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6 October 2011

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Abstract

In this paper we will study the well known problem of production functions in an operator semigroup approach. In general, semigroups can be used to solve a large class of problems commonly known as evolution equations. They are usually described by an initial value problem for a differential equation, also known as a Cauchy problem. After summarizing some of the major properties of semigroups theory, we will provide an application to the theory of production functions. Finally we present some concluding remarks.

Key words: Production functions; Partial differential equations; Semigroups

JEL classification: C22; C51; D24.

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1 Introduction

In a recent paper Chilarescu and Vaneecloo (2007) proposed a new approach of production functions and derived an explicit formula for a time-dependent production function. To arrive at this result they solved a second order linear, two-dimensional partial differential equation. We will proceed more general here regarding that equation as an initial value problem. When we recognize that we have a semigroup, instead of studying the initial value problem directly, we can study it via the semigroup and its applicable theory. In this first section of the paper we will focus on a special class of semigroups called C_0-semigroups which are semigroups of strongly continuous bounded linear operators. In the second section we will provide an application to the theory of production functions. In the final section we present our main result and some concluding remarks.

To motivate the semigroups approach of this problem we will consider a dynamic economic system evolving with time as given by the following initial value problem (IVP):

$$u'(t) = Au(t) \text{ for } t \geq 0 \tag{1}$$

$$u(0) = u_0 \tag{2}$$

where $u(t)$ describes the state at time t which changes in time at a "rate" given by the operator A. If A and u_0 are given numbers, then the solution of the above IVP is given by $u(t) = u_0 e^{At}$. Let now consider, as is more usual in applications, that A is a linear operator with domain $D(A)$ on a Banach space of functions E, suited for a particular problem. If $A \in B(E)$, the family of all bounded linear operators on E, then the IVP is solved by $u(t) = T(t)u_0$, where $T(t) := e^{tA}$. In many applications the operator A is unbounded, as in case of partial differential operator. A *classical solution* of the so called IVP associated to A is a continuously differentiable function $u : [0, \infty) \to E$ taking its values in $D(A)$ which satisfies IVP. To arrive to a similar solution as in case of the bounded linear operators we have to proceed into two steps, but for this we need some standard results from semigroups theory. Throughout in this paper E will be a Banach space, with norm $\|\cdot\|$ and $\mathscr{B}(E)$ the Banach algebra of all bounded linear operators from E into itself. $\sigma(\cdot)$ will denote the spectrum of a closed linear operator and $\rho(\cdot) = \mathbb{C} \setminus \sigma(\cdot)$ is the resolvent set of a closed linear operator. $R(\cdot, D)$ will be the resolvent map of some closed linear operator D, defined on $\rho(D)$. Also we denote by $s(\cdot)$ the set $\sup\{Re\ \lambda : \ \lambda \in \sigma(\cdot)\}$ and by $C^k(\Omega, X)$ the space of all functions which have continuous partial derivatives of degree k with $k \in I\!N \cup \{\infty\}$. Now we recall some standard definitions and results from semigroups theory.

Definition 1. *A C_0-semigroup is a mapping $T : I\!R_+ \to B(E)$ such that*

a. *$T(t)f$ is continuous for all $f \in E$;*

b. *$T(t+s) = T(t)T(s)$ for all $s, t \in I\!R_+$;*

c. *$T(0) = I$.*

It is well-known that the map $t \to T(t)f$ is continuous for all for $t \geq 0$ and every C_0-semigroup T is exponentially bounded i.e.

$$\|T(t)\| \leq Me^{\omega t}, \text{ for all } t \geq 0$$

for some $M \geq 1$ and $\omega \in I\!R$. See for instance Neerven (1996) and Pazy (1983). We denote by

$$\omega_0(T) = \inf\{\omega \in I\!R : \ \exists M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\omega t}, \ \forall\, t \geq 0\}$$

the growth bound of T.

Remark 1. *If T is a C_0-semigroup on the Banach space E, then its growth bound is given by*

$$\omega_0(T) = \lim_{t \to \infty} \frac{\ln \|T(t)\|}{t}.$$

Definition 2. *The infinitesimal generator of a C_0-semigroup T is a linear operator A with domain $D(A)$ defined by*

$$D(A) := \left\{ f \in E : \lim_{t \to 0_+} \frac{T(t)f - f}{t} \textit{ exists in } E \right\}$$

$$Af = \lim_{t \to 0_+} \frac{T(t)f - f}{t}, \ f \in D(A).$$

In other words, A is the derivative of T in 0 in the strong sense. The generator is always a closed, densely defined operator and $AT(t)f = T(t)Af$ for all $f \in D(A)$ and $t \geq 0$. Consequently, every C_0-semigroup $T \in B(E)$ has an infinitesimal generator and $\frac{d}{dt}T(t)f = AT(t)f$, for all $f \in D(A)$ and $t \geq 0$, which shows that if $u_0 \in D(A)$ the abstract Cauchy problem has a classical solution given by $u(t) = T(t)u_0$. This was the first step.

