_x$ $(\in F) > p_1^*$, such that some of the type 1 firms charge $\widehat{p}_x$ and supply a positive amount. Suppose to the contrary that such a price exists.

This implies that the total number of type 1 firms charging p_1^*, say $\tilde{n}$, can be at most $n_1^* - 1$. Otherwise the residual demand facing these firms would be exactly $\frac{d(p_1^*)}{\tilde{n}}$.[25] Since $\tilde{n} \geq n_1^*$, we have that $\frac{d(p_1^*)}{\tilde{n}} < c_1'^{-1}(p_1^*)$. Hence all

[24] Given that $\frac{d(p^*)}{\tilde{n}} < c_l'^{-1}(p^*)$, $\forall l$, all firms must be supplying at least $\frac{d(p^*)}{\tilde{n}}$. The assertion now follows from an analogue of Assumption 4(i).

[25] First note that firms of type $j > 1$, even if they charge p_1^*, would have an output of zero. Thus the residual demand facing all firms of type 1 charging p_1^* is at least $\frac{d(p_1^*)}{\tilde{n}}$

such firms would supply $\frac{d(p^*)}{\tilde{n}}$ and the residual demand at any higher price, p_x, would be zero.

Next consider some $\widehat{p}_x > p_1^*$. Clearly, the number of type 1 firms charging $\widehat{p}_x$ is less than n_{1x}. Otherwise, one of the type 1 firms would have a residual demand that is less than or equal to $\frac{d(\widehat{p}_x)}{n_{1x}}$. Since $\frac{d(\widehat{p}_x)}{n_{1x}} < c_l'^{-1}(\widehat{p}_x)$, this firm would supply at most $\frac{d(\widehat{p}_x)}{n_{1x}}$ and have a profit less than $\tilde{\pi}_1$. Hence such a firm would have an incentive to deviate to p_1^*, when it can supply at least $\frac{d(p_1^*)}{n_1^*}$ and earn $\tilde{\pi}_1$. Thus the total number of firms producing a strictly positive amount is less than $\hat{N}_2$, a contradiction.

We can then mimic step 3 of Proposition 1 to argue that all type 1 firms have an output level of $\frac{d(p^*)}{n^1}$. Finally, since type 1 firms exhaust the demand at p_1^*, producing a positive amount is not profitable for other firms. ■

Proof of the example following Proposition 6. Let there be two types of firms with $c_1(q) = q^2$ and $c_2(q) = q + q^2$, so that $c_1'(0) < c_2'(0)$. There are 2 firms of type 1 and n^2 firms of type 2. The demand function is $q = 4 - p$, and the residual demand function is as follows:

Suppose there are m firms charging p_i. Then $R_i(P,Q) = q_i$, if either $\sum_{p_j=p_i} q_j \leq \max[0, d(p_i) - \sum_{p_k<p_i} q_k]$, or $\sum_{p_j=p_i} q_j > \max[0, d(p_i) - \sum_{p_k<p_i} q_k]$ and $q_i \leq \frac{\max[0, d(p_i) - \sum_{p_k<p_i} q_k]}{m}$. Otherwise, $R_i(P,Q) \geq \frac{\max[0, d(p_i) - \sum_{p_k<p_i} q_k]}{m}$.

Let $\alpha = 0.01$, so that $p_1^* = 0.01$ and $p_2^* = 1.01$.

Step 1. We first argue that for n^2 large enough, any possible equilibrium must involve all firms of type 2 charging the price p_2^* and supplying the whole of the residual demand coming to them.

1a. We first claim that there cannot be an equilibrium where the output level of some of the type 2 firms is zero. This follows since these firms can always charge p_2^* and obtain a residual demand of at least $\frac{d(p_2^*) - 2c_2'^{-1}(p_2^*)}{n^2+2} > 0$ (follows from the nature of the residual demand function).

(from A3(ii)), A3(iv) and A4(i)). Given that $\frac{d(p_1^*)}{\tilde{n}} < c_1'^{-1}(p^*)$, all such firms of type 1 must be supplying at least $\frac{d(p_1^*)}{\tilde{n}}$. The assertion now follows from an analogue of A3(ii).

