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**Abstract**

This paper proposes a new method for approximating vector autoregressions by a finite-state Markov chain. The method is more robust to the number of discrete values and tends to outperform the existing methods over a wide range of the parameter space, especially for highly persistent vector autoregressions with roots near the unit circle.

**Keywords:** Markov Chain, Vector Autoregressive Processes, Functional Equation, Numerical Methods, Moment Matching

**JEL Codes**: C15, C60

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# 1 Introduction

The finite-state Markov chain approximation methods developed by Tauchen (1986a) and Tauchen and Hussey (1991) are widely used in solving functional equations where the state variables follow autoregressive processes. Nonlinear dynamic macroeconomic and asset pricing models often imply a set of integral equations (moment conditions) that do not admit explicit solutions. Discrete-valued approximations prove to be an effective tool for reducing the complexity of the problem. Also, there is a renewed interest in these methods for generating simulation data from nonlinear dynamic models in evaluating the sampling properties of generalized method of moments estimators (Tauchen, 1986b; Hansen, Heaton and Yaron, 1996; Stock and Wright, 2000; among others). The Markov-chain approximation methods choose discrete values for the state variables and construct transition probabilities such that the characteristics of the generated process mimic those of the underlying process. However, both Tauchen (1986a) and Tauchen and Hussey (1991) point out that these methods do not perform well for highly persistent autoregressive (AR) processes or processes with characteristic roots close to unity. Although these methods can generate a better approximation at the cost of a finer state space, that type of approach is not always feasible.

The poor approximation of the methods by Tauchen (1986a) and Tauchen and Hussey (1991) for strongly autocorrelated processes has spurred a renewed research interest given the prevalence of highly persistent shocks in dynamic macroeconomic models. Rouwenhorst (1995) proposes a Markov-chain approximation of an AR(1) process constructed by targeting its first two conditional moments. Some recent advances in the literature on Markov-chain approximation methods include Adda and Cooper (2003), Floden (2008) and Kopecky and Suen (2010). While these methods provide substantial improvements in approximating highly persistent univariate AR(1) processes, their extension to vector autoregressive (VAR) processes (as well as to higher-order autoregressive processes), which is of great practical interest to macroeconomists, is not readily available and possibly highly non-trivial. As a result, the method by Tauchen (1986a) continues to be employed almost exclusively by re-

searchers for approximating multivariate processes by finite-state Markov chains. The only alternative method that is available for approximating multivariate processes is the method proposed by Galindev and Lkhagvasuren (2010). However, this method is developed for a particular class of multivariate autoregressive processes: correlated AR(1) shocks, i.e., a set of AR(1) shocks whose innovation terms are correlated with each other. Although this method can be applied to vector autoregressions by decomposing the latter into a set of interdependent AR(1) shocks, the state space generated by the method is not finite, except for the special case of equally-persistent underlying shocks. Therefore, to the best of our knowledge, a general method for approximating VAR processes by a finite-state Markov chain with appealing approximation properties over the whole parameter region of interest (including highly persistent parameterizations) is not yet available in the literature.

This paper fills this gap and proposes a new method for approximating vector autoregressions (and higher-order scalar autoregressive processes) by a finite-state Markov chain. The main idea behind this method is to construct the Markov chain by targeting conditional moments of the underlying continuous process as in Rouwenhorst (1995), rather than directly calculating the transition probabilities using the distribution of the continuous process as in the existing methods. In this respect, we give our method a moment-matching interpretation. More specifically, we express the Markov-chain transitional probabilities as the solution of a nonparametric (empirical likelihood) problem subject to moment restrictions. To target the conditional moments in constructing the Markov chain, we use key elements of the Markov chains generated by the methods of Tauchen (1986a) and Rouwenhorst (1995). Therefore, the new method extends the methods of Tauchen (1986a) and Tauchen and Hussey (1991) to highly persistent cases and Rouwenhorst (1995) and Kopecky and Suen (2010) to vector cases, while still maintaining a finite number of states.

The new method yields accurate approximations without relying on a large number of grid points for the state variables. In particular, the method extends the finite-state Markov chain approximation to a much wider range of the parameter space. While the largest gains

of the proposed method arise when the characteristic roots of the underlying process are close to unity, it tends to outperform (in terms of bias and variance) the existing methods even when the persistence is moderate or low. Finally, the method can be readily adapted to accommodate other important features of the conditional distribution of the continuous-valued process.

The rest of the paper is organized as follows. Section 2 introduces the continuous- and discrete-valued versions of the multivariate model and the main notation. Section 3 reviews the existing approximation methods and demonstrates that they fail to deliver a reasonable approximation as the roots of the continuous-valued process approach the unit circle. The reason for this is that the existing methods calculate transition probabilities defined over *discrete* grids using *continuous* probability density functions. Therefore, the quality of the approximation deteriorates sharply when the standard deviation of the error terms becomes comparable to or smaller than the distance between the grid points. Our new approximation method is introduced in Section 4. We show that the new method matches the first two conditional moments of the underlying process and describe the construction of the transition probability matrix and the Markov chain. Section 5 investigates the numerical properties of the method in a bivariate VAR(1) process with a varying degree of persistence. Section 6 concludes. MATLAB codes for implementing the method are available from the authors upon request. The proofs and some additional theoretical results are presented in Appendices A and B.

# 2 Model

In this section we present the underlying continuous-valued vector autoregressive process and introduce the main structure and notation for the finite-state Markov chain used for approximating the continuous process.

## 2.1 Continuous VAR process

Let $\mathbf{y}_t$ be an $(M \times 1)$ vector containing the values that variables, $y_1, y_2, \cdots, y_M$, assume at date $t$. We consider the following vector autoregressive (VAR) process:

$$\mathbf{y}_t = A\mathbf{y}_{t-1} + \boldsymbol{\varepsilon}_t, \tag{1}$$

where $A$ is an $(M \times M)$ matrix with roots that lie strictly outside (although arbitrarily close to) the unit circle and the $(M \times 1)$ vector $\boldsymbol{\varepsilon}_t$ is i.i.d $\mathcal{N}(0, \Omega)$, where $\Omega$ is a diagonal matrix. Extending the analysis to a non-diagonal $\Omega$ is relatively straightforward and is discussed later in the paper.[1] Our focus on the zero-mean, first-order VAR is primarily driven by expositional and notational simplicity and deterministic terms as well as higher-order dynamics can be easily incorporated at the expense of additional notation. Let $\Sigma$ denote the unconditional covariance matrix of the process $\mathbf{y}$. Then, it is the case that

$$\Sigma = A\Sigma A' + \Omega. \tag{2}$$

Let $a_{i,j}$ denote the row $i$, column $j$ element of the matrix $A$. Also, let $\omega_i$ and $\sigma_i$ denote, respectively, the square roots of the $i$-th diagonal elements of $\Omega$ and $\Sigma$ for $i \in \{1, 2, \cdots, M\}$.

