On the existence and the number of limit cycles in evolutionary games

Moreira, Helmar Nunes and Araujo, Ricardo Azevedo

University of Brasilia

2011

On the Existence and the Number of Limit Cycles in Evolutionary Games*

Helmar Nunes Moreira

Department of Mathematics

University of Brasilia

and

Ricardo Azevedo Araujo

Department of Economics

University of Brasilia

Abstract

In this paper it is shown that an extended evolutionary system proposed by Hofbauer and Sigmund (1998) may be transformed into a Kukles system. Then a Dulac-Cherkas function related to the Kukles system is derived, which allows us to determine the number of limit cycles or its non-existence.

Keywords: limit cycles, evolutionary game theory, Kukles system, Dulac-Cherkas function.

JEL classification: C6, C7.

---

*Departament of Economics, University of Brasilia, Campus Universitário Darcy Ribeiro, Brasília, DF, Cep 70910-900. E-mails: rsaaraujo@unb.br and . h.n.moreira@mat.unb.br. Financial support from CNPq is acknowledged.

1. Introduction

The study of existence of limit cycles in economic dynamic models is a research field that has a tradition that dates back to Goodwin (1951). Since then this kind of study was performed by a number of researchers in different contexts. Chang and Smith (1971) for instance have studied the existence and persistence of cycles in a nonlinear version of Kaldor's 1940 model. Feichtinger et. al (2002) have shown that their existence is related to a wide range of applications such as renewable resources, optimal saving, politico-economic cycles etc. In evolutionary games, the existence of limit cycles is also ubiquitous. [See Hofbauer and So (1990, 1994) and Cheng (1981)].

Although the aim of finding periodic orbits is mainly related to the detection of Hopf bifurcations, there are other methods such as analytical solutions of differential systems and numerical investigations [See Feichtinger (1987, 1992)]. This paper aims to study the existence of limit cycles in evolutionary games by using a method that for the best of our knowledge was not adopted yet. We show that the study of the number of limit cycles and its existence may be reduced to the study of a differential system of Kukles type combined with the method of Dulac-Cherkas function.

Some authors such as Sáez and Szántó (2002) have studied the existence of limit cycles in the Kukles system by using the traditional Hopf bifurcation method. The method of Dulac-Cherkas was developed by Cherkas (1978) and recently extended by Cherkas et. al (2011) to study the number of limit cycles in a generalized Liénard system through the construction of Kukles system. Following this approach it is possible to derive an upper bound for the number of limit cycles of a Kukles system including criteria for the non-existence of limit cycles.

Here we intend to apply this approach to study a generalized version of a Hofbauer and Sigmund model (1998, p.224):

$$\frac{dx}{dt} = x(1-x)(a+bx+cy+dxy), \quad \textbf{(1)}$$

$$\frac{dy}{dt} = y(1-y)(a_1 + b_1x + c_1y + d_1xy),$$

on the unit square $Q = \{(x, y): 0 \leq x, y \leq 1\}$. By studying a new method of detecting limit cycles in this model allows us to tackle a number of evolutionary games in which the system (1) is a general case. Particular versions of this system may arise as the outcome of dynamic replicator of evolutionary game of well known static games such as the 'batle of sexes', the 'entry deterrence game' and the 'matching penning games'. [see Weibull (1996, p. 176)].

This paper is organized as follows: in the next section the Kukles system from system (1) is built and in section 3 we develop an algorithm to find the correspondent Dulac-Sherkas function. Section 4 concludes.

1. Construction of the Kukles system from system (1)

In order to built the Kukles system from (1) it is assumed that:

(P1) System **(1)** has exactly one critical point (x^*, y^*) in int Q. It is given by the solutions of the pair of equations:

$$(a+bx+cy+dxy) = 0, (a_1 + b_1x + c_1y + d_1xy) = 0.$$

Replacing y by $y^* =- (a_1 + b_1x^*)/(c_1 + d_1x^*)$ in the first of these equations, we obtain a quadratic equation for x:

$Ax^2 + Bx + C = 0,$

Where: $A = bd_1 - db_1, B = (bc_{1+}ad_1) - (cb_1 + da_1)$ and $C = ac_1 - ca_1$. The roots of this equation may be found using the quadratic formula: $x = (-B \pm \sqrt{B^2 - 4AC})/2A$. In particular, if $A > 0$ and $C < 0$ or $A < 0$ and $C > 0$, we have one positive root and one negative root, and we take x^* to be the positive root;

(P2) For $x > 0, cx + d > 0;$

(P3) The straight lines $x = 0$ and $y = 0$ are invariant by the flow of **(1)**.

