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Abstract

Is the paradise of effortless communication the ideal environment for knowledge creation? Or, can the development of local culture in regions raise knowledge productivity compared to a single region with a unitary culture? In other words, can a real technological increase in the cost of collaboration and the cost of public knowledge flow between regions, resulting in cultural differentiation between regions, increase welfare? In our framework, a culture is a set of ideas held exclusively by residents of a location. In general in our model, the equilibrium path generates separate cultures in different regions. When we compare this to the situation where all workers are resident in one region, R & D workers become too homogeneous and there is only one culture. As a result, equilibrium productivity in the creation of new knowledge is lower relative to the situation when there are multiple cultures and workers are more diverse.

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1 Introduction

> If everything occurred at the same time there would be no development. If everything existed in the same place there could be no particularity. Only space makes possible the particular, which then unfolds in time. Only because we are not equally near to everything; only because everything does not rush in upon us at once; only because our world is restricted, for every individual, for his people, and for mankind as a whole, can we, in our finiteness, endure at all. ... Space creates and protects us in this limitation. Particularity is the price of our existence. (Lösch, 1940, Epilogue)

Thus, as Lösch pointed out more than half a century ago, space has an economic role aside from erecting barriers to trade in commodity markets.

Rephrasing this in terms of our context, the question we ask is: *Can a real technological increase in the cost of collaboration and cost of public knowledge flow between regions increase welfare?* Does the creation of a regional culture of ideas in common among a population raise or lower productivity in the creation of new knowledge? What role is played by interregional interaction among researchers?

The deeper motivation for this work comes from three religious texts. The biblical story of the Tower of Babel is told in Genesis 11: 1-9. When the earth had only one language, residents dared to construct a tower to reach heaven and make a name for themselves. The builders were scattered and their languages confounded. Was this punishment, or a blessing in disguise?

The second religious text is Samuelson (1949). On pp. 194-195, an angel descends from heaven:

> Now suppose that an angel came down from heaven and notified some fraction of all the labour and land units producing clothing that they were to be called Americans, the rest to be called Europeans; and some different fraction of the food industry that henceforth they were to carry American passports. Obviously, just giving people and areas national labels does not alter anything: it does not change commodity or factor prices or production patterns.

Again, if separation implies no changes in commodity market equilibrium, but rather a divergence of cultures, the angel could improve welfare. The devil, of course, is in the details. This discussion emphasizes the difference between economic integration and the type of cultural integration or diversity that is our focus.

The third text, Jacobs (1961, Part Two), discusses the role of diversity in the urban context.

For illustrative purposes, suppose that there are locations, or regions, where R & D can take place. R & D workers collaborating in different regions face a discount in their productivity due to distance. There is public knowledge transmission, for example through patenting, in a region, but inter-regional public knowledge transmission is tempered by distance (lost in translation).

To get the intuition across, suppose that there is a single region in the economy, with researchers or knowledge workers living in it. At the beginning there is public knowledge transmission, for example through patenting, that occurs within the region. With this structure and a relatively effective public knowledge transmission mechanism, the path of knowledge production actually realized, called the equilibrium path, involves a pattern of work with people rapidly changing partners located in the region. Even though the capacity of researchers to absorb public information is limited, knowledge diversity within the region is small.

Suppose now that the knowledge workers are suddenly differentiated in terms of their location. That is, half the workers are separated from the other half, and all workers are presumed immobile. It becomes more costly for a researcher to work with another in the other region as opposed to their home region. Interaction between regions is open, in the sense that researchers can work with those in the other region, and public knowledge is transmitted between locations, but at a discount relative to public transmission within a region. On the new equilibrium path, it is never best to work exclusively with people in one location.

The key feature in our analysis is as follows. Knowledge diversity *between* the two regions develops over time, but does not in itself improve productivity *within* each region. Within each region, knowledge workers are relatively homogeneous. To increase productivity, they must somehow differentiate themselves from one another. To accomplish this objective, they form the *inter-regional* working groups that are the key to our results. Working groups are available for intra-regional interaction as well, but in that context, they only serve to increase the homogeneity of workers in the same working group in the region, thus decreasing their productivity. Therefore, working groups are never used by choice in the intra-regional context. In contrast, in the inter-regional context, *intensive public knowledge transfer within a small inter-regional working group can serve to differentiate the members of that group from others in the home region*, increasing heterogeneity within each region and thus increasing productivity.[1] In the end, each agent will have to strike a balance between time spent in a small inter-regional working group, and time spent working with others in their own region who are not members of the small inter-regional working group. This balance creates both diversity *within* each region as well as higher productivity. In this way, productivity in the creation of new ideas as well as the income obtained by researchers from patents rise in the two-region economy. The maximal productivity attainable is bounded by the maximum productivity of working with someone in another region.

The model we present is a two region economy in which there are equal populations of immobile knowledge workers in the regions. Each agent can produce ideas on their own with the investment of time, but they can also produce new ideas with a partner in either region. Knowledge production at a given time is dependent on the set of ideas known exclusively by one or the other partner, and the set of ideas that the two have in common. Ideas in common are important for communication, whereas ideas known exclusively by one of the partners is important for bringing originality into the potential partnership. When considering the choice of partners, the agents balance the costs and benefits of working with a partner within the same region and a partner in the other region. There is a productivity cost for working with someone in the other region, but there is a potential benefit in that their knowledge profile might be more appealing than the knowledge profile of residents of the home region since they have more exclusive ideas than residents of the home region. The agents are myopic in their choice of partners (or work in isolation) so they maximize the flow of new ideas created. We use myopic core as the solution concept.

Our results indicate that, given an initial situation where there is a high degree of homogeneity in workers, division into two regions will result in a big improvement in knowledge productivity when: 1) Heterogeneity (as opposed to homogeneity) of workers' knowledge bases is important in the produc-

[1] As an example of inter-regional working groups in the context of economic research, focus on Japan and the US. The set of researchers that are alumni of a particular university, say the University of Rochester or the University of Chicago, form groups crossing international boundaries with training and ideas in common that can promote new knowledge creation and sharing among each group's participants.

tion function for partnerships, so diversity increases productivity; 2) Inter-regional public knowledge transmission is weak (since this promotes inter-regional knowledge differentiation); 3) Public knowledge transmission within each inter-regional working group is effective, so workers can differentiate themselves from others in the same region rapidly; 4) The within-region public information transmission technology is very effective so that autarky yields too much homogeneity and thus is unproductive. The rapid recent development of information technology increases the scope of the applicability of our analysis. We shall discuss this issue further in the conclusions.

Culture comes into play in the following manner. In our framework, a culture is a set of ideas held exclusively by residents of a location. In general in our model, the equilibrium path generates separate cultures in different regions.[2] Earlier work (see for example Berliant and Fujita, 2008; Berliant and Fujita, 2009; Berliant and Fujita, 2010) did not consider regions or locations, so there is no concept of culture.[3]

The model has empirical content. Consider, for example, the Japanese economy from 1993 to the present. In terms of per capita GDP, in 1993, Japan ranked number one among OECD countries, declining to seventh place in 2003, 14th place in 2006, and 19th place in 2008.[4] The top ranked countries in 2008 were all small, northern European countries (Luxembourg, Norway, Switzerland, Denmark, Ireland, the Netherlands, Iceland, Sweden, Finland, Austria). What happened to cause this? As is well known, dense communication and social networks (nomunication, or communication with drinking) imply intensive interactions among co-workers, resulting in rapid learning from others and fast growth when the country is less developed and most of the new ideas arrive from external sources, but too much homogeneity among workers when the country is more developed and on the cutting edge of innovation. This increased homogeneity, particularly of knowledge workers, can slow innovation and thus economic growth. In contrast, the top 10 countries are small, but each has its own local language, university system, television, and more generally, culture. The total population of these top 10 countries is about half of Japan's population. The total geographic span of these countries is about

[2] Lösch (1940) calls this spatial diversity "particularity."

[3] A rather difficult extension of the model would allow endogenous migration between regions.

[4] The web site
http://www.esri.cao.go.jp/jp/sna/h20-kaku/percapita.pdf contains interesting data on per capita GDP of various OECD countries.

the same as Japan, but each of these countries has its own local cultural center. In contrast, Japan is very centralized in many respects, including media and education. In the age of the knowledge economy, this result is consistent with our conclusions.[5]

There is an interesting empirical literature on culture, diversity and growth. In this literature, diversity (or the characteristics of people) is generally taken to be exogenous, but mobile. After adjusting for various econometric problems, most obviously reverse causality in that diversity is not random across cities, Ottaviano and Peri (2006) find that cultural diversity has a positive effect on the productivity of locals using U.S. data. Bellini et al. (2008) find similar effects in European data. The effects of immigration on local rents and wages have been studied by Card (2007) and Ottaviano and Peri (2008). The empirical effects of the migration of culturally differentiated workers on innovation are studied in Agarwal et al (2008) and Kerr and Lincoln (2008). Determinants of the R & D location decisions of multinational firms are examined in Belderbos et al (2009). In contrast with all of this literature, we model diversity as endogenous and immobile, but demonstrate how diversity and multiple cultures interacting can improve productivity.

More relevant to our work is the empirical paper of Cardoso et al (2010) on international trends in economic research. They find that a country's progress in publishing in top journals is correlated with international collaborations between coauthors, consistent with our analysis.

Section 2 gives the model and notation, Section 3 gives preliminary analysis of the model, whereas Section 4 analyzes the equilibrium path of dynamics in the knowledge production sector. Section 5 gives our conclusions and suggestions for future knowledge workers. Three appendices provide the proofs of key results.

[5] In contrast with modern Japan, Tokugawa Japan (approximately 1600-1860) was partitioned into about 200 domains ruled by daimyo. They and their entourages (including samurai) were required by the shogun to make regular pilgrimages to Edo. As eloquently described by Vaporis (2008), in Edo they interacted with both the locals and the delegates from other domains, particularly scholars, artists, and artisans. In the process, they created new ideas and culture, transmitting some of it back to the residents of their home domain. This two way interaction raised the cultural level of the country as a whole.

2 The Model

The economy consists of two regions called A and B. As explained in the introduction, initially there are no differences between workers in the two regions, as there are no barriers between them and there is in reality only one region. But this notation is useful later, when workers are exogenously (and suddenly) separated into the two regions. There are N R & D workers, also called K-workers, *in each region*, and they are *immobile*. The set of K-workers in region A is denoted by the same notation A, whereas the set of K-workers in region B is denoted by B. This simplifies notation, and it should be obvious from the context which meaning applies.

Production of a new manufactured commodity requires the purchase of a patent. To keep matters simple, we do not elaborate the details of the manufacturing sector, but refer the interested reader to Berliant and Fujita (2011). These patents are produced by the R & D sector, and they are the only output of this sector. Each new patent embodies a new idea. Not all new ideas result in patents. New ideas are produced by K-workers using their prior stock of knowledge. The scheme for producing new ideas is described as a knowledge production process. Income for R & D workers is derived exclusively from the sale of patents.

The basic layout of this sector is similar to Berliant and Fujita (2008). While avoiding excessive repetition, we present below the details of the R & D process.

At any given time, each K-worker has a stock of knowledge that has some commonalities with other K-workers but some knowledge distinct from other workers. Since workers possess knowledge exclusive of others, they may wish to cooperate with each other in the knowledge production process. Heterogeneity of knowledge in a partnership brings more originality, but knowledge in common is important for communication. Thus, K-worker heterogeneity is an essential feature of the model and of the knowledge production process. The K-workers choose to work alone or with a partner, maximizing their myopic payoff, namely the value of patents produced at that time. The solution concept used is the myopic core. If they work alone, new ideas are produced as a function of the total number of ideas known by a K-worker. If a pair of workers produces new ideas together, their knowledge production is a function of their knowledge in common on the one hand and the knowledge they have that is distinct from their partner on the other. Knowledge that is produced by an agent at a given time becomes part of the stock of knowledge for that agent in the future. In addition, some of these ideas become patented and are sold to the manufacturing sector. The ideas embodied in the patents become public, and thus will be available to be learned by all the agents in the R & D sector.

The basic unit of knowledge is called an idea.[6] The number of potential ideas is infinite. In this paper, we will treat ideas symmetrically. In describing the process of knowledge production, that is either accomplished alone or in cooperation with another K-worker, the sufficient statistics about the state of knowledge of a K-worker i at a given time can be described as follows. We shall focus on K-worker i and her potential partner K-worker j. First, $n_i(t)$ represents the total stock of i's ideas at time t. Second, $n_{ij}^c(t)$ represents the total stock of ideas that i has in common with K-worker j at time t. Third, $n_{ij}^d(t)$ represents the stock of ideas that i knows but j doesn't know at time t. Finally, $n_{ji}^d(t)$ represents the stock of ideas that j knows but i doesn't know at time t.

By definition, $n_{ij}^c(t) = n_{ji}^c(t)$.[7] It also holds by definition that

$$n_i(t) = n_{ij}^c(t) + n_{ij}^d(t) \tag{1}$$

Knowledge is a set of ideas that are possessed by a person at a particular time. However, knowledge is not a static concept. New knowledge can be produced either individually or jointly, and ideas can be shared with others. But all of this activity takes time.

Now we describe the components of the rest of the model. To keep the description as simple as possible, we focus on just two agents, i and j. At each time, each agent faces a decision about whether or not to meet with others. If two agents want to meet at a particular time, a meeting will occur. If an agent decides not to meet with anyone at a given time, then the agent creates new knowledge separately, away from everyone else. If two persons do decide to meet at a given time, then they collaborate to create new knowledge together.[8]

At each moment of time, there are two mutually exclusive ways to produce new knowledge. The first way is to work alone, away from others. We denote the event that K-worker i does research alone at time t by $\delta_{ii}(t) = 1$, indicating

[6] In principle, all of these time-dependent quantities are positive integers. However, for simplicity we take them to be continuous (in $\mathbb{R}_+$) throughout the paper.

[7] In general, however, it is not necessary that $n_{ij}^d(t) = n_{ji}^d(t)$.

[8] Since there is an infinity of potential ideas, the probability that the same idea is duplicated by any K-worker or K-workers (even at different points of time) is assumed to be zero.

that i works with herself. Otherwise, $\delta_{ii}(t) = 0$. Alternatively, K-worker i can choose to work with a partner, say K-worker j in either region. We denote the event that K-worker i wishes to work with j at time t by $\delta_{ij}(t) = 1$. Otherwise, $\delta_{ij}(t) = 0$. In equilibrium, this partnership is realized at time t if $\delta_{ij}(t) = \delta_{ji}(t) = 1$.

Consider first the case where K-worker i works alone. In this case, idea production is simply a function of the stock of i's ideas at that time. Let $a_{ii}(t)$ be the rate of production of new ideas created by person i in isolation at time t. Then we assume that their creation of new knowledge during isolation is proportional to their stock of knowledge $n_i(t)$ at time t:

$$a_{ii}(t) = \alpha \cdot n_i(t) \text{ when } \delta_{ii}(t) = 1 \tag{2}$$

where α is a positive constant.

If a meeting occurs between i and j at time t ($\delta_{ij}(t) = \delta_{ji}(t) = 1$), then joint knowledge creation occurs, and it is governed by the following dynamics.

In the case where both K-worker i and K-worker j *reside in the same region* and agree to work together, namely when $\delta_{ij}(t) = \delta_{ji}(t) = 1$ for $j \neq i$, joint knowledge creation is given by:[9]

$$a_{ij}(t) = 2\beta \cdot (n_{ij}^c)^\theta \cdot (n_{ij}^d \cdot n_{ji}^d)^{\frac{1-\theta}{2}} \text{ when } i, j \in A \text{ or } i, j \in B \tag{3}$$

where $0 < \theta < 1$, $\beta > 0$. These parameters are explained just below.

In the case where K-worker i and K-worker j *reside in different regions* and agree to work together, namely when $\delta_{ij}(t) = \delta_{ji}(t) = 1$ for $j \neq i$, joint knowledge creation is given by:

$$a_{ij}(t) = \tau \cdot 2\beta \cdot (n_{ij}^c)^\theta \cdot (n_{ij}^d \cdot n_{ji}^d)^{\frac{1-\theta}{2}} \text{ when } i \in A \text{ and } j \in B, \text{ or } j \in A \text{ and } i \in B \tag{4}$$

where $0 < \tau < 1$. Due to the distance between the regions, we assume that when two K-workers live in different places, their collaborative research productivity is reduced by a factor of τ. Some time (and knowledge production) is lost when one researcher visits a collaborator in another region. Or time is lost due to differences in languages. But these are just examples. In general,

[9] We may generalize equation (3) as follows:

$$a_{ij}(t) = \max\left\{(\alpha - \varepsilon)\, n_i(t), (\alpha - \varepsilon)\, n_j(t), 2\beta \cdot (n_{ij}^c)^\theta \cdot (n_{ij}^d \cdot n_{ji}^d)^{\frac{1-\theta}{2}}\right\}$$

where $\varepsilon > 0$ represents the costs from the lack of concentration. This generalization, however, does not change the results presented in this paper in any essential way.

we are simply assuming that research productivity is reduced due to distance between collaborators.

So when two people meet, joint knowledge creation occurs at a rate proportional to the normalized product of their knowledge in common, the differential knowledge of i from j, and the differential knowledge of j from i. The parameter β represents the overall level of joint knowledge productivity. Moreover, the rate of creation of new knowledge is high when the proportions of ideas in common, ideas exclusive to person i, and ideas exclusive to person j are in balance. The parameter θ represents the weight on knowledge in common as opposed to differential knowledge in the production of new ideas. Ideas in common are necessary for communication, whereas ideas exclusive to one person or the other imply more heterogeneity or originality in the collaboration. The special case where θ tends to 1 captures the circumstance where different knowledge is unproductive; for example, knowing a theorem that potential partners don't know is unproductive relative to everyone knowing that theorem.

Income for the research sector derives from selling patents. But not all ideas are patentable. For every collection of ideas created, we assume that η proportion are patentable as blueprints of new products. Thus, they are sold to the manufacturing sector. The residual ideas, namely $1-\eta$ proportion of new ideas, becomes tacit knowledge that is only known to the creator or creators of these ideas. They are useful for future creation of yet further ideas.

Let $y_i(t)$ to be the income of K-worker i at time t, and let $\Pi(t)$ be the price of patents at time t. Then, suppressing t for notational simplicity:

$$y_i = \Pi \cdot \eta \cdot (\delta_{ii} \cdot a_{ii} + \sum_{j \neq i} \delta_{ij} \cdot a_{ij}/2) \tag{5}$$

The formula implies that the revenue from new patents is split evenly if two K-workers are producing new ideas together. The K-workers take the price Π as given at each time, so the assumption of myopia on their part implies that the price does not affect their behavior. For this reason, we do not consider explicitly the market for patents in the remainder of the paper.

Concerning the rule used by an agent to choose their best partner, to keep the model tractable in this first analysis, we assume a myopic rule. At each moment of time t, person i would like a meeting with person j in either region when her income while meeting with j is highest among all potential partners, including herself. Maximizing income at a given time amounts to choosing

$\{\delta_{ij}\}_{j=1}^{2N}$ so that the right hand side of (5) is highest, meaning that a selection is made only among the most productive partners. Loosely speaking, this interaction could be modeled as a noncooperative game, with player i choosing $\{\delta_{ij}\}_{j=1}^{2N}$ as strategies, and equilibrium implying that for each pair of players i and j, $j \neq i$, $\delta_{ij} = \delta_{ji}$, whereas $\delta_{ij} > 0$ only for those players j that yield maximal payoffs for player i.[10]

This noncooperative approach is useful for explaining the ideas behind our model, but we employ a cooperative approach for two reasons. First, it gives the same equilibrium path as the noncooperative approach but with less cumbersome notation and structure. Second, as we are attempting to model close interactions within groups, it is plausible that agents will act cooperatively. We assume that at each time, the myopic persons interacting choose a core configuration. That is, we restrict attention to configurations such that at any point in time, no coalition of persons can get together and make themselves better off *in that time period*. In essence, our solution concept at a point in time is the myopic core.

Although knowledge creation in isolation or in pairs represents the basic forms of knowledge creation, it turns out that the equilibrium path often requires a mixture of these basic forms, namely δ_{ij} takes on fractional values. The reason is that on the equilibrium path, K-workers wish to form groups where close interaction takes place in pairs within the group but there is no direct interaction between groups. This is a natural way to balance communication and diversity preservation.[11] K-workers in the same group wish to change partners within the group as frequently as possible. The purpose is to balance the proportion of different and common ideas with partners within

[10] More formally, out of equilibrium payoffs are defined and a selection or refinement of Nash equilibrium used as in Berliant et al. (2006, pp. 77-78). A refinement of Nash equilibrium is necessary to exclude some trivial equilibria, for example where nobody ever chooses to meet anyone else. Specifically, choose $1 > \epsilon > 0$ and positive constants $\{f_{ij}\}_{i=1,j<i}^{N}$ such that $\sum_{i=1}^{N}\sum_{j<i} f_{ij} = \epsilon$. Define $f_{ji} = f_{ij}$ for $j > i$. Then the payoffs for the noncooperative game are specified as follows. Fix strategies $\{\delta_{ij}\}_{i,j=1}^{2N}$. For K-workers i and j for whom $\delta_{ij} \neq \delta_{ji}$, a meeting of length $f_{ij} = f_{ji}$ occurs. For K-workers i and j for whom $\delta_{ij} = \delta_{ji}$ (excluding $j = i$) a meeting of length $(1-\epsilon)\cdot\delta_{ij}$ occurs. Work in isolation (δ_{ii}) is assigned the residual time. The Nash equilibria we select are the equilibria when $\epsilon = 0$, but that are also limits of Nash equilibria as $\epsilon \to 0$. The reader should note that the noncooperative interpretation of the myopic core is especially important in the multi-region context of this paper, where cooperation among agents is not as reasonable as in the one region context of earlier work.

[11] These groups differentiate themselves from one another so that, eventually, they have so little in common that members in different groups do not interact.

the same group as best as can be achieved. This suggests a work pattern with rapidly changing partners on the equilibrium path, that is, a work pattern where a worker rotates through fixed partners as fast as possible in order to maximize the instantaneous increase in income. For example, worker 1 chooses K-workers 2 and 3 as partners, and rotates between the two partners under equilibrium values of δ_{12} and δ_{13} such that $\delta_{12}+\delta_{13}=1$. Worker 1 might wish to work with workers 2 and 3 for half of each month, but wants to alternate between them so that worker 1 does not have the same partner on consecutive days. As time intervals in this discrete time model become shorter, the limit is a fractional δ_{1j} $(j=2,3)$ where $\delta_{12}=\delta_{13}=1/2$. Other K-workers behave analogously. In order for this type of work pattern to take place, of course, all persons must agree to follow this pattern.[12] In general, we allow $\delta_{ij}\in[0,1]$, and for all i, $\sum_{j=1}^{2N}\delta_{ij}=1$. In equilibrium, $\delta_{ij}=\delta_{ji}$ for all $i,j=1,2,...,2N$.

As noted previously, all agents take prices, in this case Π, as given, implying:

$$\max_{\{\delta_{ij}\}_{j=1}^{2N}}(\delta_{ii}\cdot a_{ii}+\sum_{j\neq i}\delta_{ij}\cdot a_{ij}/2) \tag{6}$$

subject to the obvious constraints:

$$\sum_{j=1}^{2N}\delta_{ij}=1,\ \delta_{ij}\geq 0 \text{ for } i=1,...,2N \tag{7}$$

Since n_i is a stock variable, this is equivalent to

$$\max_{\{\delta_{ij}\}_{j=1}^{2N}}(\frac{\delta_{ii}\cdot a_{ii}+\sum_{j\neq i}\delta_{ij}\cdot a_{ij}/2}{n_i}) \tag{8}$$

In order to rewrite this problem in a convenient form, we first define the total number of ideas possessed by i and j:

$$n^{ij}=n_{ij}^d+n_{ji}^d+n_{ij}^c \tag{9}$$

and define new variables

$$\begin{aligned} m_{ij}^c &\equiv m_{ji}^c=\frac{n_{ij}^c}{n^{ij}}=\frac{n_{ji}^c}{n^{ij}} \\ m_{ij}^d &= \frac{n_{ij}^d}{n^{ij}},\ m_{ji}^d=\frac{n_{ji}^d}{n^{ij}} \end{aligned}$$

[12] Although in equilibrium the income derived by person 1 from partnerships with persons 2 and 3 are the same at a given time when $\delta_{12}=\delta_{13}=1/2$, its first derivative is higher when both partnerships are active. We shall use this refinement.

By definition, m_{ij}^d represents the proportion of ideas exclusive to person i among all the ideas known by person i or person j. Similarly, m_{ij}^c represents the proportion of ideas known in common by persons i and j among all the ideas known by the pair. From (9), we obtain

$$1 = m_{ij}^d + m_{ji}^d + m_{ij}^c \tag{10}$$

whereas (9) and (1) yield

$$n_i = (1 - m_{ji}^d) \cdot n^{ij} \tag{11}$$

Using these identities and new variables, while recalling the knowledge production function (3), we obtain (see Technical Appendix a for details)

$$a_{ij} = n_i \cdot 2G(m_{ij}^d, m_{ji}^d) \text{ for } j \neq i \text{ in the same region} \tag{12}$$

$$a_{ij} = n_i \cdot \tau \cdot 2G(m_{ij}^d, m_{ji}^d) \text{ for } j \neq i \text{ in different regions} \tag{13}$$

where

$$G(m_{ij}^d, m_{ji}^d) \equiv \frac{\beta \left(1 - m_{ij}^d - m_{ji}^d\right)^{\theta} \cdot (m_{ij}^d \cdot m_{ji}^d)^{\frac{1-\theta}{2}}}{1 - m_{ji}^d} \tag{14}$$

For ease of notation, we write A_{-i} for the set of K-workers in region A less agent i. Analogous notation holds for region B.

For K-worker i in region A, using (2) and (12), we can rewrite the income function (5) as

$$y_i = \Pi \cdot \eta \cdot n_i \cdot (\delta_{ii} \cdot \alpha + \sum_{j \in A_{-i}} \delta_{ij} \cdot G(m_{ij}^d, m_{ji}^d) + \sum_{j \in B} \delta_{ij} \cdot \tau \cdot G(m_{ij}^d, m_{ji}^d)) \tag{15}$$

and the optimization problem (8) as follows:

$$\max_{\{\delta_{ij}\}_{j=1}^{2N}} (\delta_{ii} \cdot \alpha + \sum_{j \in A_{-i}} \delta_{ij} \cdot G(m_{ij}^d, m_{ji}^d) + \sum_{j \in B} \delta_{ij} \cdot \tau \cdot G(m_{ij}^d, m_{ji}^d)) \tag{16}$$

subject to the obvious constraints (7).

Suppose that for each $i = 1, 2, ..., 2N$, $\{\delta_{ij}^*\}_{j=1}^{2N}$ solves the optimization problem immediately above. Furthermore, suppose that it happens to be the case that

$$\delta_{ij}^* = \delta_{ji}^* \text{ for } i, j = 1, 2, ..., 2N$$

Then, by construction, $\{\delta_{ij}^*\}_{i,j=1}^{2N}$ must also be the solution to the following social optimization problem:

$$\max\{\sum_{i=1}^{2N} y_i \mid \sum_{j=1}^{2N} \delta_{ij} = 1, \ \delta_{ij} \geq 0, \ \delta_{ij} = \delta_{ji} \text{ for } i, j = 1, 2, ..., 2N\}$$

Thus, $\{\delta^*_{ij}\}_{i,j=1}^{2N}$ is in the myopic core.

Next we turn to the acquisition of new knowledge by each individual. There are two ways to acquire new knowledge for a K-worker: internal production of new ideas and information from public sources. The first way has the feature that ideas produced alone are attributed to that worker, whereas ideas produced in pairs are attributed to both K-workers who produce them. In either case, the new ideas are learned by exactly the people who produce them. The second source of knowledge acquisition derives from the new ideas that are patented. The patented ideas become public information immediately. Some of this public information is learned by the knowledge workers. However, their capacity for learning this public knowledge is limited. We call the constant C the learning capacity of a knowledge worker. As we shall detail next, there are 4 sources of public knowledge. Each time period for learning public information is divided into two subperiods. In the first subperiod, public knowledge generated from pairs of workers in the same region is studied. In the second subperiod, public knowledge generated from pairs of workers from different regions is studied by the knowledge workers. In each subperiod, there are two competing sources of new public knowledge. But in both subperiods, learning capacity is limited.

As we discuss next and as justified in subsection 1 of Appendix 1, we introduce the following specifications for the public knowledge absorption technology, explained in detail just below:

$$\mu = \frac{C}{\eta(N-1)} \cdot \frac{1}{1+\widetilde{\gamma}} \tag{17}$$

$$\widetilde{\mu} = \frac{C}{\eta N} \cdot \frac{\widetilde{\gamma}}{1+\widetilde{\gamma}} \tag{18}$$

$$\widehat{\mu} = \frac{\widehat{C}}{\eta \cdot 2(N-1)} \tag{19}$$

$$\overleftrightarrow{\mu} = \frac{\overleftrightarrow{C}}{2(N^*-1)} \tag{20}$$

where

$$\widehat{C} + \overleftrightarrow{C} = C \tag{21}$$

and

$$\overleftrightarrow{C} = \begin{cases} \overline{C} \cdot \frac{N^*}{\overline{N}} & \text{for } N^* < \overline{N} \\ \overline{C} & \text{for } N^* \geq \overline{N} \end{cases} \tag{22}$$

A certain proportion of patented ideas in a region, μ, are learned by all of the K-workers in that region. In general, μ will be a decreasing function of N.

Limited time and energy determine how many of these new, public ideas can be learned. Due to these limitations, the amount of information a K-worker can learn from patents in their region at a given time is, roughly, proportional to the number of new ideas she can create in that time. The number of new ideas and thus patents is proportional to the number of K-workers in that region, so as detailed in equation (17), μ will be inversely proportional to N.[13] Thus, these ideas become knowledge in common for all K-workers in that region.[14]

A second source of friction between regions, beyond the direct cost of collaboration, is in public information transmission. It is natural to assume that public knowledge is transmitted better to workers in the same region where it was created. Some is "lost in translation" in the process of communication to the other region. This could be viewed as a pure language issue, but more usefully, the creators of the knowledge possess some human capital related to the creation of the idea in their region that is not present in the other region. Some ideas are lost in translation, or some time is lost in translation so not as many of the ideas can be publicly communicated between regions as within a single region. Yet another interpretation of this idea is that questions can be asked of researchers who live within the region, thus making communication of their new discoveries easier for those who live nearby than for those who live far away.

The absorption of public knowledge transmitted from the other region, namely produced by two partners residing in the other region, is discounted by a factor $\widetilde{\gamma}$, $0 \leq \widetilde{\gamma} < 1$, relative to public knowledge produced by partners resident in one's own home region; see equation (18). This gives us $\widetilde{\mu} < \mu$. Public ideas produced by partnerships of the same type (categorized by regions of residence of the partners) are substitutes.

Next we turn to public knowledge attributable to inter-regional partnerships, namely where the partners live in different regions. In general, such public knowledge is assumed to be complementary to public knowledge produced by partners exclusively resident in one location or the other. There are

[13] In theory, it might be possible to accumulate a stock of ideas patented in past periods to learn in the future. The problem with this is that such information perpetually accumulates, and thus due to time constraints there is never an opportunity to learn the content of older patented ideas.

[14] It has been suggested that if K-workers become too homogeneous, they might learn the patented ideas selectively so as not to overlap with the knowledge acquired by other K-workers in the same fashion. However, this level of coordination, especially when N is large, seems far-fetched. It seems more likely that ideas attractive for whatever reason will be learned by all.

two types of such public knowledge, and according to equation (21) they are assumed to be substitutes for each other. The first is general public knowledge from inter-regional cooperation, represented by $\widehat{\mu}$. It is analogous to the previous concepts, namely public knowledge derived from pairs of partners from different regions, and is given by equation (19). In what follows we naturally assume $\widehat{\mu} < \mu$.[15] The final type of public information transmission is from "inter-regional working groups" consisting of people from both regions working together; these groups develop endogenously, as explained in detail in Section 4.2. For these groups, public information is transmitted *only within the group itself*, not to the general population of either region. The effectiveness of this last kind of public knowledge transmission is represented by $\overleftrightarrow{\mu}$, and is given in equation (20). For these inter-regional working groups, it is assumed in equation (22) that the effectiveness of public knowledge transmission within the inter-regional working group increases with group size N^* up to a point $(\overline{N})$, above which it is constant.

It should be evident at this point that τ on the one hand and $\widetilde{\mu}$, $\widehat{\mu}$, $\overleftrightarrow{\mu}$ and μ on the other are empirically related. The productivity of long distance collaboration is correlated with the effectiveness of public knowledge transmission between regions, but not perfectly. Public knowledge transmission can be ineffective if the library of one collaborator in region A does not subscribe to some journals published in region B for reasons of cost or language, but this does not prohibit collaborations between authors in different regions. Some correlation may derive from language differences that affect both collaboration and public knowledge transmission between regions. In what follows, we treat all of these exogenous parameters as independent.

Next, for each of the four different types of new ideas created at each moment, we calculate their number. Let us focus on agent i, as the expressions for the other agents are analogous. Let I_{AA} be the total number of ideas created at a given moment by researchers resident exclusively in region A:

$$I_{AA} = \sum_{k \in A} \delta_{kk} \cdot a_{kk} + \Big(\sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot a_{kl}\Big)/2 \tag{23}$$

Similarly, let I_{BB} be the total number of ideas created at a given moment by researchers resident exclusively in region B:

$$I_{BB} = \sum_{k \in B} \delta_{kk} \cdot a_{kk} + \Big(\sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot a_{kl}\Big)/2 \tag{24}$$

[15] Of course, this actually follows from equations (17) and (19) and the definition of $\widetilde{\gamma}$.

Next, let I_{AB} be the total number of ideas created at a given moment by pairs where one researcher is resident in A and the other is resident in B:

$$I_{AB} = \sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot a_{kl} \tag{25}$$

Finally, inter-regional interaction will occur in subsets of the population called groups. Each K-worker will belong to exactly one inter-regional group. Focusing on one particular K-worker $i \in A$, we define their group to be

$$\Gamma_i = \{\Gamma_{iA}, \Gamma_{iB}\}$$

where $i \in \Gamma_{iA}$, Γ_{iA} represents the set of people from region A to which i is associated, whereas Γ_{iB} represents the set of people from region B to which i is associated. We define ideas generated within a group Γ_i as

$$I_{\Gamma_i} = \sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot a_{kl}$$

Within each group, people work exclusively with the workers from the other region, not with the workers from their own region. (Intra-regional partnerships were already considered in (23) and (24).)