## 2.2 Finite-state Markov chain

Let $\tilde{\mathbf{y}}_t$ denote the finite-state Markov chain that approximates $\mathbf{y}_t$ in (1). Each component $\tilde{y}_{i,t}$ takes on one of the $N_i$ discrete values denoted by $\bar{y}_i^{(1)}, \bar{y}_i^{(2)}, \cdots, \bar{y}_i^{(N_i)}$. Therefore, at each point in time, the entire system will be in one of the $N^* = N_1 \times N_2 \times \cdots \times N_M$ states. Let

[1] Results for non-Gaussian errors, that target also the conditional skewness and kurtosis of the underlying process, are available from the authors upon request. Since the normality (and log-normality, in the case of modeling shocks with stochastic volatility as in Fernandez-Villaverde, Guerron-Quintana, Rubio-Ramirez and Uribe, forthcoming) assumption is routinely used in describing the properties of the shocks in macroeconomic models, the current version of the paper presents the construction of the finite-state Markov chain only for this benchmark case. We should also note that regardless of the true values of skewness and kurtosis of the error terms, the unconditional skewness and kurtosis of the process converge to those of the normal distribution when the persistence approaches the nonstationary boundary.

$\bar{y}^{(1)}, \bar{y}^{(2)}, \cdots, \bar{y}^{(N^*)}$ label these $N^*$ states and $\Pi$ denote the $N^* \times N^*$ transition matrix whose [row $j$, column $k$] element $\pi_{j,k}$ measures the probability that in the next period the system will be in state $k$ conditional on the current state $j$.

Our *goal* is to construct a finite number of grid points for each element of $\tilde{\mathbf{y}}_t$ and to calculate the associated transition probability matrix $\Pi$ so that the characteristics of the generated process closely mimic those of the underlying process $\mathbf{y}$.

Define

$$h_i(j,l) = \Pr(\tilde{y}_{i,t} = \bar{y}_i^{(l)} | \tilde{\mathbf{y}}_{t-1} = \bar{y}^{(j)}) \tag{3}$$

for $i = 1, 2, \cdots, M$, $l = 1, 2, \cdots, N_i$ and $j = 1, 2, \cdots, N^*$. For any $i$, let $L_i$ be an integer-valued function such that $\tilde{y}_i = \bar{y}_i^{(L_i(j))}$ when the system is in state $j$. Given these functions, the transition probability $\pi_{j,k}$ is given by the product of the individual transition probabilities:

$$\pi_{j,k} = \prod_{i=1}^{M} h_i(j, L_i(k)). \tag{4}$$

This means that, for each pair $(i,j)$, we need to construct $N_i$ transition probabilities

$$H_i(j) = \{h_i(j,1), h_i(j,2), \cdots, h_i(j,N_i)\} \tag{5}$$

over the grid points $\bar{y}_i^{(1)}, \bar{y}_i^{(2)}, \cdots, \bar{y}_i^{(N_i)}$. Since, for each $(i,j)$, $\sum_{l=1}^{N_i} h_i(j,l) = 1$, $H_i(j)$ can be considered as a probability mass distribution defined over the discrete values $\bar{y}_i^{(1)}$, $\bar{y}_i^{(2)}$, $\cdots$, $\bar{y}_i^{(N_i)}$.

The problem of determining the probability weights associated with this probability mass distribution can be expressed as a nonparametric likelihood problem. In particular, the nonparametric (or empirical) likelihood estimate of the transition probability matrix can be obtained as the solution to the constrained maximization problem:

$$\max_{h_i(j,l) \in [0,1]} \sum_{i=1}^{M} \sum_{j=1}^{N^*} \sum_{l=1}^{N_i} \ln(h_i(j,l)) \tag{6}$$

subject to the constraint

$$\sum_{l=1}^{N_i} h_i(j,l) = 1 \text{ for } i = 1, \cdots, M. \tag{7}$$

To avoid trivial solutions, this optimization problem needs to be augmented with additional restrictions that best describe the statistical properties of model (1) for $\mathbf{y}_t$. These restrictions typically take the form of conditional moments. Let

$$\mu_i(j) = a_{i,1}\bar{y}_1^{(L_1(j))} + a_{i,2}\bar{y}_2^{(L_2(j))} + \cdots + a_{i,M}\bar{y}_M^{(L_M(j))} \tag{8}$$

for any $i$ and $j$, denote the expected value of process $y_{i,t+1}$, conditional on $\mathbf{y}_t = \bar{y}^{(j)}$. Then, it seems natural to impose the restriction

$$\sum_{l=1}^{N_i} h_i(j,l)\bar{y}_i^{(l)} = \mu_i(j) \tag{9}$$

for $i = 1, \cdots, M$ and $j = 1, \cdots, N^*$ which requires that the Markov chain adequately approximates the conditional mean of the continuous-valued process $\mathbf{y}_t$. The new method that we propose below targets also the second conditional moment of the process $\mathbf{y}_t$ by imposing the additional restriction

$$\sum_{l=1}^{N_i} h_i(j,l)(\bar{y}_i^{(l)} - \mu_i(j))^2 = \omega_i^2 \tag{10}$$

for $i = 1, \cdots, M$ and $j = 1, \cdots, N^*$.

# 3 Existing Methods

Before outlining the new method, we first discuss the disadvantages of the existing finite-state methods for approximating vector autoregressions. We consider the method proposed by Tauchen (1986a) as a representative of these methods. The main reason is that, according to Floden (2008), the method by Tauchen (1986a) is more robust to parameters of the

underlying process than its version in Tauchen and Hussey (1991). Moreover, these existing methods share the common feature that for calculating the transition probabilities defined over discrete grids, they use continuous probability distribution functions. Finally, as argued in the introduction, the finite-state extension of the recently proposed methods for improving the Markov chain approximation in near-nonstationary region to multivariate processes is not readily available.

