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Brogi, Athos

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Athos Brogi
UniCredit SpA, Via Calabria 31, 20158 Milano, e-mail: athos.brogi@unicredit.eu

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1. Introduction

This short paper presents a time-changing volatility binomial tree suitable for the pricing of European options.

2. Binomial Tree

Time-points t_i, $i = 0,1,\ldots,n$, are equidistant, and time-periods $\Delta t = t_i - t_{i-1}$, $i = 1,\ldots,n$, and time horizon $T = n\Delta t$, which is fixed length of time of expiration of option in years. t_0 is current time-point. We also have an extra historical time-point, t_{-1}, which precedes t_0, and such that $t_0 - t_{-1} = \Delta t$.
The underlying security price can either rise or fall from one-point to the next, $i = 1,\ldots,n$:

$$S_{t_i} = S_{t_{i-1}} u_{t_i} \text{ with probability } q_i \text{ or}$$
$$S_{t_i} = S_{t_{i-1}} d_{t_i} \text{ with probability } 1 - q_i, \tag{1}$$

where u_{t_i} stands for *up*, d_{t_i} stands for *down*, and u_{t_i}, d_{t_i} are variable. q_i is the risk-neutral probability of underlying security price at t_{i-1}, $S_{t_{i-1}}$, rising to $S_{t_{i-1}} u_{t_i}$ at t_i. Further down we derive a formula for q_i.
The definition of continuously compounded return of underlying security from t_{i-1} to t_i:

$$R_{t_i} = \log S_{t_i} - \log S_{t_{i-1}}, \qquad i = 0,1,\ldots,n. \tag{2}$$

We call *current return* $R_{t_0} = \log S_{t_0} - \log S_{t_{-1}}$, where $S_{t_{-1}}$ is a known historical price, so *current return* is known too. Rearranging (1), and taking logarithms, and using (2) we define, $i = 1,\ldots,n$,

$$\log S_{t_i} / S_{t_{i-1}} \mid \sigma_{t_i} = \log u_{t_i} = R^+_{t_i} \text{ with probability } q_i \text{ or}$$
$$\log S_{t_i} / S_{t_{i-1}} \mid \sigma_{t_i} = \log d_{t_i} = R^-_{t_i} \text{ with probability } 1 - q_i, \tag{3}$$

where

$$R_{t_i}^{+} = \mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}\ , \qquad (4)$$
$$R_{t_i}^{-} = \mu\Delta t - \sigma_{t_i}\sqrt{\Delta t}\ , \qquad (5)$$

with $\mu\Delta t < \sigma_{t_i}\sqrt{\Delta t}$ for large *n*, or equivalently small Δt. $\sigma_{t_i}\sqrt{\Delta t}$ is part of a volatility process, $\{\sigma_{t_i}\sqrt{\Delta t}\}_{i=0}^{n}$, which we need to model, where σ_{t_0} is known current annual volatility.

3. Martingale Condition

Under no arbitrage, the discounted price process of the underlying security, $\{\tilde{S}_{t_i}\}_{i=0}^{n}$, must be a martingale. We now derive a formula for risk-neutral probability q_i in (1), so that $\{\tilde{S}_{t_i}\}_{i=0}^{n}$ is a martingale.
Let us introduce a sample of independent Bernoulli random variables, which are independent of $\{\tilde{S}_{t_i}\}_{i=0}^{n}$:

$$Z_i = +1 \text{ with probability } q_i \text{ or}$$
$$Z_i = -1 \text{ with probability } 1 - q_i\ ,$$

where q_i is the risk-neutral probability in (1). Then (4) and (5) can be written as one equation:

$$R_{t_i} = \mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}Z_i\ , \qquad i = 1,\dots,n\ .$$