The dynamics of the knowledge system are based on the assumption that once learned, ideas are not forgotten. Using the argument above, we obtain knowledge system dynamics. First, we provide the dynamics of the new knowledge learned by each K-worker:

$$\begin{aligned} \text{For } i &\in A: \\ \dot{n}_i &= \sum_{j \in A} \delta_{ij} \cdot a_{ij} + \sum_{j \in B} \delta_{ij} \cdot a_{ij} + \mu \cdot \eta \cdot (I_{AA} - \sum_{j \in A} \delta_{ij} \cdot a_{ij}) \\ &\quad + \widetilde{\mu} \cdot \eta \cdot I_{BB} + \widehat{\mu} \cdot \eta \cdot (I_{AB} - \sum_{j \in B} \delta_{ij} \cdot a_{ij}) + \overleftrightarrow{\mu} \cdot (I_{\Gamma_i} - \sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot a_{ij}) \end{aligned} \tag{26}$$

$$\begin{aligned} \text{For } i &\in B: \\ \dot{n}_i &= \sum_{j \in A} \delta_{ij} \cdot a_{ij} + \sum_{j \in B} \delta_{ij} \cdot a_{ij} + \mu \cdot \eta \cdot (I_{BB} - \sum_{j \in B} \delta_{ij} \cdot a_{ij}) \\ &\quad + \widetilde{\mu} \cdot \eta \cdot I_{AA} + \widehat{\mu} \cdot \eta \cdot (I_{AB} - \sum_{j \in A} \delta_{ij} \cdot a_{ij}) + \overleftrightarrow{\mu} \cdot (I_{\Gamma_i} - \sum_{j \in \Gamma_{iA}} \delta_{ij} \cdot a_{ij}) \end{aligned} \tag{27}$$

For the new knowledge in common learned by each pair of K-workers i and j, we have:

$$\begin{aligned} \text{For } i &\in A,\, j \in A: \\ j &\in \Gamma_{iA}\text{:}\ \dot{n}_{ij}^c = \delta_{ij} \cdot a_{ij} + \mu \cdot \eta \cdot (I_{AA} - \delta_{ij} \cdot a_{ij}) + \widetilde{\mu} \cdot \eta \cdot I_{BB} + \widehat{\mu} \cdot \eta \cdot I_{AB} + \overleftrightarrow{\mu} \cdot I_{\Gamma_i} \\ j &\notin \Gamma_{iA}\text{:}\ \dot{n}_{ij}^c = \delta_{ij} \cdot a_{ij} + \mu \cdot \eta \cdot (I_{AA} - \delta_{ij} \cdot a_{ij}) + \widetilde{\mu} \cdot \eta \cdot I_{BB} + \widehat{\mu} \cdot \eta \cdot I_{AB} \end{aligned} \tag{28}$$

$$\text{For } i \in A,\ j \in B: \tag{29}$$
$$\begin{aligned}
j &\in \Gamma_{iB}:\ \dot{n}^c_{ij} = \delta_{ij}\cdot a_{ij} + \widetilde{\mu}\cdot\eta\cdot I_{AA} + \widetilde{\mu}\cdot\eta\cdot I_{BB} + \widehat{\mu}\cdot\eta\cdot(I_{AB} - \delta_{ij}\cdot a_{ij}) \\
&\quad + \overleftrightarrow{\mu}\cdot(I_{\Gamma_i} - \delta_{ij}\cdot a_{ij}) \\
j &\notin \Gamma_{iB}:\ \dot{n}^c_{ij} = \delta_{ij}\cdot a_{ij} + \widetilde{\mu}\cdot\eta\cdot I_{AA} + \widetilde{\mu}\cdot\eta\cdot I_{BB} + \widehat{\mu}\cdot\eta\cdot(I_{AB} - \delta_{ij}\cdot a_{ij})
\end{aligned}$$

Finally, for each pair of K-workers i and j, we obtain the new knowledge learned exclusively by i as follows:

$$\text{For } i \in A,\ j \in A: \tag{30}$$
$$\begin{aligned}
j &\in \Gamma_{iA}:\ \dot{n}^d_{ij} = (1-\mu\cdot\eta)\cdot\sum_{k\in A_{-j}}\delta_{ik}\cdot a_{ik} + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot a_{ik} \\
j &\notin \Gamma_{iA}:\ \dot{n}^d_{ij} = (1-\mu\cdot\eta)\cdot\sum_{k\in A_{-j}}\delta_{ik}\cdot a_{ik} + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot a_{ik} \\
&\quad + \overleftrightarrow{\mu}\cdot(I_{\Gamma_i} - \sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot a_{ik})
\end{aligned}$$

$$\text{For } i \in A,\ j \in B: \tag{31}$$
$$\begin{aligned}
j &\in \Gamma_{iB}:\ \dot{n}^d_{ij} = (1-\widetilde{\mu}\cdot\eta)\cdot\sum_{k\in A}\delta_{ik}\cdot a_{ik} + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot a_{ik} \\
&\quad +(\mu\cdot\eta - \widetilde{\mu}\cdot\eta)\cdot(I_{AA} - \sum_{j\in A}\delta_{ij}\cdot a_{ij}) \\
j &\notin \Gamma_{iB}:\ \dot{n}^d_{ij} = (1-\widetilde{\mu}\cdot\eta)\cdot\sum_{k\in A}\delta_{ik}\cdot a_{ik} + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot a_{ik} \\
&\quad +(\mu\cdot\eta - \widetilde{\mu}\cdot\eta)\cdot(I_{AA} - \sum_{j\in A}\delta_{ij}\cdot a_{ij}) + \overleftrightarrow{\mu}\cdot(I_{\Gamma_i} - \sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot a_{ik})
\end{aligned}$$

$$\text{For } i \in A,\ j \in B: \tag{32}$$
$$\begin{aligned}
j &\in \Gamma_{iB}:\ \dot{n}^d_{ji} = (1-\widetilde{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{jk}\cdot a_{jk} + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in A_{-i}}\delta_{jk}\cdot a_{jk} \\
&\quad +(\mu\cdot\eta - \widetilde{\mu}\cdot\eta)\cdot(I_{BB} - \sum_{k\in B}\delta_{jk}\cdot a_{jk}) \\
j &\notin \Gamma_{iB}:\ \dot{n}^d_{ji} = (1-\widetilde{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{jk}\cdot a_{jk} + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in A_{-i}}\delta_{jk}\cdot a_{jk} \\
&\quad +(\mu\cdot\eta - \widetilde{\mu}\cdot\eta)\cdot(I_{BB} - \sum_{k\in B}\delta_{jk}\cdot a_{jk}) + \overleftrightarrow{\mu}\cdot(I_{\Gamma_j} - \sum_{k\in\Gamma_{iA}}\delta_{jk}\cdot a_{jk})
\end{aligned}$$

To give more intuition, let us explain equation (26) in detail. The left hand side of this equation represents new knowledge learned by person i. The first two terms on the right hand side represent private knowledge creation. The next two terms, $\mu \cdot \eta \cdot (I_{AA} - \sum_{j \in A} \delta_{ij} \cdot a_{ij}) + \widetilde{\mu} \cdot \eta \cdot I_{BB}$, represent the absorption of public knowledge created by partners respectively in A and in B. These two sources of public knowledge compete with each other, since the total public knowledge learning capacity from these two sources is C. The final two terms, $\widehat{\mu} \cdot \eta \cdot (I_{AB} - \sum_{j \in B} \delta_{ij} \cdot a_{ij}) + \overleftrightarrow{\mu} \cdot (I_{\Gamma_i} - \sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot a_{ij})$, represent the absorption of public knowledge created by partners in different regions. The first term represents absorption of general public knowledge that is created by all partnerships with one member from region A and the other from region B, whereas the second term represents absorption of specific public knowledge that is created within a worker's inter-regional working group. These two sources of public knowledge compete with each other, since total learning capacity from these two sources is C.

Thus, equations (26) and (27) say that the increase in the knowledge of person i is the sum of: the knowledge created in isolation, the knowledge created jointly with someone else, and the transfer of new knowledge from new patents. Equations (28) and (29) mean that the increase in the knowledge in common for persons i and j equals the new knowledge created jointly by them plus the transfer of knowledge from new patents. Finally, equations (30), (31) and (32) mean that all the knowledge created by person i either in isolation or joint with persons other than person j becomes a part of the differential knowledge of person i from person j, *except* for patented ideas that are learned by K-workers.[16]

In Section 6.2 of Appendix 1, we collect the elements of the dynamics of $\dot{n}$ and $\dot{m}_{ij}^d$, describing them in terms of n_i and m_{ij}^d $(i, j = 1, ..., 2N)$ only.

3 Knowledge Dynamics in the Pairwise Symmetric Situation

Since we are concerned with the macro behavior of the economy and the big picture in terms of culture, we make a number of simplifying assumptions. We impose the assumption that the initial state of knowledge for all K-workers is pairwise symmetric in terms of heterogeneity.

Suppose that at some given time, all K-workers across the two regions have

[16] We do not assume that when persons work together, all of their previously learned knowledge is transferred between them over time. This would lead to a kind of contagion model. We study this separately for a simplified framework in Berliant and Fujita (2009).

the same stock of ideas:

$$n_i = n_j \text{ for all } i \text{ and } j \tag{33}$$

Using equation (11), since $n^{ij} = n^{ji}$ by definition, it follows that

$$m_{ij}^d = m_{ji}^d \quad \text{for all } i \neq j \tag{34}$$

meaning that the proportions of differential knowledge are pairwise symmetric. Equation (16) is simplified as

$$\max_{\{\delta_{ij}\}_{j=1}^{2N}} (\delta_{ii} \cdot \alpha + \sum_{j \in A_{-i}} \delta_{ij} \cdot g(m_{ij}^d) + \sum_{j \in B} \delta_{ij} \cdot \tau \cdot g(m_{ij}^d)) \tag{35}$$

where the function g is defined as

$$g(m) \equiv G(m, m) \equiv \beta \frac{(1-2m)^{\theta} m^{(1-\theta)}}{1-m} \tag{36}$$

Furthermore, since $a_{ij} = a_{ji}$ by definition, substituting (34) into (12) yields

$$\frac{a_{ij}/2}{n_i} = \frac{a_{ji}/2}{n_j} = g(m_{ij}^d) \text{ for } i \text{ and } j \text{ in the same region} \tag{37}$$

$$\frac{a_{ij}/2}{n_i} = \frac{a_{ji}/2}{n_j} = \tau \cdot g(m_{ij}^d) \text{ for } i \text{ and } j \text{ in different regions} \tag{38}$$

Thus, when two K-workers i and j in the same region cooperate in knowledge production and their knowledge states are symmetric, $g(m_{ij}^d)$ represents the creation of new ideas per capita (normalized by the size of individual knowledge input, n_i). Analogously, when two K-workers i and j in different regions cooperate in knowledge production and their knowledge states are symmetric, $\tau \cdot g(m_{ij}^d)$ represents the creation of new ideas per capita (normalized by the size of individual knowledge input, n_i). In this context, condition (35) means that each K-worker wishes to engage in knowledge production in a partnership with a person (possibly including herself) leading to the highest K-productivity.

Figure 1 illustrates the graph of the intra-regional K-productivity function $g(m)$ as the upper bold curve for parameter values $\beta = 1$ and $\theta = 1/3$. In addition, it illustrates the inter-regional K-productivity function $\tau \cdot g(m)$ as the lower bold curve for the same parameters and $\tau = 0.89$.

FIGURE 1 GOES HERE

Differentiating $g(m)$ yields

$$g'(m) = g(m) \cdot \frac{(1-\theta) - (2-\theta) \cdot m}{(1-2m) \cdot m \cdot (1-m)}$$

implying that

$$g'(m) \underset{<}{\overset{>}{-}} 0 \text{ as } m \underset{>}{\overset{<}{-}} \frac{1-\theta}{2-\theta} \quad \text{for } m \in (0, \frac{1}{2}) \tag{39}$$

Thus, $g(m)$ is strictly quasi-concave on $[0, 1/2]$, achieving its maximal value at

$$m^B = \frac{1-\theta}{2-\theta} \tag{40}$$

which we call the "Bliss Point." It is the point where knowledge productivity is highest for each person. Notice that the bliss point is the same for the two curves. Also in Figure 1, we define the point m^S by the condition:

$$g(m^S) = \tau \cdot g(m^B),\ m^S < m^B \tag{41}$$

Since m^S is defined uniquely as a function of exogenous parameters, we write $m^S = m^S(\tau, \theta)$.

Substituting (36) into (15), we have the income equation for K-worker $i \in A$:

$$y_i = \Pi \cdot \eta \cdot n_i \cdot [\delta_{ii} \cdot \alpha + \sum_{j \in A_{-i}} \delta_{ij} \cdot g(m_{ij}^d) + \sum_{j \in B} \delta_{ij} \cdot \tau \cdot g(m_{ij}^d)] \tag{42}$$

At this point, it is useful to remind the reader that we are using a myopic core concept to determine equilibrium at each point in time. In fact, it is necessary to sharpen that concept in the model with $2N$ persons. *When there is more than one vector of strategies that is in the myopic core at a particular time, namely more than one vector of joint strategies implies the same, highest income for all persons, the one with the highest first derivative of income* $\dot{y}_i$ *is selected.* Furthermore, when the derivative of income is still the same among best options, agent i chooses an option that maximizes the second derivative of income, $\ddot{y}_i$, and so on. The justification for this assumption is that at each point in time, people are attempting to maximize the flow of income. The formal definition of the myopic core and proof that it is nonempty can be found in Berliant and Fujita (2008, Appendix 0). Although the theorem is general, in the remainder of this paper we shall focus on the symmetric case.

Taking the time derivative,[17]

$$\begin{aligned}\dot{y}_i &= \{\dot{\Pi}\cdot\eta\cdot n_i+\Pi\cdot\eta\cdot\dot{n}_i\}\cdot \\ &\quad [\delta_{ii}\cdot\alpha+\sum_{j\in A_{-i}}\delta_{ij}\cdot g(m_{ij}^d)+\sum_{j\in B}\delta_{ij}\cdot\tau\cdot g(m_{ij}^d)] \\ &\quad +\Pi\cdot\eta\cdot n_i[\sum_{j\in A_{-i}}\delta_{ij}\cdot g'(m_{ij}^d)\cdot\dot{m}_{ij}^d+\sum_{j\in B}\delta_{ij}\cdot\tau\cdot g'(m_{ij}^d)\cdot\dot{m}_{ij}^d]\end{aligned} \tag{43}$$

where

$$\sum_{j\in A}\delta_{ij}+\sum_{j\in B}\delta_{ij}=1 \text{ for all } i\in A\cup B$$

When the symmetry condition (34) holds, using (33) and (36), the dynamics of n_i and m_{ij}^d can be rewritten as in Section 6.3 of Appendix 1, where it is obvious that the basic rules that govern knowledge dynamics in the pairwise symmetric case are described in terms of n_i and m_{ij}^d $(i,j=1,2,...2N)$ only.

Notice that the expression for person i's income, (15), does not contain δ_{ji} for $j\neq i$. Hence equations (16) and (35) do not contain it either. But the expression (43) for $\dot{y}$ contains $\dot{m}_{ij}^d$, which in turn involves all of $\{\delta_{lk}\}_{l,k=1}^{2N}$. Thus, when person i performs the optimization problem $\max_{\{\delta_{ij}\}_{li,j=1}^{2N}}\dot{y}_i$, a crucial question is whether the feasibility constraint $\delta_{ij}=\delta_{ji}$ for each $j\neq i$ is considered as a constraint by person i or not. If so, then our subsequent expressions, particularly for $\dot{m}_{ij}^d$, feature cancellation of δ_{ij} with δ_{ji}, and our algebra becomes much simpler. Otherwise such cancellation is impossible and the analysis becomes much more complex. However, since we are dealing with myopic core rather than a noncooperative game structure, we can take a simpler approach in this work.

[17] From (35), when $\{\delta_{ij}\}_{j=1}^{2N}$ is chosen optimally by person i, we have

$$y_i=\Pi\cdot\eta\cdot n_i\cdot(\sum_{j\in A_{-i}}\delta_{ij}+\sum_{j\in B}\delta_{ij})\cdot\max\{\alpha,\max_{j\in A_{-i}}g(m_{ij}^d),\max_{j\in B}\tau\cdot g(m_{ij}^d)\}$$

where $\sum_{j\in A_{-i}}\delta_{ij}+\sum_{j\in B}\delta_{ij}=1$. Thus, in taking the time derivative of (42), except possibly on a set of measure zero, we have $\sum_{j\in A_{-i}}\dot{\delta}_{ij}+\sum_{j\in B}\dot{\delta}_{ij}=0$, and hence (43) follows.

4 The Equilibrium Path of Knowledge Dynamics

4.1 One Region

First we study the case of *one region.* This is the paradise of effortless communication, Babel before the intervention of a deity. Formally speaking, there is only one region, say region A, in this spaceless economy of $2N$ K-workers. In the dynamics, we drop all of the terms related to residents of region B, and simplify the expressions derived in Appendix 6.3 as follows:

$$\begin{aligned} \text{For } i &\in A: \\ \frac{\dot{n}_i}{n_i} &= [\delta_{ii} \cdot \alpha + \sum_{j \in A_{-i}} \delta_{ij} \cdot 2g\left(m_{ij}^d\right)] \\ &\quad + \mu \cdot \eta \cdot [\sum_{k \in A_{-i}} \delta_{kk} \cdot \alpha + \sum_{k \in A_{-i}} \sum_{l \in A_{-k}} \delta_{kl} \cdot g\left(m_{kl}^d\right)] \end{aligned} \tag{44}$$

$$\begin{aligned} \text{For } i &\in A, \\ \frac{\dot{m}_{ij}^d}{1 - m_{ij}^d} &= (1 - m_{ji}^d) \left\{ \delta_{ii} \cdot \alpha + \sum_{k \in A_{-\{i,j\}}} \delta_{ik} \cdot 2g(m_{ik}^d) \right\} \\ &\quad - m_{ij}^d \cdot \left\{ \delta_{ij} \cdot 2g(m_{ij}^d) + \mu \cdot \eta \cdot \left[\sum_{k \in A} \delta_{kk} \cdot \alpha + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot g\left(m_{kl}^d\right) \right] \right\} \\ &\quad - m_{ij}^d \cdot \left\{ \delta_{jj} \cdot \alpha + \sum_{k \in A_{-\{i,j\}}} \delta_{jk} \cdot 2g(m_{jk}^d) \right\} \end{aligned}$$

The initial state of knowledge is given by

$$n_{ij}^c(0) = n^c(0) \text{ for all } i \neq j \tag{45}$$

$$n_{ij}^d(0) = n^d(0) \text{ for all } i \neq j \tag{46}$$

implying that

$$n_i(0) = n^c(0) + n^d(0) \equiv n(0) \tag{47}$$

At the initial state, each pair of K-workers has the same number of ideas, $n^c(0)$, in common. Moreover, for any pair of K-workers, the number of ideas that one K-worker knows but the other does not know is the same and equal to $n^d(0)$. Given that the initial state of knowledge is symmetric among the K-workers, as seen below, it turns out that the equilibrium configuration at any time also maintains the basic pairwise symmetry among K-workers.

Now we are ready to investigate the actual equilibrium path, depending on the given initial composition of knowledge,

$$m_{ij}^d(0) = m^d(0) = \frac{n^d(0)}{n^c(0) + 2n^d(0)}$$

which is common for all pairs i and j $(i \neq j)$.

In the rest of paper, we assume that N is sufficiently large so that for any finite constant ξ, we can use the approximation:

$$\frac{\xi}{N} \approx 0 \tag{48}$$

In the remainder of this paper, we also assume that

$$\alpha < g(m^B) \tag{49}$$

so as to avoid the trivial case of all agents always working in isolation.

In Figure 1, let m^J and m^I be defined on the horizontal axis at the left intersection and the right intersection between the $g(m)$ curve and the horizontal line at height α, respectively.

Previous work characterized the equilibrium path of knowledge creation dynamics in a single region. The various equilibrium paths are determined by the initial heterogeneity of the K-workers. To be precise, from Berliant and Fujita (2011), we have:

Proposition 1: *Assume that the number of K-workers $2N$ is sufficiently large. The equilibrium path of K-worker interactions and the sink point of the knowledge creation process depend on the initial condition, $m^d(0)$.*

When initial heterogeneity satisfies $m^J < m^d(0) \leq m^B$, the myopic core path follows one of two subcases. Let $\widetilde{C} \equiv \frac{2\theta}{1-\theta}$.

(a) $C < \widetilde{C}$. The myopic core path consists of an initial time interval in which each K-worker is always paired with another but trades partners as rapidly as possible (with $\delta_{ij} = 1/(2N-1)$ for all i and for all $j \neq i$). When the bliss point, $m^B = \frac{1-\theta}{2-\theta}$, is attained, the agents split into groups of size $\widetilde{N}^B = 1 + \frac{1}{\theta - \frac{(1-\theta)C}{2}}$, and they remain at the bliss point.

(b) $C > \widetilde{C}$. The myopic core path has all K-workers paired with another but trading partners as rapidly as possible (with $\delta_{ij} = 1/(2N-1)$ for all i and for all $j \neq i$). This continues forever. The equilibrium path remains to the left of the bliss point, so the bliss point is never attained. The sink point is $m_{aut}^* = \frac{1}{2+\frac{C}{2}}$.

Other initial conditions for the system are possible, but we refer to Berliant and Fujita (2008, 2009, 2011) for detailed examination of the other cases. For completeness, we wish to note that long term collaboration between partners is an equilibrium outcome for some other cases, and that although we presume agents to be myopic, we have shown that the equilibrium path has surprising efficiency properties, so the results would be unchanged if we considered agents who were not myopic. With this justification, our focus is on myopic agents for reasons of simplicity.

Since we wish to examine how the knowledge creation system responds to the introduction of interaction with another region, **our focus in the remainder of the paper is on case (b).** The reason for this focus is as follows. Both cases specify initial heterogeneity to the left of the bliss point, so K-workers are too homogeneous relative to maximal productivity at the bliss point. In case (a), public knowledge transmission is relatively weak, as specified by a low value of C. Thus, even though workers start out relatively homogeneous, they can differentiate themselves from others by working with everyone else, and eventually attain the relative heterogeneity and maximal productivity of the bliss point without any sort of intervention. In contrast, for case (b), public knowledge transmission is strong, represented by a high value of C. Even though one K-worker works with all others, the state m^*_{aut} to the left of the bliss point, namely with more worker homogeneity and lower productivity than optimal, is the steady state. Try as they might, the knowledge workers cannot climb the productivity hill from the left, because the public knowledge force pushes them back. Here is where divine intervention and culture that is local in nature can improve matters.

In case (b), the dynamics imply that only one large group forms within the region, so each agent works with everyone else an equal amount of time. Heterogeneity m^d changes, approaching the sink point given by:

$$m^*_{\text{aut}} = \frac{1}{2 + \frac{C}{2}} < m^B = \frac{1-\theta}{2-\theta} \tag{50}$$

The sink point is to the left of the bliss point, so the bliss point is never reached. Intuitively, this is due to the large externality from public knowledge; it is impossible to attain sufficient heterogeneity. Without loss of generality, we assume that:[18]

$$m^J \leq m^*_{\text{aut}} < m^B$$

[18] When C is very large, it is possible that $m^*_{\text{aut}} < m^J$, implying that the actual sink point is at m^J in Figure 1, and that all K-workers eventually work in isolation. However, as long as condition (49) holds, we will have essentially the same result when the single region is split into two regions.

From Appendix 6.3, and using (17) in the case where region A has population $2N$ and $\widetilde{\gamma}$ is set to 1, we obtain the knowledge growth rate of individuals at the sink point as follows:

$$
\begin{aligned}
\text{For } i &\in A: \\
\frac{\dot{n}_i}{n_i} &= [\delta_{ii} \cdot \alpha + \sum_{j \in A_{-i}} \delta_{ij} \cdot 2g\left(m^*_{\text{aut}}\right)] \\
&\quad +\mu \cdot \eta \cdot [\sum_{k \in A_{-i}} \delta_{kk} \cdot \alpha + \sum_{k \in A_{-i}} \sum_{l \in A_{-k}} \delta_{kl} \cdot g\left(m^*_{\text{aut}}\right)] \\
&= 2g(m^*_{\text{aut}}) + \mu \cdot \eta \cdot (2N-1) \cdot g(m^*_{\text{aut}}) \\
&= 2g(m^*_{\text{aut}}) + C \cdot g(m^*_{\text{aut}}) \\
&= g(m^*_{\text{aut}}) \cdot (2 + C) \qquad (51)
\end{aligned}
$$

Thus, given the learning capacity C, the knowledge growth rate is proportional to individual K-productivity $g(m^*_{\text{aut}})$. In the case of Figure 1, for example, the sink point m^*_{aut} is far to the left of the bliss point; thus, $g(m^*_{\text{aut}})$ is much lower than $\tau \cdot g(m^B)$. This suggests that if division of the population into two regions results in greater heterogeneity of knowledge composition in each region, then the knowledge growth rate of the economy will increase; we shall discuss this in detail in the next subsection.

4.2 Differentiation Between Two Regions

When the builders of the tower are scattered and their languages confounded, one region is split into two regions, A and B, each with the same population N. Now there is friction when working with someone from the other region, namely $\tau < 1$, and public knowledge transmission between the two regions becomes more difficult than it was with only one region. Public knowledge from the other region is discounted by $\widetilde{\gamma} < 1$. We claim that after expulsion from paradise, and confounding of languages, a "New Eden" can be achieved. *In other words, introduction of real costs of communication between two regions actually can result in a welfare improvement.*

Figure 2 depicts the state of the New Eden where $m^d_{ij} = m^S$ for every active intra-regional pair i and j whereas $m^d_{ij} = m^B$ for every active inter-regional pair i and j, implying that the K-productivity of each K-worker always equals $g(m^S) = \tau g(m^B)$. Intuitively, thus, the split of one region into two produces a higher knowledge output. However, Figure 2 does not really account for public knowledge transmission (and the resulting increase in individual stocks of knowledge), so the calculations proving this are a bit more intricate. To be precise, we introduce:

Definition: A stationary state in the two-region system is called *welfare improving* when the associated knowledge growth rate for each individual is higher than that at the initial one-region state.

Based on this definition, we have the following result.

Proposition 2: *There is a nonempty, open set of exogenous parameters for which there exists a* ***welfare improving*** *myopic core stationary state (The New Eden) of the following form:*

1. *Each individual engages in both inter-regional and intra-regional pairwise interaction. The proportion of time each individual spends interacting with members of their own region is denoted by φ^* ($0 < \varphi^* < 1$). Thus, the proportion of time each individual spends interacting with members of the other region is $1 - \varphi^*$.*

2. *Inter-regional interaction takes place in groups only. All inter-regional working groups have the same composition, namely the same number N^* of members from each region. (Please refer to Figure 3.) Formally, if $\Gamma_i = \{\Gamma_{iA}, \Gamma_{iB}\}$ is the inter-regional working group for $i \in A$, the size of Γ_{iA} is the same as the size of Γ_{iB}, namely N^*.*

3. *Inter-regional interactions take the following form: For $i \in A$ and $j \in B$,*

$$\begin{aligned} j &\in \Gamma_{iB} \Longrightarrow m_{ij}^d = m^B \text{ and } \delta_{ij} = \frac{1-\varphi^*}{N^*} \\ j &\notin \Gamma_{iB} \Longrightarrow m_{ij}^d > m^B \text{ and } \delta_{ij} = 0 \end{aligned}$$

   (Please refer to Figures 2 and 3.) That is, for inter-regional interactions, a person spends an equal amount of time with every person from the other region in their working group, maintaining knowledge differential at the bliss point. A person spends no time working with people from the other region who are not in their working group since the corresponding productivity is lower than at the bliss point.

4. *Intra-regional interactions take the following form: For $i, j \in A$, defining m^S as in equation (41):*

$$\begin{aligned} j &\notin \Gamma_{iA} \Longrightarrow m_{ij}^d = m^S \text{ and } \delta_{ij} = \frac{\varphi^*}{N-N^*} \\ j &\in \Gamma_{iA} \Longrightarrow m_{ij}^d < m^S \text{ and } \delta_{ij} = 0 \end{aligned}$$

(Please refer to Figure 2.) A person spends an equal amount of time with every person from their own region not in their inter-regional working group, maintaining knowledge differential at the point where pairwise productivity is the same as the bliss point for inter-regional production. That is, for intra-regional interactions, a person spends no time working with people from their own region who are in their inter-regional working group since the corresponding productivity is lower than at the bliss point.

5. *At the New Eden, the dynamics for n_i are given by:*

$$\frac{\dot{n}_i}{n_i} = g\left(m^S\right) \cdot [2 + C] \text{ for } i \in A \cup B \tag{52}$$

Since $m^*_{aut} < m^S < m^B$, in comparison with (51), it follows that the knowledge growth rate is higher at the New Eden than under autarky.

FIGURES 2 AND 3 GO HERE

In order to have a New Eden, it is obvious from Figure 2 that exogenous parameters must satisfy:

$$m^J < m^*_{aut} \equiv \frac{1}{2 + \frac{C}{2}} < m^S \tag{53}$$

or equivalently:

$$\alpha < g(m^*_{aut}) < \tau g(m^B) \equiv g(m^S) \tag{54}$$

In terms of the original parameters, (54) means

$$\alpha < \beta \cdot \frac{(\frac{C}{2})^\theta}{1 + \frac{C}{2}} < \tau \cdot \beta \cdot \theta^\theta \cdot (1 - \theta)^{1-\theta} \tag{55}$$

Thus, in the rest of this section, we always assume that condition (53), (54), or (55) holds. Condition (55) holds when α is sufficiently small whereas C is sufficiently large.

The proof of Proposition 2 including Lemmas A1-A5 can be found in Appendix 2.

Now that we have confirmed that the New Eden exists, we turn next to characterizing further the New Eden. First we derive the iso-N^* curves for Figure 4. That is, we derive the parameter combinations that generate the same inter-regional working group size N^* at the steady state. For convenience, we give these parameter combinations in terms of m^S and m^B.[19] Referring to Section 8.4.5 in the Technical Appendix, we find that:

$$m^S = m^S(m^B, N^*) = \tag{56}$$
$$\frac{N^* \cdot \left[1 - (2 + \frac{C}{2}) \cdot m^B + (1 - m^B) \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \cdot \left(\frac{C}{2} + \frac{C}{\overleftrightarrow{C}}\right)\right] + m^B + \frac{2(1-m^B)}{\overleftrightarrow{C}}}{N^* \cdot \left[1 - (2 + \frac{C}{2}) \cdot m^B + (1 - m^B) \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \cdot \left(\frac{C}{2} + (2 + \frac{C}{2}) \cdot \frac{C}{\overleftrightarrow{C}}\right)\right] + (2 + \frac{C}{2}) \cdot \left[m^B + \frac{2(1-m^B)}{\overleftrightarrow{C}}\right]}$$

where $\overleftrightarrow{C}$ is given by (22):

$$\overleftrightarrow{C} = \begin{cases} \overline{C} \cdot \frac{N^*}{\overline{N}} & \text{for } N^* < \overline{N} \\ \overline{C} & \text{for } N^* \geq \overline{N} \end{cases}$$

For any fixed $N^* > 0$, equation (56) defines an iso-N^* curve in $m^B \times m^S$ space. Notice from (56) that iso-N^* curves are independent of parameter τ. Figure 4 shows examples of iso-N^* curves under various values of N^*, where relevant parameters are fixed at $C = \overline{C} = 32$, $\widetilde{\gamma} = 0$ and $\overline{N} = 100$.[20] The main characteristics of iso-N^* curves that can be observed from Figure 4 are summarized in Lemma 1, which can readily be derived from equation (56).

FIGURE 4 GOES HERE

Lemma 1. The iso-N^* curves defined by (56) have the following characteristics:

(i) As N^* approaches 0, the iso-N^* curve becomes a horizontal line at height m^*_{aut}:

$$\lim_{N^* \to 0} m^S(m^B, N^*) = \frac{1}{2 + \frac{C}{2}} = m^*_{aut}$$

(ii) As N^* approaches ∞, the iso-N^* curve becomes

$$\begin{aligned} \widehat{m}^S(m^B) &\equiv \lim_{N^* \to \infty} m^S(m^B, N^*) \\ &= \frac{1 - (2 + \frac{C}{2}) \cdot m^B + (1 - m^B) \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \cdot \left(\frac{C}{2} + \frac{C}{\overline{C}}\right)}{1 - (2 + \frac{C}{2}) \cdot m^B + (1 - m^B) \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \cdot \left(\frac{C}{2} + \frac{C}{\overline{C}} \cdot (2 + \frac{C}{2})\right)} \end{aligned} \tag{57}$$

called the supreme iso-N^* curve.

[19] The explicit solution of N^* in terms of the original parameters is given by Lemma A3 in Appendix 2.

[20] In Figure 7, the iso-N^* curve when $N^* = \overline{N} = 100$ and the iso-N^* curve when $N^* = \infty$ are indistinguishable, and thus both the curves are represented by the same bold curve. Mathematically speaking, however, we can readily see from (56) that except at $m^B = 0.5$, the iso-N^* curve continuously shifts upward as N^* increases from $\overline{N}$ to ∞.

(iii) All iso-N^* curves pass through the common point, $(m^B, m^S) = (\overline{m}^B, m^*_{aut})$, where

$$\overline{m}^B \equiv \frac{1 + \frac{C}{2} \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}}{2 + \frac{C}{1+\widetilde{\gamma}}} \tag{58}$$

When C is fixed, $\overline{m}^B$ decreases continuously in $\widetilde{\gamma}$ such that at the boundaries:

$$\overline{m}^B = \frac{1}{2} \text{ when } \widetilde{\gamma} = 0, \tag{59}$$

$$\overline{m}^B = \frac{1}{2 + \frac{C}{2}} = m^*_{aut} \text{ when } \widetilde{\gamma} = 1 \tag{60}$$

(iv) Each iso-N^* curve is downward sloping and strictly concave on $(0, 1/2)$.

(v) Except at $\overline{m}^B$, the iso-N^* curve shifts continuously upward as N^* increases from 0 to ∞.

By definition, no point (m^B, m^S) above the supreme iso-N^* curve is attainable as a stationary state myopic core point. Furthermore, no point (m^B, m^S) above the diagonal $m^S = m^B$ line or below the horizontal $m^S = m^*_{aut}$ line is attainable as a stationary state myopic core point. Thus, the domain of $m^B \times m^S$ space that is attainable as a New Eden is limited to the interior of the triangle delineated by the supreme N^*-curve and the two lines $m^S = m^B$ and $m^S = m^*_{aut}$. In order to investigate how this feasible domain changes with parameters, first let us focus on the effects of parameter $\widetilde{\gamma}$ (the measure of ease of public knowledge transmission between regions) on the supreme N^*-curve. Setting $C = \overline{C} = 32$, Figure 5 shows how the supreme iso-N^* curve changes as parameter $\widetilde{\gamma}$ increases from 0 to 1. Using (57), we can readily generalize the key characteristics of supreme iso-N^* curves as follows:

FIGURE 5 GOES HERE

Lemma 2. When $\widetilde{\gamma}$ changes parametrically while C and $\overline{C}$ are fixed, supreme iso-N^* curves defined by (57) have the following characteristics:

(i) All supreme iso-N^* curves pass through the common point, $(m^B, m^S) = (m^*_{aut}, \overline{m}^S)$, where

$$\overline{m}^S \equiv \frac{\frac{C}{2} + \frac{C}{\overline{C}}}{\frac{C}{2} + \frac{C}{\overline{C}} \cdot (2 + \frac{C}{2})} \tag{61}$$

When C is fixed, $\overline{m}^S$ increases continuously in $\overline{C} \leq C$ such that

$$\overline{m}^S = \frac{1}{2 + \frac{C}{2}} = m^*_{aut} \text{ when } \overline{C} = 0$$

$$\overline{m}^S = \frac{1}{2} \text{ when } \overline{C} = C.$$

(ii) For any $0 \leq \widetilde{\gamma} < 1$, the supreme iso-$N^*$ curve is downward sloping and strictly concave.

(iii) Except at $\overline{m}^S$, the supreme iso-N^* curve shifts continuously leftward as $\widetilde{\gamma}$ increases from 0 towards 1; in the limit, it becomes the vertical line: $m^B = m^*_{aut} = 1/\left[2 + \frac{C}{2}\right]$.

As we can see from Figure 5, the domain that is attainable as a New Eden is nonempty as long as $\widetilde{\gamma} < 1$ and $\overline{m}^S > m^*_{aut}$. Since

$$\overline{m}^S - m^*_{aut} = \frac{\frac{C}{2} \cdot (1 + \frac{C}{2})}{\frac{C}{2} + \frac{C}{\overline{C}} \cdot (2 + \frac{C}{2})}$$

the condition $\overline{m}^S > m^*_{aut}$ always holds provided that $\overline{C} > 0$. Hence, we can conclude as follows:

Lemma 3. In $m^B \times m^S$ space, the domain that is attainable as a stationary state myopic core point is not empty if and only if $\widetilde{\gamma} < 1$ and $\overline{C} > 0$.

However, when $\widetilde{\gamma}$ approaches 1, or when $\overline{C}$ approaches 0, the domain for a New Eden disappears in the limit. In other words, when there is no discount in the inter-regional transfer of public knowledge (i.e., $\widetilde{\gamma} = 1$), or when there is no within group externality for the inter-regional interactions in knowledge creation (i.e., $\overline{C} = 0$), it is impossible to attain a New Eden.

Given that we have characterized N^*, we can now characterize φ^* in terms of N^*. As shown in Appendix 2:

$$\varphi^* = 1 - \frac{2}{\overline{C} \cdot \frac{N^*}{\overline{N}}} \cdot \frac{(2 + \frac{C}{2}) \cdot m^S - 1}{1 - m^S} \text{ for } N^* < \overline{N}, \tag{62}$$

$$\varphi^* = 1 - \frac{2}{\overline{C}} \cdot \frac{(2 + \frac{C}{2}) \cdot m^S - 1}{1 - m^S} \text{ for } N^* \geq \overline{N} \tag{63}$$

It can be readily confirmed that $0 < \varphi^* < 1$. One interesting question concerns how τ affects the value of φ^*. We shall return to this question shortly, after Lemma 4.