Lemma 1. A planar differential system of the form **(1)** can be transformed into the following equation of the *Kukles* system

$$\frac{dX}{d\tau} = U, \frac{dU}{d\tau} = h_o(X) + h_1(X)U + h_2(X)U^2 + h_3(X)U^3, \tag{2}$$

$h_i: R \to R, 0 \leq i \leq 3$, are continuous and that $h_3(X) \neq 0$. **(3)**

Proof. We translate the interior equilibrium (x^*, y^*) to the origin by the translation $X = x - x^*, Y = y - y^*$. Thus, **(1)** is transformed into

$$\frac{dX}{dt} = (X + x^*)[1 - (X + x^*)][(b + dy^*)X + (c + dx^*)Y + dXY], \tag{4}$$

$$\frac{dY}{dt} = (Y + y^*)[1 - (Y + y^*)][(b_1 + d_1y^*)X + (c_1 + d_1x^*)Y + d_1XY],$$

where from **(P2)** follows that $c + d(X + x^*) > 0$. Hence we take the new changes of coordinates:

$X = X,\ U = (b + dy^*)X + (c + dx^*)Y + dXY,\ d\tau = (X + x^*)[1 - (X + x^*)]dt,$

which transforms **(4)** into Kukles system **(2),** where:

$$h_0(X) = (MG^2)^{-1}\{F[-HG^2 + (L-H)FG + LF^2]\},$$

$$h_1(X) = (MG^2)^{-1}\{(bM + H)\ G^2 - [dM + 2(L-H)\ ]FG - 3LF^2\},$$

$$h_2(X) = (MG^2)^{-1}\{[dM + (L-H)]G + 3LF\},$$

$$h_3(X) = -L(MG^2)^{-1}, \qquad \textbf{(5)}$$

$$M = M(X) = (X + x^*)[1 - (X + x^*)],$$

$$F = F(X) = a + b(X + x^*),$$

$$G = G(X) = c + d(X + x^*),$$

$$H = H(X) = a_1 + b_1(X + x^*),$$

$$L = L(X) = c_1 + d_1(X + x^*).$$

Remark 1: The origin is a critical point of **(2)** for which the characteristic equation has purely imaginary roots, i.e. **(a)** $h_0(0) = 0;\ (\boldsymbol{b})h_1(0) = 0, (\boldsymbol{c})\ h_0'(0) < 0.$ It follows from **(5)** that:

$$y^* = \frac{H(0)}{L(0)} = \frac{F(0)}{G(0)} => M(0)G^2(0)h_0(0) = 0; (\boldsymbol{b})\ M(0)h_1(0) = bM(0) + [l(0) + dM(0)]y^* - L(0)(y^*)^2 = tr\ A(x^*, y^*) = 0; (\boldsymbol{c}) -\ M^2(0)h_0'(0) = M(0)y^*(1 - y^*)\{[bL(0) - b_1G(0)] + [dL(0) - d_1G(0)]y^*\} = \det A(x^*, y^*) > 0,$$

where the community matrix of the system (1.1) at equilibrium is:

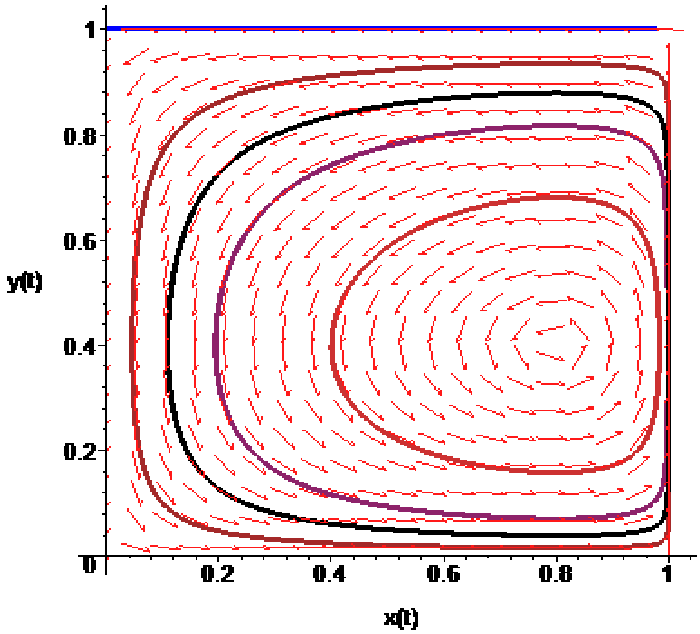
$$A(x^*, y^*) = \begin{pmatrix} (b + dy^*)M(0) & G(0)M(0) \\ (b_1 + d_1y^*)y^*(1 - y^*) & L(0)y^*(1 - y^*) \end{pmatrix},$$

whose determinant is $\det A(x^*, y^*) > 0$ and whose trace is $tr\ A(x^*, y^*) = 0.$

Example 1: Let:

$a = 0.4, b = -0.0001, c = -1, d = -0.001; a_1 = -0.4;\ b_1 = 0.5;\ c_1 = 0.000001;\ d_1 = -0.0004$

The equilibrium point is $(x^*, y^*) = (0.800024776, 0.3996003074)$



Remark 2: The case in which $b=b_1=0$ and $d=d_1=0$ was analyzed by Hofbauer and Sigmund (1998, p. 119) in the bi-dimensional case of 'Battle of sex', yielding a center. The case in which $d=d_1=0$ is proposed as an exercise to show the existence of a heteroclinic cycle on the unit square. [Hofbauer and Sigmund (1998, p. 224)]

3. *Dulac-Cherkas* function for the *Kukles* system (2)

The method of Dulac-Cherkas function may be used to derive an upper bound for the number of limit cycles of **(2)** including criteria for the non-existence of limit cycles**.** First we recall the definition of a *Dulac-Cherkas* function:

Definition 1. Consider a C^1differential system

$$\frac{dx}{dt} = P(x,y),\ \frac{dy}{dt} = Q(x,y), \tag{6}$$

in some open region $\Sigma \subset R^2$ and set $X_t = (P,Q)$. A function $\Psi \in C^1(\Sigma,R)$ is called a *Dulac-Cherkas* function of system **(6)** in Σ if there exists a real number $\kappa \neq 0$ such that

$$\phi = (grad\,\Psi, X_t) + \kappa\Psi.\,div(X_t) > 0(< 0), \qquad \text{in } \Sigma \subset R^2, \tag{7}$$

where $\Psi(x,y) = \Psi_0(x) + \Psi_1(x)y + \cdots + \Psi_n(x)y^n$. **(8)**

Instead of dealing with a general n, we present the details of the proof for the cases n=2 and 3, respectively. Let $X = x,\ U = y$ in the *Kukles* system **(2).**

Case I: Construction of a class of Dulac-Cherkas function $\Psi(x,y)$ for the case $n = 2$. We take:

$$\Psi(x,y) = \Psi_0(x) + \Psi_1(x)y + \Psi_2(x)y^2, \tag{9}$$

With: $\Psi_2(x) \neq 0$. The form of the function ϕ is:

$$\phi(x,y) = \phi_0(x) + \phi_1(x)y + \phi_2(x)y^2 + \phi_3(x)y^3 + \phi_4(x)y^4, \tag{10}$$