The construction of the transition probabilities and the Markov chain for Tauchen's (1986a) method can be described as follows. For each $i$, Tauchen (1986a) chooses equispaced grid points over the interval $[-m\sigma_i, m\sigma_i]$ for some $m > 0$.[2] Specifically, for each $i$, the grid points are chosen according to the following rule:

$$\bar{y}_i^{(l)} = -m\sigma_i + (l-1)\triangle_i, \tag{11}$$

where $\triangle_i = 2m\sigma_i/(N_i - 1)$ and $l = 1, 2, \cdots, N_i$. Note that $\triangle_i$ measures the distance between two consecutive nodes of $\tilde{y}_i$.

Given the above grid points, consider the following partition of the real line for each $i$: $C_i^{(1)} = ]-\infty, \bar{y}_i^{(1)} + \triangle_i/2]$, $C_i^{(N_i)} = ]\bar{y}_i^{(N_i)} - \triangle_i/2, \infty[$, and $C_i^{(l)} = ]\bar{y}_i^{(l)} - \triangle_i/2, \bar{y}_i^{(l)} + \triangle_i/2]$, where $l = 2, 3, \cdots, N_i - 1$. Tauchen (1986a) calculates the transition probabilities as

$$h_i(j,l) = \Pr\left(\mu_i(j) + \varepsilon_i \in C_i^{(l)}\right). \tag{12}$$

Denoting the cumulative distribution function of the standard normal variable $\varepsilon_i/\omega_i$ by $\Phi_i$, equation (12) can be rewritten as

$$h_i(j,l) = \begin{cases} \Phi_i\left(\frac{\bar{y}_i^{(1)} - \mu_i(j) + \triangle_i/2}{\omega_i}\right) & \text{if } l = 1, \\ 1 - \Phi_i\left(\frac{\bar{y}_i^{(N_i)} - \mu_i(j) - \triangle_i/2}{\omega_i}\right) & \text{if } l = N_i, \\ \Phi_i\left(\frac{\bar{y}_i^{(l)} - \mu_i(j) + \triangle_i/2}{\omega_i}\right) - \Phi_i\left(\frac{\bar{y}_i^{(l)} - \mu_i(j) - \triangle_i/2}{\omega_i}\right) & \text{otherwise.} \end{cases} \tag{13}$$

[2] For example, in Tauchen (1986a), $m$ is set to 3.

According to Tauchen (1986a), the rationale for equations (8) and (12) is that if the partitioning $\left(C_i^{(1)}, C_i^{(2)}, \cdots, C_i^{(M)}\right)$ is reasonably fine, then the conditional distribution of $\tilde{y}_{i,t}$ given state $j$ at time $t-1$ will approximate closely (in the sense of weak convergence) the conditional distribution of $y_{i,t}$ given $y_{i,t-1} = \mu_i(j)$.

Given the finite-state Markov chain $\tilde{\mathbf{y}}_t$, let

$$\tilde{\boldsymbol{\varepsilon}}_t = \tilde{\mathbf{y}}_t - A\tilde{\mathbf{y}}_{t-1}, \tag{14}$$

$\tilde{\Omega}$ be the covariance matrix of $\tilde{\boldsymbol{\varepsilon}}$ and $\tilde{\omega}_i$ denote the square root of the $i$-th diagonal element of this matrix. Since the conditional probabilities for this Markov chain are obtained by centering the density of $\boldsymbol{\varepsilon}$ over $A\tilde{\mathbf{y}}_{t-1}$, we have

$$\mathrm{E}(\tilde{\mathbf{y}}_t|\tilde{\mathbf{y}}_{t-1}, \tilde{\mathbf{y}}_{t-2}, \cdots) = A\tilde{\mathbf{y}}_{t-1}, \tag{15}$$

and

$$\mathrm{E}(\tilde{\boldsymbol{\varepsilon}}_t) = E\{E[(\tilde{\mathbf{y}}_t - A\tilde{\mathbf{y}}_{t-1})|\tilde{\mathbf{y}}_{t-1}, \tilde{\mathbf{y}}_{t-2}, \cdots]\} = 0_M. \tag{16}$$

This implies, by the law of iterated expectations, that $\tilde{\boldsymbol{\varepsilon}}_t$ is uncorrelated with $\tilde{\mathbf{y}}_{t-s}$ for $s = 1, 2, \cdots$. However, the conditional covariance matrix of $\tilde{\boldsymbol{\varepsilon}}_t$, $\mathrm{Var}(\tilde{\boldsymbol{\varepsilon}}_t|\tilde{\mathbf{y}}_{t-1}, \tilde{\mathbf{y}}_{t-2}, \cdots)$, depends on $\tilde{\mathbf{y}}_{t-1}$ and thus, $\tilde{\boldsymbol{\varepsilon}}_t$ and $\tilde{\mathbf{y}}_{t-1}$ are dependent (Anderson, 1989). This clearly suggests that targeting the second conditional moment will improve the quality of the Markov-chain approximation and it serves as a main motivation for the new method proposed in this paper.

We now show that calculating the transition probabilities using the continuous distribution functions does not always deliver meaningful approximations.

**Proposition 1.** *Let $\tilde{\omega}_i^2$ denote the conditional variance of $\tilde{\varepsilon}$ in equation (14), where $\tilde{\mathbf{y}}_t$ is the finite-state Markov chain constructed using the Tauchen's (1986a) method . Then, for any set of integers $(N_1, N_2, \cdots, N_M)$ and any arbitrarily small positive number $\epsilon$, there always exists a vector autoregressive process for which $\tilde{\omega}_i^2/\omega_i^2 < \epsilon$ for all $i$.*

*Proof.* See Appendix A. □

Proposition 1 is an extension of the results in Galindev and Lkhagvasuren (2010). The main implication of the result in Proposition 1 is that Tauchen's (1986a) method will fail to approximate the variability in $\mathbf{y}_t$ as one or more of the roots of the underlying continuous-valued VAR process $\mathbf{y}_t$ approach the unit circle. This problem arises because most of the existing approximation methods, including the method by Tauchen (1986a), target only the first conditional moment of the continuous-valued process $\mathbf{y}_t$.

A few remarks on some alternative approximation methods may seem warranted. Tauchen and Hussey (1991) modify the method of Tauchen (1986a) by choosing the grid points according to the Gaussian quadrature rule. However, Floden (2008) demonstrates that for a wide range of parameters, Tauchen's (1986a) method is more robust than the one in Tauchen and Hussey (1991). Floden (2008) uses an alternative weighting function to improve the performance of the method in Tauchen and Hussey (1991). However, some gains occur only over a limited range of the parameters of the underlying process. In fact, Galindev and Lkhagvasuren (2010) show that outside of the range considered by Floden (2008), his version fails to generate any time varying Markov chain and thus is less robust than the method by Tauchen (1986a).