Although Lemma 3 gives the domain for a possible New Eden in $m^B \times m^S$ space, the feasible combinations of m^B and m^S are actually uniquely defined by (41). In order to see the relationship between m^B and m^S in terms of the original parameters, let us rewrite (41) explicitly:

$$\frac{(1 - 2m^S)^\theta \cdot (m^S)^{1-\theta}}{1 - m^S} = \tau \cdot \frac{(1 - 2m^B)^\theta \cdot (m^B)^{1-\theta}}{1 - m^B}, \ m^S < m^B \tag{64}$$

In turn, the bliss point m^B is uniquely defined by θ from (40); or, solving (40) for θ,

$$\theta = \frac{1 - 2m^B}{1 - m^B} \text{ for } 0 < m^B < \frac{1}{2} \tag{65}$$

Hence, substituting (65) for θ in (64), for any fixed value of $\tau \in (0,1)$, equations (64) and (65) together define the feasible relationship between m^B and m^S as a unique curve in $m^B \times m^S$ space, which is called an iso-τ curve.

Figure 6 shows numerical examples of iso-τ curves. Intuitively, low τ means that the discount in productivity for working with a person in the other region, compared to working with a person in the home region, is large. For example, travel costs are high. A value of τ close to 1 means that there isn't much difference in productivity or cost for working with a person in the other region compared to the home region. This graph shows that as τ moves from 0 to 1, for a given person in region A, the knowledge differential between that person and potential partners in the home and the other region become close. Of course, when $\tau = 1$, there is no difference between partners in the home and away regions, so $m^S = m^B$ and this is represented by the upward sloping $45°$ line.

FIGURE 6 GOES HERE

Using (64) and (65), we can readily generalize the main characteristics of iso-τ curves as follows:

Lemma 4. When $\tau \in (0,1)$ *changes parametrically, iso-*τ *curves defined by (64) and (65) have the following characteristics:*

(i) All iso-τ *curves pass through the origin.*

(ii) Each iso-τ *curve is strictly convex and tangent to the vertical line at* $m^B = \frac{1}{2}$*; in the limit*

$$m^S = \frac{\tau}{1+\tau} \text{ at } m^B = \frac{1}{2} \tag{66}$$

(iii) Except at the origin, the iso-τ *curve shifts continuously upward as* τ *increases from* 0 *towards* 1*; in the limit, it coincides with the upward sloping diagonal line.*

Concerning the previous question about the impact of the value of τ on φ^*, along the supreme iso-N^* curve (for example), we can see that as m^B increases, m^S decreases and thus, applying equation (63), φ^* must increase. In combination with Figures 4 and 6, τ must decrease along this curve. In short, as inter-regional interaction becomes more costly, the equilibrium path features less inter-regional interaction.

With this preparation, we can describe Figure 7 next. To be concrete, let us choose a set of parameters as follows:

$$C = \overline{C} = 32,\ \widetilde{\gamma} = 0,\ \overline{N} = 100 \tag{67}$$

Then, using (57), we can draw the supreme iso-N^* curve ($N^* = \infty$) as in Figure 7. Also, using (50), the m^*_{aut}-line can be drawn as in Figure 7. Thus, we describe the *domain of possible stationary state myopic core points* as a large triangle delineated by the supreme iso-N^* curve, the m^*_{aut}-line and the upward sloping diagonal line. Choose any point inside this triangle, for example, point a which is at the intersection of the iso-τ curve with $\tau = 0.9$ and the vertical line at $m^B = 0.3$. From (65), $m^B = 0.3$ means $\theta = 4/7$; thus point a in Figure 7 corresponds to the parameters

$$\tau = 0.9 \text{ and } \theta = 4/7 \tag{68}$$

In turn, the equilibrium group size N^* is determined by the iso-$N^* = 30$ curve passing through point a. In this way, the set of parameters, (67) and (68), uniquely determines the New Eden as a stationary state myopic core point. At point a, since $m^S = 0.186 > m^*_{aut} = 0.056$, we have from (52) that

$$\frac{\text{Knowledge growth rate at the New Eden}}{\text{Knowledge growth rate under autarchy}} = \frac{g(m^S)}{g(m^*_{aut})} = \frac{.45799}{.28778} = 1.5915$$

implying a large improvement in welfare.

FIGURE 7 GOES HERE

As another example of a New Eden, while holding fixed the parameters in (67), let us change (68) as follows:

$$\tau = 0.606,\ \theta = 0.245$$

This is illustrated in Figure 8. Then, since $m^B = 0.43$ from (40), we have that

$$\frac{\text{Knowledge growth rate at the New Eden}}{\text{Knowledge growth rate under autarchy}} = \frac{g(m^S)}{g(m^*_{aut})} = \frac{0.347}{0.117} = 2.97$$

Thus, by breaking the one region into two, the new myopic core steady state achieves a knowledge growth rate almost 3 times higher than the one region economy.

FIGURE 8 GOES HERE

So in practical terms, when is the New Eden, the myopic core steady state generated by splitting one region into two, a big improvement over the one region case? To answer this question, notice that from equation (51) for the one region situation and from equation (52) for the two region situation, the potential improvement in K-productivity is completely determined by the size of $g(m^*_{aut})$ for the one region case relative to the size of $g(m^S) = \tau \cdot g(m^B)$ in the two region case. This comparison can be seen in terms of exogenous parameters in equation (55) by taking the ratio of the two sides of the inequality. We focus on the most favorable cases for the New Eden, namely from Lemma 2: $\widetilde{\gamma} = 0$ and $\overline{C} = C$. Referring to Figure 7, for each point (m^B, m^S) in Figure 7, we examine the ratio of the knowledge productivity in the New Eden compared to autarky. The most favorable case is the upper envelope of the domain of possible stationary state myopic core points. As is apparent in Figure 7, the upper envelope consists of two parts: the upward sloping diagonal $m^S = m^B$ up to the intersection with the supreme iso-N^* curve, and the supreme iso-N^* curve to the right of the intersection point.[21] There are two cases to consider. First, for the upward sloping diagonal in Figure 7, since $m^S = m^B$, $\tau = 1$. Hence, the ratio of interest along this segment is:

$$E(\theta; C) \equiv \frac{g(m^B)}{g(m^*_{aut})} = \frac{\theta^\theta \cdot (1-\theta)^{1-\theta}}{\frac{(\frac{C}{2})^\theta}{1+\frac{C}{2}}}$$

For the second case along the supreme iso-N^* curve, in equation (57), setting $\widetilde{\gamma} = 0$, $\overline{C} = C$, and $m^B = \frac{1-\theta}{2-\theta}$,

$$E(\theta; C) \equiv \frac{g(\widehat{m}^S(m^B))}{g(m^*_{aut})} = \frac{\left\{\left(1+\frac{C}{2}\right) \cdot (1-\theta) - 1\right\}^\theta \cdot \left\{\left(1+\frac{C}{2}\right) \cdot \theta + 1\right\}^{1-\theta}}{\left(\frac{C}{2}\right)^\theta}$$

Figure 9 illustrates $E(\theta; C)$ as a function of θ where C takes on values 8, 16, and 32.[22]

FIGURE 9 GOES HERE

For each fixed C, the function E is single peaked, and the maximum is attained along the supreme iso-N^* curve rather than along the upward sloping diagonal. For $C = 8$, the maximum is attained at $\theta = .252$ with $E = 1.67$; for $C = 16$, the maximum is attained at .262 with $E = 2.23$; for $C = 32$, the maximum is attained at $\theta = .256$ with $E = 3.21$.[23] Notice that the optimal value of θ seems to be stable at around .25 when the values of C vary.

[21] The supreme iso-N^* curve is not really feasible since it requires an infinite population, but any point below this curve and as close as desired is attainable.

[22] Along the supreme iso-N^* segment, we have neglected τ. Given m^B and thus θ, the parameter τ is completely determined by m^S, as illustrated in Figure 7, but its calculation is difficult.

[23] The corresponding values of τ are: for $C = 8$, $\tau = .834$; for $C = 16$, $\tau = .76$; for $C = 32$, $\tau = .679$.

Summarizing this analysis, we can conclude that given an initial situation where there is a high degree of homogeneity in workers, division into two regions will result in a big improvement in knowledge productivity when: 1) Inter-regional public knowledge transmission is weak ($\widetilde{\gamma}$ is small, since this promotes inter-regional knowledge differentiation); 2) Public knowledge transmission within each inter-regional working group is effective, so workers can differentiate themselves from others in the same region rapidly ($\overline{C}$ is large); 3) Heterogeneity (as opposed to homogeneity) of workers' knowledge bases is important in the production function for partnerships, so diversity increases productivity (θ is small);[24] 4) The within-region public information transmission technology is very effective (C is large) so that autarky is unproductive.

4.3 The Transition Process

Up to this point, in this section we have studied the properties of two myopic core steady states: first with one region, and then with two regions. In this short subsection, we shall discuss the transition, according to the story of the Tower of Babel, from a lower knowledge productivity steady state with one region to a higher knowledge productivity steady state with two regions. There are two transition phases between the steady states.

First, the one region autarkic economy is split into two regions. This is illustrated in Figure 10.

FIGURE 10 GOES HERE

Immediately after the division, for a given person in region A, the relative knowledge differentiation of potential partners in A and potential partners in B is essentially the same. However, the cost of working with a partner in region B is higher, since $\tau < 1$. Thus, people in each region work only with partners in their own region. However, people in the two regions become differentiated from each other over time. Given that N is large, and that people are working only with partners in the same region, they will work with all others in the region for a small amount of time, the same for every

[24] When θ is too small (to the left of the peak of E), knowledge workers try to avoid building up knowledge in common with any of their partners. If the intra-regional public knowledge transmission technology and the inter-regional working group public knowledge transmission technology are at all effective, it is hard to avoid building up knowledge in common with active partners. Thus, productivity will be lower than for values of θ closer to the peak.

partnership. They maintain the same knowledge differentiation with their active partners, namely they stay at m^*_{aut}. In this phase, the two regions are endogenously developing different areas of expertise due to the barriers to contact between regions.

The regions continue in an autarkic mode until the regions are sufficiently differentiated, featuring the same productivity for potential partners in their own region and in the other region. Then the second transition phase begins. This is illustrated in Figure 11.

FIGURE 11 GOES HERE

At this time, a person in region A begins to participate in an inter-regional working group, as described in the previous subsection for the final steady state, and with people in their own region who are not in their inter-regional working group, also as described in the previous subsection for the final steady state. However, the size of the inter-regional working group, N^*, and the total time spent working with partners in the home region, φ^*, will not be the same as at the steady state. The reason is that people do not want to maintain the bliss point, since they haven't reached it yet, but rather wish to move to the right, increasing both differentiation relative to active partners as well as productivity as fast as possible. In order to avoid building up knowledge in common with workers from the other region in their inter-regional working group (and thus slowing the rate of increase of productivity), they want to make their inter-regional working group as large as possible subject to feasibility, namely $N^* = \frac{N}{2}$. Finally, when they reach the bliss point for their partners from the other region in their inter-regional working group, they shift to N^* and φ^* that will maintain the bliss point.

Other transition processes are possible, but we stick to a description of a simple one.

5 Conclusions

We have endeavored to clarify a second role of spatial distance in the economy beyond the first and obvious role of creating a barrier to the exchange of commodities between locations. This second role is the propagation of the differentiation of agents themselves, in the sense that they form separate cultures. It can result in an increase in knowledge productivity in the entire economy relative to the situation when there is no spatial distance between agents. The key to this increase is in the ability of inter-regional working groups to form and to further differentiate agents residing in the *same* region due to knowledge spillovers within the inter-regional working group.

An interesting alternative interpretation of our model is: Rather than using A and B to denote regions, instead they denote different ethnic groups in the same region. Then, although communication between ethnicities is more difficult than within the same ethnicity, knowledge productivity is higher due to diversity. This interpretation ties nicely into Ottaviano and Peri (2006, 2008).

Our analysis has implications for the impact of the recent rapid development in information technology on the rate of global knowledge productivity. Faster knowledge transmission due to improved information technology evidently makes the dissemination of new ideas more rapid, but it also tends to create more homogeneity in the knowledge bases of researchers. Differentiation of researchers through the formation of inter-regional working groups can help to turn this disadvantage to an advantage. Generally speaking, location and knowledge creation are intertwined; for example, see Duranton and Puga (2001) and Helsley and Strange (2004).

A natural but difficult extension of our model would introduce migration of researchers between regions, providing another way to circulate knowledge. Regarding migration, the role of immigration policy and of the educational systems in various countries would be a topic worthy of further exploration.

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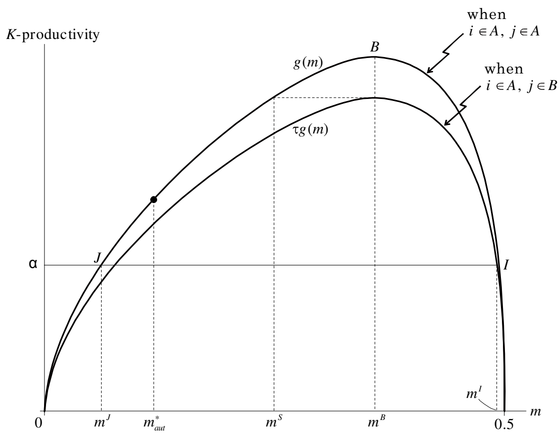

Figure 1: The intra-regional K-productivity curve $g(m)$ and the inter-regional K-productivity curve $\tau \cdot g(m)$ with the same bliss point m^B.

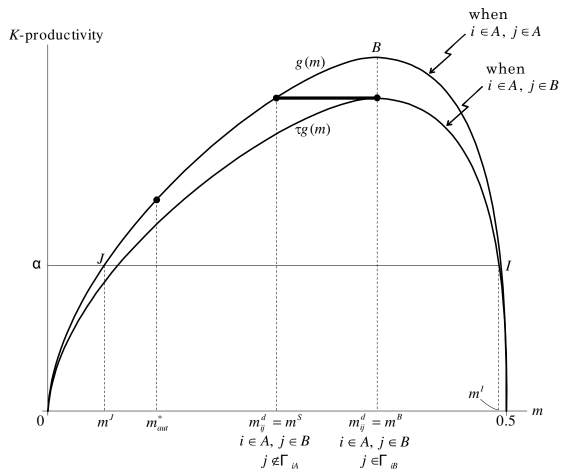

Figure 2: The New Eden: Achieving high K-productivity though diverse cultures.

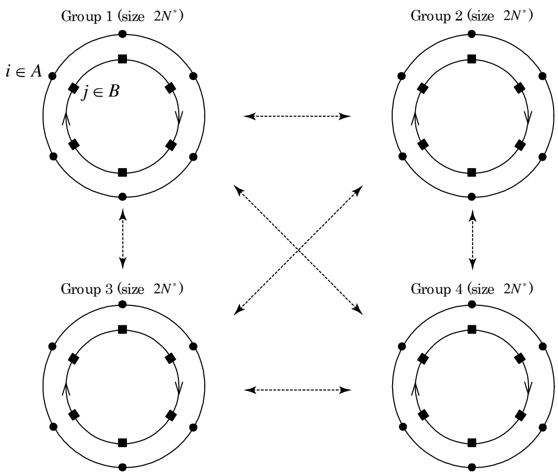

Figure 3: Inter-regional interactions at the New Eden: Tables at a Chinese restaurant.

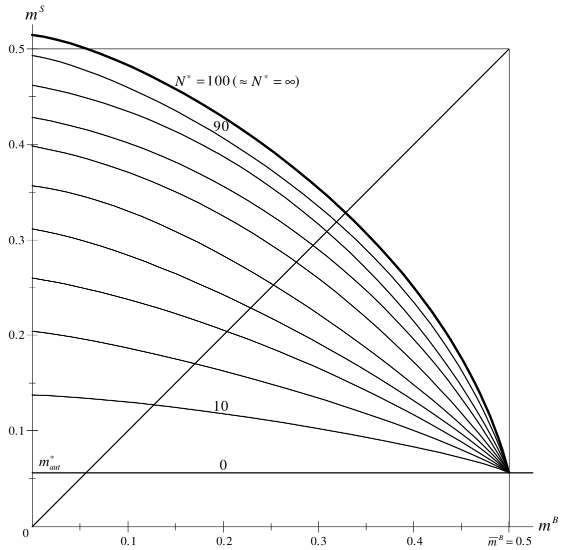

Figure 4: Iso-N^* curves ($C = \overline{C} = 32$, $\widetilde{\gamma} = 0$, $\overline{N} = 100$, $m^*_{aut} = 0.056$, $\overline{m}^B = 0.50$): curves are $N^* = 0$ (horizontal), 10, 20, 30, 40, 50, 60, 70, 80, 90, 100 (top)

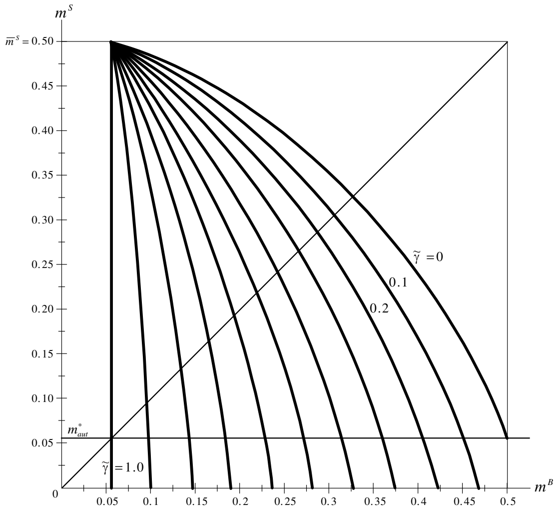

Figure 5: Change in the supreme iso-N^* curve as $\widetilde{\gamma}$ increases from 0 to 1 ($C = \overline{C} = 32$, $m^*_{aut} = 0.056$, $\overline{m}^S = 0.50$): curves are $\widetilde{\gamma} = 0$ (top), .1, .2, .3, .4, .5, .6, .7, .8, .9, 1.0 (vertical)

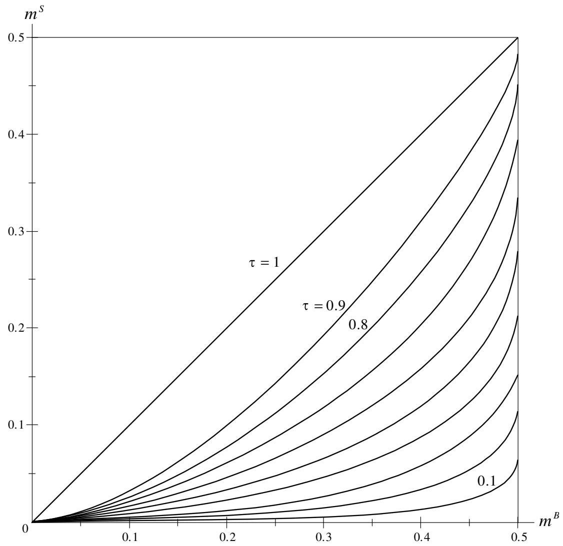

Figure 6: Iso-τ curves: $\tau = .1$ (bottom), .2, .3, .4, .5, .6, .7, .8, .9, 1.0 (top)

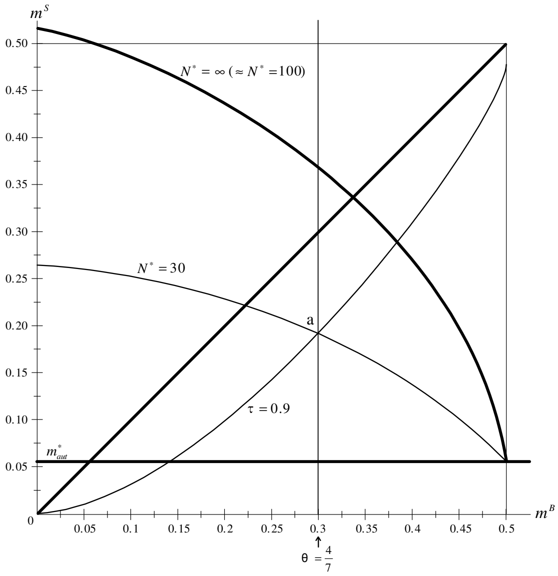

Figure 7: An interior point a inside the domain of feasible New Edens ($C = \overline{C} = 32$, $\widetilde{\gamma} = 0$, $\overline{N} = 100$, $m^*_{aut} = 1/\left(2 + \frac{C}{2}\right) = 0.056$)

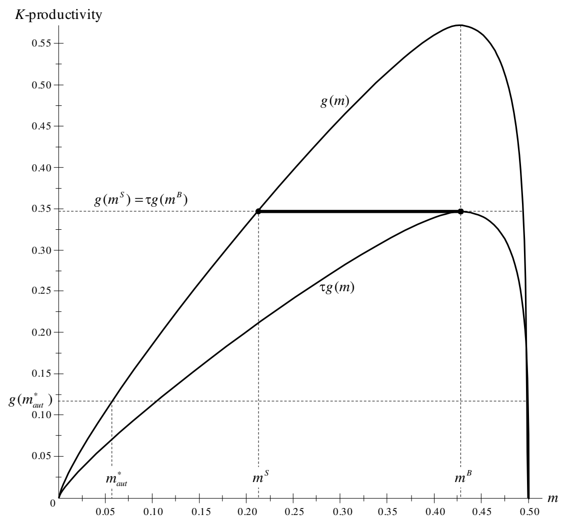

Figure 8: Achieving higher knowledge productivity through the creation of culture ($\theta = .245$, $\tau = 0.606$, $C = \overline{C} = 32$, $\widetilde{\gamma} = 0$, $\overline{N} = N^* = 100$, $m^*_{aut} = 0.056$, $m^S = 0.214$, $m^B = 0.43$, $g(m^*_{aut}) = 0.117$, $g(m^S) = \tau g(m^B) = 0.347$)

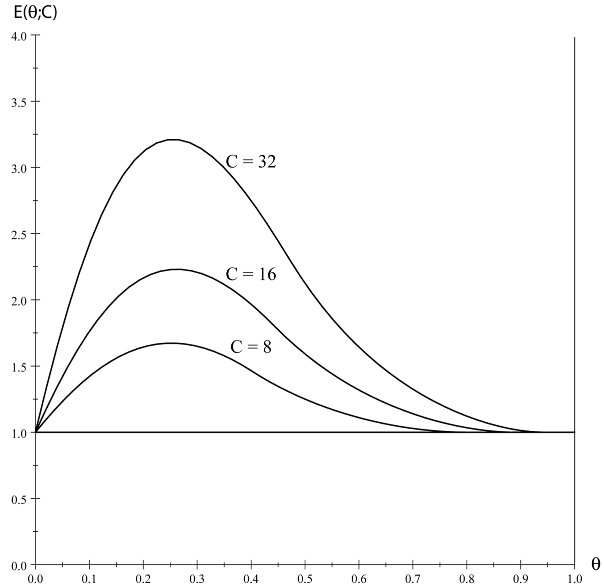

Figure 9:
$E(\theta; C) = \frac{\text{Knowledge growth rate at the New Eden}}{\text{Knowledge growth rate under autarchy}}$
as a function of θ for $C = 32$ (top), $C = 16$, $C = 8$ (bottom)

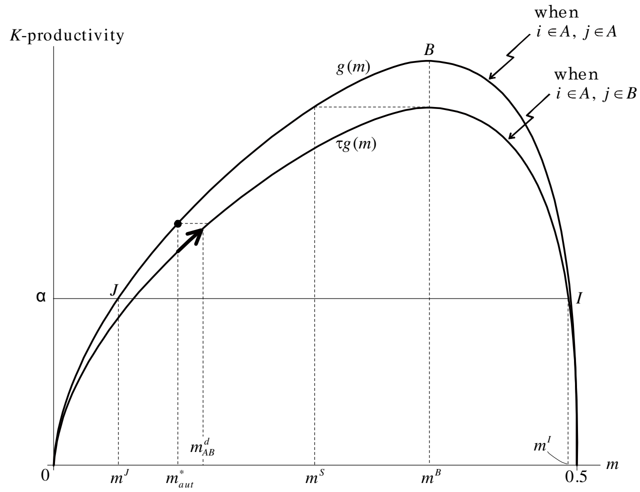

Figure 10: Transition process, Phase 1 (no inter-regional interaction): $g(m^d_{AB}) < g(m^*_{aut})$

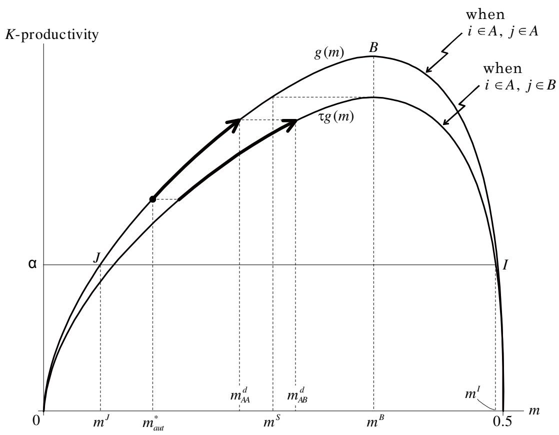

Figure 11: Transition process, Phase 2 (with inter-regional interactions): $g(m^d_{AA}) = g(m^d_{AB}) > g(m^*_{aut})$

Culture and Diversity in Knowledge Creation

Marcus Berliant and Masahisa Fujita

Appendix

Not for Publication

6 Appendix 1

6.1 Justification of the Knowledge Absorption Function

It is natural to assume that public knowledge transmission between regions is not as effective as public knowledge transmission within a region. To make this notion precise, we must differentiate between public knowledge produced by partnerships consisting of two K-workers in the same region, two from the other region, one from each region, and one from each region that are members of the same inter-regional working group. The one region model will be a special case where parameters are set so that there are no frictions between the two regions. Consider the following equalities: for $i \in A$

$$C \cdot \left(\delta_{ii} \cdot a_{ii} + \sum_{j \in A_{-i}} \delta_{ij} \cdot (a_{ij}/2) \right) = \mu \cdot \eta \cdot \left([I_{AA} - \sum_{j \in A} \delta_{ij} \cdot a_{ij}] + \widetilde{\gamma} \cdot I_{BB} \right) \tag{69}$$

$$\text{where } 0 \leq \widetilde{\gamma} < 1$$

and for $i \in A$

$$\widehat{C} \cdot \left(\sum_{j \in B} \delta_{ij} \cdot (a_{ij}/2) \right) = \widehat{\mu} \cdot \eta \cdot \left(I_{AB} - \sum_{j \in B} \delta_{ij} \cdot a_{ij} \right) \tag{70}$$

$$\overleftrightarrow{C} \cdot \left(\sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot (a_{ij}/2) \right) = \overleftrightarrow{\mu} \cdot \eta \cdot \left(I_{\Gamma_i} - \sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot a_{ij} \right) \tag{71}$$

where

$$\widehat{C} + \overleftrightarrow{C} = C \tag{72}$$

$$\overleftrightarrow{C} = \begin{cases} \overline{C} \cdot \frac{N^*}{\overline{N}} & \text{for } N^* < \overline{N} \\ \overline{C} & \text{for } N^* \geq \overline{N} \end{cases} \tag{73}$$

First we give the idea behind the overall structure of this system, and then we proceed to discuss in detail each component equation. For the purpose of explanation, consider the case where the day is divided into two subperiods. The length of these time periods is determined endogenously; this will eventually be represented by φ^* for intra-regional interactions and $1 - \varphi^*$ for inter-regional interactions, under the symmetric interactions case. The first subperiod features only intra-regional interaction, whereas the second has only inter-regional interaction. The inter-regional interaction time is further divided into time spent with persons generally from the other region, and time spent specifically with one's inter-regional interaction group. Associated with each subperiod are externalities, namely knowledge absorbed from the partners one is working with in general, at the time knowledge creation takes place.

To be specific, when a person is working with a partner in their own region, they naturally absorb a proportion of the total knowledge created in their home region at that time. At the same time, they also absorb a (lesser) proportion of the ideas created in the other region.

When a person is working with a partner from another region, they absorb a different proportion of the all the new ideas created by inter-regional interactions at that time. This proportion is potentially different from the absorption rate from the intra-regional externality absorption rate. At the same time, when a person is working with others in their inter-regional working groups, they absorb a proportion of the new ideas created within that working group at that time.

For all but the inter-regional working group externality, the knowledge absorbed through the externality becomes knowledge in common for the workers in the same region. In contrast, the inter-regional working group externality is entirely different. Ideas learned through the inter-regional working group externality become knowledge in common for *only* the members of the specific working group. That is, these ideas are *not* learned by persons in a region who are not members of the same working group, and thus they become part of the differential knowledge between members of that inter-regional working group and everyone else.

We shall explain the content of these equations piece by piece. On the right hand side of equation (69), the term in brackets $I_{AA} - \sum_{j=1}^{N} \delta_{ij} \cdot a_{ij}$ represents the new knowledge produced in region A in the first sub-period that does not involve partnerships including K-worker i. Since I_{BB} represents new knowledge created in the first sub-period by partnerships involving only workers in B, we discount it by $\widetilde{\gamma}$ due to friction ("lost in translation"). Recall that η gives the rate at which new ideas are patented, whereas μ gives the rate at which publicly revealed ideas can be absorbed by a K-worker. Therefore the right hand side of the equation represents the public knowledge revealed by patents that is absorbed by K-worker i in the first sub-period. The term in brackets on the left hand side represents new knowledge created by K-worker i at an instant in the first sub-period. In total, the equation means that the new public knowledge that can be absorbed by K-worker i is proportional to their capacity to produce new ideas. In essence, this is due to the constraint on their time and the productivity of their effort both to absorb new ideas and to produce them.

Equation (70) represents the analogous equation for inter-regional partnerships in the second sub-period. The interpretation of this equation is analogous to the previous one. But there is an additional implicit assumption when we write these two equations, (69) and (70), separately. That is, we assume that public knowledge produced by inter-regional partnerships is complementary to (in contrast to substitutes with) knowledge produced by intra-regional partnerships.

Equation (71) represents the analogous equation for partnerships from person i's inter-regional working group in the second sub-period. The right hand term in brackets $I_{\Gamma_i} - \sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot a_{ij}$ represents the total knowledge produced by partnerships in person i's inter-regional working group that do not involve partnerships including K-worker i. Recall that η gives the rate at which new ideas are patented, whereas $\overleftrightarrow{\mu}$ gives the rate at which publicly revealed ideas *within the inter-regional working group* can be absorbed by a K-worker. The term in brackets on the left hand side represents new knowledge created by K-worker i in inter-regional working group partnerships at an instant. The equation says that the rate of public knowledge absorption from person i's inter-regional working group is proportional to their capacity to produce new ideas with working group partners from the other region.[25]

For equation (72), attention is divided into the two sources of inter-regional externalities, namely the attention to the externality $\widehat{C}$ from general inter-regional partners and the attention to the externality $\overleftrightarrow{C}$ from partners in one's inter-regional working group. The total attention $\widehat{C} + \overleftrightarrow{C}$ that each person devotes to inter-regional externalities is equal to the total learning capacity C, which is the same as the constant in equation (69). As explained in equation (73), $\overleftrightarrow{C}$ and hence $\widehat{C}$ are endogenous variables determined by working group size N^*.

Equation (73) says that the larger the working group, the more attention is paid to the externality from the working group of size N^*. Consequently, less attention is paid to the general externality from inter-regional partnerships. Beyond group size $\overline{N}$, the effect of group size on attention is attenuated.

To provide more intuition and useful expressions for the analysis in the

[25] It is possible to have intra-regional working groups that are the intra-regional analogs of equation (71). However, this only serves to make people in the same region more similar, and this does not improve welfare. In other words, such groups would not be used in the myopic core.

main text of the paper, we consider a special case with symmetric knowledge composition: $n_i = n$ for all i and $m^d_{ij} = m^d$ for $i \neq j$ in the same region, whereas $m^d_{ij} = m^d_{AB}$ for i and j in different regions.[26] $\sum_{k\in A_{-i}} \delta_{ij} = \varphi$ for all $i \in A$, $\sum_{k\in B_{-i}} \delta_{ij} = \varphi$ for all $i \in B$ and $g(m^d) = \tau \cdot g(m^d_{AB}) > \alpha$. In this case, $\delta_{ii} = 0$ for all i, and $a_{ij} = n \cdot 2 \cdot g(m^d)$ for all $i \neq j$. Hence, in (69), using (23) to (25),

$$\delta_{ii} \cdot a_{ii} + \sum_{j\in A_{-i}} \delta_{ij} \cdot (a_{ij}/2) = \varphi \cdot n \cdot g(m^d)$$

$$I_{AA} - \sum_{j\in A} \delta_{ij} \cdot a_{ij} + \widetilde{\gamma} \cdot I_{BB} - \varphi \cdot 2 \cdot g(m^d) + \widetilde{\gamma} \cdot N \cdot n \cdot g(m^d)$$
$$= N \cdot \varphi \cdot n \cdot g(m^d) - \varphi \cdot n \cdot 2g(m^d) + \widetilde{\gamma} \cdot N \cdot \varphi \cdot n \cdot g(m^d)$$

Thus,

$$\begin{aligned}
\mu &= \frac{C}{\eta} \cdot \frac{\varphi \cdot n \cdot g(m^d)}{\varphi \cdot (N-2) \cdot n \cdot g(m^d) + \varphi \cdot \widetilde{\gamma} \cdot N \cdot n \cdot g(m^d)} \\
&= \frac{C}{\eta} \cdot \frac{1}{(N-2) + \widetilde{\gamma} \cdot N} \\
&= \frac{C}{\eta} \cdot \frac{1}{(N-1) \cdot (1+\widetilde{\gamma}) - (1-\widetilde{\gamma})} \\
&\approx \frac{C}{\eta(N-1)} \cdot \frac{1}{1+\widetilde{\gamma}}
\end{aligned}$$

The reason $N-1$ appears in the denominator here is because the externality excludes ideas produced by oneself, in particular for the externality in one's home region. However, for the externality from the other region, there is no need to exclude ideas produced by oneself. Thus, N appears instead of $N-1$ in the denominator.

For notational convenience, we define:

$$\widetilde{\mu} \equiv \widetilde{\gamma}\mu = \frac{C}{\eta N} \cdot \frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}$$

that represents the absorption rate for the knowledge externality from region B.

In the context of the same example, in equation (70) we have:

$$\begin{aligned}
\sum_{j\in B} \delta_{ij} \cdot (a_{ij}/2) &= (1-\varphi) \cdot \tau \cdot n \cdot 2g(m^d)/2 \\
&= (1-\varphi) \cdot n \cdot g(m^d)
\end{aligned}$$

[26] A slight generalization of this example applies on the myopic core path, as explained in Section 4.

$$
\begin{aligned}
I_{AB} - \sum_{j \in B} \delta_{ij} \cdot a_{ij} &= N \cdot (1-\varphi) \cdot \tau \cdot n \cdot 2g(m^d_{AB}) - (1-\varphi) \cdot \tau \cdot n \cdot 2g(m^d_{AB}) \\
&= 2(N-1) \cdot (1-\varphi) \cdot n \cdot 2g(m^d)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\widehat{\mu} &= \frac{\widehat{C}}{\eta} \cdot \frac{(1-\varphi) \cdot n \cdot g(m^d)}{2(N-1) \cdot (1-\varphi) \cdot n \cdot g(m^d)} \\
&= \frac{\widehat{C}}{\eta} \cdot \frac{1}{2(N-1)}
\end{aligned}
$$

Once again, in the context of the same example, in equation (71),

$$
\begin{aligned}
\sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot (a_{ij}/2) &= (1-\varphi) \cdot \frac{N^*}{N} \cdot \tau \cdot n \cdot 2g(m^d)/2 \\
&= (1-\varphi) \cdot \frac{N^*}{N} \cdot \tau \cdot n \cdot g(m^d)
\end{aligned}
$$

$$
\begin{aligned}
I_{\Gamma_i} - \sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot a_{ij} &= N^* \cdot (1-\varphi) \cdot \frac{N^*}{N} \cdot \tau \cdot n \cdot 2g(m^d_{AB}) - (1-\varphi) \cdot \frac{N^*}{N} \cdot \tau \cdot n \cdot 2g(m^d_{AB}) \\
&= 2(N^*-1) \cdot (1-\varphi) \cdot \tau \cdot \frac{N^*}{N} \cdot n \cdot g(m^d)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\overleftrightarrow{\mu} &= \frac{\overleftrightarrow{C}}{\eta} \cdot \frac{(1-\varphi) \cdot \frac{N^*}{N} \cdot \tau \cdot n \cdot g(m^d)}{2(N^*-1) \cdot (1-\varphi) \cdot \tau \cdot \frac{N^*}{N} \cdot n \cdot g(m^d)} \\
&= \frac{\overleftrightarrow{C}}{\eta} \cdot \frac{1}{2(N^*-1)}
\end{aligned}
$$

In conclusion, assuming N is sufficiently large, we employ the following specifications:

$$
\begin{aligned}
\mu &= \frac{C}{\eta(N-1)} \cdot \frac{1}{1+\widetilde{\gamma}} \\
\widetilde{\mu} &\equiv \widetilde{\gamma}\mu = \frac{C}{\eta N} \cdot \frac{\widetilde{\gamma}}{1+\widetilde{\gamma}} \\
\widehat{\mu} &= \frac{\widehat{C}}{\eta \cdot 2(N-1)} \\
\overleftrightarrow{\mu} &= \frac{\overleftrightarrow{C}}{2(N^*-1)} \\
\text{where } 0 &\leq \widetilde{\gamma} < 1,
\end{aligned}
$$

$$\widehat{C} + \overleftrightarrow{C} = C$$

and

$$\overleftrightarrow{C} = \begin{cases} \overline{C} \cdot \frac{N^*}{\overline{N}} & \text{for } N^* < \overline{N} \\ \overline{C} & \text{for } N^* \geq \overline{N} \end{cases}$$

6.2 Basic Dynamics Without Symmetry

In this appendix, we summarize the dynamics of n_i and m_{ij}^d $(i, j = 1, 2, \cdots, 2N)$.