Where:

$$\phi_4(x) = (2+3\kappa)h_3(x)\Psi_2(x),$$

$$\phi_3(x) = \Psi_2'(x) + (1+3\kappa)h_3(x)\Psi_1(x) + 2(1+\kappa)h_2(x)\Psi_2(x),$$

$$\phi_2(x) = \Psi_1'(x) + (1+2\kappa)h_2(x)\Psi_1(x) + (2+\kappa)h_1(x)\Psi_2(x) + 3\kappa h_3(x)\Psi_0(x), \tag{11}$$

$$\phi_1(x) = \Psi_0'(x) + (1+\kappa)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x) + 2\kappa h_2(x)\Psi_0(x),$$

$$\phi_0(x) = h_0(x)\Psi_1(x) + \kappa h_1(x)\Psi_1(x),$$

follows directly from **(7)** after arranging in powers of the variable y. Our goal is to determine the functions $\Psi_j(x), j = 0,1,2,$ and the real number κ in such a way that we have:

$$\phi_i(x) = 0 \text{ for } i = 1,2,3,4. \quad \textbf{(12)}$$

Then it holds

$$\phi(x,y) = \phi_0(x) = h_0(x)\Psi_1(x) + \kappa h_1(x)\Psi_0(x) \quad \textbf{(13)}$$

If we additionally require:

$$\phi(x,y) = \phi_0(x) \geq 0 \ (\leq 0) \text{ for } (x,y) \in \Sigma \quad \textbf{(14)}$$

and if $\phi(x,y) = \phi_0(x)$ vanishes only at finitely many points of x, then Ψ is *Dulac-Cherkas* function of **(2)** in Σ. From **(11)-(12)**, we obtain a system of three linear differential equations to determine the three functions $\Psi_0(x), \Psi_1(x), \Psi_2(x)$. This system reads:

$$0 = (2+3\kappa)h_3(x)\Psi_2(x),$$

$$0 = \Psi_2'(x) + (1+3\kappa)h_3(x)\Psi_1(x) + 2(1+\kappa)h_2(x)\Psi_2(x),$$

$$0 = \Psi_1'(x) + (1+2\kappa)h_2(x)\Psi_1(x) + (2+\kappa)h_1(x)\Psi_2(x) + 3\kappa h_3(x)\Psi_0(x) \quad \textbf{(15)}$$

$$0 = \Psi_0'(x) + (1+\kappa)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x) + 2\kappa h_2(x)\Psi_0(x).$$

The first equation is an algebraic equation which determines according to **(2)** and **(9)** the constant κ uniquely as $\kappa = -2/3$. Substituting in **(15)**, we obtain the following system of linear differential equations:

$$0 = \Psi_2'(x) - h_3(x)\Psi_1(x) + (\tfrac{2}{3})h_2(x)\Psi_2(x), \quad \textbf{(16)}$$

$$0 = \Psi_1'(x) - \left(\tfrac{1}{3}\right)h_2(x)\Psi_1(x) + \left(\tfrac{4}{3}\right)h_1(x)\Psi_2(x) - 2h_3(x)\Psi_0(x) \quad \textbf{(17)}$$

$$0 = \Psi_0'(x) + \left(\tfrac{1}{3}\right)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x) - \left(\tfrac{4}{3}\right)h_2(x)\Psi_0(x). \quad \textbf{(18)}$$

Taking into account **(16)** and **(17)** we obtain:

$$\Psi_1(x) = (\tfrac{1}{h_3(x)})[\Psi_2'(x) + (\tfrac{2}{3})h_2(x)\Psi_2(x)], \quad \textbf{(19)}$$

$$\Psi_0(x) = (\tfrac{1}{2h_3(x)})[\Psi_1'(x) - \left(\tfrac{1}{3}\right)h_2(x)\Psi_1(x) + \left(\tfrac{4}{3}\right)h_1(x)\Psi_2(x)]. \quad \textbf{(20)}$$

Substituting **(19)** into **(20)** we get:

$$\Psi_0(x) = H_2(x)\Psi_2''(x) + H_1(x)\Psi_2'(x) + H_0(x)\Psi_2(x), \quad \textbf{(21)}$$

Where:

$$H_2(x) = \left[\frac{1}{2h_3^2(x)}\right]$$

$$H_1(x) = (2^{-1})(h_3^{-1})'(x)h_3^{-1}(x) + (6^{-1})h_2(x)h_3^{-2}(x), \quad \textbf{(22)}$$

$$H_0(x) = (3^{-1})h_2(x)(h_3^{-1})'(x)h_3^{-1}(x) + (3^{-1})(h_3^{-2})(x)h_2'(x) +$$

$$\left(\tfrac{2}{3}\right)h_1(x)h_3^{-1}(x) - (\tfrac{1}{9})h_2^2(x)(h_3^{-2})(x)).$$

