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Abstract We are concerned with the problem of aggregating infinite utility streams and the possible adoption of consequentialist equity principles when using numerical evaluations of the streams. We find a virtually universal incompatibility between the Basu-Mitra approach (that advocates for social welfare functions and renounces continuity assumptions) and postulates that capture various forms of strict preference for a reduction in inequality like the Strong Equity Principle, the Pigou-Dalton Transfer principle, or Altruistic Equity.

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1 Introduction

This paper is primarily concerned with the problem of aggregating infinite utility streams by means of numerical evaluations, and the possible adoption of consequentialist equity principles under such position. Our objective is to argue that there is a fundamental incompatibility between the Basu-Mitra approach (that uses social welfare functions and renounces continuity assumptions) and salient postulates of strict inequality aversion like the Strong Equity Principle (cf., Bossert et al., 2007), the Pigou-Dalton Transfer principle (cf., Sakai, 2006, and Bossert et al., 2007), or Altruistic Equity (cf., Sakamoto, 2011). All these axioms have the common spirit of expressing a strict preference for distributions of utilities among generations that reduce inequality in various forms.

The essential shortfall of the approach by numerical evaluations or social welfare functions (SWFs) in the context of intergenerational equity has been brought to the fore by a number of contributions. Either if one requests anonymity-type properties (Basu and Mitra, 2003, Crespo et al., 2009), the very mild Hammond Equity for the Future (Banerjee, 2006, Alcantud and García-Sanz, 2010a), or variations of other consequentialist principles of aversion to inequality (Alcantud, 2010, 2011, and Sakamoto, 2011), relaxed –but not universally acceptable– versions of the Pareto principle like strong Pareto, weak dominance, or weak Pareto, lead to incompatibility under different specifications of the domain of utility sequences. Here we go further and prove that with the simple assumption of monotonicity, which does not impose any ethical restriction but is just an uncontroversial consistency requirement, SWFs must contradict the ethics of the Pigou-Dalton transfer principle at the level of a especially plausible generalization. This implies that other distributional axioms implying Pigou-Dalton under monotonicity, like the Strong Equity Principle or the Lorenz domination principle, also deem incompatible with the Basu-Mitra approach.

2 Notation and definitions

Let $\mathbf{X}$ denote a subset of $\mathbb{R}^{\mathbb{N}}$, that represents a domain of utility sequences or infinite-horizon utility streams. For simplicity we assume $\mathbf{X} = Y^{\mathbb{N}}$ and say that Y is the set of feasible utilities. We adopt the standard notation for utility streams: $\mathbf{x} = (x_1, ..., x_n, ...) \in \mathbf{X}$. We write $\mathbf{x} \geqslant \mathbf{y}$ if $x_i \geqslant y_i$ for each $i = 1, 2, ...$, and $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for each $i = 1, 2, ...$ Also, $\mathbf{x} > \mathbf{y}$ means $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

A *social welfare function* (SWF) is a function $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$. The analysis of intergenerational aggregation by means of SWFs is usually called the Basu-Mitra approach.

We are concerned with the following efficiency axioms:

Axiom SP (*Strong Pareto*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} > \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

The next efficiency axiom are all implied by Strong Pareto. Monotonicity is regarded as a necessary condition for efficiency thus all our results refer to monotonic SWFs:

Axiom MON (*Monotonicity*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} \geqslant \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$.

Another fairly justifiable weakening of SP is the following:

Axiom WP (*Weak Pareto*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} \gg \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

An independent weaker version of Strong Pareto is:

Axiom WD (*Weak Dominance*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and there is $j \in \mathbb{N}$ such that $x_j > y_j$, and $x_i = y_i$ for all $i \neq j$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

Now we recall some axioms that intend to prioritize more egalitarian allocations by expressing a strict preference for certain distributions of utilities among generations.

Axiom PDT (*Pigou-Dalton transfer principle*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that there is $\varepsilon > 0$ with $y_j = x_j - \varepsilon \geqslant y_k = x_k + \varepsilon$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$.

PDT is a consequentialist equity principle stating that a transfer of utility from a generation that is richer to a poorer generation must be socially beneficial provided that their relative positions do not change. It is a notion of inequality aversion in a cardinal vein that has been introduced in this literature by Bossert et al. (2007) –under the name *strict transfer principle*– and Sakai (2006). [1]

A reinforced form of both PDT and the classical Hammond Equity postulate is the following axiom (cf., Bossert et al., 2007):

Axiom SEP (*Strict Equity Principle*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j \geqslant y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$.