Next let

$$\pi' = p_2^* \frac{d(p_2^*) - 2c_2'^{-1}(p_2^*)}{n_2^* + 2} - c(\frac{d(p_2^*) - 2c_2'^{-1}(p_2^*)}{n_2^* + 2}).$$

1b. We then argue that there cannot be some $\widehat{p}_i$ $(\in F) > p_2^*$, such that some of the type 2 firms charge $\widehat{p}_i$ and supply a positive amount. Suppose to the contrary that such a price exists. This implies that the total number of type 2 firms charging p_2^*, say $\tilde{n}$, can be at most n_2^*-1. Otherwise, the residual demand facing these firms would be less than $\frac{d(p_2^*)}{\tilde{n}}$. Since $\frac{d(p_2^*)}{\tilde{n}} < c_2'^{-1}(p_2^*)$, all such firms would supply the demand coming to them and the residual demand at any higher price, $\widehat{p}_i$, would be zero.

Now consider some $\widehat{p}_i > p^*$. Clearly, if the number of firms charging $\widehat{p}_i$ is large, then some of these firms would have a profit less than π'. Hence such a firm would have an incentive to deviate to p_2^*, when it can earn at least π'. Thus, if the total number of type 2 firms is large, then all of them must be charging p_2^*.

Step 2. Next consider type 1 firms. For n^2 large, neither of these firms can be charging p_2^*, since, in that case, the profit of the type 1 firm will be small, and it can do better by charging $c_2'(0)$.

Step 3. Given step 2, the only possible equilibrium must involve both the type 1 firms charging $c_2'(0) = 1$ and supplying $c_1'^{-1}(1) = 0.5$ when they have a profit of 0.25 each.

3a. Given that all type 2 firms are charging p_2^*, in equilibrium the type 1 firms cannot be charging a price strictly greater than p_2^*, since in that case the type 1 firms will have no demand.

3b. Whereas if they charge a price strictly lower than $c_2'(0)$, then their profit will be lower compared to what they obtain from charging $c_2'(0)$. This follows since the maximum possible output of the other type 1 firm is $c_2'(0)$, so that for all $p \leq c_2'(0)$, the residual demand facing this type 1 firm is less than $c'^{-1}(0)$.

Step 4. We finally argue that both the type 1 firms charging $c_2'(0)$ cannot be an equilibrium. Since both the type 1 firms supply $c_1'^{-1}(1) = 0.5$, the total amount supplied by the type 2 firms will be 1.99 $(= 4-p_2^*-1)$. Next suppose that a type 1 firms deviates to 1.02 $(= p_2^*+\alpha)$. Given that the rationing rule is efficient, it can supply the residual demand 0.49 ($= 4 - p_2^* - \alpha - 1.99 - 0.5$) and increase its profit level to 0.2597. Hence no equilibrium exists. ■

Proof that Eq. (11) has a unique solution. Consider the negative of the Jacobian of Eq. (11), $[J]$, where the first row refers to the first firm of type 1, the second row refers to the second firm of type 1, and so on. Clearly, $j_{kk} = c_l''(q_{il}) - p^* d(p^*)\gamma_{11}(q_{il}, \sum_a \sum_b q_{ab} - q_{il})$ and $j_{kz}|_{z \neq k} = -p^* d(p^*)\gamma_{12}(q_{il}, \sum_a \sum_b q_{ab} - q_{il})$, such that k refers to the appropriate firm i of type l. Let $[J_{n-1}^{11}]$ denote the $n-1 \times n-1$ matrix obtained from $[J]$ by deleting the first row and the first column.

For the Gale-Nikaido (1965) univalence theorem to hold, it is sufficient to show that $[J]$ is positive definite. (The other condition that $\forall i, l,\ q_{il}$ is defined over a convex domain, is clearly satisfied.) Note that $\forall k,\ j_{kz} = j_k$, whenever $z \neq k$. Moreover, $\forall k,\ j_{kk} > j_k > 0$ (this follows since $\forall i, l$, $\sum_a \sum_b q_{ab} - q_{il} > q_{il} > 0$, so that $\gamma_{12}(q_i, \sum_{j \neq i} q_j) < 0$, and $\gamma_{11}(q_i, \sum_{j \neq i} q_j) - \gamma_{12}(q_i, \sum_{j \neq i} q_j) < 0$).