Despite some numerical and methodological differences across the existing Markov-chain approximations, all these methods suffer from the same problem as in Tauchen (1986a) since they calculate the transition matrices using distribution functions centered around the first conditional moment. In other words, regardless of the way the grid points are constructed, there is a non-zero distance between any two grid points and thus one can directly extend Proposition 1 to these methods.

# 4 New Method

Having demonstrated the main shortcoming of the existing methods, we now propose an alternative method for approximating a VAR process by a finite-state Markov chain.

## 4.1 Main idea

Unlike the existing finite-state methods for a multivariate process, which calculate the transition probabilities using the conditional distribution function of $\mathbf{y}_t$, the new method chooses the transition probabilities by targeting the key conditional moments of $\mathbf{y}_t$. Specifically, it approximates the underlying process by targeting the $2M \times N^*$ moment conditions given by equations (9) and (10). This means that the method chooses the grid points $\{\bar{y}_i^{(1)}, \bar{y}_i^{(2)}, \cdots, \bar{y}_i^{(N_i)}\}_{i=1}^M$ and the associated probability mass functions $\{H_i(1),\ H_i(2),\ \cdots,\ H_i(N^*)\}_{i=1}^M$ so that, for all $i$ and $j$, the mean and the variance of distribution $H_i(j)$ target, respectively, $\mu_i(j)$ and $\omega_i^2$.

The grid points and the probability mass functions are constructed using the method of Rouwenhorst (1995). This particular choice of grid points and transition probabilities generates a Markov chain which *perfectly* matches the conditional mean and conditional variance of the underlying scalar AR(1) process.[3] While incorporating information about the conditional variance is expected to deliver efficiency gains compared to Tauchen's (1986a) method over the whole permissible parameter range, we also show that this method has some appealing properties when the underlying AR(1) process is highly persistent or near unit root.

Although the method developed by Rouwenhorst (1995) tends to perform well in ap-

[3]In particular, for an AR(1) process $y_t = \rho y_{t-1} + \varepsilon_t$,where $|\rho| < 1$, $\varepsilon_t$ is i.i.d. $\mathcal{N}(0, (1-\rho^2)\sigma^2)$ with $\sigma^2 = \text{Var}(y_t)$, it can be shown that $\text{E}(\tilde{y}_t) = 0$, $\text{Var}(\tilde{y}_t) = \sigma^2$ and $\text{Corr}(\tilde{y}_t, \tilde{y}_{t-1}) = \rho$ as well as $\text{E}(\tilde{y}_t|\tilde{y}_{t-1} = \bar{y}^{(k)}) = \rho\bar{y}^{(k)}$ and $\text{Var}(\tilde{y}_t|\tilde{y}_{t-1} = \bar{y}^{(k)}) = (1-\rho^2)\sigma^2$. One important feature of this result is that the persistence (autocorrelation) parameter for our Markov chain method is independent of the number of grid points. This stands in sharp contrast with the existing methods (including Tauchen's (1986a) method) which are very sensitive to the number of grid points in approximating near unit root processes.

proximating highly persistent scalar autoregressive processes,[4] it is not readily applicable to the vector case. As pointed out in the introduction, the method by Galindev and Lkhagvasuren (2010) is tailored for approximating correlated AR(1) shocks and does not deliver a finite-state space when adapted for approximating vector autoregressions.[5] By contrast, the new method developed in this paper delivers a finite-state approximation regardless of the degree of persistence of the different components of $\mathbf{y}$ while using only $M$ state variables (i.e., $\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_M$). This could potentially offer substantial computational gains when solving functional equations.

## 4.2 Markov chain construction

We use the following procedure to construct the Markov chain $\tilde{\mathbf{y}}$:

1. For each $i$, set $n = N_i$, $s = \sigma_i$ and $\rho = \sqrt{1 - \omega_i^2/\sigma_i^2}$.

2. Following Rouwenhorst (1995), construct $n$ discrete values $\{\bar{y}_i^{(1)}, \bar{y}_i^{(2)}, \cdots, \bar{y}_i^{(n)}\}$ that are computed as

$$\bar{y}_i^{(k)} = -s\sqrt{n-1} + \frac{2s}{\sqrt{n-1}}(k-1) \tag{17}$$

   for $k = 1, 2, \cdots, n$, and let these be the grid points of $\tilde{y}_i$.

3. Construct the probability transition matrix of an $n$-state (scalar) Markov process with persistence $\rho$. Let $P^{(n)}(\rho)$ denote this matrix. When $n = 2$, the probability transition matrix is given by

$$P^{(2)}(\rho) = \begin{pmatrix} \frac{1+\rho}{2} & \frac{1-\rho}{2} \\ \frac{1-\rho}{2} & \frac{1+\rho}{2} \end{pmatrix}. \tag{18}$$

[4] See, for example, Galindev and Lkhagvasuren (2010) and Kopecky and Suen (2010) for a numerical comparison of the method of Rouwenhorst (1995) with some commonly-used existing methods including those of Tauchen (1986a) and Tauchen and Hussey (1991).

[5] In order to approximate an $M$-variate process $\mathbf{y_t}$ given by equation (1), in general, the method by Galindev and Lkhagvasuren (2010) uses $2M - 1$ state variables, of which $M - 1$ are continuous. Therefore, their method requires an additional approximation step, $M - 1$ dimensional numerical interpolation, when applied to functional equations.

For higher values of $n$, the transition probability matrix is constructed recursively using the elements of matrix $P^{(2)}(\rho)$ (for details, see Rouwenhorst (1995)). Let $p_{k,l}^{(n)}(\rho)$, the row $k$, column $l$ element of $P^{(n)}(\rho)$, be the probability that the $n$-state process switches from $\bar{y}_i^{(k)}$ to $\bar{y}_i^{(l)}$. Consequently, the $k$-th row of the matrix $P^{(n)}(\rho)$ is given by

$$p_k^{(n)}(\rho) = \{p_{k,1}^{(n)}(\rho), p_{k,2}^{(n)}(\rho), \cdots, p_{k,n}^{(n)}(\rho)\}. \tag{19}$$

Since $\sum_{l=1}^{n} p_{k,l}^{(n)}(\rho) = 1$, this row can be interpreted as a probability mass function associated with the nodes $\{\bar{y}_i^{(1)}, \bar{y}_i^{(2)}, \cdots, \bar{y}_i^{(n)}\}$. Note that the mean and the variance of the probability mass distribution $p_k^{(n)}(\rho)$ are, respectively, $\rho\bar{y}_i^{(k)}$ and $\omega_i^2$.