Thus, by using **(19)** and **(21)**, we get from **(18)** the linear ordinary differential equation:

$$\Psi_2'''(x) + r_2(x)\Psi_2''(x) + r_1(x)\Psi_2'(x) + r_0(x)\Psi_2(x) = 0, \quad \textbf{(23)}$$

Where:

$$r_2(x) = \left(\frac{1}{H_2(x)}\right)\left[H_2'(x) + H_1 - \left(\frac{4}{3}\right)h_2(x)H_2(x)\right],$$

$$r_1(x) = \left(\frac{1}{H_2(x)}\right)\left[H_0(x) + H_1'(x) + \left(\frac{1}{3}\right)h_1(x)h_3^{-1}(x) - \left(\frac{4}{3}\right)h_2(x)H_1(x)\right], \qquad \textbf{(24)}$$

$$r_0(x) = \left(\frac{1}{H_2(x)}\right)[H_0'(x) + \left(\frac{2}{9}\right)h_1(x)h_2(x)h_3^{-1}(x) + 2h_0(x) - (\frac{4}{3})h_2(x)H_0(x)$$

From **(23)-(24),** we obtain:

Proposition 1. Consider system **(2)** under conditions **(3)** and **(5)**. Additionally we assume:

$$\phi(x,y) = \phi_0(x) = h_0(x)\Psi_1(x) - \left(\frac{2}{3}\right)h_1(x)\Psi_0(x) > 0(< 0),$$

for $\forall x \epsilon R$, where Ψ_1 is defined by **(19)** and Ψ_0 is defined by **(21)**. Then the function $\Psi(x,y)$ with the form **(9)** is a *Dulac-Cherkas* function of system **(2).** Hence, from Cherkas et al. (2011) we conclude that system **(2)** has at most one limit cycle, and if it exists it is hyperbolic.

Case II: Construction of a class of Dulac-Cherkas function $\Psi(x,y)$ in case $n = 3$. We take:

$$\Psi(x,y) = \Psi_0(x) + \Psi_1(x) + \Psi_2(x)y^2 + \Psi_3(x)y^3, \qquad \textbf{(25)}$$

With: $\Psi_3(x) \neq 0$. The form of the function ϕ is:

$$\phi(x,y) = \phi_0(x) + \phi_1(x)y + \phi_2(x)y^2 + \phi_3(x)y^3 + \phi_4(x)y^4 + \phi_5(x)y^5 \qquad \textbf{(26)}$$

Where:

$$\phi_5(x) = 3(1+\kappa)h_3(x)\Psi_3(x),$$

$$\phi_4(x) = \Psi_3'(x) + (3+2\kappa)h_2(x)\Psi_3(x) + (2+3\kappa)h_3(x)\Psi_2(x),$$

$$\phi_3(x) = \Psi_2'(x) + (2+2\kappa)h_2(x)\Psi_2(x) + (3+\kappa)h_1(x)\Psi_3(x) +$$

$$+(1+3\kappa)h_3(x)\Psi_1(x), \qquad \textbf{(27)}$$

$$\phi_2(x) = \Psi_1'(x) + (1+2\kappa)h_2(x)\Psi_1(x) + 3\kappa h_3(x)\Psi_0(x) +$$

$$+(2+\kappa)h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x),$$

$$\phi_1(x) = \Psi_0'(x) + 2\kappa h_2(x)\Psi_0(x) + (1+\kappa)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x),$$

$$\phi_0(x) = h_0(x)\Psi_1(x) + \kappa h_1(x)\Psi_0(x).$$

In the same way of **(12)-(15)**, we obtain a system of three linear differential equations to determine the three functions $\Psi_0(x), \Psi_1(x), \Psi_2(x)$. This system reads:

$$0 = 3(1+\kappa)h_3(x)\Psi_3(x),$$

$$0 = \Psi_3'(x) + (3+2\kappa)h_2(x)\Psi_3(x) + (2+3\kappa)h_3(x)\Psi_2(x),$$

$$0 = \Psi_2'(x) + (2+2\kappa)h_2(x)\Psi_2(x) + (3+\kappa)h_1(x)\Psi_3(x) +$$

$$+(1+3\kappa)h_3(x)\Psi_1(x) \qquad \textbf{(28)}$$

$$0 = \Psi_1'(x) + (1+2\kappa)h_2(x)\Psi_1(x) + 3\kappa h_3(x)\Psi_0(x) +$$

$$+(2+\kappa)h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x),$$

$$0 = \Psi_0'(x) + 2\kappa h_2(x)\Psi_0(x) + (1+\kappa)h_1(x)\Psi_1(x) + 2h_0(x)\Psi_2(x),$$