First, using (26), we have (see Technical Appendix b)

$$\begin{aligned}
\text{For } i &\in A: \\
\frac{\dot{n}_i}{n_i} &= (1 - \mu \cdot \eta) \cdot [\delta_{ii} \cdot \alpha + \sum_{j \in A_{-i}} \delta_{ij} \cdot 2G\left(m_{ij}^d, m_{ji}^d\right)] + (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{j \in B} \delta_{ij} \cdot \tau \cdot 2G\left(m_{ij}^d, m_{ji}^d\right) \\
&\quad + \mu \cdot \eta \cdot [\sum_{k \in A} \delta_{kk} \cdot \frac{n_k}{n_i} \cdot \alpha + (\sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m_{kl}^d, m_{lk}^d\right))] \\
&\quad + \widetilde{\mu} \cdot \eta \cdot [\sum_{k \in B} \delta_{kk} \cdot \frac{n_k}{n_i} \cdot \alpha + (\sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m_{kl}^d, m_{lk}^d\right))] \\
&\quad + \widehat{\mu} \cdot \eta \cdot [\sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m_{kl}^d, m_{lk}^d\right)] \\
&\quad + \overleftrightarrow{\mu} \cdot [\sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m_{kl}^d, m_{lk}^d\right) - \sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot \tau \cdot 2G\left(m_{ij}^d, m_{ji}^d\right)]
\end{aligned} \tag{74}$$

Next, by definition,

$$\begin{aligned}
\dot{m}_{ij}^d &= \frac{d\left(n_{ij}^d / n^{ij}\right)}{dt} \\
&= \frac{\dot{n}_{ij}^d}{n^{ij}} - \frac{n_{ij}^d}{n^{ij}} \cdot \frac{\dot{n}^{ij}}{n^{ij}} \\
&= \frac{\dot{n}_{ij}^d}{n^{ij}} - m_{ij}^d \cdot \frac{\dot{n}^{ij}}{n^{ij}} \\
&= \frac{\dot{n}_{ij}^d}{n^{ij}} - m_{ij}^d \cdot \left(\frac{\dot{n}_{ij}^c}{n^{ij}} + \frac{\dot{n}_{ij}^d}{n^{ij}} + \frac{\dot{n}_{ji}^d}{n^{ij}}\right) \\
&= \left(1 - m_{ij}^d\right) \cdot \frac{\dot{n}_{ij}^d}{n^{ij}} - m_{ij}^d \cdot \left(\frac{\dot{n}_{ij}^c}{n^{ij}} + \frac{\dot{n}_{ji}^d}{n^{ij}}\right)
\end{aligned} \tag{75}$$

Using this identity, for each different combination of i and j, we can obtain the dynamics $\dot{m}_{ij}^d$ as follows (see Technical Appendix c):

$$
\begin{aligned}
\text{For } i &\in A,\ j \in A: \\
\text{for } j &\in \Gamma_{iA},\ \dot{m}_{ij}^d = \left(1-m_{ij}^d\right)\cdot\frac{\dot{n}_{ij}^d}{n^{ij}} - m_{ij}^d\cdot\left(\frac{\dot{n}_{ij}^c}{n^{ij}}+\frac{\dot{n}_{ji}^d}{n^{ij}}\right) \\
&= (1-m_{ij}^d)(1-m_{ji}^d)\left\{(1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot 2G(m_{ik}^d,m_{ki}^d)]\right. \\
&\quad \left.+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot\tau\cdot 2G(m_{ik}^d,m_{ki}^d)\right\} \\
&\quad -m_{ij}^d\cdot(1-m_{ji}^d)\left\{\qquad(1-\mu\cdot\eta)\cdot\delta_{ij}\cdot 2G(m_{ij}^d,m_{ji}^d)\right. \\
&\quad +\mu\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha\cdot\frac{n_k}{n_i}+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^d,m_{lk}^d\right)\right] \\
&\quad +\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha\cdot\frac{n_k}{n_i}+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^d,m_{lk}^d\right)\right] \\
&\quad +\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G\left(m_{kl}^d,m_{lk}^d\right) \\
&\quad \left.+\overleftrightarrow{\mu}\cdot\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G\left(m_{kl}^d,m_{lk}^d\right)\right\} \\
&\quad -m_{ij}^d\cdot\left(1-m_{ij}^d\right)\cdot\left\{(1-\mu\cdot\eta)\cdot[\delta_{jj}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{jk}\cdot 2G(m_{jk}^d,m_{kj}^d)]\right. \\
&\quad \left.+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{jk}\cdot\tau\cdot 2G(m_{jk}^d,m_{kj}^d)\right\} \qquad (76)
\end{aligned}
$$

$$
\begin{aligned}
\text{For } i &\in A,\ j \in A: \\
\text{for } j &\notin \Gamma_{iA},\ \dot{m}_{ij}^{d} = \left(1-m_{ij}^{d}\right)\cdot\frac{\dot{n}_{ij}^{d}}{n^{ij}} - m_{ij}^{d}\cdot\left(\frac{\dot{n}_{ij}^{c}}{n^{ij}}+\frac{\dot{n}_{ji}^{d}}{n^{ij}}\right) \\
&= (1-m_{ij}^{d})(1-m_{ji}^{d})\left\{(1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot 2G(m_{ik}^{d},m_{ki}^{d})]\right. \\
&\quad +(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot\tau\cdot 2G(m_{ik}^{d},m_{ki}^{d}) \\
&\quad \left.+\overleftrightarrow{\mu}\cdot\left(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G(m_{kl}^{d},m_{lk}^{d})-\sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2G(m_{ik}^{d},m_{ki}^{d})\right)\right\} \\
&\quad -m_{ij}^{d}\cdot(1-m_{ji}^{d})\left\{\qquad(1-\mu\cdot\eta)\cdot\delta_{ij}\cdot 2G(m_{ij}^{d},m_{ji}^{d})\right. \\
&\quad +\mu\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha\cdot\frac{n_k}{n_i}+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^{d},m_{lk}^{d}\right)\right] \\
&\quad +\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha\cdot\frac{n_k}{n_i}+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^{d},m_{lk}^{d}\right)\right] \\
&\quad +\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G\left(m_{kl}^{d},m_{lk}^{d}\right) \\
&\quad \left.+\overleftrightarrow{\mu}\cdot\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G\left(m_{kl}^{d},m_{lk}^{d}\right)\right\} \\
&\quad -m_{ij}^{d}\cdot\left(1-m_{ij}^{d}\right)\cdot\left\{(1-\mu\cdot\eta)\cdot[\delta_{jj}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{jk}\cdot 2G(m_{jk}^{d},m_{kj}^{d})]\right. \\
&\quad +(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{jk}\cdot\tau\cdot 2G(m_{jk}^{d},m_{kj}^{d}) \\
&\quad \left.+\overleftrightarrow{\mu}\cdot\left(\sum_{k\in\Gamma_{jA}}\sum_{l\in\Gamma_{jB}}\delta_{kl}\cdot\frac{n_k}{n_j}\cdot\tau\cdot 2G(m_{kl}^{d},m_{lk}^{d})-\sum_{k\in\Gamma_{jB}}\delta_{jk}\cdot\tau\cdot 2G(m_{jk}^{d},m_{kj}^{d})\right)\right\}
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
\text{For } i &\in A,\ j \in B: \\
\text{for } j &\in \Gamma_{iB},\ \dot{m}_{ij}^{d} = \left(1-m_{ij}^{d}\right)\cdot\frac{\dot{n}_{ij}^{d}}{n^{ij}} - m_{ij}^{d}\cdot\left(\frac{\dot{n}_{ij}^{c}}{n^{ij}}+\frac{\dot{n}_{ji}^{d}}{n^{ij}}\right) \\
&= (1-m_{ij}^{d})(1-m_{ji}^{d})\left\{(1-\widetilde{\mu}\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-i}}\delta_{ik}\cdot 2G(m_{ik}^{d},m_{ki}^{d})]\right. \\
&\quad +(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot\tau\cdot 2G(m_{ik}^{d},m_{ki}^{d})+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha\frac{n_k}{n_i}\right. \\
&\quad \left.\left.+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G(m_{kl}^{d},m_{lk}^{d})-\delta_{ii}\cdot\alpha-\sum_{k\in A_{-i}}\delta_{ik}\cdot 2G(m_{ik}^{d},m_{ki}^{d})\right]\right\} \\
&\quad -m_{ij}^{d}\cdot(1-m_{ji}^{d})\left\{\qquad(1-\widehat{\mu}\cdot\eta)\cdot\delta_{ij}\cdot\tau\cdot 2G(m_{ij}^{d},m_{ji}^{d})\right. \\
&\quad +\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha\cdot\frac{n_k}{n_i}+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^{d},m_{lk}^{d}\right)\right] \\
&\quad +\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha\cdot\frac{n_k}{n_i}+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^{d},m_{lk}^{d}\right)\right] \\
&\quad +\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G\left(m_{kl}^{d},m_{lk}^{d}\right) \\
&\quad \left.+\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G\left(m_{kl}^{d},m_{lk}^{d}\right)-\delta_{ij}\cdot\tau\cdot 2G\left(m_{ij}^{d},m_{ji}^{d}\right))\right\} \\
&\quad -m_{ij}^{d}\cdot\left(1-m_{ij}^{d}\right)\cdot\left\{(1-\widetilde{\mu}\cdot\eta)\cdot[\delta_{jj}\cdot\alpha+\sum_{k\in B_{-j}}\delta_{jk}\cdot 2G(m_{jk}^{d},m_{kj}^{d})]\right. \\
&\quad +(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in A_{-i}}\delta_{jk}\cdot\tau\cdot 2G(m_{jk}^{d},m_{kj}^{d})+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha\frac{n_k}{n_j}\right. \\
&\quad \left.\left.+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{n_k}{n_j}\cdot G(m_{kl}^{d},m_{lk}^{d})-\delta_{jj}\cdot\alpha-\sum_{k\in B_{-j}}\delta_{jk}\cdot 2G(m_{jk}^{d},m_{kj}^{d})\right]\right\}
\end{aligned}
\tag{78}
$$

For $i \in A,\ j \in B$:

$$\text{for } j \notin \Gamma_{iB},\ \dot{m}^d_{ij} = \left(1 - m^d_{ij}\right) \cdot \frac{\dot{n}^d_{ij}}{n^{ij}} - m^d_{ij} \cdot \left(\frac{\dot{n}^c_{ij}}{n^{ij}} + \frac{\dot{n}^d_{ji}}{n^{ij}}\right)$$

$$\begin{aligned}
= \ & (1 - m^d_{ij})(1 - m^d_{ji}) \Bigg\{ (1 - \widetilde{\mu} \cdot \eta) \cdot [\delta_{ii} \cdot \alpha + \sum_{k \in A_{-i}} \delta_{ik} \cdot 2G(m^d_{ik}, m^d_{ki})] \\
& + (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{k \in B_{-j}} \delta_{ik} \cdot \tau \cdot 2G(m^d_{ik}, m^d_{ki}) + (\mu \cdot \eta - \widetilde{\mu} \cdot \eta) \cdot \Bigg[\sum_{k \in A} \delta_{kk} \cdot \alpha \frac{n_k}{n_i} \\
& + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G(m^d_{kl}, m^d_{lk}) - \delta_{ii} \cdot \alpha - \sum_{k \in A_{-i}} \delta_{ik} \cdot 2G(m^d_{ik}, m^d_{ki}) \Bigg] \\
& + \overleftrightarrow{\mu} \cdot (\sum_{k \in \Gamma_{jA}} \sum_{l \in \Gamma_{jB}} \delta_{kl} \cdot \frac{n_k}{n_j} \cdot \tau \cdot 2G(m^d_{kl}, m^d_{lk}) - \sum_{k \in \Gamma_{jB}} \delta_{jk} \cdot \tau \cdot 2G(m^d_{jk}, m^d_{kj})) \Bigg\} \\
& - m^d_{ij} \cdot (1 - m^d_{ji}) \Bigg\{ \quad (1 - \widehat{\mu} \cdot \eta) \cdot \delta_{ij} \cdot \tau \cdot 2G(m^d_{ij}, m^d_{ji}) \\
& + \widetilde{\mu} \cdot \eta \cdot \Bigg[\sum_{k \in A} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m^d_{kl}, m^d_{lk}\right) \Bigg] \\
& + \widetilde{\mu} \cdot \eta \cdot \Bigg[\sum_{k \in B} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m^d_{kl}, m^d_{lk}\right) \Bigg] \\
& + \widehat{\mu} \cdot \eta \cdot \sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m^d_{kl}, m^d_{lk}\right) \\
& + \overleftrightarrow{\mu} \cdot (\sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m^d_{kl}, m^d_{lk}\right) - \delta_{ij} \cdot \tau \cdot 2G\left(m^d_{ij}, m^d_{ji}\right)) \Bigg\} \\
& - m^d_{ij} \cdot \left(1 - m^d_{ij}\right) \cdot \Bigg\{ (1 - \widetilde{\mu} \cdot \eta) \cdot [\delta_{jj} \cdot \alpha + \sum_{k \in B_{-j}} \delta_{jk} \cdot 2G(m^d_{jk}, m^d_{kj})] \\
& + (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{k \in A_{-i}} \delta_{jk} \cdot \tau \cdot 2G(m^d_{jk}, m^d_{kj}) + (\mu \cdot \eta - \widetilde{\mu} \cdot \eta) \cdot \Bigg[\sum_{k \in B} \delta_{kk} \cdot \alpha \frac{n_k}{n_j} \\
& + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_j} \cdot G(m^d_{kl}, m^d_{lk}) - \delta_{jj} \cdot \alpha - \sum_{k \in B_{-j}} \delta_{jk} \cdot 2G(m^d_{jk}, m^d_{kj}) \Bigg] \\
& + \overleftrightarrow{\mu} \cdot (\sum_{k \in \Gamma_{jA}} \sum_{l \in \Gamma_{jB}} \delta_{kl} \cdot \frac{n_k}{n_j} \cdot \tau \cdot 2G(m^d_{kl}, m^d_{lk}) - \sum_{k \in \Gamma_{jB}} \delta_{jk} \cdot \tau \cdot 2G(m^d_{jk}, m^d_{kj})) \Bigg\}
\end{aligned} \tag{79}$$

6.3 Basic Dynamics Under Symmetry

When condition (34) holds, using (36) and (33), the dynamics (74) can be written as

$$\begin{aligned}
\text{For } i &\in A: && (80)\\
\frac{\dot{n}_i}{n_i} &= (1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\alpha + \sum_{j\in A_{-i}}\delta_{ij}\cdot 2g\left(m_{ij}^d\right)] + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{j\in B}\delta_{ij}\cdot\tau\cdot 2g\left(m_{ij}^d\right)\\
&\quad +\mu\cdot\eta\cdot[\sum_{k\in A_{-i}}\delta_{kk}\cdot\alpha + (\sum_{k\in A_{-i}}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right))]\\
&\quad +\widetilde{\mu}\cdot\eta\cdot[\sum_{k\in B}\delta_{kk}\cdot\alpha + (\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right))]\\
&\quad +\widehat{\mu}\cdot\eta\cdot[\sum_{k\in A_{-i}}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^d\right)]\\
&\quad +\overleftrightarrow{\mu}\cdot[\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^d\right) - \sum_{j\in\Gamma_{iB}}\delta_{ij}\cdot\tau\cdot 2g\left(m_{ij}^d\right)]
\end{aligned}$$

Likewise, the dynamics (76) to (79), respectively, can be written as follows:

$$
\begin{aligned}
&\text{For } i \in A,\ j \in A\text{: for } j \in \Gamma_{iA}, \qquad\qquad (81)\\
\frac{\dot{m}_{ij}^{d}}{1-m_{ij}^{d}} &= (1-m_{ji}^{d})\left\{(1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot 2g(m_{ik}^{d})]\right.\\
&\left.+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot\tau\cdot 2g(m_{ik}^{d})\right\}\\
&-m_{ij}^{d}\cdot\left\{\qquad(1-\mu\cdot\eta)\cdot\delta_{ij}\cdot 2g(m_{ij}^{d})\right.\\
&+\mu\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m_{kl}^{d}\right)\right]\\
&+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m_{kl}^{d}\right)\right]\\
&+\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^{d}\right)\\
&\left.+\overleftrightarrow{\mu}\cdot\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^{d}\right)\right\}\\
&-m_{ij}^{d}\cdot\left\{(1-\mu\cdot\eta)\cdot[\delta_{jj}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{jk}\cdot 2g(m_{jk}^{d})]\right.\\
&\left.+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{jk}\cdot\tau\cdot 2g(m_{jk}^{d})\right\}
\end{aligned}
$$

$$
\begin{aligned}
\text{For } i \quad &\in \quad A,\, j \in A\text{:} \quad \text{for } j \notin \Gamma_{iA}, \qquad (82)\\
\frac{\dot{m}_{ij}^{d}}{(1-m_{ij}^{d})} \quad &= \quad (1-m_{ji}^{d})\Bigg\{(1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot 2g(m_{ik}^{d})]\\
&\quad +(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot\tau\cdot 2g(m_{ik}^{d})\\
&\quad +\overleftrightarrow{\mu}\cdot\left(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g(m_{kl}^{d})-\sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2g(m_{ik}^{d})\right)\Bigg\}\\
&\quad -m_{ij}^{d}\cdot\Bigg\{\qquad(1-\mu\cdot\eta)\cdot\delta_{ij}\cdot 2g(m_{ij}^{d})\\
&\quad +\mu\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m_{kl}^{d}\right)\right]\\
&\quad +\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m_{kl}^{d}\right)\right]\\
&\quad +\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^{d}\right)\Bigg\}\\
&\quad -m_{ij}^{d}\cdot\Bigg\{(1-\mu\cdot\eta)\cdot[\delta_{jj}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{jk}\cdot 2g(m_{jk}^{d})]\\
&\quad +(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{jk}\cdot\tau\cdot 2g(m_{jk}^{d})\\
&\quad +\overleftrightarrow{\mu}\cdot\left(\sum_{k\in\Gamma_{jA}}\sum_{l\in\Gamma_{jB}}\delta_{kl}\cdot\tau\cdot 2g(m_{kl}^{d})-\sum_{k\in\Gamma_{jB}}\delta_{jk}\cdot\tau\cdot 2g(m_{jk}^{d})\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\text{For } i &\in A,\ j \in B\text{:}\quad \text{for } j \in \Gamma_{iB}, \qquad (83)\\
\frac{\dot{m}^d_{ij}}{1-m^d_{ij}} &= (1-m^d_{ji})\Bigg\{(1-\widetilde{\mu}\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m^d_{ik})]\\
&+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot\tau\cdot 2g(m^d_{ik})+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in A}\delta_{kk}\cdot\alpha\\
&+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g(m^d_{kl})-\delta_{ii}\cdot\alpha-\sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m^d_{ik})\Bigg]\Bigg\}\\
&-m^d_{ij}\Bigg\{\qquad(1-\widehat{\mu}\cdot\eta)\cdot\delta_{ij}\cdot\tau\cdot 2g(m^d_{ij})\\
&+\widetilde{\mu}\cdot\eta\cdot\Bigg[\sum_{k\in A}\delta_{kk}\cdot\alpha+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right)\Bigg]\\
&+\widetilde{\mu}\cdot\eta\cdot\Bigg[\sum_{k\in B}\delta_{kk}\cdot\alpha+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right)\Bigg]\\
&+\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right)\\
&+\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right)-\delta_{ij}\cdot\tau\cdot 2g\left(m^d_{ij}\right))\Bigg\}\\
&-m^d_{ij}\cdot\Bigg\{(1-\widetilde{\mu}\cdot\eta)\cdot[\delta_{jj}\cdot\alpha+\sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m^d_{jk})]\\
&+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in A_{-i}}\delta_{jk}\cdot\tau\cdot 2g(m^d_{jk})+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in B}\delta_{kk}\cdot\alpha\\
&+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g(m^d_{kl})-\delta_{jj}\cdot\alpha-\sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m^d_{jk})\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\text{For } i \in A,\ j \in B\text{: for } j \notin \Gamma_{iB}, \qquad (84)\\
\frac{\dot{m}^d_{ij}}{1-m^d_{ij}} &= (1-m^d_{ji})\Bigg\{(1-\widetilde{\mu}\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m^d_{ik})]\\
&+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot\tau\cdot 2g(m^d_{ik})+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in A}\delta_{kk}\cdot\alpha\\
&+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g(m^d_{kl})-\delta_{ii}\cdot\alpha-\sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m^d_{ik})\Bigg]\\
&+\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g(m^d_{kl})-\sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2g(m^d_{jk}))\Bigg\}\\
&-m^d_{ij}\cdot\Bigg\{\qquad(1-\widehat{\mu}\cdot\eta)\cdot\delta_{ij}\cdot\tau\cdot 2g(m^d_{ij})\\
&+\widetilde{\mu}\cdot\eta\cdot\Bigg[\sum_{k\in A}\delta_{kk}\cdot\alpha+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right)\Bigg]\\
&+\widetilde{\mu}\cdot\eta\cdot\Bigg[\sum_{k\in B}\delta_{kk}\cdot\alpha+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right)\Bigg]\\
&+\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right)\Bigg\}\\
&-m^d_{ij}\cdot\Bigg\{(1-\widetilde{\mu}\cdot\eta)\cdot[\delta_{jj}\cdot\alpha+\sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m^d_{jk})]\\
&+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in A_{-i}}\delta_{jk}\cdot\tau\cdot 2g(m^d_{jk})+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in B}\delta_{kk}\cdot\alpha\\
&+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g(m^d_{kl})-\delta_{jj}\cdot\alpha-\sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m^d_{jk})\Bigg]\\
&+\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{jA}}\sum_{l\in\Gamma_{jB}}\delta_{kl}\cdot\tau\cdot 2g(m^d_{kl})-\sum_{k\in\Gamma_{jB}}\delta_{jk}\cdot\tau\cdot 2g(m^d_{jk}))\Bigg\}
\end{aligned}
$$

Assuming that N is sufficiently large, we use the following approximations:

$$1-\mu\cdot\eta\approx 1,\ 1-\widetilde{\mu}\cdot\eta\approx 1,\ 1-\widehat{\mu}\cdot\eta\approx 1$$

Plugging these into equations (80) - (84), we obtain:

$$
\begin{aligned}
\text{For } i &\in A: \qquad (85)\\
\frac{\dot{n}_i}{n_i} &= [\delta_{ii}\cdot\alpha+\sum_{j\in A_{-i}}\delta_{ij}\cdot 2g\left(m^d_{ij}\right)]+\sum_{j\in B}\delta_{ij}\cdot\tau\cdot 2g\left(m^d_{ij}\right)\\
&+\mu\cdot\eta\cdot[\sum_{k\in A_{-i}}\delta_{kk}\cdot\alpha+(\sum_{k\in A_{-i}}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right))]\\
&+\widetilde{\mu}\cdot\eta\cdot[\sum_{k\in B}\delta_{kk}\cdot\alpha+(\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right))]\\
&+\widehat{\mu}\cdot\eta\cdot[\sum_{k\in A_{-i}}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right)]\\
&+\overleftrightarrow{\mu}\cdot[\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right)-\sum_{j\in\Gamma_{iB}}\delta_{ij}\cdot\tau\cdot 2g\left(m^d_{ij}\right)]
\end{aligned}
$$

$$
\begin{aligned}
\text{For } i &\in A,\, j\in A\text{: for } j\in\Gamma_{iA}, \qquad (86)\\
\frac{\dot{m}^d_{ij}}{1-m^d_{ij}} &= (1-m^d_{ji})\Bigg\{[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot 2g(m^d_{ik})]\\
&\quad+\sum_{k\in B}\delta_{ik}\cdot\tau\cdot 2g(m^d_{ik})\Bigg\}\\
&\quad-m^d_{ij}\cdot\Bigg\{\qquad\delta_{ij}\cdot 2g(m^d_{ij})\\
&\quad+\mu\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right)\right]\\
&\quad+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right)\right]\\
&\quad+\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right)\\
&\quad+\overleftrightarrow{\mu}\cdot\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right)\Bigg\}\\
&\quad-m^d_{ij}\cdot\Bigg\{[\delta_{jj}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{jk}\cdot 2g(m^d_{jk})]\\
&\quad+\sum_{k\in B}\delta_{jk}\cdot\tau\cdot 2g(m^d_{jk})\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\text{For } i &\in A,\ j \in A\text{:} \quad \text{for } j \notin \Gamma_{iA}, \qquad (87)\\
\frac{\dot{m}_{ij}^d}{(1-m_{ij}^d)} &= (1-m_{ji}^d)\Bigg\{[\delta_{ii}\cdot\alpha + \sum_{k\in A_{-\{i,j\}}} \delta_{ik}\cdot 2g(m_{ik}^d)]\\
&\quad + \sum_{k\in B} \delta_{ik}\cdot\tau\cdot 2g(m_{ik}^d)\\
&\quad + \overleftrightarrow{\mu}\cdot\left(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}} \delta_{kl}\cdot\tau\cdot 2g(m_{kl}^d) - \sum_{k\in\Gamma_{iB}} \delta_{ik}\cdot\tau\cdot 2g(m_{ik}^d)\right)\Bigg\}\\
&\quad - m_{ij}^d\cdot\Bigg\{ \qquad \delta_{ij}\cdot 2g(m_{ij}^d)\\
&\quad + \mu\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha + \sum_{k\in A}\sum_{l\in A_{-k}} \delta_{kl}\cdot g\left(m_{kl}^d\right)\right]\\
&\quad + \widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha + \sum_{k\in B}\sum_{l\in B_{-k}} \delta_{kl}\cdot g\left(m_{kl}^d\right)\right]\\
&\quad + \widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^d\right)\Bigg\}\\
&\quad - m_{ij}^d\cdot\Bigg\{[\delta_{jj}\cdot\alpha + \sum_{k\in A_{-\{i,j\}}} \delta_{jk}\cdot 2g(m_{jk}^d)]\\
&\quad + \sum_{k\in B} \delta_{jk}\cdot\tau\cdot 2g(m_{jk}^d)\\
&\quad + \overleftrightarrow{\mu}\cdot\left(\sum_{k\in\Gamma_{jA}}\sum_{l\in\Gamma_{jB}} \delta_{kl}\cdot\tau\cdot 2g(m_{kl}^d) - \sum_{k\in\Gamma_{jB}} \delta_{jk}\cdot\tau\cdot 2g(m_{jk}^d)\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\text{For } i \;&\in\; A,\, j \in B\text{:}\quad \text{for } j \in \Gamma_{iB}, \qquad\qquad (88)\\
\frac{\dot{m}^d_{ij}}{1-m^d_{ij}} \;&=\; (1-m^d_{ji})\Bigg\{[\delta_{ii}\cdot\alpha + \sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m^d_{ik})]\\
&+\sum_{k\in B_{-j}}\delta_{ik}\cdot\tau\cdot 2g(m^d_{ik}) + (\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in A}\delta_{kk}\cdot\alpha\\
&+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g(m^d_{kl}) - \delta_{ii}\cdot\alpha - \sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m^d_{ik})\Bigg]\Bigg\}\\
&-m^d_{ij}\Bigg\{\qquad \delta_{ij}\cdot\tau\cdot 2g(m^d_{ij})\\
&+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha + \sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right)\right]\\
&+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha + \sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right)\right]\\
&+\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right)\\
&+\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right) - \delta_{ij}\cdot\tau\cdot 2g\left(m^d_{ij}\right))\Bigg\}\\
&-m^d_{ij}\cdot\Bigg\{[\delta_{jj}\cdot\alpha + \sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m^d_{jk})]\\
&+\sum_{k\in A_{-i}}\delta_{jk}\cdot\tau\cdot 2g(m^d_{jk}) + (\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in B}\delta_{kk}\cdot\alpha\\
&+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g(m^d_{kl}) - \delta_{jj}\cdot\alpha - \sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m^d_{jk})\Bigg]\Bigg\}
\end{aligned}
$$

$$\text{For } i \in A,\ j \in B\text{: for } j \notin \Gamma_{iB}, \tag{89}$$

$$\begin{aligned}
\frac{\dot{m}_{ij}^d}{1-m_{ij}^d} &= (1-m_{ji}^d)\Bigg\{[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m_{ik}^d)] \\
&+\sum_{k\in B_{-j}}\delta_{ik}\cdot\tau\cdot 2g(m_{ik}^d)+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in A}\delta_{kk}\cdot\alpha \\
&+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g(m_{kl}^d)-\delta_{ii}\cdot\alpha-\sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m_{ik}^d)\Bigg] \\
&+\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g(m_{kl}^d)-\sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2g(m_{jk}^d))\Bigg\} \\
&-m_{ij}^d\cdot\Bigg\{\qquad\delta_{ij}\cdot\tau\cdot 2g(m_{ij}^d) \\
&+\widetilde{\mu}\cdot\eta\cdot\Bigg[\sum_{k\in A}\delta_{kk}\cdot\alpha+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right)\Bigg] \\
&+\widetilde{\mu}\cdot\eta\cdot\Bigg[\sum_{k\in B}\delta_{kk}\cdot\alpha+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right)\Bigg] \\
&+\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^d\right)\Bigg\} \\
&-m_{ij}^d\cdot\Bigg\{[\delta_{jj}\cdot\alpha+\sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m_{jk}^d)] \\
&+\sum_{k\in A_{-i}}\delta_{jk}\cdot\tau\cdot 2g(m_{jk}^d)+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in B}\delta_{kk}\cdot\alpha \\
&+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g(m_{kl}^d)-\delta_{jj}\cdot\alpha-\sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m_{jk}^d)\Bigg] \\
&+\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{jA}}\sum_{l\in\Gamma_{jB}}\delta_{kl}\cdot\tau\cdot 2g(m_{kl}^d)-\sum_{k\in\Gamma_{jB}}\delta_{jk}\cdot\tau\cdot 2g(m_{jk}^d))\Bigg\}
\end{aligned}$$

7 Appendix 2: Proof of Proposition 2

To prove Proposition 2, we find the stationary state of the form given in Proposition 2 that is consistent with the maximization of individual income. For each $i \in A$, the dynamics $\left\{\dot{m}_{ij}^d\right\}_{j=1}^{2N}$ take the following form, namely that of a stationary state attaining the New Eden:

$$
\begin{aligned}
m_{ij}^d &= m^S \text{ for } i,j \in A,\ j \notin \Gamma_{iA} && (90)\\
m_{ij}^d &= m_{ji}^d = m^d < m^S \text{ for } i,j \in A,\ j \in \Gamma_{iA} && (91)\\
m_{ij}^d &= m^B \text{ for } i \in A, j \in B, j \in \Gamma_{iB} && (92)\\
m_{ij}^d &> m^B \text{ for } i \in A, j \in B, j \in \Gamma_{iB} && (93)
\end{aligned}
$$

Then, under condition (54), maximizing income y_i defined by (42) yields

$$
\begin{aligned}
\delta_{ii} &= 0 \text{ for } i \in A && (94)\\
\delta_{ij} &= 0 \text{ for } i,j \in A,\ j \in \Gamma_{iA} && (95)\\
\delta_{ij} &= 0 \text{ for } i \in A,\ j \in B,\ j \notin \Gamma_{iB} && (96)
\end{aligned}
$$

In order to get the equilibrium values of $\{\delta_{ij}\}_{j=1}^{2N}$ that are not shown in (94) to (96) above, let us focus on a specific person, $i \in A$, and assume that

$$
\begin{aligned}
&\text{For } i \in A:\\
&\sum_{j \in A_{-i}} \delta_{ij} = \sum_{j \in A,\ j \notin \Gamma_{iA}} \delta_{ij} = \varphi_i && (97)\\
&\sum_{j \in B} \delta_{ij} = \sum_{j \in \Gamma_{iB}} \delta_{ij} = 1 - \varphi_i && (98)\\
&\sum_{l \in A} \delta_{kl} = \sum_{l \in A_{-k}} \delta_{kl} = \sum_{l \in A,\ l \notin \Gamma_{kA}} \delta_{kl} = \varphi^* \text{ for } k \in A_{-i} && (99)\\
&\sum_{l \in B} \delta_{kl} = \sum_{l \in B_{-k}} \delta_{kl} = \sum_{l \in B,\ l \notin \Gamma_{kB}} \delta_{kl} = \varphi^* \text{ for } k \in B && (100)\\
&\sum_{l \in B} \delta_{kl} = \sum_{l \in B_{-k}} \delta_{kl} = 1 - \varphi^* \text{ for } k \in A_{-i} && (101)\\
&\sum_{l \in B} \delta_{kl} = \sum_{l \in \Gamma_{iB}} \delta_{kl} = 1 - \varphi^* \text{ for } k \in \Gamma_{iA},\ k \neq i && (102)
\end{aligned}
$$

That is, except for person $i \in A$, all persons are assumed to have chosen symmetrically the equilibrium values of $\{\delta_{kl}\}$ in the form of (99) to (102). We then investigate below: For what values of φ^* will the equilibrium value of φ_i coincide with φ^*.