4. For each $j \in \{1, 2, \cdots, N^*\}$, we consider the following four distinct cases:

   (a) If $\mu_i(j) < \rho\bar{y}_i^{(1)}$, set

   $$H_i(j) \equiv p_1^{(n)}(\rho). \tag{20}$$

   (b) If $\mu_i(j) > \rho\bar{y}_i^{(n)}$, set

   $$H_i(j) \equiv p_n^{(n)}(\rho). \tag{21}$$

   In these two cases, the conditional variance $\omega_i^2$ is matched while the conditional mean attains the value closest to $\mu_i(j)$ given the grid points.

   (c) If $\mu_i(j) = \rho\bar{y}_i^{(k)}$ for some $k \in \{1, 2, \cdots, n\}$, set

   $$H_i(j) \equiv p_k^{(n)}(\rho). \tag{22}$$

   In this case, both the conditional mean $\mu_i(j)$ and conditional variance $\omega_i^2$ are matched.[6]

   (d) If any of the above three conditions is not met, there must be an integer $k$ such

[6] This case includes the special sub-case $\sigma_i = \omega_i$. Therefore, when $\rho = \sqrt{1 - \omega_i^2/\sigma_i^2} = 0$, set $H_i(j) = p_1^{(n)}(0)$ for all $j$.

that $1 \leq k \leq n-1$ and $\rho\bar{y}_i^{(k)} < \mu_i(j) < \rho\bar{y}_i^{(k+1)}$. Then, consider the following mixture distribution defined over $\{\bar{y}_i^{(1)}, \bar{y}_i^{(2)}, \cdots, \bar{y}_i^{(n)}\}$ for some $0 < \lambda < 1$:

$$\tilde{p}_k^{(n)}(\rho) = \lambda p_k^{(n)}(\rho) + (1-\lambda) p_{k+1}^{(n)}(\rho), \tag{23}$$

where

$$\lambda = \frac{\rho\bar{y}_i^{(k+1)} - \mu_i(j)}{\rho\bar{y}_i^{(k+1)} - \rho\bar{y}_i^{(k)}}. \tag{24}$$

The mean and variance of this new probability mass distribution are, respectively,

$$\tilde{\mu}_k(\rho) = \mu_i(j) \tag{25}$$

and

$$\tilde{\omega}_k^2(\rho) = (1-\rho^2)s^2 + \rho^2\lambda(1-\lambda)\left(\bar{y}_i^{(k+1)} - \bar{y}_i^{(k)}\right)^2. \tag{26}$$

Since $0 < \lambda < 1$, $\tilde{\omega}_k^2(\rho) > (1-\rho^2)s^2 = \omega_i^2$.[7] This means that although the mean of the mixture distribution $\tilde{p}_k^{(n)}(\rho)$ hits its target $\mu_i(j)$, the variance of the distribution is greater than our target $\omega_i^2$. It is easy to observe that $\tilde{\omega}_k^2(\rho)$ goes down as $\rho$ increases. Therefore, we repeat the calculations in Steps 3 and 4(d) for higher values of $\rho$ and choose the one which minimizes the distance $|\tilde{\omega}_k^2(\rho) - \omega_i^2|$. Let $\rho^*$ be the value of $\rho$ found in such a manner. Then, set

$$H_i(j) \equiv \tilde{p}_k^{(n)}(\rho^*). \tag{27}$$

In this case, the conditional mean $\mu_i(j)$ is matched, but the conditional variance achieves the best possible value given the grid points.

[7] Note that since $\rho < 1$ and $\lambda(1-\lambda) \leq 0.25$, it follows that the effect of the second term on the right-hand side of equation (26) converges to zero as the number of grid points increases.

Repeating this procedure for $i = 1, 2, \cdots, M$, we construct the grid points $\{\bar{y}_i^{(1)}, \bar{y}_i^{(2)}, \cdots, \bar{y}_i^{(N_i)}\}_{i=1}^{M}$ and the probability mass functions $\{H_i(1), H_i(2), \cdots, H_i(N^*)\}_{i=1}^{M}$. Given these probability mass functions, we construct the transition probability matrix of the Markov chain $\tilde{\mathbf{y}}$, $\Pi$, using equations (4) and (5). The asymptotic validity of the method is discussed in Appendix B.

While the procedure above is developed under the assumption of a diagonal covariance matrix $\Omega$, the proposed method can be easily extended to the case of a non-diagonal covariance matrix. Suppose now that the underlying continuous-valued process follows

$$\mathbf{x}_t = b + B\mathbf{x}_{t-1} + \eta_t, \tag{28}$$

where $\eta_t$ is i.i.d $(0, \Psi)$ and $\Psi$ is a non-diagonal matrix.[8] Let $G$ be a lower triangular matrix such that $\Omega = G\Psi G^{-1}$ is a diagonal matrix. Define the transformations (Tauchen, 1986b),

$$\mathbf{x}_t \to G[\mathbf{y}_t - (I_M - B)^{-1}b], \tag{29}$$

$$B \to A = GBG^{-1}, \tag{30}$$

and

$$\eta_t \to G\varepsilon_t. \tag{31}$$

Then, we have the same model as in equation (1). After computing the discrete Markov-chain approximation for this modified model, we reverse the transformations above in order to obtain the discrete process corresponding to equation (28).

Furthermore, since any stationary AR($p$) process can be expressed in a companion form as a VAR(1) process, our method effectively extends the method by Rouwenhorst (1995) to higher-order scalar autoregressive processes. Another appealing feature of our method is

[8] See also Terry and Knotek II (2011), who extend Tauchen (1986a) to processes with a non-diagonal covariance matrix using multivariate integration. Since Terry and Knotek II (2011) calculate the transition matrices using distribution functions, the issue with highly persistent shocks remains in their extension.

its flexibility which allows us to deal with possibly nonlinear VAR processes and targeting additional moments of the conditional distribution such as skewness and kurtosis.

Below we examine numerically how well the method works in terms of approximating autoregressive processes for various degrees of persistence of the discrete space. We show that the new method outperforms the method by Tauchen (1986a), especially for the processes whose characteristic roots are close to one.

# 5 Numerical Evaluation

For our main numerical evaluation, we consider the bivariate VAR(1) case ($M = 2$) with

$$\boldsymbol{\varepsilon}_t \sim \text{i.i.d } \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}\right) \tag{32}$$

and $A = A_0^K$, where

$$A_0 = \begin{pmatrix} 0.995619 & 0.005335 \\ 0.003557 & 0.992063 \end{pmatrix} \tag{33}$$

and $K$ is a positive integer set to 1, 10 and 100.[9] It is straightforward to see that higher values of $K$ imply lower persistence. As in Tauchen (1986a), we choose nine grid points for each component: $\tilde{N} = N_1 = N_2 = 9$. We also consider another case in which the state space is much finer: $\tilde{N} = 19$.