The first equation is an algebraic equation which determines according to **(2)** and **(9)** the constant κ uniquely as $\kappa = -1$. Substituting in **(28)**, we obtain the following system of linear differential equations:

$$0 = \Psi_3'(x) + h_2(x)\Psi_3(x) - h_3(x)\Psi_2(x), \qquad \textbf{(29)}$$

$$0 = \Psi_2'(x) + 2h_1(x)\Psi_3(x) - 2h_3(x)\Psi_1(x), \qquad \textbf{(30)}$$

$$0 = \Psi_1'(x) - h_2(x)\Psi_1(x) - 3h_3(x)\Psi_0(x) + h_1(x)\Psi_2(x) + 3h_0(x)\Psi_3(x), \qquad \textbf{(31)}$$

$$0 = \Psi_0'(x) - 2h_2(x)\Psi_0(x) + 2h_0(x)\Psi_2(x). \qquad \textbf{(32)}$$

Taking into account **(29)** and **(30)** we obtain:

$$\Psi_2(x) = (h_3^{-1})(x)[\Psi_3'(x) + h_2(x)\Psi_3(x)], \qquad \textbf{(33)}$$

$$\Psi_1(x) = 2(h_3^{-1})(x)[\Psi_2'(x) + 2h_1(x)\Psi_3(x)]. \qquad \textbf{(34)}$$

By substituting $\Psi_2'(x)$ obtained from **(33)** into **(34),** we have:

$$\Psi_1(x) = G_2(x)\Psi_3''(x) + G_1(x)\Psi_3'(x) + G_0(x)\Psi_3(x), \qquad \textbf{(35)}$$

Where:

$$G_2(x) = \left[\frac{1}{2h_3^2(x)}\right]$$

$$G_1(x) = (2^{-1})(h_3^{-1})'(x)(x)h_3^{-1}(x) + (2^{-1})h_2(x)h_3^{-2}(x), \qquad \textbf{(36)}$$

$$G_0(x) = (2^{-1})h_2'(x)h_3^{-2}(x) + (2^{-1})h_2(x)h_3^{-1}(x)(h_3^{-1})'(x) +$$

$$+ h_1(x)h_3^{-1}(x).$$

From **(31),** we get:

$$\Psi_0(x) = (3^{-1})(h_3^{-1})(x)[\Psi_1'(x) - h_2(x)\Psi_1(x) + h_1(x)\Psi_2(x) +$$

$$+3h_0(x)\Psi_3(x)], \quad \textbf{(37)}$$

By substitutingthe equations **(33)** and **(36)** into **(37),** we have:

$$3h_3(x)\Psi_0(x) = G_2\Psi_3''' + [G_1 + G_2' + h_2G_2]\Psi_3'' +$$

$$+[G_0 + G_1' - h_2G_1 + h_1h_3^{-1}]\Psi_3' + [G_0' - h_2G_0 + h_1h_2h_3^{-1} + 3h_0]\Psi_3, \quad \textbf{(38)}$$

with $h_3(x) \neq 0$. Thus, from **(38):**

$$\Psi_0(x) = L_3(x)\Psi_3''' + L_2(x)\Psi_3'' + L_1(x)\Psi_3' + L_0(x)\Psi_3. \quad \textbf{(39)}$$

Substituting **(33), (39)** and $\Psi_0'(x)$ in equation **(32),** we get the linear ordinary differential equation:

$$a_4(x)\Psi_3^{(iv)}(x) + a_3(x)\Psi_3'''(x) + a_2(x)\Psi_3''(x) + a_1(x)\Psi_3'(x) + a_0(x)\Psi_3(x) = 0, \quad \textbf{(40)}$$