Using the specification (90) to (102) above, the terms inside the square brackets of the income equation (42) for $i \in A$ simplify as follows:

$$\begin{aligned}
\delta_{ii} \cdot \alpha + \sum_{j \in A_{-i}} \delta_{ij} \cdot g(m_{ij}^d) &= \delta_{ii} \cdot \alpha + \sum_{j \in \Gamma_{iA},\, j \neq i} \delta_{ij} \cdot g(m_{ij}^d) + \sum_{j \in A_{-i},\, j \notin \Gamma_{iA}} \delta_{ij} \cdot g(m_{ij}^d) \\
&= 0 + 0 + (\sum_{j \in A_{-i},\, j \notin \Gamma_{iA}} \delta_{ij}) \cdot g(m^S) \\
&= \varphi_i \cdot g(m^S)
\end{aligned}$$

$$\begin{aligned}
\sum_{j \in B} \delta_{ij} \cdot \tau \cdot g(m_{ij}^d) &= \sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot \tau \cdot g(m_{ij}^d) + \sum_{j \in B,\, j \notin \Gamma_{iB}} \delta_{ij} \cdot \tau \cdot g(m_{ij}^d) \\
&= (\sum_{j \in \Gamma_{iB}} \delta_{ij}) \cdot \tau \cdot g(m^B) + 0 \\
&= (1 - \varphi_i) \cdot g(m^S)
\end{aligned}$$

Thus, the income equation becomes

$$\begin{aligned}
y_i &= \Pi \cdot \eta \cdot n_i \cdot [\varphi_i \cdot g(m^S) + (1 - \varphi_i) \cdot g(m^S)] \\
&= \Pi \cdot \eta \cdot n_i \cdot g(m^S)
\end{aligned}$$

that is independent of the choice variables $\{\delta_{ij}\}_{j=1}^{2N}$ of person i. Therefore, we consider the change in income, equation (43), as the objective function for person i:

$$\begin{aligned}
\dot{y}_i &= \{\dot{\Pi} \cdot \eta \cdot n_i + \Pi \cdot \eta \cdot \dot{n}_i\} \cdot \\
&\quad [\delta_{ii} \cdot \alpha + \sum_{j \in A_{-i}} \delta_{ij} \cdot g(m_{ij}^d) + \sum_{j \in B} \delta_{ij} \cdot \tau \cdot g(m_{ij}^d)] \\
&\quad + \Pi \cdot \eta \cdot n_i [\sum_{j \in A_{-i}} \delta_{ij} \cdot g'(m_{ij}^d) \cdot \dot{m}_{ij}^d + \sum_{j \in B} \delta_{ij} \cdot \tau \cdot g'(m_{ij}^d) \cdot \dot{m}_{ij}^d] \\
&= \{\dot{\Pi} \cdot \eta \cdot n_i + \Pi \cdot \eta \cdot \dot{n}_i\} \cdot g(m^S) \\
&\quad + \Pi \cdot \eta \cdot n_i \cdot F_i
\end{aligned} \tag{103}$$

where

$$F_i \equiv \sum_{j \in A_{-i}} \delta_{ij} \cdot g'(m_{ij}^d) \cdot \dot{m}_{ij}^d + \sum_{j \in B} \delta_{ij} \cdot \tau \cdot g'(m_{ij}^d) \cdot \dot{m}_{ij}^d \tag{104}$$

In order to evaluate this equation, as shown in Section 8.4.1 in the Technical Appendix, we obtain the following dynamics of n_i and m_{ij}^d:

$$\begin{gathered}
\text{For } i \in A: \\
\dot{n}_i = n_i \cdot g\left(m^S\right) \cdot [2 + C]
\end{gathered} \tag{105}$$

$$\begin{aligned}
\text{For } i &\in A,\ j \in A\text{: for } j \in \Gamma_{iA}, \qquad (106)\\
\dot{m}_{ij}^{d} &= (1-m^{d}) \cdot 2g(m^{S}) \cdot \left\{ 1-(2+\frac{C}{2}) \cdot m^{d} - m^{d} \cdot \left[\frac{\overleftrightarrow{C}}{2(N^{*}-1)} \cdot (1-\varphi_{i}) \right] \right\}\\
\text{where } m_{ij}^{d} &= m_{ji}^{d} = m^{d}
\end{aligned}$$

$$\begin{aligned}
\text{For } i &\in A,\ j \in A\text{: for } j \notin \Gamma_{iA}, \qquad (107)\\
\dot{m}_{ij}^{d} &= (1-m^{S}) \cdot 2g(m^{S}) \cdot \left\{ 1-(2+\frac{C}{2})m^{S} + (1-m^{S}) \cdot \frac{\overleftrightarrow{C}}{2} \cdot (1-\varphi^{*}) \right.\\
&\quad \left. -(1-m^{S}) \cdot \delta_{ij} \right\}\\
\text{where } m_{ij}^{d} &= m_{ji}^{d} = m^{S}
\end{aligned}$$

$$\begin{aligned}
&\text{For } i \in A,\ j \in B\text{: for } j \in \Gamma_{iB}, \qquad (108)\\
&\dot{m}_{ij}^{d} = (1-m^{B}) \cdot 2g(m^{S}) \cdot \left\{ 1-(2+\frac{C}{2}) \cdot m^{B} + (1-m^{B}) \cdot \frac{C}{2} \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \cdot \varphi^{*} \right.\\
&\quad \left. -(1-m^{B}) \cdot \delta_{ij} - m^{B} \cdot \overleftrightarrow{\mu} \cdot (1-\varphi_{i}-\delta_{ij}) \right\}\\
&\text{where } m_{ij}^{d} = m_{ji}^{d} = m^{B}, \text{ and } g(m^{S}) = \tau \cdot g(m^{B})
\end{aligned}$$

$$\begin{aligned}
\text{For } i &\in A,\ j \in B\text{: for } j \notin \Gamma_{iB}, \qquad (109)\\
\dot{m}_{ij}^{d} &= (1-m^{d}) \cdot 2g(m^{S}) \cdot\\
&\left\{ 1-(2+\frac{C}{2}) \cdot m^{d} + (1-m^{d}) \cdot \left[\frac{C}{2} \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \cdot \varphi^{*} + \frac{\overleftrightarrow{C}}{2} \cdot (1-\varphi^{*}) \right] \right.\\
&\left. -m^{d} \cdot \overleftrightarrow{\mu} \cdot (1-\varphi_{i}) \right\}\\
\text{where } m_{ij}^{d} &= m_{ji}^{d} = m^{d}, \text{ and } g(m^{S}) = \tau \cdot g(m^{B})
\end{aligned}$$

Since from (105), $\dot{n}_i$ is independent of the choice variables of person i, the only term remaining from the expression for $\dot{y}_i$ that is dependent on the choice variables for person i at the time they are chosen is F_i. In other words, the maximization problem for person i:

$$\max_{\{\delta_{ij}\}_{j=1}^{2N}} \dot{y}_i$$
where $\dot{y}_i$ is given by (103)

reduces to:

$$\max_{\{\delta_{ij}\}_{j=1}^{2N}} F_i$$
where F_i is given by (104)

Using (90) - (102) and (107), F_i simplifies as follows (please refer to Section 8.4.2 of the Technical Appendix for calculations):

$$
\begin{aligned}
F_i &= g'(m^S) \cdot \sum_{j \in A,\ j \notin \Gamma_{iA}} \delta_{ij} \cdot \dot{m}_{ij}^d \qquad (110) \\
&= g'(m^S) \cdot (1 - m^S) \cdot 2g(m^S) \cdot \\
& \left\{ \varphi_i \cdot \left[1 - (2 + \frac{C}{2}) m^S + (1 - m^S) \cdot \frac{\overleftrightarrow{C}}{2} \cdot (1 - \varphi^*) \right] - (1 - m^S) \cdot \sum_{j \in A,\ j \notin \Gamma_{iA}} \delta_{ij}^2 \right\}
\end{aligned}
$$

Thus, the optimization problem above further reduces to:

$$
\max_{\{\delta_{ij} |\ j \in A,\ j \notin \Gamma_{iA}\}} F_i
$$
$$
\text{where } F_i \text{ is given by (110)}
$$

We examine this problem in two steps. In the first step, we fix in (97) any φ_i, $0 < \varphi_i \leq 1$, and consider the problem:

$$
\max_{\{\delta_{ij} |\ j \in A,\ j \notin \Gamma_{iA}\}} F_i \text{ subject to } \sum_{j \in A,\ j \notin \Gamma_{iA}} \delta_{ij} = \varphi_i \qquad (111)
$$
$$
\text{where } F_i \text{ is given by (110)}
$$

In the second step, we consider the choice of φ_i. As shown in Section 8.4.2 of the Technical Appendix, the first step yields the following result:

Lemma A1. The optimization problem (111) has the solution:

$$
\delta_{ij}^* = \frac{\varphi_i}{N - N^*} \text{ for } j \in A,\ j \notin \Gamma_{iA} \qquad (112)
$$

and F_i defined by (110) becomes

$$
F_i = g'(m^S) \cdot (1 - m^S) \cdot 2g(m^S) \cdot \left[1 - (2 + \frac{C}{2}) m^S + (1 - m^S) \cdot \frac{\overleftrightarrow{C}}{2} \cdot (1 - \varphi^*) \right] \cdot \varphi_i \qquad (113)
$$

where $g'(m^S) \cdot (1 - m^S) \cdot 2g(m^S) > 0$ since $m^S < m^B$.

In the second step, we consider the choice of φ_i that maximizes F_i given by (113). Since $g'(m^S) \cdot (1 - m^S) \cdot 2g(m^S) > 0$ because $m^S < m^B$, there are 3 different cases:

(i) when the term in square brackets in (113) is positive;

(ii) when the term in square brackets in (113) is negative;

(iii) when the term in square brackets in (113) is zero.

Note that in any of these three cases, since we have been considering a representative person $i \in A$, if φ_i^* is a solution to the maximization problem, then the definition of the myopic core implies that

$$\varphi^* = \varphi_i^* \text{ for all } i \in A \tag{114}$$

As shown in Section 8.4.3 of the Technical Appendix, we can readily see that in cases (i) and (ii), condition (114) leads to a contradiction of either the assumption concerning the sign of the term in the square brackets in (113) given by the particular case, or to a contradiction of the definition of a steady state. Hence, only case (iii) can occur at the myopic core, meaning that

$$1-(2+\frac{C}{2})m^S+(1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)=0$$

leading to:

Lemma A2. At the myopic core stationary state,

$$1-\varphi^*=\frac{2}{\overleftrightarrow{C}}\cdot\frac{(2+\frac{C}{2})\cdot m^S-1}{1-m^S} \tag{115}$$

Hence

$$\begin{aligned}\varphi^* &= 1-\frac{2}{\overleftrightarrow{C}}\cdot\frac{(2+\frac{C}{2})\cdot m^S-1}{1-m^S} \qquad (116)\\ &= \frac{2}{\overleftrightarrow{C}\cdot(1-m^S)}\cdot\left[\frac{\overleftrightarrow{C}}{2}\cdot(1-m^S)-(2+\frac{C}{2})\cdot m^S+1\right]\end{aligned}$$

where

$$0<\varphi^*<1 \tag{117}$$

We can prove (117) as follows. Since $m^*_{\text{aut}}=\frac{1}{2+\frac{C}{2}}<m^S$, (115) means that $1-\varphi^*>0$ and thus $\varphi^*<1$. By the following reasoning, it must also be the case that $\varphi^*>0$ at the steady state. From (108),

For $i \in A$, $j \in B$: for $j \in \Gamma_{iB}$, when $\varphi_i=\varphi^*$ for all $i \in A$:

$$\begin{aligned}\dot{m}^d_{ij} = (1-m^B)\cdot 2g(m^S)\cdot\Bigg\{&1-(2+\frac{C}{2})\cdot m^B+(1-m^B)\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*\\ &-(1-m^B)\cdot\delta_{ij}-m^B\cdot\overleftrightarrow{\mu}\cdot(1-\varphi^*-\delta_{ij})\Bigg\}\end{aligned}$$

Since $m^B>m^S$ and (53) imply $1-(2+\frac{C}{2})\cdot m^B<0$, it follows that $\dot{m}^d_{ij}<0$ whenever $\varphi^*\le 0$, which is inconsistent with the steady state condition (92). Hence, whenever we have a solution for the steady state, it follows that $\varphi^*>0$.

Furthermore, we can readily confirm (please refer to Section 8.4.4 in the Technical Appendix) that setting $\delta_{ij} = \delta^*_{ij}$ given by (112) and using $1-\varphi^*$ given by (115), dynamics (107) yields

$$\dot{m}^d_{ij} = 0 \text{ for all } i \in A,\, j \in A,\, j \notin \Gamma_{iA} \tag{118}$$

as expected from (90). In dynamics (106), setting $1-\varphi_i = 1-\varphi^*$ and using (115), we can also confirm (please refer to Section 8.4.4) that

$$\begin{aligned} \text{for } i,\, j &\in A,\, j \in \Gamma_{iA}\text{: once } m^d_{ij} \leq m^S\text{, then} \\ m^d_{ij} &< m^S \text{ forever after that time} \end{aligned} \tag{119}$$

as expected from (91). Likewise, in dynamics (109), setting $1-\varphi_i = 1-\varphi^*$ given by (115), we can show that

$$\begin{aligned} \text{for } i &\in A,\, j \in B,\, j \notin \Gamma_{iB}\text{: once } m^d_{ij} \geq m^B\text{, then} \\ m^d_{ij} &> m^B \text{ forever after that time} \end{aligned} \tag{120}$$

as expected from (92).

Notice that since $\overleftrightarrow{C}$ is defined by (22), φ^* given by (116) involves another unknown N^*. The other relationship for determining φ^* and N^* simultaneously can be obtained from another steady state condition, (92), as follows.

Setting $\varphi_i = \varphi^*$ in (108) and arranging terms yields:

$$\begin{aligned} &\text{For } i \in A,\, j \in B\text{: for } j \in \Gamma_{iB}, \\ &\dot{m}^d_{ij} = (1-m^B)\cdot 2g(m^S)\cdot\Big\{1-(2+\frac{C}{2})\cdot m^B + (1-m^B)\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* \\ &\quad -m^B\cdot\overleftrightarrow{\mu}\cdot(1-\varphi^*) - (1-m^B-m^B\cdot\overleftrightarrow{\mu})\cdot\delta_{ij}\ \Big\} \end{aligned}$$

where φ^* is given in (116). A necessary condition for a steady state at $m^d_{ij} = m^d_{ji} = m^B$ is $\dot{m}^d_{ij} = 0$ for $j \in \Gamma_{iB}$, or

$$1-(2+\frac{C}{2})\cdot m^B + (1-m^B)\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* = m^B\cdot\overleftrightarrow{\mu}\cdot(1-\varphi^*) + (1-m^B-m^B\cdot\overleftrightarrow{\mu})\cdot\delta_{ij}$$

An immediate implication is that δ_{ij} is the same for all $j \in \Gamma_{iB}$, and hence:

$$\delta^*_{ij} = \frac{1-\varphi^*}{N^*} \text{ for all } j \in \Gamma_{iB} \tag{121}$$

Thus, using (22),

$$\begin{aligned}
& 1-(2+\frac{C}{2})\cdot m^{B}\ +(1-m^{B})\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^{*} \\
= \;& m^{B}\ \cdot\overleftrightarrow{\mu}\cdot(1-\varphi^{*})+(1-m^{B}-m^{B}\ \cdot\overleftrightarrow{\mu})\cdot\frac{1-\varphi^{*}}{N^{*}} \\
= \;& m^{B}\ \cdot\overleftrightarrow{\mu}\cdot(1-\varphi^{*})\cdot(1-\frac{1}{N^{*}})+(1-m^{B})\cdot\frac{1-\varphi^{*}}{N^{*}} \\
= \;& m^{B}\ \cdot\frac{\overleftrightarrow{C}}{2N^{*}}\cdot(1-\varphi^{*})+(1-m^{B})\cdot\frac{1-\varphi^{*}}{N^{*}} \\
= \;& (1-m^{B}+m^{B}\ \cdot\frac{\overleftrightarrow{C}}{2})\cdot\frac{1-\varphi^{*}}{N^{*}}
\end{aligned}$$

In short,

$$\begin{aligned}
& 1-(2+\frac{C}{2})\cdot m^{B}\ +(1-m^{B})\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^{*} \\
= \;& (1-m^{B}+m^{B}\ \cdot\frac{\overleftrightarrow{C}}{2})\cdot\frac{1-\varphi^{*}}{N^{*}}
\end{aligned} \tag{122}$$

implying that

$$N^{*}=\frac{(1-m^{B}+m^{B}\ \cdot\frac{\overleftrightarrow{C}}{2})\cdot(1-\varphi^{*})}{1-(2+\frac{C}{2})\cdot m^{B}\ +(1-m^{B})\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^{*}} \tag{123}$$

Now we consider two cases. Either setting $\overleftrightarrow{C}=\overline{C}$ in (22) for $N^{*}\geq\overline{N}$,

$$\begin{aligned}
N^{*} \;&=\; \frac{(1-m^{B}+m^{B}\ \cdot\frac{\overline{C}}{2})\cdot(1-\varphi^{*})}{1-(2+\frac{C}{2})\cdot m^{B}\ +(1-m^{B})\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^{*}}\geq\overline{N} \\
& \text{and} \\
\varphi^{*} \;&=\; 1-\frac{2}{\overline{C}}\cdot\frac{(2+\frac{C}{2})\cdot m^{S}-1}{1-m^{S}}
\end{aligned} \tag{124}$$

or setting $\overleftrightarrow{C}=\overline{C}\cdot N^{*}/\overline{N}$ in (22) for $N^{*}<\overline{N}$, and then solving (123) for N^{*}:

$$\overline{N}>N^{*}=\frac{1-m^{B}}{1-(2+\frac{C}{2})\cdot m^{B}\ +(1-m^{B})\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^{*}-\frac{m^{B}\overline{C}}{2\overline{N}}\cdot(1-\varphi^{*})} \tag{125}$$

and

$$\varphi^{*}=1-\frac{2}{\overline{C}\cdot\frac{N^{*}}{\overline{N}}}\cdot\frac{(2+\frac{C}{2})\cdot m^{S}-1}{1-m^{S}} \tag{126}$$

In the first case, substituting for φ^{*} in (124), the solution is represented explicitly by (127) in Lemma A3 below. In the second case, we have two equations in the two unknowns φ^* and N^*. Substituting for φ^* in (124) and solving the quadratic equation for N^*, we can obtain (128) below:

Lemma A3. At the myopic core stationary state, we have that

$$N^* = E(m^B, m^S) \text{ when } E(m^B, m^S) \geq \overline{N} \tag{127}$$

where

$$E(m^B, m^S) \equiv \frac{[m^B \; + \frac{2}{\overline{C}} \cdot (1 - m^B)] \cdot \frac{(2+\frac{C}{2}) \cdot m^S - 1}{1 - m^S}}{1 - (2 + \frac{C}{2}) \cdot m^B \; + (1 - m^B) \cdot \frac{C}{2} \cdot \frac{1 - \widetilde{\gamma}}{1 + \widetilde{\gamma}} \cdot (1 - \frac{2}{\overline{C}} \cdot \frac{(2+\frac{C}{2}) \cdot m^S - 1}{1 - m^S})}$$

or

$$\begin{aligned} N^* &= D(m^B, m^S) \text{ when } D(m^B, m^S) < \overline{N} \\ &\quad \text{where} \\ D(m^B, m^S) &\equiv H(m^B, m^S) + \sqrt{H(m^B, m^S)^2 + J(m^B, m^S)} \\ H(m^B, m^S) &\equiv \frac{\left[m^B + (1 - m^B) \cdot \frac{C}{\overline{C}} \cdot \overline{N} \cdot \frac{1 - \widetilde{\gamma}}{1 + \widetilde{\gamma}}\right] \cdot \frac{(2+\frac{C}{2}) \cdot m^S - 1}{1 - m^S}}{2 \cdot \left[1 - (2 + \frac{C}{2}) \cdot m^B + (1 - m^B) \cdot \frac{C}{2} \cdot \frac{1 - \widetilde{\gamma}}{1 + \widetilde{\gamma}}\right]} \\ J(m^B, m^S) &\equiv \frac{\frac{2\overline{N}}{\overline{C}} \cdot (1 - m^B) \cdot \frac{(2+\frac{C}{2}) \cdot m^S - 1}{1 - m^S}}{1 - (2 + \frac{C}{2}) \cdot m^B + (1 - m^B) \cdot \frac{C}{2} \cdot \frac{1 - \widetilde{\gamma}}{1 + \widetilde{\gamma}}} \end{aligned} \tag{128}$$

We can readily show that:

$$D(m^B, m^S) = \overline{N} \implies E(m^B, m^S) = \overline{N}$$

Hence, (127) and (128) together define N^* consistently.

Having determined all the endogenous variables (as functions of the exogenous variables) at the New Eden, we now proceed to show that the New Eden is in the myopic core. In general, the myopic core path will depend on initial conditions, but here we focus on the steady state at the New Eden. Obviously, we can fix a time t and examine payoffs for agents at that time since agents are myopic. Fix an agent i. Much of the work in this subsection has been to show that, starting at the New Eden state, if person i can choose $\{\delta_{ij}\}_{j=1}^{2N}$ where δ_{ji} is set to δ_{ij}, they will choose the New Eden. This immediately implies that no one or two person coalition can do better than the New Eden at a given time t, as y_i is independent of person i's choice variables, and the selection of $\{\delta_{ij}\}_{j=1}^{2N}$ to maximize $\dot{y}_i$ is optimal for each person i. More generally, we must consider larger coalitions. Recall that this is a non-transferable utility game;

side payments are not allowed. Notice that the calculations deriving the New Eden all include the public knowledge transmission externality terms. Thus, if person i could be dictator, they might for example wish to change δ_{kj} for some $k \neq i$, $j \neq i$ in order to increase person i's own payoff due to a change in the public information transmission externality. However, due to the symmetry of the solution (that is derived, not imposed), this would clearly lead to a loss of utility for both persons k and j, who would then decline to join the coalition. Since this is true for every person, and the New Eden is a feasible strategy, it is not possible for any coalition to do better, and thus the New Eden is in the myopic core. Summarizing the argument in this paragraph:

Lemma A4. The New Eden stationary state, described in the statement of Proposition 2, is in the myopic core.

In order to show that the set of parameters generating a New Eden is nonempty and open, consider the point a in Figure 7, corresponding to the parameters given in (67) and (68). The point a is in the feasible domain. Now, let us change the chosen parameters given by (67) just a little. Then, the supreme iso-N^* curve from (57) and the m^*_{aut}-line from (50) shift only a little. Thus, point a in Figure 7 remains inside the domain of a possible New Eden. Therefore, when we change parameters τ and θ marginally from (68), the point a moves only marginally, remaining inside the feasible domain for a New Eden. From this observation and recalling Lemma 3, we can conclude as follows:

Lemma A5. There is a nonempty, open set of exogenous parameters for which there exists a welfare improving myopic core stationary state.

Putting together Lemmas A1-A5, we obtain Proposition 2.

8 Appendix 3: Technical Appendix

8.1 Appendix a: Derivation of Equation (12)

Using (3) and (11),

$$\begin{aligned}
& \frac{a_{ij}/2}{n_i} \\
= \; & \frac{n^{ij}}{n_i} \cdot \frac{a_{ij}/2}{n^{ij}} \\
= \; & \frac{1}{1-m_{ji}^d} \cdot \beta (m_{ij}^c)^\theta \cdot (m_{ij}^d \cdot m_{ji}^d)^{\frac{1-\theta}{2}} \\
= \; & \frac{\beta \left(1-m_{ij}^d - m_{ji}^d\right)^\theta \cdot (m_{ij}^d \cdot m_{ji}^d)^{\frac{1-\theta}{2}}}{1-m_{ji}^d} \\
= \; & G(m_{ij}^d, m_{ji}^d)
\end{aligned}$$

which leads to (12).

8.2 Appendix b: Derivation of Equation (74)

From (26) we have:

For $i \in A$:

$$\begin{aligned}
\frac{\dot{n}_i}{n_i} = \; & \sum_{j \in A} \delta_{ij} \cdot \frac{a_{ij}}{n_i} + \sum_{j \in B} \delta_{ij} \cdot \frac{a_{ij}}{n_i} + \mu \cdot \eta \cdot (\frac{I_{AA}}{n_i} - \sum_{j \in A} \delta_{ij} \cdot \frac{a_{ij}}{n_i}) \\
& + \widetilde{\mu} \cdot \eta \cdot \frac{I_{BB}}{n_i} + \widehat{\mu} \cdot \eta \cdot (\frac{I_{AB}}{n_i} - \sum_{j \in B} \delta_{ij} \cdot \frac{a_{ij}}{n_i}) + \overleftrightarrow{\mu} \cdot (\frac{I_{\Gamma_i}}{n_i} - \sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot \frac{a_{ij}}{n_i}) \\
= \; & (1 - \mu \cdot \eta) \cdot (\sum_{j \in A} \delta_{ij} \cdot \frac{a_{ij}}{n_i}) + (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{j \in B} \delta_{ij} \cdot \frac{a_{ij}}{n_i} + \mu \cdot \eta \cdot \frac{I_{AA}}{n_i} \\
& + \widetilde{\mu} \cdot \eta \cdot \frac{I_{BB}}{n_i} + \widehat{\mu} \cdot \eta \cdot \frac{I_{AB}}{n_i} + \overleftrightarrow{\mu} \cdot (\frac{I_{\Gamma_i}}{n_i} - \sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot \frac{a_{ij}}{n_i}) \\
= \; & (1 - \mu \cdot \eta) \cdot [\delta_{ii} \cdot \frac{a_{ii}}{n_i} + \sum_{j \in A_{-i}} \delta_{ij} \cdot \frac{a_{ij}}{n_i}] + (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{j \in B} \delta_{ij} \cdot \frac{a_{ij}}{n_i} \\
& + \mu \cdot \eta \cdot [\sum_{k \in A} \delta_{kk} \cdot \frac{a_{kk}}{n_i} + (\sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{a_{kl}}{n_i})/2 - \sum_{j \in A} \delta_{ij} \cdot \frac{a_{ij}}{n_i}] \\
& + \widetilde{\mu} \cdot \eta \cdot [\sum_{k \in B} \delta_{kk} \cdot \frac{a_{kk}}{n_i} + (\sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{a_{kl}}{n_i})/2] \\
& + \widehat{\mu} \cdot \eta \cdot (\sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{a_{kl}}{n_i}) + \overleftrightarrow{\mu} \cdot (\sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot \frac{a_{kl}}{n_i} - \sum_{j \in \Gamma_{iB}} \delta_{ij} \cdot \frac{a_{ij}}{n_i})
\end{aligned}$$

$$
\begin{aligned}
= \quad & (1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\frac{a_{ii}}{n_i}+\sum_{j\in A_{-i}}\delta_{ij}\cdot\frac{a_{ij}}{n_i}]+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{j\in B}\delta_{ij}\cdot\frac{a_{ij}}{n_i} \\
& +\mu\cdot\eta\cdot[\sum_{k\in A}\delta_{kk}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kk}}{n_k}+(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kl}}{n_k})/2] \\
& +\widetilde{\mu}\cdot\eta\cdot[\sum_{k\in B}\delta_{kk}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kk}}{n_k}+(\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kl}}{n_k})/2] \\
& +\widehat{\mu}\cdot\eta\cdot(\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kl}}{n_k}) \\
& +\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kl}}{n_k}-\sum_{j\in\Gamma_{iB}}\delta_{ij}\cdot\frac{a_{ij}}{n_i})
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{j\in A_{-i}}\delta_{ij}\cdot 2G\left(m_{ij}^d,m_{ji}^d\right)]+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{j\in B}\delta_{ij}\cdot\tau\cdot 2G\left(m_{ij}^d,m_{ji}^d\right) \\
& +\mu\cdot\eta\cdot[\sum_{k\in A}\delta_{kk}\cdot\frac{n_k}{n_i}\cdot\alpha+(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^d,m_{lk}^d\right))] \\
& +\widetilde{\mu}\cdot\eta\cdot[\sum_{k\in B}\delta_{kk}\cdot\frac{n_k}{n_i}\cdot\alpha+(\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^d,m_{lk}^d\right))] \\
& +\widehat{\mu}\cdot\eta\cdot[\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G\left(m_{kl}^d,m_{lk}^d\right)] \\
& +\overleftrightarrow{\mu}\cdot[\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G\left(m_{kl}^d,m_{lk}^d\right)-\sum_{j\in\Gamma_{iB}}\delta_{ij}\cdot\tau\cdot 2G\left(m_{ij}^d,m_{ji}^d\right)]
\end{aligned}
$$

which leads to (74).

8.3 Appendix c: Derivation of Equations (76) to (79)

In order to calculate $\dot{m}_{ij}^d$ by using equation (75), we first obtain $\dot{n}_{ij}^d$ for $i \in A$ and $j \in A$. Using (30) and (11), we have

$$
\begin{aligned}
\text{For } i &\in A,\ j \in A: \\
\text{for } j &\in \Gamma_{iA},\ \frac{\dot{n}_{ij}^d}{n^{ij}} = \frac{(1-\mu\cdot\eta)\cdot\sum_{k\in A_{-j}}\delta_{ik}\cdot a_{ik}}{n^{ij}} + \frac{(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot a_{ik}}{n^{ij}} \\
&= (1-\mu\cdot\eta)\cdot[\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} + \sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot\frac{a_{ik}}{n^{ij}}] + \frac{(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot a_{ik}}{n^{ij}} \\
&= (1-\mu\cdot\eta)\cdot[\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} + \sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ik}}{n_i}] + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ik}}{n_i} \\
&= \frac{n_i}{n^{ij}}\cdot\left\{(1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\alpha + \sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot\frac{a_{ik}}{n_i}] + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot\frac{a_{ik}}{n_i}\right\} \\
&= \left(1-m_{ji}^d\right)\cdot\left\{(1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\alpha + \sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot 2G(m_{ik}^d, m_{ki}^d)]\right. \\
&\qquad \left. +(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot\tau\cdot 2G(m_{ik}^d, m_{ki}^d)\right\}
\end{aligned}
$$

For $i \in A$, $j \in A$:

$$\text{for } j \notin \Gamma_{iA},\ \frac{\dot{n}_{ij}^d}{n^{ij}} = \frac{(1-\mu\cdot\eta)\cdot\sum_{k\in A_{-j}}\delta_{ik}\cdot a_{ik}}{n^{ij}} + \frac{(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot a_{ik}}{n^{ij}}$$

$$+\frac{\overleftrightarrow{\mu}\cdot(I_{\Gamma_i}-\sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot a_{ik})}{n^{ij}}$$

$$= (1-\mu\cdot\eta)\cdot[\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} + \sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot\frac{a_{ik}}{n^{ij}}] + \frac{(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot a_{ik}}{n^{ij}}$$

$$+\overleftrightarrow{\mu}\cdot\left(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\frac{\delta_{kl}\cdot a_{kl}}{n^{ij}} - \frac{\sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot a_{ik}}{n^{ij}}\right)$$

$$= (1-\mu\cdot\eta)\cdot[\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} + \sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ik}}{n_i}] + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ik}}{n_i}$$

$$+\overleftrightarrow{\mu}\cdot\left(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_i}{n^{ij}}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kl}}{n_k} - \sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ik}}{n_i}\right)$$

$$= \frac{n_i}{n^{ij}}\cdot\left[\left\{(1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\alpha + \sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot\frac{a_{ik}}{n_i}] + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot\frac{a_{ik}}{n_i}\right\}\right.$$

$$\left.+\overleftrightarrow{\mu}\cdot\left(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G(m_{kl}^d, m_{lk}^d) - \sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2G(m_{ik}^d, m_{ki}^d)\right)\right]$$

$$= (1-m_{ji}^d)\cdot\left[\left\{(1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\alpha + \sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot 2G(m_{ik}^d, m_{ki}^d)]\right.\right.$$

$$\left.+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot\tau\cdot 2G(m_{ik}^d, m_{ki}^d)\right\}$$

$$\left.+\overleftrightarrow{\mu}\cdot\left(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G(m_{kl}^d, m_{lk}^d) - \sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2G(m_{ik}^d, m_{ki}^d)\right)\right]$$

Similarly,

For $i \in A$, $j \in A$:

$$\text{for } j \in \Gamma_{iA},\ \frac{\dot{n}_{ji}^d}{n^{ij}} = (1-m_{ij}^d)\cdot\left\{(1-\mu\cdot\eta)\cdot[\delta_{jj}\cdot\alpha + \sum_{k\in A_{-\{i,j\}}}\delta_{jk}\cdot 2G(m_{jk}^d, m_{kj}^d)]\right.$$

$$\left.+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{jk}\cdot\tau\cdot 2G(m_{jk}^d, m_{kj}^d)\right\}$$

$$\text{For } i \notin \Gamma_{jA},\ \frac{\dot{n}^d_{ji}}{n^{ij}} = \left(1 - m^d_{ij}\right) \cdot \left[\left\{(1-\mu\cdot\eta)\cdot[\delta_{jj}\cdot\alpha + \sum_{k\in A_{-\{i,j\}}} \delta_{jk}\cdot 2G(m^d_{jk}, m^d_{kj})]\right.\right.$$

$$\left. + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{jk}\cdot\tau\cdot 2G(m^d_{jk}, m^d_{kj})\right\}$$

$$\left. + \overleftrightarrow{\mu}\cdot\left(\sum_{k\in\Gamma_{jA}}\sum_{l\in\Gamma_{jB}}\delta_{kl}\cdot\frac{n_k}{n_j}\cdot\tau\cdot 2G(m^d_{kl}, m^d_{lk}) - \sum_{k\in\Gamma_{jB}}\delta_{jk}\cdot\tau\cdot 2G(m^d_{jk}, m^d_{kj})\right)\right]$$

Next, we obtain $\dot{n}^c_{ij}$ for $i \in A$ and $j \in A$. Using the second equation in (28) yields

$$\text{For } i \in A,\ j \in A:$$

$$\text{for } j \notin \Gamma_{iA},\ \frac{\dot{n}^c_{ij}}{n^{ij}} = \frac{\delta_{ij}\cdot a_{ij}}{n^{ij}} + \mu\cdot\eta\cdot(\frac{I_{AA}}{n^{ij}} - \delta_{ij}\cdot\frac{a_{ij}}{n^{ij}}) + \widetilde{\mu}\cdot\eta\cdot\frac{I_{BB}}{n^{ij}} + \widehat{\mu}\cdot\eta\cdot\frac{I_{AB}}{n^{ij}}$$

$$= \delta_{ij}\cdot\frac{a_{ij}}{n^{ij}} + \mu\cdot\eta\cdot\left[\sum_{k\in A}\frac{\delta_{kk}\cdot a_{kk}}{n^{ij}} + \left(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{a_{kl}}{n^{ij}}\right)/2 - \delta_{ij}\cdot\frac{a_{ij}}{n^{ij}}\right]$$

$$+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\frac{\delta_{kk}\cdot a_{kk}}{n^{ij}} + \left(\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{a_{kl}}{n^{ij}}\right)/2\right]$$

$$+\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\frac{a_{kl}}{n^{ij}}$$

$$= \delta_{ij}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ij}}{n_i} + \mu\cdot\eta\cdot\left[\sum_{k\in A}\frac{\delta_{kk}\cdot\alpha\cdot n_k}{n^{ij}} + \left(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n^{ij}}\cdot\frac{a_{kl}}{n_k}\right)/2 - \delta_{ij}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ij}}{n_i}\right]$$

$$+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\frac{\delta_{kk}\cdot\alpha\cdot n_k}{n^{ij}} + \left(\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{n_k}{n^{ij}}\cdot\frac{a_{kl}}{n_k}\right)/2\right]$$

$$+\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\frac{n_k}{n^{ij}}\cdot\frac{a_{kl}}{n_k}$$

Thus, using equations (2), (3), (12), and (11), we have:

For $i \in A$, $j \in A$:

$$\text{for } j \notin \Gamma_{iA},\ \frac{\dot{n}_{ij}^c}{n^{ij}} = \delta_{ij} \cdot (1 - m_{ji}^d) \cdot 2G(m_{ij}^d, m_{ji}^d) + \mu \cdot \eta \cdot \left[\sum_{k \in A} \delta_{kk} \cdot \alpha \cdot \frac{n_i}{n^{ij}} \cdot \frac{n_k}{n_i} \right.$$
$$\left. + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_i}{n^{ij}} \cdot \frac{n_k}{n_i} \cdot G\left(m_{kl}^d, m_{lk}^d\right) - \delta_{ij} \cdot \frac{n_i}{n^{ij}} \cdot 2G(m_{ij}^d, m_{ji}^d) \right]$$
$$+ \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in B} \delta_{kk} \cdot \alpha \cdot \frac{n_i}{n^{ij}} \cdot \frac{n_k}{n_i} + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_i}{n^{ij}} \cdot \frac{n_k}{n_i} \cdot G\left(m_{kl}^d, m_{lk}^d\right) \right]$$
$$+ \widehat{\mu} \cdot \eta \cdot \sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{n_i}{n^{ij}} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m_{kl}^d, m_{lk}^d\right)$$
$$= (1 - m_{ji}^d) \cdot \left\{ \qquad (1 - \mu \cdot \eta) \cdot \delta_{ij} \cdot 2G(m_{ij}^d, m_{ji}^d) \right.$$
$$+ \mu \cdot \eta \cdot \left[\sum_{k \in A} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m_{kl}^d, m_{lk}^d\right) \right]$$
$$+ \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in B} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m_{kl}^d, m_{lk}^d\right) \right]$$
$$\left. + \widehat{\mu} \cdot \eta \cdot \sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m_{kl}^d, m_{lk}^d\right) \right\}$$

Likewise, using the first equation in (28) yields:

For $i \in A$, $j \in A$:

$$\text{for } j \in \Gamma_{iA},\ \frac{\dot{n}_{ij}^c}{n^{ij}} = \frac{\delta_{ij} \cdot a_{ij}}{n^{ij}} + \mu \cdot \eta \cdot (\frac{I_{AA}}{n^{ij}} - \delta_{ij} \cdot \frac{a_{ij}}{n^{ij}}) + \widetilde{\mu} \cdot \eta \cdot \frac{I_{BB}}{n^{ij}} + \widehat{\mu} \cdot \eta \cdot \frac{I_{AB}}{n^{ij}} + \frac{\overleftrightarrow{\mu} \cdot I_{\Gamma i}}{n^{ij}}$$

Thus, following similar logic, we obtain:

$$
\begin{aligned}
\text{for } j \in \Gamma_{iA}, \frac{\dot{n}^c_{ij}}{n^{ij}} = (1 - m^d_{ji}) \cdot \Bigg\{ \quad & (1 - \mu \cdot \eta) \cdot \delta_{ij} \cdot 2G(m^d_{ij}, m^d_{ji}) \\
& + \mu \cdot \eta \cdot \left[\sum_{k \in A} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m^d_{kl}, m^d_{lk}\right) \right] \\
& + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in B} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m^d_{kl}, m^d_{lk}\right) \right] \\
& + \widehat{\mu} \cdot \eta \cdot \sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m^d_{kl}, m^d_{lk}\right) \\
& \overleftrightarrow{\mu} \cdot \sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m^d_{kl}, m^d_{lk}\right) \Bigg\}
\end{aligned}
$$

Putting everything together, we have:

For $i \in A$, $j \in A$:

$$
\begin{aligned}
\text{for } j &\in \Gamma_{iA},\ \dot{m}_{ij}^{d} = \left(1-m_{ij}^{d}\right)\cdot\frac{\dot{n}_{ij}^{d}}{n^{ij}} - m_{ij}^{d}\cdot\left(\frac{\dot{n}_{ij}^{c}}{n^{ij}}+\frac{\dot{n}_{ji}^{d}}{n^{ij}}\right) \\
&= (1-m_{ij}^{d})(1-m_{ji}^{d})\left\{(1-\mu\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot 2G(m_{ik}^{d},m_{ki}^{d})]\right. \\
&\quad \left. +(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{ik}\cdot\tau\cdot 2G(m_{ik}^{d},m_{ki}^{d})\right\} \\
&\quad -m_{ij}^{d}\cdot(1-m_{ji}^{d})\left\{\qquad(1-\mu\cdot\eta)\cdot\delta_{ij}\cdot 2G(m_{ij}^{d},m_{ji}^{d})\right. \\
&\quad +\mu\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha\cdot\frac{n_k}{n_i}+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^{d},m_{lk}^{d}\right)\right] \\
&\quad +\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha\cdot\frac{n_k}{n_i}+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^{d},m_{lk}^{d}\right)\right] \\
&\quad +\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G\left(m_{kl}^{d},m_{lk}^{d}\right) \\
&\quad \left. +\overleftrightarrow{\mu}\cdot\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G\left(m_{kl}^{d},m_{lk}^{d}\right)\right\} \\
&\quad -m_{ij}^{d}\cdot\left(1-m_{ij}^{d}\right)\cdot\left\{(1-\mu\cdot\eta)\cdot[\delta_{jj}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{jk}\cdot 2G(m_{jk}^{d},m_{kj}^{d})]\right. \\
&\quad \left. +(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B}\delta_{jk}\cdot\tau\cdot 2G(m_{jk}^{d},m_{kj}^{d})\right\}
\end{aligned}
$$

This yields equation (76).