The martingale condition is

$$\mathrm{E}\left(\tilde{S}_{t_i} \mid \tilde{S}_{t_{i-1}}, \tilde{S}_{t_{i-2}},\dots\right) = \tilde{S}_{t_{i-1}}\ , \qquad i = 1,\dots,n\ ,$$
$$\mathrm{E}\left(e^{-ir\Delta t} S_{t_{i-1}} e^{R_{t_i}} \mid \tilde{S}_{t_{i-1}}, \tilde{S}_{t_{i-2}},\dots\right) = e^{-(i-1)r\Delta t} S_{t_{i-1}}$$
$$\mathrm{E}\left(e^{\mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}Z_i} \mid \tilde{S}_{t_{i-1}}, \tilde{S}_{t_{i-2}},\dots\right) = e^{r\Delta t}\ ,$$

where *r* is the risk-free rate of interest, which is constant during time horizon *T*, and

$$e^{\mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}Z_i} \mid \sigma_{t_i} = e^{\mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}} \text{ with probability } q_i \text{ or}$$
$$e^{\mu\Delta t + \sigma_{t_i}\sqrt{\Delta t}Z_i} \mid \sigma_{t_i} = e^{\mu\Delta t - \sigma_{t_i}\sqrt{\Delta t}} \text{ with probability } 1 - q_i\ ,$$

so that

$$q_i e^{\mu\Delta t+\sigma_{t_i}\sqrt{\Delta t}} + (1-q_i)e^{\mu\Delta t-\sigma_{t_i}\sqrt{\Delta t}} = e^{r\Delta t}.$$

Hence,

$$q_i = \frac{e^{r\Delta t} - e^{\mu\Delta t-\sigma_{t_i}\sqrt{\Delta t}}}{e^{\mu\Delta t+\sigma_{t_i}\sqrt{\Delta t}} - e^{\mu\Delta t-\sigma_{t_i}\sqrt{\Delta t}}}.$$

In risk-neutral pricing we set $\mu = r$, so that

$$q_i = \frac{1-e^{-\sigma_{t_i}\sqrt{\Delta t}}}{e^{\sigma_{t_i}\sqrt{\Delta t}} - e^{-\sigma_{t_i}\sqrt{\Delta t}}}, \qquad i = 1,\ldots,n. \tag{6}$$

For large *n*, or equivalently small Δt, substituting the exponentials in (6) by their series expansions ignoring terms of order $(\Delta t)^{3/2}$ or higher, we get

$$q_i = \frac{1}{2} - \frac{1}{4}\sigma_{t_i}\sqrt{\Delta t}, \qquad i = 1,\ldots,n. \tag{7}$$

So, the risk-neutral probability of $S_{t_{i-1}}$ rising is less than for $S_{t_{i-1}}$ falling. This is true for any *n*, or equivalently any Δt.

$$\mathrm{E}(R_{t_i} \mid \sigma_{t_i}) = \mu\Delta t + (2q_i - 1)\sigma_{t_i}\sqrt{\Delta t} \tag{8}$$

$$\mathrm{var}(R_{t_i} \mid \sigma_{t_i}) = 4q_i(1-q_i)\sigma_{t_i}^2\Delta t. \tag{9}$$

Notice that if $q_i = 1/2$ (which it is not), then

$$\mathrm{E}(R_{t_i} \mid \sigma_{t_i}) = \mu\Delta t$$

$$\mathrm{var}(R_{t_i} \mid \sigma_{t_i}) = \sigma_{t_i}^2\Delta t.$$

Setting q_i as in (6) is an artificial device which forces $\{\tilde{S}_{t_i}\}_{i=0}^{n}$ to be a martingale.