[9]The matrix $A_0$ is chosen for comparison purposes: when $K = 100$,

$$A = A_0^{100} = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.5 \end{pmatrix}$$

and thus the vector autoregressive process becomes the one considered in Tauchen (1986a).

## 5.1 The accuracy of the approximation

Let $\{\tilde{y}_t\}_{t=1}^{\tau}$ denote simulated time series either from the Markov chain approximation by Tauchen (1986a) or the method proposed in this paper. The accuracy of the two approximations can then be examined by estimating the key parameters of the initial process in equation (1). The parameters of interest are the unconditional variances of $y_1$ and $y_2$ (denoted by $\sigma_1^2$ and $\sigma_2^2$), the cross-correlation coefficient between $y_1$ and $y_2$, and the persistence measures $1-\xi_1$ and $1-\xi_2$, where $\xi_1$ and $\xi_2$ are the two roots (eigenvalues) of matrix $A$. As in Tauchen (1986a) and Tauchen and Hussey (1991), the simulated counterpart of $A$, $\hat{A}$, is obtained by fitting the linear autoregressive model in equation (1) to $\{\tilde{y}_t\}_{t=1}^{\tau}$. The summary of the approximation accuracy is based on 1,000 Monte Carlo replications of length $\tau = 2,000,000$. Tables 1 and 2 report the root mean squared errors (RMSE) as well as the biases and standard deviations of these parameters relative to their true values.

The results suggest that the new method dominates the method by Tauchen (1986a) in terms of bias and RMSE for all parameters of interest across all degrees of persistence. For example, for the least persistent case ($K = 100$), the relative bias for $\tilde{N} = 9$ of the estimated $1-\xi_1$, $\sigma_1^2$ and $\sigma_2^2$, using data generated by Tauchen's (1986a) method, is 3.5%, 6.6% and 4.4%, respectively, whereas the corresponding biases for the new method are 0.9%, -0.8% and -0.5%. For the moderate degree of persistence ($K = 10$), the biases for the method of Tauchen (1986a) become -19.3%, 35.6% and 28.7%, while those of the new method remain almost constant at 1.7%, -0.7% and -0.9%, respectively. However, the significant advantages of our method become particularly striking for the high persistence case ($K = 1$). For this degree of persistence, Tauchen's (1986a) method fails to produce any time variation in the approximate Markov chain process, which is consistent with our theoretical results in Proposition 1. For example, the average probability of switching from the current state to any other state (with $\tilde{N} = 9$) is only 0.03% for the method by Tauchen (1986a). This results in substantially large biases and inflated RMSEs for the parameters of interest. At the same time, our method continues to perform extremely well with very low biases and

RMSEs. Increasing the number of grid points from 9 to 19 improves the performance of Tauchen's (1986a) method in the less persistent cases but its numerical properties in the highly persistent case remain rather poor.

Kopecky and Suen (2010) prove that the invariant distribution of the Markov chain constructed by Rouwenhorst's (1995) method is a binomial distribution. A direct consequence of this result is that the invariant distribution of the Markov chain constructed by Rouwenhorst's (1995) method converges asymptotically (as the number of states goes to infinity) to a normal distribution. This is not surprising because the method by Rouwenhorst (1995) targets only the first two conditional moments of the underlying process. Therefore, it might be instructive to see how our new method and Tauchen's (1986a) method approximate the higher-order moments (skewness and excess kurtosis) of the continuously-valued process. The results (not reported here to conserve space) show that the higher-order moments of the two methods do not differ much when the persistence is low. When persistence is high, the new method outperforms (often substantially) Tauchen's method by generating skewness and excess kurtosis much closer to their true values. In highly persistent cases, the method by Tauchen (1986a) often fails to generate any variation in some of the components of $\tilde{y}$ (see Proposition 1) and thus their higher-order moments are not defined.

## 5.2 Conditional moments

The evaluation of the approximation accuracy in Tables 1 and 2 is based on unconditional moments of the underlying and simulated processes. Potentially important information about the quality of the approximation is also contained in the conditional moments. Hence, it would be interesting also to report the first two moments, conditional on the state of the process.

Given the constructed grid points and transition probabilities, the implied conditional mean and variance are $\hat{\mu}_i(j) = \sum_{l=1}^{N_i} h_i(j,l)\bar{y}_i^{(l)}$ and $\hat{\omega}_i^2(j) = \sum_{l=1}^{N_i} h_i(j,l)(\bar{y}_i^{(l)} - \hat{\mu}_i(j))^2$, where $i \in \{1, 2, \cdots, M\}$ and $j \in \{1, 2, \cdots, N^*\}$. Then, for each $i$ and $j$, the distances between

the targeted and the generated conditional moments can be measured by $|\hat{\mu}_i(j) - \mu_i(j)|$ and $|\hat{\omega}_i^2(j)/\omega_i^2 - 1|$. To assess the overall accuracy of the conditional moments, we consider the weighted averages of these distances across the $N^*$ states using the frequencies of each state as weights. The weights are constructed from a simulated process of length $\tau = 2,000,000$. The results are presented in Table 3 and show that the new method performs extremely well across all parameterizations. Again, this is not surprising since, by construction, this method targets the first two conditional moments of the underlying process. More importantly, the results show that calculating the transition probabilities using the conditional distribution, as in Tauchen (1986a), generates a substantial bias in the conditional moments. This numerical finding also lends support to our theoretical result in Proposition 1.

# 6 Conclusion

This paper proposes a new method for approximating vector autoregressions by a finite-state Markov chain. The main idea behind this method is to construct the Markov chain by targeting a set of conditional moments of the underlying process rather than calculating the transition probabilities directly from an assumed distribution, centered around the first conditional moment, as in the existing methods. The new method yields accurate approximations for a wide range of the parameter space, without relying on a large number of grid points for the state variables. The improved approximation accuracy of the proposed method is expected to have important quantitative implications for solving dynamic stochastic models as well as multivariate functional equations.

# A Proof of Proposition 1

We establish the result in Proposition 1 for process $i$, we parameterize the high persistence as local-to-unity (see Phillips, 1987, for example).