Under SEP, any utility sacrice made by a richer generation that is rewarded by any utility gain by a poorer generation is socially beneficial when their relative positions do not change. As a variant of the ethics supporting PDT, Hara et al. (2006) propose a more acceptable postulate in the following form:

Axiom AE (*Altruistic Equity*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that there are $\varepsilon > \delta > 0$ with $y_j = x_j - \delta \geqslant y_k = x_k + \varepsilon$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$.

Compared to PDT, the ethical principle under AE is less demanding: It only claims that reductions in welfare for the rich that are accompanied by increases to the poor must be socially preferred when the gain of the poor is

[1] The formulation in Bossert et al. (2007) is different but equivalent: if $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j \geqslant y_k > x_k$ and $x_j + x_k = y_j + y_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$.

greater than the loss of the rich. [2] It is simple to check that monotonic SWFs that verify either MON or WD and PDT must verify AE (v., Hara et al., 2006, for a related fact).

Finally, our next axiom is the usual "equal treatment of all generations" postulate à-la-Sidgwick and Diamond.

Axiom AN (*Anonymity*). Any finite permutation of a utility stream produces a socially indifferent utility stream.

3 Pigou-Dalton, Altruistic Equity, and the existence of monotonic Social Welfare Functions

Although the Pigou-Dalton transfer principle and anonymity can be combined with weak dominance under representability of the social evaluation (cf., Alcantud 2010, Proposition 5), we proceed to show that this does not reconciles PDT with the Basu-Mitra approach: Under extremely weak technical assumptions on the structure of the set of feasible utilities, every monotonic SWF must contradict the weaker AE. In order to anticipate the reasons for our very mild technical restrictions, observe that PDT and AE hold vacuously unless $\mathbf{X}$ has some specific composition. In particular, since $\mathbf{X} = Y^{\mathbb{N}}$ we must request that in order for PDT to impose real egalitarian restrictions on the evaluations the following must be true: there are $x_1, x_1 + t, x_1 + 2t \in Y$ for some $t > 0$. If there are $x_1, x_1 + t, x_1 + 2t, x_1 + 3t \in Y$ for some $t > 0$ then AE is non-trivial too.

Therefore in order to state our main result, not only we need to refer to the cardinality or the ordinal properties of the set of feasible utilities but also to its intrinsic specification. In this we separate from other related antecedents. For example, when studying SWFs on domains of utility streams with the form $\mathbf{X} = Y^{\mathbb{N}}$, the following facts are known. Basu and Mitra (2003) proves that AN and SP are incompatible as long as Y has at least 2 different elements. Theorem 2 in Alcantud and García-Sanz (2010b) assures that HE and SP are incompatible as long as Y has at least 4 different elements. Dubey and Mitra (2011) characterized the restrictions on Y for which AN and WP are compatible. These are precisely the sets Y that do not contain any set of the *order-type* of the set of integer numbers.

Theorem 1 *Suppose Y is a set of real numbers such that there are $x_1 \in Y$ and $t > 0$ with $x_1 + kt \in Y$, $k = 1, ..., 6$. Then there are not SWFs on $\boldsymbol{X} = Y^{\mathbb{N}}$ that verify MON and AE.*

Proof We assume without loss of generality that $\{0, 1, 2, 3, 4, 5, 6\} \subseteq Y$. We use a standard construction to produce a suitable uncountable collection $\{E_i\}_{i \in I}$

[2] Sakamoto (2011) uses the term Altruistic Equity-1 or AE-1 instead. He also uses another variant of PDT that is called Altruistic Equity-2 or AE-2. Because we are maintaining the basic principle of monotonicity throughout, and AE-2 and SEP are equivalent under MON, we do not need to refer to Sakamoto's AE-2 in our paper.

of infinite proper subsets of $\mathbb{N}$. We request that $\forall i, j \in I$ $[i < j \Rightarrow E_i \subsetneq E_j$ and $E_j - E_i$ is infinite$]$. We do not lose generality if we assume $\{1,2\} \subseteq E_i$ for every index $i \in I$. In order to justify that such collection exists, we take $\{r_1, r_2, ...\}$ an enumeration of the rational numbers in $I = (0,1)$, set $E'(i) = \{n \in \mathbb{N} : r_n < i\}$ for each $i \in I$ and then $E(i) = E'(i) \cup \{1,2\}$.

Let us define the following two utility streams associated with each $i \in I$:

$$r(i)_p = \begin{cases} 5 & \text{if } \ p \in \{1,2\} \\ 3 & \text{if } \ p \in E_i - \{1,2\} \\ 0 & \quad \text{otherwise} \end{cases}$$

$$l(i)_p = \begin{cases} 6 & \text{if } \ p = 1 \\ 3 & \text{if } \ p \in E_i - \{1\} \\ 0 & \quad \text{otherwise} \end{cases}$$

By AE, the open interval $(\mathbf{W}(l(i)), \mathbf{W}(r(i)))$ is not empty: When passing from $l(i)$ to $r(i)$, one unit is lost by generation 1 while generation 2 gains two units, and both have the same endowment at $r(i)$, namely 5.