Define $[\tilde{J}]$, such that, $\tilde{j}_{11} = j_1$, and, $\forall kl \neq 11,\ \tilde{j}_{kl} = j_{kl}$. Let $[\tilde{J}_{n-1}^{22}]$ denote the $n-1 \times n-1$ matrix obtained by deleting the second row and second column of $[\tilde{J}]$. Moreover, $\forall i$, define $c_i = j_{ii} - j_i > 0$. Finally, let $[Z]$ denote the matrix where, $\forall a \neq 1, 2,\ z_{ab} = j_{ab}$ and, $\forall b,\ z_{1b} = z_{2b} = 1$. Clearly, $|Z| = 0$.

The proof is by induction on the size of the matrix.

Induction Hypothesis: All principal minor of order m of $[J]$ are positive. Moreover, all principal minor of order m of $[\tilde{J}]$ such that the first row is kept unchanged, are positive.

Clearly the induction hypothesis holds for $m = 2$. Next suppose that it holds for $m \leq n-1$. To show that it holds for $m = n$. Note that $|\tilde{J}_n| = c_2|\tilde{J}_{n-1}^{22}| + j_1 j_2 |Z| > 0$, where the inequality follows from the induction hypothesis and the fact that $|Z| = 0$. Next, $|J_n| = c_1|J_{n-1}^{11}| + |\tilde{J}_n| > 0$, where the inequality follows from the induction hypothesis and the previous step. The argument for the principal minors of $[J]$, and principal minors of $[\tilde{J}]$ where the first row is kept unchanged, are similar. ■

Proof of Lemma $1'$. Notice that

$$\begin{aligned}\lim_{n\to\infty} p^* d(p^*)\gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) &= p^* d(p^*)\frac{1}{d(p^*)} \text{ (from Assumption 6(iv))} \\ &= p^* > c_l'(0) = \lim_{n\to\infty} c_l'(\frac{d(p^*)}{n-1}). \qquad \blacksquare\end{aligned}$$

Proof of Lemma $2'$. Suppose not, i.e. there is some l such that

$$(n^l - 1)q_l' + \sum_{j\neq l} n^j q_j' \leq d(p^*). \tag{18}$$

Then

$$q_l' \leq \frac{d(p^*)}{(n^l-1)} \leq \frac{d(p^*)}{n-1}. \tag{19}$$

Next observe that

$$\begin{aligned} & p^* d(p^*)\gamma_1(q_l', (n^l-1)q_l' + \sum_{j\neq l} n^j q_j') \\ \geq\ & p^* d(p^*)\gamma_1(q_l', d(p^*)) \text{ (from Eq. (18) and since in this case } \gamma_{12} < 0) \\ \geq\ & p^* d(p^*)\gamma_1(\frac{d(p^*)}{n-1}, d(p^*)) \text{ (from Eq. (19) and the fact that } \gamma_{11} < 0) \\ >\ & c_l'(\frac{d(p^*)}{n-1}) \text{ (since } n \geq \widehat{M}_1) \\ \geq\ & c_l'(q_l') \text{ (from Eq. (19)).} \end{aligned} \tag{20}$$

However, this violates Eq. (12). ■

Proof of Lemma $3'$. Let the number of type l firms charging $\hat{p}_i$ be k, where $k \geq \hat{n}_{il}$. First consider the case where none of the other firms charge prices that are less than $\hat{p}_i$. Clearly, if all the type l firms charging $\hat{p}_i$ produce identical amounts then the maximum profit of all such firms is $\frac{\hat{p}_i d(\hat{p}_i)}{k} - c(\frac{d(\hat{p}_i)}{k})$. Since $k \geq \hat{n}_i il$, this is less than $\hat{\pi}$.