For $i \in A$, $j \in A$:

$$\text{for } j \notin \Gamma_{iA},\ \dot{m}^d_{ij} = \left(1 - m^d_{ij}\right) \cdot \frac{\dot{n}^d_{ij}}{n^{ij}} - m^d_{ij} \cdot \left(\frac{\dot{n}^c_{ij}}{n^{ij}} + \frac{\dot{n}^d_{ji}}{n^{ij}}\right)$$

$$= (1 - m^d_{ij})(1 - m^d_{ji}) \Bigg\{ (1 - \mu \cdot \eta) \cdot [\delta_{ii} \cdot \alpha + \sum_{k \in A_{-\{i,j\}}} \delta_{ik} \cdot 2G(m^d_{ik}, m^d_{ki})]$$

$$+ (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{k \in B} \delta_{ik} \cdot \tau \cdot 2G(m^d_{ik}, m^d_{ki})$$

$$+ \overleftrightarrow{\mu} \cdot \left(\sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G(m^d_{kl}, m^d_{lk}) - \sum_{k \in \Gamma_{iB}} \delta_{ik} \cdot \tau \cdot 2G(m^d_{ik}, m^d_{ki}) \right) \Bigg\}$$

$$- m^d_{ij} \cdot (1 - m^d_{ji}) \Bigg\{ \quad (1 - \mu \cdot \eta) \cdot \delta_{ij} \cdot 2G(m^d_{ij}, m^d_{ji})$$

$$+ \mu \cdot \eta \cdot \left[\sum_{k \in A} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m^d_{kl}, m^d_{lk}\right) \right]$$

$$+ \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in B} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m^d_{kl}, m^d_{lk}\right) \right]$$

$$+ \widehat{\mu} \cdot \eta \cdot \sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m^d_{kl}, m^d_{lk}\right) \Bigg\}$$

$$- m^d_{ij} \cdot \left(1 - m^d_{ij}\right) \cdot \Bigg\{ (1 - \mu \cdot \eta) \cdot [\delta_{jj} \cdot \alpha + \sum_{k \in A_{-\{i,j\}}} \delta_{jk} \cdot 2G(m^d_{jk}, m^d_{kj})]$$

$$+ (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{k \in B} \delta_{jk} \cdot \tau \cdot 2G(m^d_{jk}, m^d_{kj})$$

$$+ \overleftrightarrow{\mu} \cdot \left(\sum_{k \in \Gamma_{jA}} \sum_{l \in \Gamma_{jB}} \delta_{kl} \cdot \frac{n_k}{n_j} \cdot \tau \cdot 2G(m^d_{kl}, m^d_{lk}) - \sum_{k \in \Gamma_{jB}} \delta_{jk} \cdot \tau \cdot 2G(m^d_{jk}, m^d_{kj}) \right) \Bigg\}$$

This gives us equation (77).

Next consider the case $i \in A$, $j \in B$. Using the first equation in (31) yields:

For $i \in A$, $j \in B$:

$$\text{for } j \in \Gamma_{iB},\ \frac{\dot{n}^d_{ij}}{n^{ij}} = \frac{(1-\widetilde{\mu}\cdot\eta)\cdot\sum_{k\in A}\delta_{ik}\cdot a_{ik}}{n^{ij}} + \frac{(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot a_{ik}}{n^{ij}}$$

$$+\frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot(I_{AA}-\sum_{j\in A}\delta_{ij}\cdot a_{ij})}{n^{ij}}$$

$$= (1-\widetilde{\mu}\cdot\eta)\cdot[\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} + \sum_{k\in A_{-i}}\delta_{ik}\cdot\frac{a_{ik}}{n^{ij}}] + \frac{(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot a_{ik}}{n^{ij}}$$

$$+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\left[\sum_{k\in A}\frac{\delta_{kk}\cdot\alpha\cdot n_k}{n^{ij}}\right.$$

$$\left.+\left(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{a_{kl}}{n^{ij}}\right)/2-\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}}-\frac{\sum_{k\in A-i}\delta_{ik}\cdot a_{ik}}{n^{ij}}\right]$$

$$= (1-\widetilde{\mu}\cdot\eta)\cdot[\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} + \sum_{k\in A_{-i}}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ik}}{n_i}] + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ik}}{n_i}$$

$$+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\left[\sum_{k\in A}\frac{\delta_{kk}\cdot\alpha\cdot n_k}{n^{ij}}\right.$$

$$\left.+\left(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n^{ij}}\frac{a_{kl}}{n_k}\right)/2-\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}}-\sum_{k\in A_{-i}}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\frac{a_{ik}}{n_i}\right]$$

$$= \frac{n_i}{n^{ij}}\cdot\left\{(1-\widetilde{\mu}\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-i}}\delta_{ik}\cdot\frac{a_{ik}}{n_i}] + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot\frac{a_{ik}}{n_i}\right.$$

$$+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha\frac{n_k}{n_i}\right.$$

$$\left.\left.+\left(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\frac{a_{kl}}{n_k}\right)/2-\delta_{ii}\cdot\alpha-\sum_{k\in A_{-i}}\delta_{ik}\cdot\frac{a_{ik}}{n_i}\right]\right\}$$

$$= \left(1-m^d_{ji}\right)\cdot\left\{(1-\widetilde{\mu}\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-i}}\delta_{ik}\cdot 2G(m^d_{ik},m^d_{ki})]\right.$$

$$+(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot\tau\cdot 2G(m^d_{ik},m^d_{ki})$$

$$+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha\frac{n_k}{n_i}\right.$$

$$\left.\left.+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G(m^d_{kl},m^d_{lk})-\delta_{ii}\cdot\alpha-\sum_{k\in A_{-i}}\delta_{ik}\cdot 2G(m^d_{ik},m^d_{ki})\right]\right\}$$

Similarly, using the first equation in (32), we have:

For $i \in A$, $j \in B$:

$$\text{for } j \in \Gamma_{iB},\ \frac{\dot{n}^d_{ji}}{n^{ij}} = \left(1 - m^d_{ij}\right) \cdot \Bigg\{ (1 - \widetilde{\mu} \cdot \eta) \cdot [\delta_{jj} \cdot \alpha + \sum_{k \in B_{-j}} \delta_{jk} \cdot 2G(m^d_{jk}, m^d_{kj})]$$

$$+ (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{k \in A_{-i}} \delta_{jk} \cdot \tau \cdot 2G(m^d_{jk}, m^d_{kj})$$

$$+ (\mu \cdot \eta - \widetilde{\mu} \cdot \eta) \cdot \Bigg[\sum_{k \in B} \delta_{kk} \cdot \alpha \frac{n_k}{n_j}$$

$$+ \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_j} \cdot G(m^d_{kl}, m^d_{lk}) - \delta_{jj} \cdot \alpha - \sum_{k \in B_{-j}} \delta_{jk} \cdot 2G(m^d_{jk}, m^d_{kj}) \Bigg] \Bigg\}$$

Next, using the second equation in (31) yields:

For $i \in A$, $j \in B$:

$$
\begin{aligned}
\text{for } j \notin \Gamma_{iB},\ \frac{\dot{n}_{ij}^{d}}{n^{ij}} &= \frac{(1-\widetilde{\mu}\cdot\eta)\cdot\sum_{k\in A}\delta_{ik}\cdot a_{ik}}{n^{ij}} + \frac{(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot a_{ik}}{n^{ij}} \\
&\quad + \frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot(I_{AA}-\sum_{j\in A}\delta_{ij}\cdot a_{ij})}{n^{ij}} + \frac{\overleftrightarrow{\mu}\cdot(I_{\Gamma_i}-\sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot a_{ik})}{n^{ij}} \\
&= (1-\widetilde{\mu}\cdot\eta)\cdot[\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} + \sum_{k\in A_{-i}}\delta_{ik}\cdot\frac{a_{ik}}{n^{ij}}] + \frac{(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot a_{ik}}{n^{ij}} \\
&\quad +(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\left[\sum_{k\in A}\frac{\delta_{kk}\cdot\alpha\cdot n_k}{n^{ij}}\right. \\
&\quad \left. + \left(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{a_{kl}}{n^{ij}}\right)/2 - \frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} - \frac{\sum_{k\in A-i}\delta_{ik}\cdot a_{ik}}{n^{ij}}\right] \\
&\quad + \frac{\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot a_{kl}-\sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot a_{ik})}{n^{ij}} \\
&= (1-\widetilde{\mu}\cdot\eta)\cdot[\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} + \sum_{k\in A_{-i}}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ik}}{n_i}] + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ik}}{n_i} \\
&\quad +(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\left[\sum_{k\in A}\frac{\delta_{kk}\cdot\alpha\cdot n_k}{n^{ij}}\right. \\
&\quad \left. + \left(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n^{ij}}\frac{a_{kl}}{n_k}\right)/2 - \frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} - \sum_{k\in A_{-i}}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\frac{a_{ik}}{n_i}\right] \\
&\quad +\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_i}{n^{ij}}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kl}}{n_k} - \sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ik}}{n_i}) \\
&= \frac{n_i}{n^{ij}}\cdot\left\{(1-\widetilde{\mu}\cdot\eta)\cdot[\delta_{ii}\cdot\alpha + \sum_{k\in A_{-i}}\delta_{ik}\cdot\frac{a_{ik}}{n_i}] + (1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot\frac{a_{ik}}{n_i}\right. \\
&\quad +(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha\frac{n_k}{n_i} + \left(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\frac{a_{kl}}{n_k}\right)/2 - \delta_{ii}\cdot\alpha - \sum_{k\in A_{-i}}\delta_{ik}\cdot\frac{a_{ik}}{n_i}\right] \\
&\quad \left. +\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kl}}{n_k} - \sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\frac{a_{ik}}{n_i})\right\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \left(1-m_{ji}^{d}\right)\cdot\Bigg\{(1-\widetilde{\mu}\cdot\eta)\cdot[\delta_{ii}\cdot\alpha+\sum_{k\in A_{-i}}\delta_{ik}\cdot 2G(m_{ik}^{d},m_{ki}^{d})] \\
& +(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in B_{-j}}\delta_{ik}\cdot\tau\cdot 2G(m_{ik}^{d},m_{ki}^{d}) \\
& +(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in A}\delta_{kk}\cdot\alpha\frac{n_k}{n_i} \\
& +\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G(m_{kl}^{d},m_{lk}^{d})-\delta_{ii}\cdot\alpha-\sum_{k\in A_{-i}}\delta_{ik}\cdot 2G(m_{ik}^{d},m_{ki}^{d})\Bigg] \\
& +\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G(m_{kl}^{d},m_{lk}^{d})-\sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2G(m_{ik}^{d},m_{ki}^{d}))\Bigg\}
\end{aligned}
$$

Similarly, using the second equation in (32), we have:

For $i \in A$, $j \in B$:

$$
\begin{aligned}
\text{for } j \notin \Gamma_{iB},\ \frac{\dot{n}_{ji}^{d}}{n^{ij}} = & \left(1-m_{ij}^{d}\right)\cdot\Bigg\{(1-\widetilde{\mu}\cdot\eta)\cdot[\delta_{jj}\cdot\alpha+\sum_{k\in B_{-j}}\delta_{jk}\cdot 2G(m_{jk}^{d},m_{kj}^{d})] \\
& +(1-\widehat{\mu}\cdot\eta)\cdot\sum_{k\in A_{-i}}\delta_{jk}\cdot\tau\cdot 2G(m_{jk}^{d},m_{kj}^{d}) \\
& +(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in B}\delta_{kk}\cdot\alpha\frac{n_k}{n_j} \\
& +\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{n_k}{n_j}\cdot G(m_{kl}^{d},m_{lk}^{d})-\delta_{jj}\cdot\alpha-\sum_{k\in B_{-j}}\delta_{jk}\cdot 2G(m_{jk}^{d},m_{kj}^{d})\Bigg] \\
& +\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{jA}}\sum_{l\in\Gamma_{jB}}\delta_{kl}\cdot\frac{n_k}{n_j}\cdot\tau\cdot 2G(m_{kl}^{d},m_{lk}^{d})-\sum_{k\in\Gamma_{jB}}\delta_{jk}\cdot\tau\cdot 2G(m_{jk}^{d},m_{kj}^{d}))\Bigg\}
\end{aligned}
$$

Next, for $i \in A$ and $j \in B$, we calculate $\dot{n}_{ij}^{c}$. Using the second equation in

(29) yields:

For $i \in A$, $j \in B$:

$$\text{for } j \notin \Gamma_{iB},\ \frac{\dot{n}_{ij}^c}{n^{ij}} = \delta_{ij}\cdot\frac{a_{ij}}{n^{ij}} + \widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in A}\frac{\delta_{kk}\cdot a_{kk}}{n^{ij}} + \left(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{a_{kl}}{n^{ij}}\right)/2\right]$$
$$+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\frac{\delta_{kk}\cdot a_{kk}}{n^{ij}} + \left(\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{a_{kl}}{n^{ij}}\right)/2\right]$$
$$+\widehat{\mu}\cdot\eta\cdot\left[\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\frac{a_{kl}}{n^{ij}} - \delta_{ij}\cdot\frac{a_{ij}}{n^{ij}}\right]$$
$$= \delta_{ij}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ij}}{n_i} + \widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\frac{n_i}{n^{ij}}\cdot\frac{\alpha\cdot n_k}{n_i} + \left(\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_i}{n^{ij}}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kl}}{n_k}\right)/2\right]$$
$$+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\frac{n_i}{n^{ij}}\cdot\frac{\alpha\cdot n_k}{n_i} + \left(\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{n_i}{n^{ij}}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kl}}{n_k}\right)/2\right]$$
$$+\widehat{\mu}\cdot\eta\cdot\left[\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\frac{n_i}{n^{ij}}\cdot\frac{n_k}{n_i}\cdot\frac{a_{kl}}{n_k} - \delta_{ij}\cdot\frac{n_i}{n^{ij}}\cdot\frac{a_{ij}}{n_i}\right]$$

Thus, using equations (2), (3), (12), and (11), we have:

For $i \in A$, $j \in B$:

$$\text{for } j \notin \Gamma_{iB},\ \frac{\dot{n}_{ij}^c}{n^{ij}} = (1-m_{ji}^d)\cdot\Bigg\{\quad(1-\widehat{\mu}\cdot\eta)\cdot\delta_{ij}\cdot\tau\cdot 2G(m_{ij}^d, m_{ji}^d)$$
$$+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha\cdot\frac{n_k}{n_i} + \sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^d, m_{lk}^d\right)\right]$$
$$+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha\cdot\frac{n_k}{n_i} + \sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G\left(m_{kl}^d, m_{lk}^d\right)\right]$$
$$+\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\tau\cdot 2G\left(m_{kl}^d, m_{lk}^d\right)\Bigg\}$$

Likewise, using the first equation in (29), we have:

For $i \in A$, $j \in B$:

$$\begin{aligned}
\text{for } j \in \Gamma_{iB},\ \frac{\dot{n}^c_{ij}}{n^{ij}} &= \delta_{ij} \cdot \frac{a_{ij}}{n^{ij}} + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in A} \frac{\delta_{kk} \cdot a_{kk}}{n^{ij}} + \left(\sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{a_{kl}}{n^{ij}} \right) /2 \right] \\
&\quad + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in B} \frac{\delta_{kk} \cdot a_{kk}}{n^{ij}} + \left(\sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{a_{kl}}{n^{ij}} \right) /2 \right] \\
&\quad + \widehat{\mu} \cdot \eta \cdot \left[\sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{a_{kl}}{n^{ij}} - \delta_{ij} \cdot \frac{a_{ij}}{n^{ij}} \right] + \frac{\overleftrightarrow{\mu} \cdot (I_{\Gamma_i} - \delta_{ij} \cdot a_{ij})}{n^{ij}} \\
&= \delta_{ij} \cdot \frac{n_i}{n^{ij}} \cdot \frac{a_{ij}}{n_i} + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in A} \delta_{kk} \cdot \frac{n_i}{n^{ij}} \cdot \frac{\alpha \cdot n_k}{n_i} + \left(\sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_i}{n^{ij}} \cdot \frac{n_k}{n_i} \cdot \frac{a_{kl}}{n_k} \right) /2 \right] \\
&\quad + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in B} \delta_{kk} \cdot \frac{n_i}{n^{ij}} \cdot \frac{\alpha \cdot n_k}{n_i} + \left(\sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_i}{n^{ij}} \cdot \frac{n_k}{n_i} \cdot \frac{a_{kl}}{n_k} \right) /2 \right] \\
&\quad + \widehat{\mu} \cdot \eta \cdot \left[\sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{n_i}{n^{ij}} \cdot \frac{n_k}{n_i} \cdot \frac{a_{kl}}{n_k} - \delta_{ij} \cdot \frac{n_i}{n^{ij}} \cdot \frac{a_{ij}}{n_i} \right] \\
&\quad + \overleftrightarrow{\mu} \cdot \Big(\sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot \frac{n_i}{n^{ij}} \cdot \frac{n_k}{n_i} \cdot \frac{a_{kl}}{n_k} - \delta_{ij} \cdot \frac{n_i}{n^{ij}} \cdot \frac{a_{ij}}{n_i} \Big)
\end{aligned}$$

Thus, using equations (2), (3), (12), and (11), we have:

For $i \in A$, $j \in B$:

$$\begin{aligned}
\text{for } j \in \Gamma_{iB},\ \frac{\dot{n}^c_{ij}}{n^{ij}} &= (1 - m^d_{ji}) \cdot \Big\{ \quad (1 - \widehat{\mu} \cdot \eta) \cdot \delta_{ij} \cdot \tau \cdot 2G(m^d_{ij}, m^d_{ji}) \\
&\quad + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in A} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m^d_{kl}, m^d_{lk}\right) \right] \\
&\quad + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in B} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m^d_{kl}, m^d_{lk}\right) \right] \\
&\quad + \widehat{\mu} \cdot \eta \cdot \sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m^d_{kl}, m^d_{lk}\right) \\
&\quad + \overleftrightarrow{\mu} \cdot \Big(\sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m^d_{kl}, m^d_{lk}\right) - \delta_{ij} \cdot \tau \cdot 2G\left(m^d_{ij}, m^d_{ji}\right) \Big) \Big\}
\end{aligned}$$

Therefore, putting all of this together, we obtain:

$$
\begin{aligned}
\text{For } i &\in A,\, j \in B: \\
\text{for } j &\in \Gamma_{iB},\, \dot{m}_{ij}^d = \left(1 - m_{ij}^d\right) \cdot \frac{\dot{n}_{ij}^d}{n^{ij}} - m_{ij}^d \cdot \left(\frac{\dot{n}_{ij}^c}{n^{ij}} + \frac{\dot{n}_{ji}^d}{n^{ij}}\right) \\
&= (1 - m_{ij}^d)(1 - m_{ji}^d) \Bigg\{ (1 - \widetilde{\mu} \cdot \eta) \cdot [\delta_{ii} \cdot \alpha + \sum_{k \in A_{-i}} \delta_{ik} \cdot 2G(m_{ik}^d, m_{ki}^d)] \\
&\quad + (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{k \in B_{-j}} \delta_{ik} \cdot \tau \cdot 2G(m_{ik}^d, m_{ki}^d) + (\mu \cdot \eta - \widetilde{\mu} \cdot \eta) \cdot \Bigg[\sum_{k \in A} \delta_{kk} \cdot \alpha \frac{n_k}{n_i} \\
&\quad + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G(m_{kl}^d, m_{lk}^d) - \delta_{ii} \cdot \alpha - \sum_{k \in A_{-i}} \delta_{ik} \cdot 2G(m_{ik}^d, m_{ki}^d) \Bigg] \Bigg\} \\
&\quad - m_{ij}^d \cdot (1 - m_{ji}^d) \Bigg\{ \qquad (1 - \widehat{\mu} \cdot \eta) \cdot \delta_{ij} \cdot \tau \cdot 2G(m_{ij}^d, m_{ji}^d) \\
&\quad + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in A} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m_{kl}^d, m_{lk}^d\right) \right] \\
&\quad + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in B} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m_{kl}^d, m_{lk}^d\right) \right] \\
&\quad + \widehat{\mu} \cdot \eta \cdot \sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m_{kl}^d, m_{lk}^d\right) \\
&\quad + \overleftrightarrow{\mu} \cdot \Big(\sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m_{kl}^d, m_{lk}^d\right) - \delta_{ij} \cdot \tau \cdot 2G\left(m_{ij}^d, m_{ji}^d\right) \Big) \Bigg\} \\
&\quad - m_{ij}^d \cdot \left(1 - m_{ij}^d\right) \cdot \Bigg\{ (1 - \widetilde{\mu} \cdot \eta) \cdot [\delta_{jj} \cdot \alpha + \sum_{k \in B_{-j}} \delta_{jk} \cdot 2G(m_{jk}^d, m_{kj}^d)] \\
&\quad + (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{k \in A_{-i}} \delta_{jk} \cdot \tau \cdot 2G(m_{jk}^d, m_{kj}^d) + (\mu \cdot \eta - \widetilde{\mu} \cdot \eta) \cdot \Bigg[\sum_{k \in B} \delta_{kk} \cdot \alpha \frac{n_k}{n_j} \\
&\quad + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_j} \cdot G(m_{kl}^d, m_{lk}^d) - \delta_{jj} \cdot \alpha - \sum_{k \in B_{-j}} \delta_{jk} \cdot 2G(m_{jk}^d, m_{kj}^d) \Bigg] \Bigg\}
\end{aligned}
$$

and that gives us (78).

For $i \in A$, $j \in B$:

$$\text{for } j \notin \Gamma_{iB},\ \dot{m}_{ij}^{d} = \left(1 - m_{ij}^{d}\right) \cdot \frac{\dot{n}_{ij}^{d}}{n^{ij}} - m_{ij}^{d} \cdot \left(\frac{\dot{n}_{ij}^{c}}{n^{ij}} + \frac{\dot{n}_{ji}^{d}}{n^{ij}}\right)$$

$$\begin{aligned}
= \ & (1 - m_{ij}^{d})(1 - m_{ji}^{d}) \Bigg\{ (1 - \widetilde{\mu} \cdot \eta) \cdot [\delta_{ii} \cdot \alpha + \sum_{k \in A_{-i}} \delta_{ik} \cdot 2G(m_{ik}^{d}, m_{ki}^{d})] \\
& + (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{k \in B_{-j}} \delta_{ik} \cdot \tau \cdot 2G(m_{ik}^{d}, m_{ki}^{d}) + (\mu \cdot \eta - \widetilde{\mu} \cdot \eta) \cdot \Bigg[\sum_{k \in A} \delta_{kk} \cdot \alpha \frac{n_k}{n_i} \\
& + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G(m_{kl}^{d}, m_{lk}^{d}) - \delta_{ii} \cdot \alpha - \sum_{k \in A_{-i}} \delta_{ik} \cdot 2G(m_{ik}^{d}, m_{ki}^{d}) \Bigg] \\
& + \overleftrightarrow{\mu} \cdot \Big(\sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot \frac{n_k}{n_j} \cdot \tau \cdot 2G(m_{kl}^{d}, m_{lk}^{d}) - \sum_{k \in \Gamma_{iB}} \delta_{ik} \cdot \tau \cdot 2G(m_{ik}^{d}, m_{ki}^{d}) \Big) \Bigg\} \\
& - m_{ij}^{d} \cdot (1 - m_{ji}^{d}) \Bigg\{ \quad (1 - \widehat{\mu} \cdot \eta) \cdot \delta_{ij} \cdot \tau \cdot 2G(m_{ij}^{d}, m_{ji}^{d}) \\
& + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in A} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m_{kl}^{d}, m_{lk}^{d}\right) \right] \\
& + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in B} \delta_{kk} \cdot \alpha \cdot \frac{n_k}{n_i} + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G\left(m_{kl}^{d}, m_{lk}^{d}\right) \right] \\
& + \widehat{\mu} \cdot \eta \cdot \sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot \tau \cdot 2G\left(m_{kl}^{d}, m_{lk}^{d}\right) \Bigg\} \\
& - m_{ij}^{d} \cdot \left(1 - m_{ij}^{d}\right) \cdot \Bigg\{ (1 - \widetilde{\mu} \cdot \eta) \cdot [\delta_{jj} \cdot \alpha + \sum_{k \in B_{-j}} \delta_{jk} \cdot 2G(m_{jk}^{d}, m_{kj}^{d})] \\
& + (1 - \widehat{\mu} \cdot \eta) \cdot \sum_{k \in A_{-i}} \delta_{jk} \cdot \tau \cdot 2G(m_{jk}^{d}, m_{kj}^{d}) + (\mu \cdot \eta - \widetilde{\mu} \cdot \eta) \cdot \Bigg[\sum_{k \in B} \delta_{kk} \cdot \alpha \frac{n_k}{n_j} \\
& + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot \frac{n_k}{n_j} \cdot G(m_{kl}^{d}, m_{lk}^{d}) - \delta_{jj} \cdot \alpha - \sum_{k \in B_{-j}} \delta_{jk} \cdot 2G(m_{jk}^{d}, m_{kj}^{d}) \Bigg] \\
& + \overleftrightarrow{\mu} \cdot \Big(\sum_{k \in \Gamma_{jA}} \sum_{l \in \Gamma_{jB}} \delta_{kl} \cdot \frac{n_k}{n_j} \cdot \tau \cdot 2G(m_{kl}^{d}, m_{lk}^{d}) - \sum_{k \in \Gamma_{jB}} \delta_{jk} \cdot \tau \cdot 2G(m_{jk}^{d}, m_{kj}^{d}) \Big) \Bigg\}
\end{aligned}$$

and that gives us (79).

8.4 Technical Appendix for Proposition 2

8.4.1 Derivation of Dynamic Equations (105)-(109)

Focus on $i \in A$. We will write down the dynamics of n_i and m_{ij}^d associated with equation (103). Setting

$$\begin{aligned}
\delta_{ii} &= 0 & (129)\\
\delta_{ij} &= 0 \text{ for } i, j \in A,\ j \in \Gamma_{iA} & (130)\\
\delta_{ij} &= 0 \text{ for } i \in A,\ j \in B,\ j \notin \Gamma_{iB} & (131)\\
m_{ij}^d &= m^S \text{ for } i, j \in A,\ j \notin \Gamma_{iA} & (132)\\
m_{ij}^d &= m_{ji}^d = m^d < m^S \text{for } i, j \in A,\ j \in \Gamma_{iA} & (133)\\
m_{ij}^d &= m^B \text{ for } i \in A, j \in B, j \in \Gamma_{iB} & (134)
\end{aligned}$$

in (85):

$$\begin{aligned}
\text{For } i &\in A: \\
\frac{\dot{n}_i}{n_i} &= [(\sum_{\substack{j \in A \\ j \notin \Gamma_{iA}}} \delta_{ij}) \cdot 2g\left(m^S\right)] + (\sum_{j \in \Gamma_{iB}} \delta_{ij}) \cdot \tau \cdot 2g\left(m^B\right) \\
&+ \mu \cdot \eta \cdot [(\sum_{k \in A_{-i}} \sum_{l \in A_{-k}} \delta_{kl}) \cdot g\left(m^S\right)] \\
&+ \widetilde{\mu} \cdot \eta \cdot [(\sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl}) \cdot g\left(m^S\right))] \\
&+ \widehat{\mu} \cdot \eta \cdot [(\sum_{k \in A_{-i}} \sum_{l \in B} \delta_{kl}) \cdot \tau \cdot 2g\left(m^B\right)] \\
&+ \overleftrightarrow{\mu} \cdot [(\sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl}) \cdot \tau \cdot 2g\left(m^B\right) - (\sum_{j \in \Gamma_{iB}} \delta_{ij}) \cdot \tau \cdot 2g\left(m^B\right)]
\end{aligned}$$

Now setting

$$
\begin{aligned}
\text{For } i &\in A: \\
\sum_{j\in A_{-i}} \delta_{ij} &= \sum_{j\in A,\, j\notin\Gamma_{iA}} \delta_{ij} = \varphi_i && (135)\\
\sum_{j\in B} \delta_{ij} &= \sum_{j\in\Gamma_{iB}} \delta_{ij} = 1-\varphi_i && (136)\\
\sum_{l\in A} \delta_{kl} &= \sum_{l\in A_{-k}} \delta_{kl} = \sum_{l\in A,\, l\notin\Gamma_{kA}} \delta_{kl} = \varphi^* \text{ for } k\in A_{-i} && (137)\\
\sum_{l\in B} \delta_{kl} &= \sum_{l\in B_{-k}} \delta_{kl} = \sum_{l\in B,\, l\notin\Gamma_{kB}} \delta_{kl} = \varphi^* \text{ for } k\in B && (138)\\
\sum_{l\in B} \delta_{kl} &= \sum_{l\in B_{-k}} \delta_{kl} = 1-\varphi^* \text{ for } k\in A_{-i} && (139)\\
\sum_{l\in B} \delta_{kl} &= \sum_{l\in\Gamma_{iB}} \delta_{kl} = 1-\varphi^* \text{ for } k\in \Gamma_{iA},\, k\neq i && (140)
\end{aligned}
$$

and using (41) we have:

$$
\begin{aligned}
\text{For } i &\in A: \\
\frac{\dot{n}_i}{n_i} &= \varphi_i\cdot 2g\left(m^S\right) + (1-\varphi_i)\cdot 2g\left(m^S\right) \\
&\quad +\mu\cdot\eta\cdot(N-1)\cdot\varphi^*\cdot g\left(m^S\right) \\
&\quad +\widetilde{\mu}\cdot\eta\cdot N\cdot\varphi^*\cdot g\left(m^S\right) \\
&\quad +\widehat{\mu}\cdot\eta\cdot(N-1)\cdot(1-\varphi^*)\cdot 2g\left(m^S\right) \\
&\quad +\overleftrightarrow{\mu}\cdot\left[N^*\cdot(1-\varphi^*)\cdot 2g\left(m^S\right) - (1-\varphi^*)\cdot 2g\left(m^S\right)\right] \\
&= 2g\left(m^S\right) + [\mu\cdot\eta\cdot(N-1)\cdot\varphi^* + \widetilde{\mu}\cdot\eta\cdot N\cdot\varphi^* \\
&\quad +2\widehat{\mu}\cdot\eta\cdot(N-1)\cdot(1-\varphi^*) + 2\overleftrightarrow{\mu}\cdot(N^*-1)\cdot(1-\varphi^*)]\cdot g\left(m^S\right)
\end{aligned}
$$

Using (17), (18), (19), and (20):

$$
\begin{aligned}
\text{For } i &\in A: \\
\frac{\dot{n}_i}{n_i} &= 2g\left(m^S\right) + [C\cdot\frac{1}{1+\widetilde{\gamma}}\cdot\varphi^* + C\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* \\
&\quad +\widehat{C}\cdot(1-\varphi^*) + \overleftrightarrow{C}\cdot(1-\varphi^*)]\cdot g\left(m^S\right) \\
&= [2 + C\cdot\varphi^* + (\widehat{C}+\overleftrightarrow{C})\cdot(1-\varphi^*)]\cdot g\left(m^S\right)
\end{aligned}
$$

Then using (21):

$$
\begin{aligned}
\text{For } i &\in A: \\
\frac{\dot{n}_i}{n_i} &= g\left(m^S\right)\cdot[2+C]
\end{aligned}
$$

which leads to (105).