Let $\tilde{y}_T^{(n)} = (\tilde{y}_1^{(n)}, \tilde{y}_2^{(n)}, \cdots, \tilde{y}_t^{(n)}, \cdots, \tilde{y}_T^{(n)})$ be a realization of the $n$-state Markov chain of length $T$ approximated over $n$ grid points. In what follows, we keep $n$ fixed and perform the analysis as $T \to \infty$. In the local-to-unity framework, the autoregressive parameter in the AR process is reparameterized as a function of $T$ as

$$\rho = 1 - \frac{c}{T}, \tag{A.1}$$

where $c > 0$ is a fixed constant. This is an artificial statistical device in which the parameter space is a shrinking neighborhood of one as $T$ increases. This parameterization proves to be a very convenient tool to study the properties of strongly dependent processes as $T \to \infty$.

First note that using this reparameterization, the variance of the innovation for the continuous-valued process can be expressed as

$$\omega^2 = (1 - \rho^2)\sigma^2 = \frac{2c\sigma^2}{T} - \frac{c^2\sigma^2}{T^2}. \tag{A.2}$$

For Tauchen's (1986a) method, the probability that the process switches from state $j$ (corresponding to grid point $\bar{y}^{(j)}$) to any other state is given by

$$1 - \pi_{j,j} = 1 - \Pr\left(\left|\varepsilon - \frac{c\bar{y}^{(j)}}{T}\right| \leq 2\triangle\right) \tag{A.3}$$

where $\pi_{j,j}$ is the $j$-th diagonal element of $\Pi$. As $T \to \infty$, the persistence of the process increases and $0 < \bar{y}^{(j)}/T < 2\triangle$ (for all $j$) with probability approaching one. Therefore,

$$1 - \pi_{j,j} \leq 1 - \Pr\left(|\varepsilon| \leq 2\triangle\right) = 2\left(1 - \Phi\left(\frac{2\triangle}{\sqrt{\sigma^2(1-\rho^2)}}\right)\right) < 2\left(1 - \Phi\left(\frac{2\triangle\sqrt{T}}{\sqrt{2c\sigma^2}}\right)\right) \tag{A.4}$$

and

$$\frac{1-\pi_{j,j}}{\omega^2} < \frac{2\left(1-\Phi\left(\frac{\triangle\sqrt{2T}}{\sigma\sqrt{c}}\right)\right)}{2c\sigma^2/T - c^2\sigma^2/T^2} \tag{A.5}$$

for all $j$. Since

$$\Phi\left(\frac{\triangle\sqrt{2T}}{\sigma\sqrt{c}}\right) \to 1 \text{ as } T \to \infty, \tag{A.6}$$

by l'Hopital's rule,

$$\lim_{T\to\infty} \frac{1-\pi_{j,j}}{\omega^2} = \frac{\triangle}{2\sigma^3 c^{3/2}\pi^{1/2}} \frac{1}{(1/T^{3/2} - c/T^{5/2})\exp(2c\,\triangle^2\,T/\sigma^2)} = 0. \tag{A.7}$$

Hence, since the limiting behavior of the conditional variance of the Markov-chain approximation is determined by the limiting behavior of $1-\pi_{j,j}$,

$$\frac{\tilde{\omega}^2}{\omega^2} \to 0 \text{ as } T \to \infty. \tag{A.8}$$

This completes the proof of Proposition 1.

# B Asymptotic validity of the approximation method

Consider the following scalar continuous-valued AR(1) process $y' = \rho y + \varepsilon$ ($|\rho| < 1$) with conditional density $f(y'|y)$ and the function

$$e_g(y) = \int g(y')f(y'|y)dy, \tag{B.1}$$

where $g(y) \in C_0[a,b]$ and $C_0[a,b]$ denotes the space of continuous functions on $[a,b]$ with $a < b$ and both $a$ and $b$ are finite. Assume that the support of $f(y'|y)$ is a subset of $[a,b]\times[a,b]$ and $f(y'|y)$ is jointly continuous in $y'$ and $y$. Let $\tilde{y}$ denote the $n$-state Markov-chain approximation proposed that takes on the discrete values $\{\bar{y}^{(1)}, \bar{y}^{(2)}, \cdots, \bar{y}^{(n)}\}$ and

transition probabilities $\pi_{j,k}^{(n)} = \Pr(\tilde{y}' = \bar{y}^{(k)} | \tilde{y} = \bar{y}^{(j)})$. Let

$$e_{gn}(y) = \sum_{k=1}^{n} g(\bar{y}^{(k)}) \pi_{j,k}^{(n)}. \tag{B.2}$$

Following Tauchen and Hussey (1991), we need to show the uniform convergence result

$$\sup_{y \in [a,b]} |e_{gn}(y) - e_g(y)| \xrightarrow{p} 0 \tag{B.3}$$

as $n \to \infty$.

The pointwise convergence of the conditional distribution of the Markov chain $\tilde{y}'$ given $\tilde{y} = \bar{y}^{(j)}$ to the conditional distribution of $y'$ given $y = \mu(j)$ can be inferred from noting that the transition probability matrix for our method can be expressed in a polynomial form (see Kopecky and Suen, 2010) and by appealing to the Stone-Weierstrass approximation theorem. Finally, the condition that $e_{gn}(y)$ is uniformly bounded converts the pointwise convergence into uniform convergence. As a result, $e_{gn}(y)$ is equicontinuous which is a sufficient condition for the uniform convergence result

$$\sup_{y \in [a,b]} |e_{gn}(y) - e_g(y)| \xrightarrow{p} 0 \text{ as } n \to \infty. \tag{B.4}$$

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**Table 1.** Approximation Accuracy: RMSE

| | | $\hat{\sigma}_1^2$ | | $\hat{\sigma}_2^2$ | | $\text{Corr}(\tilde{y}_1, \tilde{y}_2)$ | | $1 - \hat{\xi}_1$ | | $1 - \hat{\xi}_2$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $K$ | $\tilde{N}$ | Tau. | New | Tau. | New | Tau. | New | Tau. | New | Tau. | New |
| 100 | 9 | 0.066 | 0.008 | 0.044 | 0.005 | 0.017 | 0.006 | 0.035 | 0.010 | 0.003 | 0.001 |
| | 19 | 0.036 | 0.002 | 0.025 | 0.002 | 0.002 | 0.002 | 0.003 | 0.003 | 0.001 | 0.001 |
| 10 | 9 | 0.356 | 0.010 | 0.287 | 0.011 | 0.047 | 0.006 | 0.193 | 0.019 | 0.121 | 0.003 |
| | 19 | 0.278 | 0.008 | 0.195 | 0.006 | 0.004 | 0.003 | 0.008 | 0.008 | 0.004 | 0.003 |
| 1 | 9 | 0.993 | 0.025 | 0.215 | 0.021 | NA | 0.010 | 216.099 | 0.032 | 0.993 | 0.010 |
| | 19 | 0.634 | 0.025 | 0.585 | 0.020 | 0.079 | 0.009 | 0.963 | 0.026 | 0.748 | 0.010 |