Now consider the case where the output level of all the firms charging $\hat{p}_i$ are not the same. Clearly, if the aggregate production by all such firms is less than equal to $d(\hat{p}_i)$, then some of the firms would be producing and selling less than $\frac{d(\hat{p}_i)}{k}$, and consequently would have a profit less than $\frac{\hat{p}_i d(\hat{p}_i)}{k} - c_l(\frac{d(\hat{p}_i)}{k}) < \hat{\pi}$. Whereas, if the aggregate production of such firms is greater than $d(\hat{p}_i)$, then some firms would sell less than $\frac{d(\hat{p}_i)}{k}$, while their production would be larger. Again their profit would be less than $\frac{\hat{p}_i d(\hat{p}_i)}{k} - c_l(\frac{d(\hat{p}_i)}{k})$.

Finally, if some of the other firms charge less than $\hat{p}_i$, then the residual demand at $\hat{p}_i$ would be even less than $d(\hat{p}_i)$. We can now mimic the earlier argument to claim that some of the firms charging $\overline{p}_i$ would have a profit less than $\frac{\hat{p}_i d(\hat{p}_i)}{k} - c_l(\frac{d(\hat{p}_i)}{k})$. ■

Proof of Proposition 7. Step 1. We first argue that all the firms must be producing strictly positive amounts in equilibrium. Suppose to the contrary that firm i of type l has an output level of zero.

(i) If the total production by the firms charging p^* is less than $d(p^*)$ then firm i of type l can charge p^*. Since $p^* > c'(0)$, its profit would increase if it produces a sufficiently small amount.

(ii) Next consider the case where the total production by the firms charging p^* is greater than $d(p^*)$. Without loss of generality let firms 1 to m charge p^*, and, moreover, let firm 1 (of type k) have a strictly positive level of output, i.e. $q_{1k} > 0$. Note that

$$\begin{aligned}
\frac{\partial \pi_{il}(q_{il}, \sum_a \sum_b q_{ab} - q_{il})}{\partial q_{il}}|_{q_{il}=0} &= p^* d(p^*)\gamma_1(0, \sum_{j=1}^{m} q_j) - c_l'(0) \\
&= p^* d(p^*)\gamma_1(0, \sum_{j=1}^{m} q_j) - c_k'(0) \text{ (since } c_l'(0) = c_m'(0)) \\
&> p^* d(p^*)\gamma_1(q_{1k}, \sum_{j=2}^{m} q_j) - c_k'(q_{1k}) \text{ (since } \gamma_{11} - \gamma_{12} < 0), \\
&= \frac{\partial \pi_{1k}(q_{1k}, \sum_a \sum_b q_{ab} - q_{1k})}{\partial q_{1k}} = 0.
\end{aligned}$$

But then firm i of type l can charge p^* and, by producing a sufficiently small level of output, increase its profit level.

Step 2. We then argue that there cannot be some $\widehat{p}_i$ $(\in F) > p^*$ such that some firms charge $\widehat{p}_i$ and supply a positive amount.

Suppose to the contrary that such a price exists. Then the total number of firms charging p^* can be at most $n^{**}-1$. Suppose not. Clearly, the aggregate production by these firms be less than $d(p^*)$. Hence, all firms of type l must be producing q_l^*. But this implies that total production is greater than $d(p^*)$. (This follows from the definition of n^{**}). But this is a contradiction. Thus the total number of firms is at most $n^{**}-1$. Moreover, the aggregate output of these firms can be at most $\tilde{q}$ (this follows from the definition of $\tilde{q}$).

Now consider some $\widehat{p}_i > p^*$. Clearly, the number of firms charging $\widehat{p}_i$ is less than $\max_l \hat{n}_{il}$. Since otherwise some of the firms charging $\widehat{p}_i$ would have a profit less than $\overline{\pi}$. But such a firm can ensure a profit of at least $\overline{\pi}$ by charging p^* (since the aggregate output of the firms charging p^* is at most $\tilde{q}$). Thus the total number of firms producing a strictly positive amount is less than $\widehat{M}_2$, thus contradicting step 1. Hence all firms charge p^*.

Step 3. We can mimic step 3 of the uniqueness part of Proposition 2 to claim that any equilibrium must be symmetric. Then, assuming that $\forall l$, $n^l \geq 2$, we have that $\forall i, l$, $\sum_a \sum_b q_{ab} - q_{il} > q_{il}$.