Where:

$$a_4(x) = L_3(x),$$

$$a_3(x) = L_2(x) + L_3'(x) - 2h_2(x)L_3(x),$$

$$a_2(x) = L_1(x) + L_2'(x) - 2h_2(x)L_2(x), \quad \textbf{(41)}$$

$$a_1(x) = L_0(x) + L_1'(x) - 2h_2(x)L_1(x) + 2h_0(x)h_3^{-1}(x),$$

$$a_0(x) = L_0'(x) - 2h_2(x)L_0(x) + 2h_0(x)h_2(x)h_3^{-1}(x).$$

From **(40)-(41),** we have:

Proposition 2. Consider system **(2)** under condition **(3)**. Additionally we assume $\phi(x,y) = \phi_0(x) = h_0(x)\Psi_1(x) - h_1(x)\Psi_0(x) > 0(< 0)$, for $\forall x \epsilon R$, where Ψ_1 is defined by **(35)** and Ψ_0 is defined by **(39)**. Then function $\Psi(x,y)$ with the form **(25)** is a *Dulac-Cherkas* function of system **(2).** Hence, from Cherkas et al. (2011) we conclude that system **(2)** has at most one limit cycle, and if it exists it is hyperbolic.

4. Concluding Remarks

In this paper, we have approached an extended version of an ordinary differential system that arises from evolving game theory proposed by Hofbauer and Sigmund (1998) by using a new method. We have shown that the study of the existence and number of limit cycles in this system may be carried out through transforming the original system into a Kukles system and then derive the Dulac-Cherkas function for this new system. This approach allows us to study the existence of limit cycles in varied cases of evolutionary games such as the 'battle of sexes'.

References

Chang, W. and Smith, D. 1971. The existence and persistence of cycles in a nonlinear model: Kaldor's 1940 model re-examined. *Review of Economic Studies* 38, 37 – 44.

Cheng, K. 1981. Uniqueness of a Limit Cycle for Predator-Prey System. *SIAM Journal of Mathematical Analysis and Applications* 12, 115 – 126.

Cherkas, L. 1978. Conditions for the equation $yy' = \sum_{i=0}^{3} p_i(x)y^i$ to have a center. *Differential Equations* 14, 1133-1138.

Cherkas, L., Grin, A. and Schneider,K. 2011. Dulac-Cherkas functions for generalized Liénard systems. *Electronic Journal of Qualitative theory of Differential Equations* 351-23.

Dockner, E. and Feichtinger. On the optimality of Limit Cycles in Dynamic Economic Systems. *Journal of Economics* 53(1), 31 – 50.

Feichtinger, G. 1987. Limit cycles in economic control models. *Optimal Control: Lecture Notes in Control and Information Sciences* 95, 46-55.

Feichtinger, G. 1992. Limit cycles in dynamic economic systems *Annals of Operations Research* 37(1), 313-344.

Feichtinger, G., Novak, A. and Wirl, F. 2002. Limit cycles in intertemporal adjustment models: Theory and applications. *Journal of Economic Dynamics and Control* 18(2), 353 – 380.

Gasull, A. and Giacomini, H. 2006. Upper bounds for the number of limit cycles through linear differential equations. *Pacific Journal of Mathematics* 23, 6277 – 296.

Goodwin, R. 1951. The Nonlinear Accelerator and the Persistence of Business Cycles. *Econometrica* 19, 1 – 17.

Hofbauer,J. and Sigmund,K. 1998. *Evolutionary Games and Population Dynamics*. Cambridge University Press, Cambridge.

Hofbauer, J. and So, J. 1990. Multiple Limit Cycles for Predator-Prey Model. *Mathematical Biosciences* 99, 71 – 75.

Hofbauer, J. and So, J. 1994. Multiple Limit Cycles in three Dimensional Lotka-Volterra Equations. *Applied Mathematics Letters* 7(6), 59 – 63.

Sáez, E. and Szántó, I. 2008. Coexistence of Algebraic and Nonalgebraic Limit Cycles in Kukles Systems. *Periodica Mathematica Hungarica*, 56 (1), 137–142

Weibull, J. 1996. *Evolutionary Game Theory*. The MIT Press. Cambridge, Massachusetts.