We intend to check that $j < i \Rightarrow \mathbf{W}(l(i)) > \mathbf{W}(r(j))$, which is impossible because an uncountable number of distinct rational numbers would be obtained. Select $k \in E_i - E_j$, thus $1 \neq k \neq 2$. We make use of an intermediate stream, namely $\mathbf{z} \in \mathbf{X}$ defined as follows:

$$z_p = \begin{cases} 6 & \text{if } \ p = 1 \\ 5 & \text{if } \ p = 2 \\ 3 & \text{if } \ p \in E_i - \{1,2,k\} \\ 0 & \quad \text{otherwise} \end{cases}$$

AE ensues $\mathbf{W}(l(i)) > \mathbf{W}(\mathbf{z})$: When passing from $\mathbf{z}$ to $l(i)$, two units are lost by generation 2 and generation k gains three units, and both have the same endowment at $l(i)$, namely 3.

Now MON implies $\mathbf{W}(\mathbf{z}) \geqslant \mathbf{W}(r(j))$ and the thesis follows. $\lhd$

Obviously our results have similar implications for distributional axioms implying AE under MON, like SEP, PDT, or the Lorenz domination principle (LD) which is stronger than PDT (cf., e.g., Hara et al., 2008).

Corollary 1 *In the conditions of Theorem 1, there are not SWFs that verify MON and SEP, resp., PDT, LD.*

In fact, when there are $x_1 \in Y$ and $t > 0$ with $x_1 + kt \in Y$, $k = 1,2,3$ there is no SWFs on $\boldsymbol{X} = Y^{\mathbb{N}}$ that verifies MON and PDT.

Proof The first statement is trivial because SEP $\Rightarrow$ PDT $\Rightarrow$ AE under MON, and LD $\Rightarrow$ PDT. The second one requires a straightforward modification of the argument above. The key facts are: Assume without loss of generality that $\{0,1,2,3\} \subseteq Y$, and proceed with the following definitions.

$$r(i)_p = \begin{cases} 2 & \text{if } \ p \in \{1,2\} \\ 1 & \text{if } \ p \in E_i - \{1,2\} \\ 0 & \quad \text{otherwise} \end{cases} \qquad l(i)_p = \begin{cases} 3 & \text{if } \ p = 1 \\ 1 & \text{if } \ p \in E_i - \{1\} \\ 0 & \quad \text{otherwise} \end{cases}$$

$$z_p = \begin{cases} 3 & \text{if} \;\; p = 1 \\ 2 & \text{if} \;\; p = 2 \\ 1 & \text{if} \;\; p \in E_i - \{1, 2, k\} \\ 0 & \quad\;\; \text{otherwise} \end{cases}$$

$\triangleleft$

We also observe that the restriction on the form of Y is met by the most usual requirements:

Corollary 2 *If $\mathbb{N} \subseteq Y$ or $Y = [0, 1]$, there are not SWFs on $\boldsymbol{X} = Y^{\mathbb{N}}$ that verify MON and SEP, resp., PDT, LD, AE.*

4 Conclusion

Theorems 1 and 2 in Bossert et al. (2007) prove that both PDT and SEP are compatible with orderings on $\mathbb{R}^{\infty}$ that verify Strong Pareto and anonymity. Nevertheless the literature on egalitarianism in the evaluation of infinite streams of utilities has provided evidences that the Pigou-Dalton transfer principle, as well as the Lorenz domination principle, conflict with weak forms of continuity and rationality even in the absence of Paretian restrictions (cf., Hara et al. 2008). More precisely, denote by l_+^{∞} the set of all bounded infinite sequences of non-negative real numbers. Then, provided that $\mathbf{X}$ verifies a technical condition including $l_+^{\infty} \subseteq \mathbf{X}$, there exists no acyclic social evaluation satisfying PDT, resp. LD, and P-upper or lower semicontinuity (each of which is implied by Diamond's continuity). Our result compares to this statement in that without the appeal to any controversial form of the Paretian axiom, but with virtually no requirement on the domain of utility streams, a principle that relaxes the Pigou-Dalton transfer principle into an ethically more acceptable form is incompatible with a numerical evaluation of the streams. These arguments speak for the difficulty of implementing the ethics of strict preference for a reduction in inequality in the intergenerational welfare analysis.

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