Note that given steps 1 and 3, we can restrict attention to q_{il} such that, $\forall i, l$, $\sum_a \sum_b q_{ab} - q_{il} > q_{il} > 0$, while solving Eq. (11).

Step 4. Next consider the game where all firms charge p^* and compete over quantities. Note that the profit function of the i-th firm of type l, $p^* d(p^*) \gamma(q_{il}, \sum_k \sum_m q_{km} - q_{il}) - c_l(q_{il})$, is continuous in the output levels and strictly concave in q_{il}. Moreover, note that we can restrict attention to strategy spaces of the form $[0, \hat{q}_l]$, where $\hat{q}_l$ is such that $p^* d(p^*) - c_l(q) < 0$, $\forall q > \hat{q}_l$. Since these are non-empty, compact and convex subsets of Euclidean spaces, we can use the Debreu (A social equilibrium existence theorem, Proceedings of the National Academy of Sciences 38, 1952, 886-893.) fixed point theorem to argue that this game has a solution in pure strategies. From step 1, this equilibrium must be interior. Moreover, given that $\sum_l n^l \geq \widehat{M}_1$, the equilibrium must involve an aggregate output greater than $d(p^*)$, and hence will be characterized by Eq. (11). Therefore Eq. (11) has a solution. Moreover, the solution is symmetric and characterized by Eq. (12).

Step 5. We finally argue that the outcome described in Proposition 7 indeed constitutes an equilibrium.

(i) Given Lemma $2'$, no firm can increase its price and gain, as the deviating firm will have no residual demand. Clearly, undercutting is not profitable either.

(ii) We then argue that none of the firms can change its output level and gain. Suppose firm i of type l produces q_{il}, while the other firms stick to the suggested output. We can argue as before that

$$\pi_{il}(q_{il}, (n^l - 1)q'_l + \sum_{j\neq l} n^j q'_j) = p^* d(p^*)\gamma(q_{il}, (n^l - 1)q'_l + \sum_{j\neq l} n^j q'_j) - c_l(q_{il})$$

is concave in q_{il} and $\frac{\partial \pi_{il}(q_{il}, (n^l-1)q'_l + \sum_{j\neq l} n^j q'_j)}{\partial q_{il}}|_{q_{il}=0} > 0$. Then

$$\frac{\partial \pi_{il}(q_{il}, (n^l - 1)q'_l + \sum_{j\neq l} n^j q'_j)}{\partial q_{il}}|_{q_{il}=q'_l} = p^* d(p^*)\gamma_1(q'_l, (n^l-1)q'_l + \sum_{j\neq l} n^j q'_j) - c'_l(q'_l). \tag{21}$$

Given Eq. (12), the firms have no incentive to change their output. Finally, given that $\gamma(q_i, \sum_{j\neq i} q_j)$ is symmetric, all the firms of the same type must be selling an identical amount. ■

Proof of Lemma $4'$. Suppose not, i.e. $\exists l$ such that $\lim_{r\to\infty} q'_l(r) = D > 0$. Then we have that

$$\begin{aligned} & \lim_{r\to\infty} \gamma_1(q'_l(r), (rn^l - 1)q'_l(r) + \sum_{j\neq l} rn^j q'_j(r)) \\ \leq\ & \lim_{r\to\infty} \gamma_1(q'_l(r), (rn^l - 1)q'_l(r)) \text{ (since for } rn^l - 1 > 1,\ \gamma_{12} < 0) \\ =\ & 0 \text{ (from Assumption 6(vi))} \\ <\ & c'_l(D) = \lim_{r\to\infty} c'_l(q'_l(r)). \end{aligned}$$

However, this is a contradiction. ■

Proof of Proposition 8. Recall, from Lemma $4'$, that $\lim_{r\to\infty} q'_l(r) = 0$, $\forall l$. Hence $\lim_{r\to\infty} \sum_j rn^j q'_j(r) = \lim_{r\to\infty}(rn^l - 1)q'_l(r) + \sum_{j\neq l} rn^j q'_j(r)$, $\forall l$.