4. Modeling Volatility

As regards the modeling of $\sigma_{t_i}\sqrt{\Delta t}$, Black (1976) already noticed a negative correlation between returns and volatility, i.e. when returns are high, volatility is low, and when returns are low, volatility is high. Such negative correlation can be captured by the following equation:

$$\sigma_{t_i}\sqrt{\Delta t} = \sigma_{t_{i-1}}\sqrt{\Delta t} - \alpha(R_{t_{i-1}} - \mu\Delta t), \qquad i = 1,\ldots,n, \tag{10}$$

where $0 < \alpha < 1$. It is clear that according to (10) volatility of returns, $\sigma_{t_i}\sqrt{\Delta t}$, can never be negative, because, recalling (4) and (5), if $R_{t_{i-1}} = R^{+}_{t_{i-1}}$, then

$$\sigma_{t_i}\sqrt{\Delta t} = \sigma_{t_{i-1}}\sqrt{\Delta t} - \alpha\left(\sigma_{t_{i-1}}\sqrt{\Delta t}\right), \qquad i = 1,\ldots,n,$$

$0 < \alpha < 1$. Alternatively, if $R_{t_{i-1}} = R^{-}_{t_{i-1}}$, then

$$\sigma_{t_i}\sqrt{\Delta t} = \sigma_{t_{i-1}}\sqrt{\Delta t} + \alpha\left(\sigma_{t_{i-1}}\sqrt{\Delta t}\right), \qquad i = 1,\ldots,n,$$

$0 < \alpha < 1$.

$$\begin{aligned} \mathrm{E}\left(\sigma_{t_i}\sqrt{\Delta t} \mid \sigma_{t_{i-1}}\right) &= \sigma_{t_{i-1}}\sqrt{\Delta t} - \alpha\left(\mathrm{E}\left(R_{t_{i-1}} \mid \sigma_{t_{i-1}}\right) - \mu\Delta t\right) \\ &= \sigma_{t_{i-1}}\sqrt{\Delta t} - \alpha\left(2q_{i-1} - 1\right)\sigma_{t_{i-1}}\sqrt{\Delta t} \end{aligned}, \qquad (11)$$

where, as in (6),

$$q_0 = \frac{1 - e^{-\sigma_{t_0}\sqrt{\Delta t}}}{e^{\sigma_{t_0}\sqrt{\Delta t}} - e^{-\sigma_{t_0}\sqrt{\Delta t}}}, \qquad (12)$$

which is known.
From (10) we see that

$$\sigma_{t_i}\sqrt{\Delta t} = \sigma_{t_0}\sqrt{\Delta t} - \alpha\sum_{j=0}^{i-1}\left(R_{t_j} - \mu\Delta t\right), \qquad i = 1,\ldots,n.$$

Hence,

$$\mathrm{E}\left(\sigma_{t_i}\sqrt{\Delta t} \mid \sigma_{t_{i-1}}, \sigma_{t_{i-2}}, \ldots\right) = \sigma_{t_0}\sqrt{\Delta t} - \alpha\sum_{j=0}^{i-1}\left(2q_j - 1\right)\sigma_{t_j}\sqrt{\Delta t}.$$

At t_n, dropping $\sqrt{\Delta t}$,

$$\mathrm{E}\left(\sigma_{t_n} \mid \sigma_{t_{n-1}}, \sigma_{t_{n-2}}, \ldots\right) = \sigma_{t_0} - \alpha\sum_{j=0}^{n-1}\left(2q_j - 1\right)\sigma_{t_j},$$

and, using (7) for q_j,

$$\lim_{n\to\infty}\mathrm{E}\left(\sigma_{t_n} \mid \sigma_{t_{n-1}}, \sigma_{t_{n-2}}, \ldots\right) = \sigma_{t_0} - \alpha\lim_{n\to\infty}\sum_{j=0}^{n-1}\left(-\frac{1}{2}\sigma_{t_j}\sqrt{\Delta t}\right)\sigma_{t_j} = \infty,$$

because each term in the sum is of order $\sqrt{\Delta t}$ (order $1/\sqrt{n}$), but extra terms are added to the sum at rate *n*. So, the expected value of σ_{t_n} has no finite limit.

$$\mathrm{var}\left(\sigma_{t_i}\sqrt{\Delta t} \mid \sigma_{t_{i-1}}\right) = \alpha^2 \,\mathrm{var}\left(R_{t_{i-1}} \mid \sigma_{t_{i-1}}\right) = 4\alpha^2 q_{i-1}(1-q_{i-1})\sigma_{t_{i-1}}^2 \Delta t\,. \tag{13}$$

Looking at (13), we note that the greater the *α*, the greater the variance of volatility of returns, and the greater the variance of volatility of returns, the greater the kurtosis of the distribution of returns.