Next, from (86),

$$
\begin{aligned}
\text{For } i &\in A,\ j \in A\text{: for } j \in \Gamma_{iA}, \qquad (141)\\
\frac{\dot{m}_{ij}^d}{1-m_{ij}^d} &= (1-m_{ji}^d)\left\{[\delta_{ii}\cdot\alpha + \sum_{k\in A_{-\{i,j\}}} \delta_{ik}\cdot 2g(m_{ik}^d)]\right.\\
&\left. + \sum_{k\in B} \delta_{ik}\cdot\tau\cdot 2g(m_{ik}^d)\right\}\\
&-m_{ij}^d\cdot\left\{\qquad \delta_{ij}\cdot 2g(m_{ij}^d)\right.\\
&+\mu\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha + \sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right)\right]\\
&+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha + \sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right)\right]\\
&+\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^d\right)\\
&\left.+\overleftrightarrow{\mu}\cdot\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^d\right)\right\}\\
&-m_{ij}^d\cdot\left\{[\delta_{jj}\cdot\alpha + \sum_{k\in A_{-\{i,j\}}}\delta_{jk}\cdot 2g(m_{jk}^d)]\right.\\
&\left.+\sum_{k\in B}\delta_{jk}\cdot\tau\cdot 2g(m_{jk}^d)\right\}
\end{aligned}
$$

Using (129)-(134) and (135)-(140), we calculate each term in (141) separately:

- $$
\begin{aligned}
[\delta_{ii}\cdot\alpha + \sum_{k\in A_{-\{i,j\}}}\delta_{ik}\cdot 2g(m_{ik}^d)] &= 0 + \sum_{k\in\Gamma_{iA},\ k\neq i,j}\delta_{ik}\cdot 2g(m_{ik}^d) + \sum_{k\in A,\ k\notin\Gamma_{iA}}\delta_{ik}\cdot 2g(m_{ik}^d)\\
&= 0 + 0 + (\sum_{k\in A,\ k\notin\Gamma_{iA}}\delta_{ik})\cdot 2g(m^S)\\
&= \varphi_i\cdot 2g(m^S)
\end{aligned}
$$

- $$\begin{aligned}\sum_{k\in B}\delta_{ik}\cdot\tau\cdot 2g(m_{ik}^d) &= \sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2g(m_{ik}^d)+\sum_{k\in B,\,k\notin\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2g(m_{ik}^d)\\ &= (\sum_{k\in\Gamma_{iB}}\delta_{ik})\cdot\tau\cdot 2g(m^B)+0\\ &= (1-\varphi_i)\cdot 2g(m^S)\end{aligned}$$

- $$\delta_{ij}\cdot 2g(m_{ij}^d)=0$$

- $$\begin{aligned}\sum_{k\in A}\delta_{kk}\cdot\alpha+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right) &= 0+\sum_{k\in A_{-i}}(\sum_{l\in A_{-k},\,l\notin\Gamma_{kA}}\delta_{kl})\cdot g\left(m^S\right)+\sum_{l\in A_{-k}}\delta_{il}\cdot g\left(m_{il}^d\right)\\ &= \sum_{k\in A_{-i}}\varphi^*\cdot g\left(m^S\right)+\varphi_i\cdot g\left(m^S\right)\\ &= [(N-1)\cdot\varphi^*+\varphi_i]\cdot g\left(m^S\right)\end{aligned}$$

- $$\sum_{k\in B}\delta_{kk}\cdot\alpha+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right)=0+N\cdot\varphi^*\cdot g\left(m^S\right)$$

- $$\begin{aligned}\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^d\right) &= \sum_{k\in A_{-i}}(1-\varphi^*)\cdot\tau\cdot 2g\left(m^B\right)+\sum_{l\in B}\delta_{il}\cdot\tau\cdot 2g\left(m_{il}^d\right)\\ &= (N-1)\cdot(1-\varphi^*)\cdot\tau\cdot 2g\left(m^B\right)+(1-\varphi_i)\cdot\tau\cdot 2g\left(m^B\right)\\ &= [(N-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\cdot\tau\cdot 2g\left(m^B\right)\\ &= [(N-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\cdot 2g\left(m^S\right)\end{aligned}$$

- $$\begin{aligned}\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^d\right) &= (\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl})\cdot\tau\cdot 2g\left(m^B\right)\\ &= \{\sum_{k\in\Gamma_{iA},\,k\neq i}(1-\varphi^*)+(1-\varphi_i)\}\cdot 2g\left(m^S\right)\\ &= [(N^*-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\cdot 2g\left(m^S\right)\end{aligned}$$

- $$\begin{aligned}[\delta_{jj}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{jk}\cdot 2g(m_{jk}^d)] &= 0+\sum_{k\in A_{-j}}\delta_{jk}\cdot 2g(m_{jk}^d)-\delta_{ji}\cdot 2g(m_{ji}^d)\\ &= 0+\varphi^*\cdot 2g\left(m^S\right)-0=\varphi^*\cdot 2g\left(m^S\right)\end{aligned}$$

•

$$\sum_{k\in B}\delta_{jk}\cdot\tau\cdot 2g(m_{jk}^d)=(1-\varphi^*)\cdot\tau\cdot 2g\left(m^B\right)=(1-\varphi^*)\cdot 2g\left(m^S\right)$$

Substituting these terms into equation (141), we have:

$$\begin{aligned}
\text{For } i &\in A,\, j\in A\text{:}\;\; \text{for } j\in\Gamma_{iA}, \qquad (142)\\
\frac{\dot{m}_{ij}^d}{1-m^d} &= (1-m^d)\left\{\varphi_i\cdot 2g(m^S)+(1-\varphi_i)\cdot 2g(m^S)\right\}\\
&\quad -m^d\cdot\{\;0+\mu\cdot\eta\cdot[(N-1)\cdot\varphi^*+\varphi_i]\cdot g\left(m^S\right)\\
&\quad +\widetilde{\mu}\cdot\eta\cdot\left[N\cdot\varphi^*\cdot g\left(m^S\right)\right]\\
&\quad +\widehat{\mu}\cdot\eta\cdot[(N-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\cdot 2g\left(m^S\right)\\
&\quad +\overleftrightarrow{\mu}\cdot[(N^*-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\cdot 2g\left(m^S\right)\}\\
&\quad -m^d\cdot\left\{\varphi^*\cdot 2g\left(m^S\right)+(1-\varphi^*)\cdot 2g\left(m^S\right)\right\}\\
&= (1-m^d)\cdot 2g(m^S)\\
&\quad -m^d\cdot g\left(m^S\right)\{\;\mu\cdot\eta\cdot[(N-1)\cdot\varphi^*+\varphi_i]\\
&\quad +\widetilde{\mu}\cdot\eta\cdot N\cdot\varphi^*\\
&\quad +2\widehat{\mu}\cdot\eta\cdot[(N-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\\
&\quad +2\overleftrightarrow{\mu}\cdot[(N^*-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\}\\
&\quad -m^d\cdot 2g\left(m^S\right)
\end{aligned}$$

$$\begin{aligned}
&= (1-2m^d)\cdot 2g(m^S)\\
&\quad -m^d\cdot g\left(m^S\right)\{\;\mu\cdot\eta\cdot(N-1)\cdot\varphi^*+\;\mu\cdot\eta\cdot\varphi_i\\
&\quad +\widetilde{\mu}\cdot\eta\cdot N\cdot\varphi^*\\
&\quad +2\widehat{\mu}\cdot\eta\cdot(N-1)\cdot(1-\varphi^*)+2\widehat{\mu}\cdot\eta\cdot(1-\varphi_i)\\
&\quad +2\overleftrightarrow{\mu}\cdot(N^*-1)\cdot(1-\varphi^*)+2\overleftrightarrow{\mu}\cdot(1-\varphi_i)\}
\end{aligned}$$

Using (17)-(20) and using the approximations $\mu \cdot \eta \cdot \varphi_i = 0$ and $\widehat{\mu} \cdot \eta \cdot (1 - \varphi_i) = 0$,

$$
\begin{aligned}
\text{For } i &\in A,\ j \in A\text{:}\ \ \text{for } j \in \Gamma_{iA}, \\
\frac{\dot{m}_{ij}^d}{1 - m^d} &= (1 - 2m^d) \cdot 2g(m^S) \\
&\quad - m^d \cdot g(m^S) \cdot \left\{ \frac{C}{1 + \widetilde{\gamma}} \cdot \varphi^* + \frac{C \cdot \widetilde{\gamma}}{1 + \widetilde{\gamma}} \cdot \varphi^* + 0 \right. \\
&\quad + \widehat{C} \cdot (1 - \varphi^*) + 0 \\
&\quad \left. + \overleftrightarrow{C} \cdot (1 - \varphi^*) + \frac{\overleftrightarrow{C}}{N^* - 1} \cdot (1 - \varphi_i) \right\} \\
&= (1 - 2m^d) \cdot 2g(m^S) \\
&\quad - m^d \cdot g(m^S) \cdot \left\{ C \cdot \varphi^* + (\widehat{C} + \overleftrightarrow{C}) \cdot (1 - \varphi^*) \right. \\
&\quad \left. + \frac{\overleftrightarrow{C}}{N^* - 1} \cdot (1 - \varphi_i) \right\} \\
&= (1 - 2m^d) \cdot 2g(m^S) \\
&\quad - m^d \cdot g(m^S) \cdot \left\{ C \cdot \varphi^* + C \cdot (1 - \varphi^*) + \frac{\overleftrightarrow{C}}{N^* - 1} \cdot (1 - \varphi_i) \right\} \\
&= (1 - 2m^d) \cdot 2g(m^S) - m^d \cdot g(m^S) \cdot \left\{ C + \frac{\overleftrightarrow{C}}{N^* - 1} \cdot (1 - \varphi_i) \right\} \\
&= 2g(m^S) \cdot \left\{ (1 - 2m^d) - m^d \cdot \left[\frac{C}{2} + \frac{\overleftrightarrow{C}}{2(N^* - 1)} \cdot (1 - \varphi_i) \right] \right\}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\text{For } i &\in A,\ j \in A\text{:}\ \ \text{for } j \in \Gamma_{iA}, \qquad (143) \\
\dot{m}_{ij}^d &= (1 - m^d) \cdot 2g(m^S) \cdot \left\{ 1 - (2 + \frac{C}{2}) \cdot m^d - m^d \cdot \left[\frac{\overleftrightarrow{C}}{2(N^* - 1)} \cdot (1 - \varphi_i) \right] \right\} \\
\text{where } m_{ij}^d &= m_{ji}^d = m^d
\end{aligned}
$$

which gives us (106).

Next, from (87),

$$
\begin{aligned}
\text{For } i &\in A,\, j \in A\text{: for } j \notin \Gamma_{iA}, \qquad (144)\\
\frac{\dot{m}^d_{ij}}{(1-m^d_{ij})} &= (1-m^d_{ji}) \cdot \Bigg\{ [\delta_{ii} \cdot \alpha + \sum_{k \in A_{-\{i,j\}}} \delta_{ik} \cdot 2g(m^d_{ik})] \\
&\quad + \sum_{k \in B} \delta_{ik} \cdot \tau \cdot 2g(m^d_{ik}) \\
&\quad + \overleftrightarrow{\mu} \cdot \left(\sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot \tau \cdot 2g(m^d_{kl}) - \sum_{k \in \Gamma_{iB}} \delta_{ik} \cdot \tau \cdot 2g(m^d_{ik}) \right) \Bigg\} \\
&\quad - m^d_{ij} \cdot \Bigg\{ \qquad \delta_{ij} \cdot 2g(m^d_{ij}) \\
&\quad + \mu \cdot \eta \cdot \left[\sum_{k \in A} \delta_{kk} \cdot \alpha + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot g\left(m^d_{kl}\right) \right] \\
&\quad + \widetilde{\mu} \cdot \eta \cdot \left[\sum_{k \in B} \delta_{kk} \cdot \alpha + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot g\left(m^d_{kl}\right) \right] \\
&\quad + \widehat{\mu} \cdot \eta \cdot \sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \tau \cdot 2g\left(m^d_{kl}\right) \Bigg\} \\
&\quad - m^d_{ij} \cdot \Bigg\{ [\delta_{jj} \cdot \alpha + \sum_{k \in A_{-\{i,j\}}} \delta_{jk} \cdot 2g(m^d_{jk})] \\
&\quad + \sum_{k \in B} \delta_{jk} \cdot \tau \cdot 2g(m^d_{jk}) \\
&\quad + \overleftrightarrow{\mu} \cdot \left(\sum_{k \in \Gamma_{jA}} \sum_{l \in \Gamma_{jB}} \delta_{kl} \cdot \tau \cdot 2g(m^d_{kl}) - \sum_{k \in \Gamma_{jB}} \delta_{jk} \cdot \tau \cdot 2g(m^d_{jk}) \right) \Bigg\}
\end{aligned}
$$

where

-
$$
\begin{aligned}
[\delta_{ii} \cdot \alpha + \sum_{k \in A_{-\{i,j\}}} \delta_{ik} \cdot 2g(m^d_{ik})] &= 0 + (\sum_{k \in A,\, k \notin \Gamma_{iA}} \delta_{ik} - \delta_{ij}) \cdot 2g(m^S) \\
&= (\varphi_i - \delta_{ij}) \cdot 2g(m^S)
\end{aligned}
$$

-
$$
\begin{aligned}
\sum_{k \in B} \delta_{ik} \cdot \tau \cdot 2g(m^d_{ik}) &= (\sum_{k \in \Gamma_{iB}} \delta_{ik}) \cdot \tau \cdot 2g(m^B) \\
&= (1 - \varphi_i) \cdot 2g(m^S)
\end{aligned}
$$

- $$\begin{aligned}
&\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g(m^d_{kl})-\sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2g(m^d_{ik})\\
=\;&[\sum_{k\in\Gamma_{iA},\,k\neq i}(1-\varphi^*)\cdot\tau\cdot 2g(m^B)+\sum_{l\in\Gamma_{iB}}\delta_{il}\cdot\tau\cdot 2g(m^B)]-\sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2g(m^B)\\
=\;&\sum_{k\in\Gamma_{iA},\,k\neq i}(1-\varphi^*)\cdot\tau\cdot 2g(m^B)\\
=\;&(N^*-1)\cdot(1-\varphi^*)\cdot\tau\cdot 2g(m^B)=(N^*-1)\cdot(1-\varphi^*)\cdot 2g(m^S)
\end{aligned}$$

- $$\delta_{ij}\cdot 2g(m^d_{ij})=\delta_{ij}\cdot 2g(m^S)$$

- $$\begin{aligned}
&\sum_{k\in A}\delta_{kk}\cdot\alpha+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right)\\
=\;&0+\sum_{k\in A_{-i}}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right)+\sum_{l\in A_{-i}}\delta_{il}\cdot g\left(m^d_{il}\right)\\
=\;&\sum_{k\in A_{-i}}\sum_{l\in A,\,l\notin\Gamma_{kA}}\delta_{kl}\cdot g\left(m^S\right)+\sum_{l\in A,\,l\notin\Gamma_{iA}}\delta_{il}\cdot g\left(m^S\right)\\
=\;&\sum_{k\in A_{-i}}\varphi^*\cdot g\left(m^S\right)+\varphi_i\cdot g\left(m^S\right)\\
=\;&[(N-1)\cdot\varphi^*+\varphi_i]\cdot g\left(m^S\right)
\end{aligned}$$

- $$\begin{aligned}
\sum_{k\in B}\delta_{kk}\cdot\alpha+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right)&=0+\sum_{k\in B}\sum_{l\in B,\,l\notin\Gamma_{kB}}\delta_{kl}\cdot g\left(m^S\right)\\
&=\sum_{k\in B}\varphi^*\cdot g\left(m^S\right)=N\cdot\varphi^*\cdot g\left(m^S\right)
\end{aligned}$$

- As shown earlier,

  $$\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right)=[(N-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\cdot 2g\left(m^S\right)$$

- $$\begin{aligned}
[\delta_{jj}\cdot\alpha+\sum_{k\in A_{-\{i,j\}}}\delta_{jk}\cdot 2g(m^d_{jk})]&=0+\sum_{k\in A_{-,j}}\delta_{jk}\cdot 2g(m^d_{jk})-\delta_{ji}\cdot 2g(m^d_{ji})\\
&=0+\varphi^*\cdot 2g(m^S)-\delta_{ji}\cdot 2g(m^S)\\
&=(\varphi^*-\delta_{ji})\cdot 2g(m^S)
\end{aligned}$$

- As shown earlier,

$$\sum_{k\in B}\delta_{jk}\cdot\tau\cdot 2g(m^d_{jk}) = (1-\varphi^*)\cdot 2g\left(m^S\right)$$

- $$\begin{aligned}
& \sum_{k\in\Gamma_{jA}}\sum_{l\in\Gamma_{jB}}\delta_{kl}\cdot\tau\cdot 2g(m^d_{kl}) - \sum_{k\in\Gamma_{jB}}\delta_{jk}\cdot\tau\cdot 2g(m^d_{jk}) \\
= & \ [\sum_{k\in\Gamma_{jA},\ k\neq j}(1-\varphi^*)\cdot\tau\cdot 2g(m^B) + \sum_{l\in\Gamma_{jB}}\delta_{jl}\cdot\tau\cdot 2g(m^B)] - \sum_{k\in\Gamma_{jB}}\delta_{jk}\cdot\tau\cdot 2g(m^B) \\
= & \sum_{k\in\Gamma_{jA},\ k\neq j}(1-\varphi^*)\cdot\tau\cdot 2g(m^B) \\
= & \ (N^*-1)\cdot(1-\varphi^*)\cdot\tau\cdot 2g(m^B) = (N^*-1)\cdot(1-\varphi^*)\cdot 2g(m^S)
\end{aligned}$$

Substituting these terms into equation (144), and setting $m^d_{ij} = m^d_{ji} = m^S$ we have:

$$\begin{aligned}
\text{For } i \in & \ A,\ j\in A\text{: for } j\notin\Gamma_{iA}, \\
\frac{\dot{m}^d_{ij}}{1-m^S} = & \ (1-m^S)\cdot\{\ (\varphi_i-\delta_{ij})\cdot 2g(m^S) \\
& +(1-\varphi_i)\cdot 2g(m^S) \\
& +\overleftrightarrow{\mu}\cdot(N^*-1)\cdot(1-\varphi^*)\cdot 2g(m^S)\} \\
& -m^S\cdot\{\ \delta_{ij}\cdot 2g(m^S) \\
& +\mu\cdot\eta\cdot[[(N-1)\cdot\varphi^*+\varphi_i]\cdot g(m^S)] \\
& +\widetilde{\mu}\cdot\eta\cdot[N\cdot\varphi^*\cdot g(m^S)] \\
& +\widehat{\mu}\cdot\eta\cdot[(N-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\cdot 2g(m^S)\ \} \\
& -m^S\cdot\{\ (\varphi^*-\delta_{ji})\cdot 2g(m^S) \\
& +(1-\varphi^*)\cdot 2g(m^S) \\
& +\overleftrightarrow{\mu}\cdot(N^*-1)\cdot(1-\varphi^*)\cdot 2g(m^S)\}
\end{aligned}$$

$$
\begin{aligned}
= \;& 2g(m^S) \cdot \Big\{ \;\; (1 - m^S) \,\{\, (\varphi_i - \delta_{ij}) \\
& +(1 - \varphi_i) \\
& + \overleftrightarrow{\mu} \cdot (N^* - 1) \cdot (1 - \varphi^*) \} \\
& -m^S \cdot \{\, \delta_{ij} \\
& + \frac{\mu \cdot \eta}{2} \cdot [(N-1) \cdot \varphi^* + \varphi_i] \\
& + \frac{\widetilde{\mu} \cdot \eta}{2} \cdot [N \cdot \varphi^*] \\
& + \widehat{\mu} \cdot \eta \cdot [(N-1) \cdot (1 - \varphi^*) + (1 - \varphi_i)] \,\} \\
& -m^S \cdot \{\, (\varphi^* - \delta_{ji}) \\
& +(1 - \varphi^*) \\
& + \overleftrightarrow{\mu} \cdot (N^* - 1) \cdot (1 - \varphi^*) \} \;\; \}
\end{aligned}
$$

$$
\begin{aligned}
= \;& 2g(m^S) \cdot \Big\{ \;\; (1 - m^S) \cdot \{\, (1 - \delta_{ij}) \\
& + \overleftrightarrow{\mu} \cdot (N^* - 1) \cdot (1 - \varphi^*) \} \\
& -m^S \cdot \{\, \delta_{ij} \\
& + \frac{\mu \cdot \eta}{2} \cdot (N-1) \cdot \varphi^* + \frac{\mu \cdot \eta}{2} \cdot \varphi_i \\
& + \frac{\widetilde{\mu} \cdot \eta}{2} \cdot [N \cdot \varphi^*] \\
& + \widehat{\mu} \cdot \eta \cdot (N-1) \cdot (1 - \varphi^*) + \widehat{\mu} \cdot \eta \cdot (1 - \varphi_i) \\
& + 1 - \delta_{ji} \\
& + \overleftrightarrow{\mu} \cdot (N^* - 1) \cdot (1 - \varphi^*) \} \;\; \}
\end{aligned}
$$

Using (17)-(20) and using the approximations $\mu{\cdot}\eta{\cdot}\varphi_i = 0$ and $\widehat{\mu}{\cdot}\eta{\cdot}(1-\varphi_i) = 0$,

$$
\begin{aligned}
\text{For } i &\in A,\ j \in A\text{:}\quad \text{for } j \notin \Gamma_{iA},\\
\frac{\dot{m}^d_{ij}}{1-m^S} &= 2g(m^S)\cdot\Bigg\{ \quad (1-m^S)\Bigg\{ \quad (1-\delta_{ij}) \\
&\quad + \frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)\Bigg\} \\
&\quad - m^S\cdot\Bigg\{ \quad \delta_{ij} +\ 1-\delta_{ji} \\
&\quad + \frac{C}{2}\cdot\frac{1}{1+\widetilde{\gamma}}\cdot\varphi^* + 0 \\
&\quad + \frac{C}{2}\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* \\
&\quad + \frac{\widehat{C}}{2}\cdot(1-\varphi^*) + 0 \\
&\quad + \frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)\Bigg\}\ \Bigg\}
\end{aligned}
$$

From equation (21), $\widehat{C} + \overleftrightarrow{C} = C$, and by definition of myopic core, $\delta_{ij} = \delta_{ji}$. Hence,

$$
\begin{aligned}
\text{For } i &\in A,\ j \in A\text{:}\quad \text{for } j \notin \Gamma_{iA},\\
\frac{\dot{m}^d_{ij}}{1-m^S} &= 2g(m^S)\cdot\Bigg[(1-m^S)\cdot\Bigg\{ (1-\delta_{ij}) + \frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)\Bigg\} \\
&\quad - m^S\cdot\Bigg\{1 + \frac{C}{2}\cdot\varphi^* + \frac{C}{2}\cdot(1-\varphi^*)\Bigg\}\Bigg] \\
&= 2g(m^S)\cdot\Bigg\{ 1 - m^S - (1-m^S)\cdot\delta_{ij} + (1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*) \\
&\quad - m^S - \frac{C}{2}\cdot m^S \ \Bigg\} \\
&= 2g(m^S)\cdot\Bigg\{ 1 - 2m^S + (1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*) \\
&\quad - \frac{C}{2}\cdot m^S \ - (1-m^S)\cdot\delta_{ij}\Bigg\}
\end{aligned}
$$

Hence,

$$\begin{aligned}
\text{For } i &\in A,\ j \in A\text{:}\ \ \text{for } j \notin \Gamma_{iA}, \qquad (145)\\
\dot{m}_{ij}^{d} &= (1-m^S)\cdot 2g(m^S)\cdot\left\{ 1-(2+\frac{C}{2})m^S+(1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)\right.\\
&\quad \left. -(1-m^S)\cdot\delta_{ij}\right\}\\
\text{where } m_{ij}^{d} &= m_{ji}^{d}=m^S
\end{aligned}$$

which gives us (107).

Likewise, from (88),

$$
\begin{aligned}
&\text{For } i \in A,\ j \in B\text{: for } j \in \Gamma_{iB}, \qquad (146)\\
\frac{\dot{m}_{ij}^d}{1-m_{ij}^d} &= (1-m_{ji}^d)\Bigg\{[\delta_{ii}\cdot\alpha + \sum_{k\in A_{-i}} \delta_{ik}\cdot 2g(m_{ik}^d)] \\
&+ \sum_{k\in B_{-j}} \delta_{ik}\cdot\tau\cdot 2g(m_{ik}^d) + (\mu\cdot\eta - \widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in A}\delta_{kk}\cdot\alpha \\
&+ \sum_{k\in A}\sum_{l\in A_{-k}} \delta_{kl}\cdot g(m_{kl}^d) - \delta_{ii}\cdot\alpha - \sum_{k\in A_{-i}} \delta_{ik}\cdot 2g(m_{ik}^d)\Bigg]\Bigg\} \\
&- m_{ij}^d\Bigg\{ \quad \delta_{ij}\cdot\tau\cdot 2g(m_{ij}^d) \\
&+ \widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha + \sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right)\right] \\
&+ \widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha + \sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right)\right] \\
&+ \widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^d\right) \\
&+ \overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^d\right) - \delta_{ij}\cdot\tau\cdot 2g\left(m_{ij}^d\right))\Bigg\} \\
&- m_{ij}^d\cdot\Bigg\{[\delta_{jj}\cdot\alpha + \sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m_{jk}^d)] \\
&+ \sum_{k\in A_{-i}}\delta_{jk}\cdot\tau\cdot 2g(m_{jk}^d) + (\mu\cdot\eta - \widetilde{\mu}\cdot\eta)\cdot\Bigg[\sum_{k\in B}\delta_{kk}\cdot\alpha \\
&+ \sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g(m_{kl}^d) - \delta_{jj}\cdot\alpha - \sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m_{jk}^d)\Bigg]\Bigg\}
\end{aligned}
$$

Using (129)-(134) and (135)-(140), we calculate each term in (146) separately:

- $$
\begin{aligned}
\delta_{ii}\cdot\alpha + \sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m_{ik}^d) &= 0 + \sum_{k\in\Gamma_{iA},\ k\neq i}\delta_{ik}\cdot 2g(m_{ik}^d) + \sum_{k\in A,\ k\notin\Gamma_{iA}}\delta_{ik}\cdot 2g(m_{ik}^d) \\
&= 0 + 0 + (\sum_{k\in A,\ k\notin\Gamma_{iA}}\delta_{ik})\cdot 2g(m^S) \\
&= \varphi_i\cdot 2g(m^S)
\end{aligned}
$$

- $$\begin{aligned}\sum_{k\in B_{-j}} \delta_{ik}\cdot\tau\cdot 2g(m^d_{ik}) &= \sum_{k\in B}\delta_{ik}\cdot\tau\cdot 2g(m^d_{ik}) - \delta_{ij}\cdot\tau\cdot 2g(m^d_{ik})\\ &= (1-\varphi_i)\cdot 2g(m^S) - \delta_{ij}\cdot\tau\cdot 2g(m^B)\\ &= (1-\varphi_i)\cdot 2g(m^S) - \delta_{ij}\cdot 2g(m^S)\\ &= [(1-\varphi_i)-\delta_{ij}]\cdot 2g(m^S)\end{aligned}$$

- As shown earlier,

$$\begin{aligned}&\sum_{k\in A}\delta_{kk}\cdot\alpha + \sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g(m^d_{kl}) - \delta_{ii}\cdot\alpha - \sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m^d_{ik})\\ =\ & [(N-1)\cdot\varphi^* + \varphi_i]\cdot g(m^S) - 0 - \varphi_i\cdot 2g(m^S)\\ =\ & [(N-1)\cdot\varphi^* - \varphi_i]\cdot g(m^S)\end{aligned}$$

- $$\delta_{ij}\cdot\tau\cdot 2g(m^d_{ij}) = \delta_{ij}\cdot\tau\cdot 2g(m^B) = \delta_{ij}\cdot 2g(m^S)$$

- As shown earlier,

$$\sum_{k\in A}\delta_{kk}\cdot\alpha + \sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right) = [(N-1)\cdot\varphi^* + \varphi_i]\cdot g(m^S)$$

- As shown earlier,

$$\sum_{k\in B}\delta_{kk}\cdot\alpha + \sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m^d_{kl}\right) = N\cdot\varphi^*\cdot g(m^S)$$

- As shown earlier,

$$\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right) = [(N-1)\cdot(1-\varphi^*) + (1-\varphi_i)]\cdot 2g(m^S)$$

- As shown earlier,

$$\begin{aligned}&\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g\left(m^d_{kl}\right) - \delta_{ij}\cdot\tau\cdot 2g\left(m^d_{ij}\right)\\ =\ & [(N^*-1)\cdot(1-\varphi^*) + (1-\varphi_i)]\cdot 2g(m^S) - \delta_{ij}\cdot\tau\cdot 2g\left(m^B\right)\\ =\ & [(N^*-1)\cdot(1-\varphi^*) + (1-\varphi_i)]\cdot 2g(m^S) - \delta_{ij}\cdot 2g\left(m^S\right)\end{aligned}$$

- $$\delta_{jj}\cdot\alpha + \sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m^d_{jk}) = 0 + \varphi^*\cdot 2g\left(m^S\right) = \varphi^*\cdot 2g\left(m^S\right)$$

- $$\sum_{k\in A_{-i}} \delta_{jk}\cdot\tau\cdot 2g(m_{jk}^d) = (\sum_{k\in\Gamma_{iA},\, k\neq i} \delta_{jk})\cdot\tau\cdot 2g(m^B) = (1-\varphi^* - \delta_{ji})\cdot\tau\cdot 2g(m^B)$$
$$= (1-\varphi^* - \delta_{ji})\cdot 2g\left(m^S\right)$$

- $$\begin{aligned}&\sum_{k\in B}\delta_{kk}\cdot\alpha + \sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g(m_{kl}^d) - \delta_{jj}\cdot\alpha - \sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m_{jk}^d)\\ =\ & [(N-1)\cdot\varphi^* + \varphi_j]\cdot g(m^S) - 0 - \varphi_j\cdot 2g(m^S)\\ =\ & [(N-1)\cdot\varphi^* + \varphi^*]\cdot g(m^S) - \varphi^*\cdot 2g(m^S)\\ =\ & (N-2)\cdot\varphi^*\cdot g(m^S)\end{aligned}$$

Substituting each term into (146) and setting $m_{ij}^d = m_{ji}^d = m^B$,

$$\begin{aligned}\text{For } i &\in A,\ j\in B\text{: for } j\in\Gamma_{iB},\\ \frac{\dot m_{ij}^d}{1-m^B} &= (1-m^B)\left\{\ [\varphi_i\cdot 2g(m^S)]\right.\\ &\quad +[(1-\varphi_i)-\delta_{ij}]\cdot 2g(m^S) + (\mu\cdot\eta - \widetilde{\mu}\cdot\eta)\cdot[(N-1)\cdot\varphi^* - \varphi_i]\cdot g(m^S)\ \left.\right\}\\ &\quad -m^B\left\{\ \delta_{ij}\cdot 2g(m^S)\right.\\ &\quad +\widetilde{\mu}\cdot\eta\cdot[(N-1)\cdot\varphi^* + \varphi_i]\cdot g(m^S)\\ &\quad +\widetilde{\mu}\cdot\eta\cdot N\cdot\varphi^*\cdot g(m^S)\\ &\quad +\widehat{\mu}\cdot\eta\cdot[(N-1)\cdot(1-\varphi^*) + (1-\varphi_i)]\cdot 2g(m^S)\\ &\quad +\overleftrightarrow{\mu}\cdot\left([(N^*-1)\cdot(1-\varphi^*) + (1-\varphi_i)]\cdot 2g(m^S) - \delta_{ij}\cdot 2g\left(m^S\right)\right)\left.\right\}\\ &\quad -m^B\cdot\left\{\varphi^*\cdot 2g\left(m^S\right)\right.\\ &\quad +(1-\varphi^*-\delta_{ji})\cdot 2g\left(m^S\right) + (\mu\cdot\eta - \widetilde{\mu}\cdot\eta)\cdot(N-2)\cdot\varphi^*\cdot g(m^S)\ \left.\right\}\end{aligned}$$

Thus,

$$
\begin{aligned}
\frac{\dot{m}_{ij}^{d}}{1-m^B} \;=\;& 2g(m^S)\cdot\Bigg\{ \quad (1-m^B)\Bigg\{ \quad \varphi_i \\
& +[(1-\varphi_i)-\delta_{ij}]+\frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)}{2}\cdot[(N-1)\cdot\varphi^*-\varphi_i]\Bigg\} \\
& -m^B\,\{\ \delta_{ij} \\
& +\frac{\widetilde{\mu}\cdot\eta}{2}\cdot[(N-1)\cdot\varphi^*+\varphi_i] \\
& +\frac{\widetilde{\mu}\cdot\eta}{2}\cdot N\cdot\varphi^* \\
& +\widehat{\mu}\cdot\eta\cdot[(N-1)\cdot(1-\varphi^*)+(1-\varphi_i)] \\
& +\overleftrightarrow{\mu}\cdot([(N^*-1)\cdot(1-\varphi^*)+(1-\varphi_i)]-\delta_{ij}) \\
& \varphi^*+(1-\varphi^*-\delta_{ji})+\frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)}{2}\cdot(N-2)\cdot\varphi^* \quad \Bigg\}\ \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\dot{m}_{ij}^{d}}{1-m^B} \;=\;& 2g(m^S)\cdot\Bigg\{ \quad (1-m^B)\ \Bigg\{1-\delta_{ij}+\frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)}{2}\cdot[(N-1)\cdot\varphi^*-\varphi_i]\Bigg\} \\
& -m^B\,\{\ \delta_{ij}-\delta_{ji} \\
& +\frac{\widetilde{\mu}\cdot\eta}{2}\cdot[(N-1)\cdot\varphi^*+\varphi_i] \\
& +\frac{\widetilde{\mu}\cdot\eta}{2}\cdot N\cdot\varphi^* \\
& +\widehat{\mu}\cdot\eta\cdot[(N-1)\cdot(1-\varphi^*)+(1-\varphi_i)] \\
& +\overleftrightarrow{\mu}\cdot([(N^*-1)\cdot(1-\varphi^*)+(1-\varphi_i)]-\delta_{ij}) \\
& +1+\frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)}{2}\cdot(N-2)\cdot\varphi^*\Bigg\}\ \Bigg\}
\end{aligned}
$$

By definition of myopic core, $\delta_{ij}=\delta_{ji}$. Using approximations $\frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)}{2}\cdot\varphi_i=0$, $\frac{\widetilde{\mu}\cdot\eta}{2}\cdot\varphi_i=0$, and $\widehat{\mu}\cdot\eta\cdot(1-\varphi_i)=0$, we have

$$
\begin{aligned}
\frac{\dot{m}_{ij}^{d}}{1-m^B} \;=\;& 2g(m^S)\cdot\Bigg\{ \quad (1-m^B)\ \Bigg\{1-\delta_{ij}+\frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)}{2}\cdot(N-1)\cdot\varphi^*\Bigg\} \\
& -m^B\ \Bigg\{\frac{\widetilde{\mu}\cdot\eta}{2}\cdot(N-1)\cdot\varphi^*+\frac{\widetilde{\mu}\cdot\eta}{2}\cdot N\cdot\varphi^* \\
& +\widehat{\mu}\cdot\eta\cdot(N-1)\cdot(1-\varphi^*) \\
& +\overleftrightarrow{\mu}\cdot(N^*-1)\cdot(1-\varphi^*)+\overleftrightarrow{\mu}\cdot(1-\varphi_i-\delta_{ij}) \\
& +1+\frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)}{2}\cdot(N-2)\cdot\varphi^*\Bigg\}\ \Bigg\}
\end{aligned}
$$

Using (17)-(20), and using the approximations

$$
\frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)}{2}=\frac{1}{2}\left(\frac{C}{N-1}\cdot\frac{1}{1+\widetilde{\gamma}}-\frac{C}{N}\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\right)\approx\frac{C}{2(N-1)}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}
$$

and

$$\frac{N-1}{N} \approx 1,\ \frac{N-2}{N-1} \approx 1$$

$$\begin{aligned}
\frac{\dot{m}_{ij}^d}{1-m^B} = \ & 2g(m^S) \cdot \Bigg\{ \quad (1-m^B) \ \left\{1-\delta_{ij}+\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*\right\} \\
& -m^B \ \Bigg\{\frac{C}{2}\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* + \frac{C}{2}\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* \\
& +\frac{\widehat{C}}{2}\cdot(1-\varphi^*) \\
& +\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*) + \overleftrightarrow{\mu}\cdot(1-\varphi_i-\delta_{ij}) \\
& +1+\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*\Bigg\} \ \Bigg\}
\end{aligned}$$

Using (21),

$$\begin{aligned}
\frac{\dot{m}_{ij}^d}{1-m^B} = \ & 2g(m^S) \cdot \Bigg\{ \quad (1-m^B) \ \left\{1-\delta_{ij}+\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*\right\} \\
& -m^B \ \Bigg\{C\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* \\
& +\frac{C}{2}\cdot(1-\varphi^*) \\
& +\overleftrightarrow{\mu}\cdot(1-\varphi_i-\delta_{ij}) \\
& +1+\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*\Bigg\} \ \Bigg\}
\end{aligned}$$

$$\begin{aligned}
\frac{\dot{m}_{ij}^d}{1-m^B} = \ & 2g(m^S) \cdot \Bigg\{\left[1-m^B \ -(1-m^B)\cdot\delta_{ij}+\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* - m^B\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*\right] \\
& -m^B \ \left[C\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* + \frac{C}{2}\cdot(1-\varphi^*) + \overleftrightarrow{\mu}\cdot(1-\varphi_i-\delta_{ij}) + 1 + \frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* \ \right]\Bigg\}
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\dot{m}_{ij}^d}{1-m^B} = \ & 2g(m^S) \cdot \Bigg\{\left[1-2m^B \ -(1-m^B)\cdot\delta_{ij}+\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*\right] \\
& -m^B \ \left[\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* + C\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* + \frac{C}{2}\cdot(1-\varphi^*) + \frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* \ \right] \\
& -m^B \ \cdot\overleftrightarrow{\mu}\cdot(1-\varphi_i-\delta_{ij}) \quad \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
= \quad & 2g(m^S) \cdot \Bigg\{ \left[1 - 2m^B \ - (1 - m^B) \cdot \delta_{ij} + \frac{C}{2} \cdot \frac{1 - \widetilde{\gamma}}{1 + \widetilde{\gamma}} \cdot \varphi^* \right] \\
& -m^B \ \left[\frac{C}{2} \cdot \frac{1 - \widetilde{\gamma}}{1 + \widetilde{\gamma}} \cdot \varphi^* + \frac{C}{2} \right] \\
& -m^B \ \cdot \overleftrightarrow{\mu} \cdot (1 - \varphi_i - \delta_{ij}) \quad \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= 2g(m^S) \cdot \Bigg\{ & 1 - 2m^B \ + (1 - m^B) \cdot \frac{C}{2} \cdot \frac{1 - \widetilde{\gamma}}{1 + \widetilde{\gamma}} \cdot \varphi^* - \frac{C}{2} \cdot m^B \\
& -(1 - m^B) \cdot \delta_{ij} - m^B \ \cdot \overleftrightarrow{\mu} \cdot (1 - \varphi_i - \delta_{ij}) \quad \Bigg\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \text{For } i \in A,\ j \in B\text{:} \quad \text{for } j \in \Gamma_{iB}, \\
& \dot{m}^d_{ij} = (1 - m^B) \cdot 2g(m^S) \cdot \Bigg\{ 1 - (2 + \frac{C}{2}) \cdot m^B \ + (1 - m^B) \cdot \frac{C}{2} \cdot \frac{1 - \widetilde{\gamma}}{1 + \widetilde{\gamma}} \cdot \varphi^* \\
& \qquad -(1 - m^B) \cdot \delta_{ij} - m^B \ \cdot \overleftrightarrow{\mu} \cdot (1 - \varphi_i - \delta_{ij}) \Bigg\}
\end{aligned} \tag{147}
$$

where $m^d_{ij} = m^d_{ji} = m^B$, and $g(m^S) = \tau \cdot g(m^B)$

which gives us (108).