(i) Suppose to the contrary that for some l, $< (rn^l-1)q'_l(r) + \sum_{j\neq l} rn^j q'_j(r) >$ does not diverge. Then there is a convergent subsequence $< r_k >$ such that

$\lim_{k\to\infty}(r_k n^l - 1)q_l'(r_k) + \sum_{j\neq l} r_k n^j q_j'(r_k) = X$, where X is finite. Then

$$\begin{aligned} & \lim_{k\to\infty} p^* d(p^*)\gamma_1(q_l'(r_k), (r_k n^l - 1)q_l'(r_k) + \sum_{j\neq l} r_k n^j q_j'(r_k)) \\ = & \frac{p^* d(p^*)}{X} \text{ (from Assumption 6(iv))} \\ > & 0 = c'(0) = \lim_{k\to\infty} c_l'(q_l'(r_k)), \end{aligned}$$

where the last equality follows from Lemma 4′. But this is a contradiction.

(ii) We show that for all l, there cannot be any subsequence of $< (rn^l - 1)q_l'(r) + \sum_{j\neq l} rn^j q_j'(r) >$ that either diverges, or converges to some limit different from $d(p^*)\frac{p^*}{c'(0)}$. To begin with suppose that for some l, $\lim_{r\to\infty}(rn^l - 1)q_l'(r) + \sum_{j\neq l} rn^j q_j'(r)$ diverges to ∞. From Eq. (12) it follows that

$$\lim_{r\to\infty} p^* d(p^*)\gamma_1(q_l'(r), (rn^l - 1)q_l'(r) + \sum_{j\neq l} rn^j q_j'(r)) = \lim_{r\to\infty} c_l'(q_l'(r)).$$

Given Assumption 6(v) and Lemma 4′, the above equation implies that $c'(0) = 0$, which is a contradiction. We can then mimic the earlier argument to claim that all convergent subsequences of $< (rn^l - 1)q_l'(r) + \sum_{j\neq l} rn^j q_j'(r) >$ converge to $d(p^*)\frac{p^*}{c'(0)}$. ∎

Proof of Proposition 9. The idea of the proof is very similar to that in Proposition 2.

Step A. Existence. Notice that since $\alpha < c_2'(0) - c_1'(0)$, it follows that $p_1^* < c_i'(0)$, for all $i \geq 2$. Thus no firm of type j, where $j \geq 2$ can profitably charge a price of p_1^*. For type 1 firms we can simply mimic the proof in Proposition 2 to claim that they cannot have a profitable deviation.

Step B. We first argue that in equilibrium all firms of type 1 charge p^*, produce $q_1'(n^1 - 1)$ and sell $\frac{d(p_1^*)}{n^1}$.

must be producing strictly positive amounts in equilibrium. The proof is in several steps.

Step 1. We first argue that all the firms of type 1 must be producing strictly positive amounts in equilibrium. Suppose to the contrary that firm i (of type 1) has an output level of zero. Consider the aggregate output produced by all the firms charging p_1^*.

(i) Suppose its less than $d(p_1^*)$. Let the i-th firm charge p_1^*. Since $p_1^* > c'(0)$, for a sufficiently small output level, the profit of firm i would increase.

(ii) Next consider the case where the total production by the firms charging p_1^* is greater than $d(p_1^*)$. Without loss of generality let these firms be $1, \cdots, m$, where $m < i$, and let $q_1 > 0$. Note that

$$\begin{aligned}
\frac{\partial \pi_i}{\partial q_i}|_{q_i=0} &= p_1^* d(p_1^*) \gamma_1(0, \sum_{j=1}^{m} q_j) - c'(0) \\
&> p_1^* d(p_1^*) \gamma_1(q_1, \sum_{j=2}^{m} q_j) - c'(q_1) \text{ (since } \gamma_{11} - \gamma_{12} < 0), \\
&= \frac{\partial \pi_1}{\partial q_1} = 0.
\end{aligned}$$

But this implies that firm i can increase its output slightly and gain.

Step 2. We then argue that there cannot be some $\widehat{p}_i$ $(\in F) > p_1^*$ such that some firms of type 1 charge p_i and supply a positive amount.