5. Option Pricing Algorithm

The above tree can be implemented easily to price European call or put options. In fact, thousands of paths along the tree are simulated, and for each path the payoff of our European option is calculated, and the price of our European option equals the arithmetic mean of the thousands of discounted payoffs.

In detail, first, the (fixed) length of time *T* between now, t_0, and when the European option expires, t_n, needs to be determined. Second, *n* is chosen and fixed, so that *T* is split into *n* smaller time periods $\Delta t = T/n = t_i - t_{i-1}$, $i = 1,\ldots,n$. For example, Δt could be one day.

Now, step-by-step, starting at t_0, given *α*:

1. Calculate previously defined *current return*, $R_{t_0} = \log S_{t_0} - \log S_{t_{-1}}$.
2. With R_{t_0} obtain $\sigma_{t_1}\sqrt{\Delta t}$ from (10), where σ_{t_0} is quoted or estimated annual volatility of returns.
3. Calculate q_1 from (6).
4. In order to determine whether S_{t_0} rises or falls, draw a uniformly distributed random number in $[0,1)$. If drawn number is in $[0,q_1)$, then S_{t_0} rises, and so input $\sigma_{t_1}\sqrt{\Delta t}$ in (4) to obtain $R_{t_1}^+$. If drawn number is in $[q_1,1)$, then S_{t_0} falls, and so input $\sigma_{t_1}\sqrt{\Delta t}$ in (5) to obtain $R_{t_1}^-$.
5. With either $R_{t_1}^+ = \log u_{t_1}$ or $R_{t_1}^- = \log d_{t_1}$ calculate either $S_{t_1} = S_{t_0}u_{t_1}$ or $S_{t_1} = S_{t_0}d_{t_1}$.
6. With S_{t_1} return to 1. shifting forward one time-point, thus calculating R_{t_1}, and repeat 1. to 5. shifting forward one time-point at each repetition until S_{t_n} is calculated.
7. With S_{t_n} calculate European call option payoff, X^c, or put payoff, X^p,

$$X^c = \max\left(S_{t_n} - k, 0\right) \tag{14}$$
$$X^p = \max\left(k - S_{t_n}, 0\right), \tag{15}$$

where *k* is the strike.

8. Repeat 1. to 7. thousands of times, thus simulating thousands of paths along the tree, and attaining thousands of payoffs.

9. Calculate price of European call, $\pi\left(X^c\right)$, or put, $\pi\left(X^p\right)$,

$$\pi\left(X^c\right)=e^{-rT}\left(\frac{1}{m}\sum_{i=1}^{m}X_i^c\right) \tag{16}$$

$$\pi\left(X^p\right)=e^{-rT}\left(\frac{1}{m}\sum_{i=1}^{m}X_i^p\right), \tag{17}$$

where m is the number of paths simulated, and payoffs calculated. r is the risk-free rate of interest, which is constant during T.

10. As a measure of the accuracy of estimate (16) or (17), calculate its standard error, given by $s/\sqrt{m}$, where s is the sample standard deviation of the m discounted payoffs obtained.

6. Advantages of the Model

- Easy to implement.
- Apart from μ and σ_{t_0}, volatility process has only one parameter, α.
- Relatively easy to calibrate. α parameter can be calibrated by trial and error, given a sensible estimate of σ_{t_0}.
- Numerically stable. It can model the implied volatility (implied σ_{t_0}) surface, where three option expiry dates with three months between them are considered, without the need to change the value of α parameter.

References

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