Finally, from (89),

$$
\begin{aligned}
&\text{For } i \in A,\ j \in B\text{: for } j \notin \Gamma_{iB}, && (148)\\
\frac{\dot{m}_{ij}^d}{1-m_{ij}^d} &= (1-m_{ji}^d)\left\{[\delta_{ii}\cdot\alpha + \sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m_{ik}^d)]\right.\\
&+\sum_{k\in B_{-j}}\delta_{ik}\cdot\tau\cdot 2g(m_{ik}^d) + (\mu\cdot\eta - \widetilde{\mu}\cdot\eta)\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha\right.\\
&\left.+\sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g(m_{kl}^d) - \delta_{ii}\cdot\alpha - \sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m_{ik}^d)\right]\\
&\left.+\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{iA}}\sum_{l\in\Gamma_{iB}}\delta_{kl}\cdot\tau\cdot 2g(m_{kl}^d) - \sum_{k\in\Gamma_{iB}}\delta_{ik}\cdot\tau\cdot 2g(m_{ik}^d))\right\}\\
&-m_{ij}^d\cdot\left\{\quad \delta_{ij}\cdot\tau\cdot 2g(m_{ij}^d)\right.\\
&+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in A}\delta_{kk}\cdot\alpha + \sum_{k\in A}\sum_{l\in A_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right)\right]\\
&+\widetilde{\mu}\cdot\eta\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha + \sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g\left(m_{kl}^d\right)\right]\\
&\left.+\widehat{\mu}\cdot\eta\cdot\sum_{k\in A}\sum_{l\in B}\delta_{kl}\cdot\tau\cdot 2g\left(m_{kl}^d\right)\right\}\\
&-m_{ij}^d\cdot\left\{[\delta_{jj}\cdot\alpha + \sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m_{jk}^d)]\right.\\
&+\sum_{k\in A_{-i}}\delta_{jk}\cdot\tau\cdot 2g(m_{jk}^d) + (\mu\cdot\eta - \widetilde{\mu}\cdot\eta)\cdot\left[\sum_{k\in B}\delta_{kk}\cdot\alpha\right.\\
&\left.+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g(m_{kl}^d) - \delta_{jj}\cdot\alpha - \sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m_{jk}^d)\right]\\
&\left.+\overleftrightarrow{\mu}\cdot(\sum_{k\in\Gamma_{jA}}\sum_{l\in\Gamma_{jB}}\delta_{kl}\cdot\tau\cdot 2g(m_{kl}^d) - \sum_{k\in\Gamma_{jB}}\delta_{jk}\cdot\tau\cdot 2g(m_{jk}^d))\right\}
\end{aligned}
$$

- As shown earlier,

$$\delta_{ii}\cdot\alpha + \sum_{k\in A_{-i}}\delta_{ik}\cdot 2g(m_{ik}^d) = \varphi_i\cdot 2g(m^S)$$

- As shown earlier,

$$\sum_{k \in B_{-j}} \delta_{ik} \cdot \tau \cdot 2g(m_{ik}^d) = (1 - \varphi_i) \cdot 2g(m^S)$$

- As shown earlier

$$\begin{aligned} & \sum_{k \in A} \delta_{kk} \cdot \alpha + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot g(m_{kl}^d) - \delta_{ii} \cdot \alpha - \sum_{k \in A_{-i}} \delta_{ik} \cdot 2g(m_{ik}^d) \\ = \quad & [(N-1) \cdot \varphi^* - \varphi_i] \cdot g(m^S) \end{aligned}$$

- As shown earlier,

$$\begin{aligned} & \sum_{k \in \Gamma_{iA}} \sum_{l \in \Gamma_{iB}} \delta_{kl} \cdot \tau \cdot 2g(m_{kl}^d) - \sum_{k \in \Gamma_{iB}} \delta_{ik} \cdot \tau \cdot 2g(m_{ik}^d) \\ = \quad & [(N^* - 1) \cdot (1 - \varphi^*) + (1 - \varphi_i)] \cdot 2g(m^S) - (1 - \varphi_i) \cdot \tau \cdot 2g\left(m^B\right) \\ = \quad & [(N^* - 1) \cdot (1 - \varphi^*) + (1 - \varphi_i)] \cdot 2g(m^S) - (1 - \varphi_i) \cdot 2g\left(m^S\right) \\ = \quad & (N^* - 1) \cdot (1 - \varphi^*) \cdot 2g(m^S) \end{aligned}$$

- $$\delta_{ij} \cdot \tau \cdot 2g(m_{ij}^d) = 0$$

- As shown earlier

$$\sum_{k \in A} \delta_{kk} \cdot \alpha + \sum_{k \in A} \sum_{l \in A_{-k}} \delta_{kl} \cdot g\left(m_{kl}^d\right) = [(N-1) \cdot \varphi^* + \varphi_i] \cdot g(m^S)$$

- As shown earlier,

$$\sum_{k \in B} \delta_{kk} \cdot \alpha + \sum_{k \in B} \sum_{l \in B_{-k}} \delta_{kl} \cdot g\left(m_{kl}^d\right) = N \cdot \varphi^* \cdot g(m^S)$$

- $$\sum_{k \in A} \sum_{l \in B} \delta_{kl} \cdot \tau \cdot 2g\left(m_{kl}^d\right) = [(N-1) \cdot (1 - \varphi^*) + (1 - \varphi_i)] \cdot 2g(m^S)$$

- $$\delta_{jj} \cdot \alpha + \sum_{k \in B_{-j}} \delta_{jk} \cdot 2g(m_{jk}^d) = \varphi^* \cdot 2g(m^S)$$

- $$\begin{aligned} \sum_{k \in A_{-i}} \delta_{jk} \cdot \tau \cdot 2g(m_{jk}^d) &= (1 - \varphi^*) \cdot \tau \cdot 2g(m^B) \\ &= (1 - \varphi^*) \cdot 2g(m^S) \end{aligned}$$

- As shown before,

$$
\begin{aligned}
&\sum_{k\in B}\delta_{kk}\cdot\alpha+\sum_{k\in B}\sum_{l\in B_{-k}}\delta_{kl}\cdot g(m^d_{kl})-\delta_{jj}\cdot\alpha-\sum_{k\in B_{-j}}\delta_{jk}\cdot 2g(m^d_{jk})\\
=\;&(N-2)\cdot\varphi^*\cdot g(m^S)
\end{aligned}
$$

-

$$
\begin{aligned}
&\sum_{k\in\Gamma_{jA}}\sum_{l\in\Gamma_{jB}}\delta_{kl}\cdot\tau\cdot 2g(m^d_{kl})-\sum_{k\in\Gamma_{jB}}\delta_{jk}\cdot\tau\cdot 2g(m^d_{jk})\\
=\;&[(N^*-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\cdot 2g(m^S)
\end{aligned}
$$

Substituting each term into (148) and setting $m^d_{ij}=m^d_{ji}=m^d$,

$$
\begin{aligned}
\text{For } i &\in A,\ j\in B\text{: for } j\notin\Gamma_{iB},\\
\frac{\dot m^d_{ij}}{1-m^d} &= (1-m^d)\left\{\varphi_i\cdot 2g(m^S)\right.\\
&+(1-\varphi_i)\cdot 2g(m^S)+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot[(N-1)\cdot\varphi^*-\varphi_i]\cdot g(m^S)\\
&\left.+\overleftrightarrow{\mu}\cdot((N^*-1)\cdot(1-\varphi^*)\cdot 2g(m^S))\right\}\\
&-m^d\cdot\left\{\qquad\widetilde{\mu}\cdot\eta\cdot[(N-1)\cdot\varphi^*+\varphi_i]\cdot g(m^S)\right.\\
&+\widetilde{\mu}\cdot\eta\cdot N\cdot\varphi^*\cdot g(m^S)\\
&\left.+\widehat{\mu}\cdot\eta\cdot[(N-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\cdot 2g(m^S)\right\}\\
&-m^d\cdot\left\{[\varphi^*\cdot 2g(m^S)\right.\\
&+(1-\varphi^*)\cdot 2g(m^S)+(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)\cdot\left[(N-2)\cdot\varphi^*\cdot g(m^S)\right]\\
&\left.+\overleftrightarrow{\mu}\cdot([(N^*-1)\cdot(1-\varphi^*)+(1-\varphi_i)]\cdot 2g(m^S))\right\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 2g(m^S) \cdot \Big\{ \quad (1-m^d)\,\{\varphi_i \\
& +(1-\varphi_i) + \frac{\mu \cdot \eta - \widetilde{\mu} \cdot \eta}{2} \cdot [(N-1) \cdot \varphi^* - \varphi_i] \\
& +\overleftrightarrow{\mu} \cdot (N^*-1) \cdot (1-\varphi^*)\} \\
& -m^d \cdot \Big\{ \frac{\widetilde{\mu} \cdot \eta}{2} \cdot [(N-1) \cdot \varphi^* + \varphi_i] \\
& +\frac{\widetilde{\mu} \cdot \eta}{2} \cdot N \cdot \varphi^* \\
& +\widehat{\mu} \cdot \eta \cdot [(N-1) \cdot (1-\varphi^*) + (1-\varphi_i)] \quad \Big\} \\
& -m^d \cdot \{[\varphi^* \\
& +(1-\varphi^*) + \frac{\mu \cdot \eta - \widetilde{\mu} \cdot \eta}{2} \cdot [(N-2) \cdot \varphi^*] \\
& +\overleftrightarrow{\mu} \cdot [(N^*-1) \cdot (1-\varphi^*) + (1-\varphi_i)]\} \quad \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 2g(m^S) \cdot \Big\{ \quad (1-m^d) \cdot \Big\{ 1 + \frac{\mu \cdot \eta - \widetilde{\mu} \cdot \eta}{2} \cdot [(N-1) \cdot \varphi^* - \varphi_i] \\
& +\overleftrightarrow{\mu} \cdot (N^*-1) \cdot (1-\varphi^*)\} \\
& -m^d \cdot \Big\{ \frac{\widetilde{\mu} \cdot \eta}{2} \cdot [(N-1) \cdot \varphi^* + \varphi_i] \\
& +\frac{\widetilde{\mu} \cdot \eta}{2} \cdot N \cdot \varphi^* \\
& +\widehat{\mu} \cdot \eta \cdot [(N-1) \cdot (1-\varphi^*) + (1-\varphi_i)]\} \\
& -m^d \cdot \{[\varphi^* \\
& +(1-\varphi^*) + \frac{\mu \cdot \eta - \widetilde{\mu} \cdot \eta}{2} \cdot (N-2) \cdot \varphi^* \\
& +\overleftrightarrow{\mu} \cdot [(N^*-1) \cdot (1-\varphi^*) + (1-\varphi_i)]\} \quad \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 2g(m^S) \cdot \Big\{ \quad (1-m^d) \cdot \Big\{ 1 + \frac{\mu \cdot \eta - \widetilde{\mu} \cdot \eta}{2} \cdot [(N-1) \cdot \varphi^* - \varphi_i] \\
& +\overleftrightarrow{\mu} \cdot (N^*-1) \cdot (1-\varphi^*)\} \\
& -m^d \cdot \Big\{ \frac{\widetilde{\mu} \cdot \eta}{2} \cdot [(N-1) \cdot \varphi^* + \varphi_i] \\
& +\frac{\widetilde{\mu} \cdot \eta}{2} \cdot N \cdot \varphi^* \\
& +\widehat{\mu} \cdot \eta \cdot [(N-1) \cdot (1-\varphi^*) + (1-\varphi_i)] \\
& +1 + \frac{\mu \cdot \eta - \widetilde{\mu} \cdot \eta}{2} \cdot (N-2) \cdot \varphi^* \\
& +\overleftrightarrow{\mu} \cdot [(N^*-1) \cdot (1-\varphi^*) + (1-\varphi_i)] \quad \} \quad \Big\}
\end{aligned}
$$

Using approximations $\frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)}{2}\cdot\varphi_i=0$, $\frac{\widetilde{\mu}\cdot\eta}{2}\cdot\varphi_i=0$, and $\widehat{\mu}\cdot\eta\cdot(1-\varphi_i)=0$, we have

$$
\begin{aligned}
= \; & 2g(m^S)\cdot\Big\{ \quad (1-m^d)\cdot\Big\{1+\frac{\mu\cdot\eta-\widetilde{\mu}\cdot\eta}{2}\cdot(N-1)\cdot\varphi^* \\
& +\overleftrightarrow{\mu}\cdot(N^*-1)\cdot(1-\varphi^*)\} \\
& -m^d\cdot\Big\{\frac{\widetilde{\mu}\cdot\eta}{2}\cdot(N-1)\cdot\varphi^* \\
& +\frac{\widetilde{\mu}\cdot\eta}{2}\cdot N\cdot\varphi^* \\
& +\widehat{\mu}\cdot\eta\cdot(N-1)\cdot(1-\varphi^*) \\
& +1+\frac{\mu\cdot\eta-\widetilde{\mu}\cdot\eta}{2}\cdot(N-2)\cdot\varphi^* \\
& +\overleftrightarrow{\mu}\cdot[(N^*-1)\cdot(1-\varphi^*)+(1-\varphi_i)] \quad \} \quad \Big\}
\end{aligned}
$$

Using (17)-(20), and using the approximations

$$
\frac{(\mu\cdot\eta-\widetilde{\mu}\cdot\eta)}{2}=\frac{1}{2}\left(\frac{C}{N-1}\cdot\frac{1}{1+\widetilde{\gamma}}-\frac{C}{N}\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\right)\approx\frac{C}{2(N-1)}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}
$$

and

$$
\frac{N-1}{N}\approx 1,\ \frac{N-2}{N-1}\approx 1
$$

$$
\begin{aligned}
= \; & 2g(m^S)\cdot\Big\{ \quad (1-m^d)\cdot\Big\{1+\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* \\
& +\overleftrightarrow{\mu}\cdot(N^*-1)\cdot(1-\varphi^*)\} \\
& -m^d\cdot\Big\{\frac{C}{2}\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*+\frac{C}{2}\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*+\frac{\widehat{C}}{2}\cdot(1-\varphi^*) \\
& +1+\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* \\
& +\overleftrightarrow{\mu}\cdot[(N^*-1)\cdot(1-\varphi^*)+(1-\varphi_i)] \quad \} \quad \Big\}
\end{aligned}
$$

Using (21), we have

$$
\begin{aligned}
= \; & 2g(m^S)\cdot\Big\{ \quad (1-m^d)\cdot\Big\{1+\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* \\
& +\overleftrightarrow{\mu}\cdot(N^*-1)\cdot(1-\varphi^*)\} \\
& -m^d\cdot\Big\{C\cdot\frac{\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*+\frac{\widehat{C}}{2}\cdot(1-\varphi^*) \\
& +1+\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^* \\
& +\overleftrightarrow{\mu}\cdot[(N^*-1)\cdot(1-\varphi^*)+(1-\varphi_i)] \quad \} \quad \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \;& 2g(m^S) \cdot \Bigg\{ \quad (1-m^d) \cdot \Bigg\{ 1 + \frac{C}{2} \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \cdot \varphi^* \\
& + \overleftrightarrow{\mu} \cdot (N^*-1) \cdot (1-\varphi^*) \} \\
& - m^d \cdot \Bigg\{ \frac{C}{2} \cdot \varphi^* + \frac{\widehat{C}}{2} \cdot (1-\varphi^*) + 1 \\
& + \overleftrightarrow{\mu} \cdot [(N^*-1) \cdot (1-\varphi^*) + (1-\varphi_i)] \quad \} \quad \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \;& 2g(m^S) \cdot \Bigg\{ \quad (1-m^d) \cdot \Bigg\{ 1 + \frac{C}{2} \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \cdot \varphi^* \\
& + \frac{\overleftrightarrow{C}}{2} \cdot (1-\varphi^*) \Bigg\} \\
& - m^d \cdot \Bigg\{ \frac{C}{2} \cdot \varphi^* + \frac{\widehat{C}}{2} \cdot (1-\varphi^*) + 1 \\
& + \frac{\overleftrightarrow{C}}{2} \cdot (1-\varphi^*) + \overleftrightarrow{\mu} \cdot (1-\varphi_i) \quad \Bigg\} \quad \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \;& 2g(m^S) \cdot \Bigg\{ \quad (1-m^d) \cdot \Bigg\{ 1 + \frac{C}{2} \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \cdot \varphi^* \\
& + \frac{\overleftrightarrow{C}}{2} \cdot (1-\varphi^*) \quad \Bigg\} \\
& - m^d \cdot \Bigg\{ \frac{C}{2} \cdot \varphi^* + \frac{C}{2} \cdot (1-\varphi^*) + 1 + \overleftrightarrow{\mu} \cdot (1-\varphi_i) \Bigg\} \quad \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \;& 2g(m^S) \cdot \Bigg\{ \quad (1-m^d) \cdot \Bigg\{ 1 + \frac{C}{2} \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \cdot \varphi^* \\
& + \frac{\overleftrightarrow{C}}{2} \cdot (1-\varphi^*) \quad \Bigg\} \\
& - m^d \cdot \Bigg\{ \frac{C}{2} + 1 + \overleftrightarrow{\mu} \cdot (1-\varphi_i) \Bigg\} \quad \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \;& 2g(m^S) \cdot \Bigg\{ \quad 1 - m^d + (1-m^d) \cdot \left[\frac{C}{2} \cdot \frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \cdot \varphi^* + \frac{\overleftrightarrow{C}}{2} \cdot (1-\varphi^*) \right] \\
& - m^d \cdot \frac{C}{2} - m^d - m^d \cdot \overleftrightarrow{\mu} \cdot (1-\varphi_i) \quad \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \; 2g(m^S)\cdot\Bigg\{ \quad 1-(2+\frac{C}{2})\cdot m^d+(1-m^d)\cdot\left[\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*+\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)\right] \\
&\quad -m^d\cdot\overleftrightarrow{\mu}\cdot(1-\varphi_i) \;\; \Bigg\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{For } i &\in A,\, j\in B\text{:} \quad \text{for } j\notin\Gamma_{iB}, \qquad (149)\\
\dot{m}_{ij}^d &= (1-m^d)\cdot 2g(m^S)\cdot \\
&\Bigg\{ \quad 1-(2+\frac{C}{2})\cdot m^d+(1-m^d)\cdot\left[\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\varphi^*+\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)\right] \\
&-m^d\cdot\overleftrightarrow{\mu}\cdot(1-\varphi_i) \;\; \Bigg\} \\
\text{where } m_{ij}^d &= m_{ji}^d=m^d, \text{ and } g(m^S)=\tau\cdot g(m^B)
\end{aligned}
$$

which gives us (109).

8.4.2 Derivation of (110) and the proof of Lemma A1

Using (94)-(102), F_i defined by (104) becomes

$$
\begin{aligned}
F_i &= \sum_{j\in\Gamma_{iA},\, j\neq i}\delta_{ij}\cdot g'(m_{ij}^d)\cdot\dot{m}_{ij}^d+\sum_{j\in A,\, j\notin\Gamma_{iA}}\delta_{ij}\cdot g'(m_{ij}^d)\cdot\dot{m}_{ij}^d \\
&\quad +\sum_{j\in\Gamma_{iB}}\delta_{ij}\cdot\tau\cdot g'(m_{ij}^d)\cdot\dot{m}_{ij}^d+\sum_{j\in B,\, j\notin\Gamma_{iB}}\delta_{ij}\cdot\tau\cdot g'(m_{ij}^d)\cdot\dot{m}_{ij}^d \\
&= \sum_{j\in\Gamma_{iA},\, j\neq i}0\cdot g'(m_{ij}^d)\cdot\dot{m}_{ij}^d+\sum_{j\in A,\, j\notin\Gamma_{iA}}\delta_{ij}\cdot g'(m^S)\cdot\dot{m}_{ij}^d \\
&\quad +\sum_{j\in\Gamma_{iB}}\delta_{ij}\cdot\tau\cdot g'(m^B)\cdot\dot{m}_{ij}^d+\sum_{j\in B,\, j\notin\Gamma_{iB}}\delta_{ij}\cdot\tau\cdot g'(m^d)\cdot\dot{m}_{ij}^d \\
&= 0+\sum_{j\in A,\, j\notin\Gamma_{iA}}\delta_{ij}\cdot g'(m^S)\cdot\dot{m}_{ij}^d+\sum_{j\in\Gamma_{iB}}\delta_{ij}\cdot\tau\cdot g'(m^B)\cdot\dot{m}_{ij}^d+0 \\
&= \sum_{j\in A,\, j\notin\Gamma_{iA}}\delta_{ij}\cdot g'(m^S)\cdot\dot{m}_{ij}^d
\end{aligned}
$$

Using (107),

$$\begin{aligned}
F_i &= g'(m^S) \cdot \sum_{j\in A,\ j\notin\Gamma_{iA}} \delta_{ij} \cdot \dot{m}^d_{ij} \\
&= g'(m^S) \cdot (1-m^S) \cdot 2g(m^S) \cdot \\
&\quad \sum_{j\in A,\ j\notin\Gamma_{iA}} \delta_{ij} \cdot \left\{ 1 - (2+\frac{C}{2})m^S + (1-m^S) \cdot \frac{\overleftrightarrow{C}}{2} \cdot (1-\varphi^*) - (1-m^S) \cdot \delta_{ij} \right\} \\
&= g'(m^S) \cdot (1-m^S) \cdot 2g(m^S) \cdot \\
&\quad \left\{ \varphi_i \cdot \left[1 - (2+\frac{C}{2})m^S + (1-m^S) \cdot \frac{\overleftrightarrow{C}}{2} \cdot (1-\varphi^*) \right] - (1-m^S) \cdot \sum_{j\in A,\ j\notin\Gamma_{iA}} \delta^2_{ij} \right\}
\end{aligned}$$

which leads to (110).

Since $g'(m^S) > 0$, the optimization problem (111) is equivalent to:

$$\min_{\{\delta_{ij}|\ j\in A,\ j\notin\Gamma_{iA}\}} \sum_{j\in A,\ j\notin\Gamma_{iA}} \delta^2_{ij} \text{ subject to } \sum_{j\in A,\ j\notin\Gamma_{iA}} \delta_{ij} = \varphi_i$$

that has solution:

$$\delta^*_{ij} = \frac{\varphi_i}{N-N^*} \text{ for } j \in A,\ j \notin \Gamma_{iA}$$

which gives us (112). Thus,

$$\begin{aligned}
\sum_{j\in A,\ j\notin\Gamma_{iA}} \delta^2_{ij} &= (N-N^*) \cdot \left(\frac{\varphi_i}{N-N^*} \right)^2 \\
&= \frac{\varphi_i^2}{N-N^*}
\end{aligned}$$

Since $N^* \leq \frac{N}{2}$ at a symmetric equilibrium,

$$\frac{\varphi_i^2}{N-N^*} \leq \frac{\varphi_i^2}{N/2} \leq \frac{1}{N/2}$$

Thus, for N large, we have

$$\frac{\varphi_i^2}{N-N^*} \approx 0$$

and thus

$$F_i = g'(m^S) \cdot (1-m^S) \cdot 2g(m^S) \cdot \left[1 - (2+\frac{C}{2})m^S + (1-m^S) \cdot \frac{\overleftrightarrow{C}}{2} \cdot (1-\varphi^*) \right] \cdot \varphi_i$$

which leads to (113).

8.4.3 Showing that the term in the square brackets in equation (113) must be zero

(i) If $1-(2+\frac{C}{2})m^S+(1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)>0$, then according to (113), F_i is maximized at $\varphi_i=1$. In the myopic core, this holds for all i, so

$$\varphi^*=\varphi_i=1 \text{ for all } i\in A$$

In that situation,

$$\begin{aligned} & 1-(2+\frac{C}{2})m^S+(1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*) \\ = \; & 1-(2+\frac{C}{2})m^S \end{aligned} \tag{150}$$

Recall from (53) that

$$m^*_{\text{aut}}=\frac{1}{2+\frac{C}{2}}<m^S \tag{151}$$

Hence from (150), $1-(2+\frac{C}{2})m^S+(1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)<0$, a contradiction. So this case cannot occur.

(ii) If $1-(2+\frac{C}{2})m^S+(1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)<0$, then according to (113), F_i is maximized when $\varphi_i=0$. Since in the myopic core this holds for all $i\in A$,

$$\varphi^*=\varphi_i=0 \text{ for all } i\in A$$

Setting $\varphi_i=\varphi^*=0$ in (108):

$$\begin{aligned} & \text{For } i\in A,\ j\in B\text{: for } j\in\Gamma_{iB}, \\ & \dot{m}^d_{ij}=(1-m^B)\cdot 2g(m^S)\cdot\Big\{1-(2+\frac{C}{2})\cdot m^B \\ & \quad -(1-m^B)\cdot\delta_{ij}-m^B\cdot\overleftrightarrow{\mu}\cdot(1-\varphi^*-\delta_{ij})\ \Big\} \\ & \quad =(1-m^B)\cdot 2g(m^S)\cdot\Big\{1-(2+\frac{C}{2})\cdot m^B \\ & \quad -m^B\cdot\overleftrightarrow{\mu}-(1-m^B-m^B\cdot\overleftrightarrow{\mu})\cdot\delta_{ij}\ \Big\} \\ & \quad =(1-m^B)\cdot 2g(m^S)\cdot\Big\{1-(2+\frac{C}{2})\cdot m^B \\ & \quad -m^B\cdot\overleftrightarrow{\mu}\cdot(1-\delta_{ij})-(1-m^B)\cdot\delta_{ij}\ \Big\} \\ & \quad <0 \end{aligned}$$

The last line follows because $m^B>m^S$ and (151) imply $1-(2+\frac{C}{2})\cdot m^B<0$. From (92), a necessary condition for a steady state at $m^d_{ij}=m^d_{ji}=m^B$ is $\dot{m}^d_{ij}=0$ for $j\in\Gamma_{iB}$, and hence we have a contradiction.

(iii) Hence, at the myopic core, the following case must hold:

$$1-(2+\frac{C}{2})m^S+(1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)=0$$

8.4.4 Derivation of (118) to (120)

For i, $j\in A$, $j\notin\Gamma_{iA}$, using $1-\varphi^*$ given by (115) and using the approximation $\delta^*_{ij}=\frac{\varphi_i}{N-N^*}\approx 0$, we have from (107):

$$\begin{aligned}
\dot{m}^d_{ij} &= (1-m^S)\cdot 2g(m^S)\cdot\left\{1-(2+\frac{C}{2})m^S+(1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)-(1-m^S)\cdot\delta_{ij}\right\}\\
&= (1-m^S)\cdot 2g(m^S)\cdot\left\{1-(2+\frac{C}{2})m^S+(1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)\right\}\\
&= (1-m^S)\cdot 2g(m^S)\cdot\left\{1-2m^S+(1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*)-\frac{C}{2}\cdot m^S\right\}\\
&= (1-m^S)\cdot 2g(m^S)\cdot\left\{1-2m^S+(1-m^S)\cdot\frac{\overleftrightarrow{C}}{2}\cdot\left(\frac{2}{\overleftrightarrow{C}}\cdot\frac{(2+\frac{C}{2})\cdot m^S-1}{1-m^S}\right)\right.\\
&\quad\left.-\frac{C}{2}\cdot m^S\right\}\\
&= (1-m^S)\cdot 2g(m^S)\cdot\left\{1-2m^S+(2+\frac{C}{2})\cdot m^S-1-\frac{C}{2}\cdot m^S\right\}\\
&= (1-m^S)\cdot 2g(m^S)\cdot\{0\}=0
\end{aligned}$$

which leads to (118).

Next, for i, $j\in A$, $j\in\Gamma_{iA}$, in (106), setting $1-\varphi_i=1-\varphi^*$ from (115):

For $i\in A$, $j\in A$: for $j\in\Gamma_{iA}$,

$$\begin{aligned}
\dot{m}^d_{ij} &= (1-m^d)\cdot 2g(m^S)\cdot\left\{1-(2+\frac{C}{2})\cdot m^d-m^d\cdot\left[\frac{\overleftrightarrow{C}}{2(N^*-1)}\cdot(1-\varphi^*)\right]\right\}\\
&= (1-m^d)\cdot 2g(m^S)\cdot\left\{1-(2+\frac{C}{2})\cdot m^d-m^d\cdot\left[\frac{\overleftrightarrow{C}}{2(N^*-1)}\cdot\frac{2}{\overleftrightarrow{C}}\cdot\frac{(2+\frac{C}{2})\cdot m^S-1}{1-m^S}\right]\right\}\\
&= (1-m^d)\cdot 2g(m^S)\cdot\left\{1-2m^d-m^d\cdot\left[\frac{C}{2}+\frac{(2+\frac{C}{2})\cdot m^S-1}{(N^*-1)\cdot(1-m^S)}\right]\right\}
\end{aligned}$$

Since $\frac{1}{2+\frac{C}{2}}\equiv m^*_{aut}<m^S$ by assumption, $\dot{m}^d_{ij}<0$ whenever $m^d_{ij}\geq m^*_{aut}$. From this, (119) follows.

Finally consider the case where the two potential collaborators are from different regions, but not in the same inter-regional work group. We examine the difference between differential knowledge growth for people in the other region who *are not* members of the same inter-regional working group, as compared to people in the other region who *are* members of the same working group. We evaluate this difference at the bliss point $m^d = m^B$. That is, from (109) and (108):

$$\text{For } i \in A,\ j,k \in B\text{: for } j \notin \Gamma_{iB}\text{, for } k \in \Gamma_{iB}$$

$$\dot{m}^d_{ij} - \dot{m}^d_{ik}\Big|_{m^d=m^B} = (1-m^B)\cdot 2g(m^S)\cdot$$

$$\left\{(1-m^B)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*) - m^B\cdot\overleftrightarrow{\mu}\cdot(1-\varphi_i) + (1-m^B)\cdot\delta_{ik} + m^B\cdot\overleftrightarrow{\mu}\cdot(1-\varphi_i-\delta_{ik})\right\}$$

$$\text{where } g(m^S) = \tau\cdot g(m^B)$$

Setting $\varphi_i = \varphi^*$,

$$= (1-m^B)\cdot 2g(m^S)\cdot$$

$$\left\{(1-m^B)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*) - m^B\cdot\overleftrightarrow{\mu}\cdot(1-\varphi^*) + (1-m^B)\cdot\delta_{ik} + m^B\cdot\overleftrightarrow{\mu}\cdot(1-\varphi^*-\delta_{ik})\right\}$$

$$= (1-m^B)\cdot 2g(m^S)\cdot$$

$$\left\{(1-m^B)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*) + (1-m^B)\cdot\delta_{ik} - m^B\cdot\overleftrightarrow{\mu}\cdot\delta_{ik}\right\}$$

Using (20),

$$= (1-m^B)\cdot 2g(m^S)\cdot$$

$$\left\{(1-m^B)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*) + (1-m^B)\cdot\delta_{ik} - m^B\cdot\frac{\overleftrightarrow{C}}{2(N^*-1)}\cdot\delta_{ik}\right\}$$

and now (121):

$$= (1-m^B)\cdot 2g(m^S)\cdot$$

$$\left\{(1-m^B)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*) + (1-m^B)\cdot\frac{1-\varphi^*}{N^*} - m^B\cdot\frac{\overleftrightarrow{C}}{2(N^*-1)}\cdot\frac{1-\varphi^*}{N^*}\right\}$$

Provided that $N^* \geq 2$, $(N^*-1)\cdot N^* > 1$ so $\frac{1}{(N^*-1)\cdot N^*} < 1$. In this case,

$$\begin{aligned} &> (1-m^B)\cdot 2g(m^S)\cdot\left\{(1-2m^B)\cdot\frac{\overleftrightarrow{C}}{2}\cdot(1-\varphi^*) + (1-m^B)\cdot\frac{1-\varphi^*}{N^*}\right\} \\ &> 0 \end{aligned}$$

In summary,

$$\text{For } i \in A,\ j,k \in B\text{:}\ \ \text{for } j \notin \Gamma_{iB},\ \text{for } k \in \Gamma_{iB}$$

$$\dot{m}_{ij}^{d} - \dot{m}_{ik}^{d}\Big|_{m^d = m^B} > 0$$

Hence, once $m_{ij}^d = m^d \geq m^B$, then since $m_{ik}^d = m^B$ for $k \in \Gamma_{iB}$ always, it follows that $m_{ij}^d = m^d > m^B$ forever after that time, implying (120).

8.4.5 Derivation of (56)

Substituting for φ^* from (116),

$$\begin{aligned}
& 1 - (2+\frac{C}{2})\cdot m^B \\
& +(1-m^B)\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\frac{2}{\overleftrightarrow{C}\cdot(1-m^S)}\cdot\left[\frac{\overleftrightarrow{C}}{2}\cdot(1-m^S)-(2+\frac{C}{2})\cdot m^S+1\right] \\
= \ & (1-m^B+m^B\cdot\frac{\overleftrightarrow{C}}{2})\cdot\frac{\frac{2}{\overleftrightarrow{C}}\cdot\frac{(2+\frac{C}{2})\cdot m^S-1}{1-m^S}}{N^*}
\end{aligned}$$

So

$$\begin{aligned}
& 1 - m^S - (2+\frac{C}{2})\cdot m^B\cdot(1-m^S) \\
& +(1-m^B)\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\cdot\frac{2}{\overleftrightarrow{C}}\cdot\left[\frac{\overleftrightarrow{C}}{2}\cdot(1-m^S)-(2+\frac{C}{2})\cdot m^S+1\right] \\
= \ & (1-m^B+m^B\cdot\frac{\overleftrightarrow{C}}{2})\cdot\frac{\frac{2}{\overleftrightarrow{C}}\cdot[(2+\frac{C}{2})\cdot m^S-1]}{N^*}
\end{aligned}$$

$$\begin{aligned}
& 1 - (2+\frac{C}{2})\cdot m^B + (1-m^B)\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} + (1-m^B)\cdot\frac{C}{\overleftrightarrow{C}}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \\
& -m^S\cdot\left\{1-(2+\frac{C}{2})\cdot m^B+(1-m^B)\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}+(2+\frac{C}{2})\cdot(1-m^B)\cdot\frac{C}{\overleftrightarrow{C}}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\right\} \\
= \ & -\frac{2}{\overleftrightarrow{C}\cdot N^*}\cdot(1-m^B+m^B\cdot\frac{\overleftrightarrow{C}}{2})+m^S\cdot(1-m^B+m^B\cdot\frac{\overleftrightarrow{C}}{2})\cdot\frac{2\cdot(2+\frac{C}{2})}{\overleftrightarrow{C}\cdot N^*}
\end{aligned}$$

$$\begin{aligned}
& 1 - (2+\frac{C}{2})\cdot m^B + (1-m^B)\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}} \\
& +(1-m^B)\cdot\frac{C}{\overleftrightarrow{C}}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}+\frac{2}{\overleftrightarrow{C}\cdot N^*}\cdot(1-m^B+m^B\cdot\frac{\overleftrightarrow{C}}{2}) \\
= \ & m^S\cdot\left\{1-(2+\frac{C}{2})\cdot m^B+(1-m^B)\cdot\frac{C}{2}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}\right. \\
& \left.+(2+\frac{C}{2})\cdot(1-m^B)\cdot\frac{C}{\overleftrightarrow{C}}\cdot\frac{1-\widetilde{\gamma}}{1+\widetilde{\gamma}}+(1-m^B+m^B\cdot\frac{\overleftrightarrow{C}}{2})\cdot\frac{2\cdot(2+\frac{C}{2})}{\overleftrightarrow{C}\cdot N^*}\right\}
\end{aligned}$$

$$
\begin{aligned}
& N^* \cdot \left[1 - (2 + \frac{C}{2}) \cdot m^B + (1 - m^B) \cdot \frac{1 - \widetilde{\gamma}}{1 + \widetilde{\gamma}} \cdot \left(\frac{C}{2} + \frac{C}{\overleftrightarrow{C}}\right)\right] \\
& + \frac{2}{\overleftrightarrow{C}} \cdot (1 - m^B + m^B \ \cdot \frac{\overleftrightarrow{C}}{2}) \\
= \ & m^S \cdot \left\{N^* \cdot \left[1 - (2 + \frac{C}{2}) \cdot m^B + (1 - m^B) \cdot \frac{1 - \widetilde{\gamma}}{1 + \widetilde{\gamma}} \cdot \left(\frac{C}{2} + (2 + \frac{C}{2}) \cdot \frac{C}{\overleftrightarrow{C}}\right)\right]\right. \\
& \left. + (1 - m^B + m^B \ \cdot \frac{\overleftrightarrow{C}}{2}) \cdot \frac{2 \cdot (2 + \frac{C}{2})}{\overleftrightarrow{C}}\right\}
\end{aligned}
$$

which leads to (56).