Suppose to the contrary that such a price exists. This implies that the total number of type 1 firms charging p_1^*, say $\tilde{n}$, can be at most $n_1^* - 1$. Suppose not, i.e. let the number of such type 1 firms be n_1^* or more. In that case, if the aggregate production by these $\tilde{n}$ firms is less than $d(p_1^*)$, then all $\tilde{n}$ firms must be producing q_1^*. But this implies that total production is greater than $d(p_1^*)$. (This follows from the definition of n_1^*). But this is a contradiction.

Now consider some $\widehat{p}_i > p_1^*$. Clearly, the number of type 1 firms charging p_1^* is less than $\hat{n}_i^1$. Thus the total number of type 1 firms producing a strictly positive amount is less than M_2^1, thus contradicting step 1. Hence all firms of type 1 must be charging p_1^*.

Step 3. Let $\tilde{q}$, denote the equilibrium output vector of type 1 firms. We first establish that this vector must be symmetric. Suppose not, and without loss of generality let $\tilde{q}_2 > \tilde{q}_1 > 0$, where both the firms are of type 1. Then,

$$\begin{aligned}
\frac{\partial \pi_1}{\partial q_1}|_{\tilde{q}} &= p_1^* d(p_1^*) \gamma_1(\tilde{q}_1, \sum_{i \neq 1} \tilde{q}_i) - c'(\tilde{q}_1) \\
&> p_1^* d(p_1^*) \gamma_1(\tilde{q}_2, \sum_{i \neq 2} \tilde{q}_i) - c'(\tilde{q}_2) \text{ (since } \gamma_{11} - \gamma_{12} < 0)
\end{aligned}$$

$$= \frac{\partial \pi_2}{\partial q_2}|_{\tilde{q}}. \tag{22}$$

This, however, is a contradiction, since in equilibrium $\frac{\partial \pi_1}{\partial q_1}|_{\tilde{q}} = 0 = \frac{\partial \pi_2}{\partial q_2}|_{\tilde{q}}$.

Step 4. Finally, we argue that there cannot be another symmetric equilibrium where the (common) output level of the firms is different from $q_1'(n^1 - 1)$. Clearly, in any symmetric equilibrium, the production level of all the firms must satisfy

$$p_1^* d(p_1^*)\gamma_1(q, (n^1 - 1)q) = c_1'(q).$$

It is easy to see that this equation has a unique solution. The argument is similar to that for the uniqueness of $q'(n-1)$.

Finally, since type 1 firms exhaust the demand at p_1^*, the output level of all firms of other types must be zero. ■

Proof of Proposition 10. Existence. We first argue that the quantity decisions are optimal. Let p' denote the lowest price in $\{P\}$. Given A3(v), $R^{p'}(P, \overline{Q}_{p'}(P))$ is well defined. Next, given the quantity decisions of the other firms, the output decisions of the firms charging p' are clearly optimal. Next from A3(iv), $R^{p_i}(P, Q_{p_i})$ is well defined. Moreover, given the quantity decisions of the other firms, the output decisions of the firms charging p_i are clearly optimal.

Next note that the stage 2 strategy implies that if, in stage 1, all the firms charge p^*, then, in stage 2, all the firms produce $\frac{d(p^*)}{n}$. Similarly, if in stage 1, $(n-1)$ of the firms charge p^*, while one of the firms charges a price strictly greater than p^*, then, in stage 2, the firms charging p^* produce $\frac{d(p^*)}{n-1}$, while the output level of the other firm is zero.

The pricing decision is also optimal since if any of the firms increase its price then, in stage 2, the output level of the other firms are such that the deviant firm has zero residual demand.

Uniqueness. It is easy to see that we cannot have an equilibrium where the output level of some of the firms is zero, since it can always charge p^* in stage 1 and supply $\frac{d(p^*)}{n}$ in stage 2.

Next observe that the definitions of $\tilde{\pi}$, n_i and n^* are valid for this case also. Hence we can mimic step 2 of the uniqueness part of Proposition 1 to argue that the only price that is sustainable in equilibrium is p^*. Finally, we can mimic step 3 of Proposition 2 to argue that all firms supply exactly $\frac{d(p^*)}{n}$. ■

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