*Notes.* This table reports the root mean squared error (RMSE) of the key parameters of the bivariate VAR(1) model relative to their true values. (See Section 5 for details). "Tau." denotes the approximation obtained by the method of Tauchen (1986a) whereas "New" denotes the Markov chain approximation method developed in this paper. Higher values of $K$ imply less persistence. $\tilde{N}$ stands for the number of grid points used for each component of $\mathbf{y}$. $\hat{\sigma}_i^2$ denote the simulated unconditional variance of $\tilde{y}_i$ where $i \in \{1, 2\}$. $\text{Corr}(\tilde{y}_1, \tilde{y}_2)$ is the cross-correlation coefficient between $\tilde{y}_1$ and $\tilde{y}_2$. $\hat{\xi}_1$ and $\hat{\xi}_2$ are the eigenvalues of matrix $\hat{A}$. NA indicates that, in some cases, there is no variation in $\tilde{y}_1$ and, therefore, $\hat{\sigma}_1^2 = 0$ and the correlation coefficient $\text{Corr}(\tilde{y}_1, \tilde{y}_2)$ is not defined. The fact that the RMSE of $\hat{\sigma}_1^2$ relative to its true value is very close to 1 indicates that, for most of the Monte Carlo experiments, there is no variation in $\tilde{y}_1$.

**Table 2.** Approximation Accuracy: Bias and Standard Deviation

| | | $\hat{\sigma}_1^2$ | | $\hat{\sigma}_2^2$ | | $\text{Corr}(\tilde{y}_1, \tilde{y}_2)$ | | $1 - \hat{\xi}_1$ | | $1 - \hat{\xi}_2$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $K$ | $\tilde{N}$ | Tau. | New | Tau. | New | Tau. | New | Tau. | New | Tau. | New |
| *Bias* | | | | | | | | | | | |
| 100 | 9 | 0.066 | -0.008 | 0.044 | -0.005 | -0.016 | -0.005 | 0.035 | 0.009 | 0.003 | 0.000 |
| | 19 | 0.036 | -0.000 | 0.025 | 0.000 | 0.000 | 0.000 | 0.002 | 0.000 | 0.000 | -0.000 |
| 10 | 9 | 0.356 | -0.007 | 0.287 | -0.009 | -0.046 | -0.005 | -0.193 | 0.017 | -0.121 | 0.001 |
| | 19 | 0.277 | -0.000 | 0.195 | -0.000 | 0.003 | -0.000 | 0.002 | 0.000 | -0.003 | 0.000 |
| 1 | 9 | -0.993 | -0.001 | -0.167 | -0.006 | NA | -0.005 | 67.092 | 0.018 | -0.993 | 0.001 |
| | 19 | 0.604 | -0.001 | 0.578 | -0.001 | -0.071 | -0.001 | -0.963 | 0.002 | -0.748 | -0.000 |
| *Standard Deviation* | | | | | | | | | | | |
| 100 | 9 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.003 | 0.001 | 0.001 |
| | 19 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.003 | 0.003 | 0.001 | 0.001 |
| 10 | 9 | 0.010 | 0.007 | 0.007 | 0.006 | 0.003 | 0.003 | 0.006 | 0.008 | 0.003 | 0.003 |
| | 19 | 0.010 | 0.008 | 0.007 | 0.006 | 0.003 | 0.003 | 0.008 | 0.008 | 0.003 | 0.003 |
| 1 | 9 | 0.025 | 0.025 | 0.135 | 0.020 | NA | 0.009 | 205.42 | 0.026 | 0.001 | 0.010 |
| | 19 | 0.192 | 0.025 | 0.091 | 0.020 | 0.035 | 0.009 | 0.005 | 0.026 | 0.005 | 0.009 |

*Notes.* This table reports the bias and the standard deviation of the parameters relative to their true values. For the bias, the numbers that are much smaller than 0.0005 (0.05%) in absolute terms are denoted by 0.000 with their appropriate signs. See also notes to Table 1.

**Table 3.** The Distance between Generated and True Conditional Moments

| | | $\hat{\mu}_1 - \mu_1$ | | $\hat{\mu}_2 - \mu_2$ | | $\hat{\omega}_1^2/\omega_1^2 - 1$ | | $\hat{\omega}_2^2/\omega_2^2 - 1$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $K$ | $\tilde{N}$ | Tau. | New | Tau. | New | Tau. | New | Tau. | New | |
| 100 | 9 | 0.0010 | 0.0002 | 0.0009 | 0.0001 | 0.1164 | 0.0000 | 0.0599 | 0.0000 | |
| | 19 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0425 | 0.0000 | 0.0233 | 0.0000 | |
| 10 | 9 | 0.0179 | 0.0001 | 0.0041 | 0.0001 | 0.0524 | 0.0117 | 0.3428 | 0.0005 | |
| | 19 | 0.0010 | 0.0000 | 0.0001 | 0.0000 | 0.3201 | 0.0001 | 0.1634 | 0.0000 | |
| 1 | 9 | 0.0163 | 0.0000 | 0.0240 | 0.0000 | 1.0000 | 0.0217 | 0.9852 | 0.0032 | |
| | 19 | 0.0171 | 0.0000 | 0.0139 | 0.0000 | 0.9515 | 0.0010 | 0.4298 | 0.0000 | |

*Notes.* This table reports the overall distance between generated and true conditional moments. Specifically, for $i \in \{1, 2\}$, the numbers in column $\hat{\mu}_i - \mu_i$, are the weighted average of $|\hat{\mu}_i(j) - \mu_i(j)|$ which uses the frequencies of states $j = 1, 2, ..., N^*$ as weights. The frequencies are constructed using a simulated process of length $\tau = 2,000,000$. Similarly, the numbers in column $\hat{\omega}_i^2/\omega_i^2 - 1$ for $i \in \{1, 2\}$ are the weighted average of $|\hat{\omega}_i^2(j)/\omega_i^2 - 1|$ which uses the same frequencies as in columns $\hat{\mu}_i - \mu_i$. The numbers that are smaller than 0.00005 are denoted by 0.0000.