Given now an operator A it is desirable to find criteria which imply that A is the generator of a C_0-semigroup. Most characterizations are based on conditions on the resolvent of the operator. The set

$$\rho(A) = \left\{\lambda \in \mathbb{C} : \lambda I - A : D(A) \to E \text{ is bijective and } (\lambda I - A)^{-1} \in B(E)\right\}$$

is called the *resolvent set* of A. For $\lambda \in \rho(A)$, the operator

$$R(\lambda, A) = (\lambda I - A)^{-1} \in B(E)$$

is called the resolvent of A in λ. In fact, since a C_0-semigroup is always exponentially bounded, the Laplace transform always exists and it turns out to be the resolvent of the operator.

Proposition 1. *If A generates a C_0-semigroup T and if $\lambda > \omega_0(T)$, then $\lambda \in \rho(A)$ and*

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f dt, \quad (f \in E).$$

It is not difficult to show that any C_0-semigroup T can be transformed in a C_0-semigroup of contraction S. Indeed if we choose

$$\|f\|_c =: \sup_{t \geq 0} e^{-\omega t} \|T(t)f\|$$

where $\omega > \omega_0(T)$ and define a C_0-semigroup S by

$$S(t) = e^{-\omega t} T(t)$$

then $\|\cdot\|$ and $\|\cdot\|_c$ are equivalent and $\|S(t)\|_c \leq 1$. The following theorem solves the problem of finding criteria which imply that A is the generator of a C_0-semigroup (for proof see Nagel (1986)).

Theorem 1. (Hille-Yosida) *Let $A : D(A) \subset E \to E$ be a densely defined operator on E. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if $(0, \infty) \subset \rho(A)$ and $\|R(\lambda, A)\| \leq \lambda^{-1}$ for all $\lambda > 0$.*

2 An Application to Production Functions

In order to arrive to our problem we consider that the production function $F(L(t), K(t), t)$ is assumed to be homogenous of degree one. If we denote by $x(t) = K(t)/L(t)$ we can say $y = f(x(t), t)$. Now we suppose that $x(t)$ is the solution to the following stochastic differential equation:

$$dx(t) = x(t)\left[adt + bd\omega(t)\right] \tag{3}$$

where $\omega(t)$ is a standard Brownian motion, a and b are constants. We denote by $f(x,t)$ the value of the production function at any instant t, $t \geq 0$. Using Itô's lemma (see Karatzas and Shreve (1991)), we can write:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 \tag{4}$$

Putting (3) and (4) together, we find that

$$df = \left[\frac{\partial f}{\partial t} + ax\frac{\partial f}{\partial x} + \frac{b^2}{2}x^2\frac{\partial^2 f}{\partial x^2}\right]dt + bx\frac{\partial f}{\partial x}d\omega(t) \tag{5}$$

Here we suppose that the production function can be written as follows

$$f(x,t) = f_d(x,t) + \Delta x(t) \tag{6}$$

where Δ is unknown to be determined such that

$$df_d(x,t) = rf_d(x,t)dt$$

and r is a real positive constant. If we choose $\Delta = \frac{\partial f}{\partial x}$ then the stochastic term vanish, and we arrive at

$$\frac{\partial f}{\partial t} + rx\frac{\partial f}{\partial x} + \frac{b^2}{2}x^2\frac{\partial^2 f}{\partial x^2} - rf = 0 \tag{7}$$

which is a second order linear, two-dimensional partial differential equation. Under the following change of variable

$$\tau = \frac{b^2(T-t)}{2}$$

we obtain

$$\frac{\partial f}{\partial \tau} = x^2 \frac{\partial^2 f}{\partial x^2} + \frac{2r}{b^2} x \frac{\partial f}{\partial x} - \frac{2r}{b^2} f, \text{ for all } \tau > 0 \tag{8}$$

Now we suppose that the dynamics of production $f(x, \tau)$ is described by the following initial value problem

$$\begin{cases} f'_\tau = x^2 f''_{x^2} + 2rb^{-2} x f'_x - 2rb^{-2} f, \; 0 \leq x < k, \; \tau > 0 \\ \\ f_0 = f(0) = \begin{cases} \delta x^\alpha, \; 0 \leq x < k, \; \alpha \in (0, \; 1), \; \delta > 0 \\ \\ 0, \; x \geq k \end{cases} \end{cases} \tag{9}$$

Our main aim is to prove that the unbounded linear operator A, defined by:

$$(Af)(x) = x^2 \frac{\partial^2 f(x)}{\partial x^2} + \frac{2r}{b^2} x \frac{\partial f(x)}{\partial x} - \frac{2r}{b^2} f(x), \; f \in D(A).$$

can generate a C_0-semigroup. For this we consider the following linear operators defined on the Hilbert space $L^2(0, \infty)$:

$$\begin{aligned} (Au)(x) &= x^2 u''(x) + \frac{2r}{b^2} x u'(x) - \frac{2r}{b^2} u(x), \\ (Bu)(x) &= x u'(x) + \gamma u(x), \; \gamma = \frac{r}{b^2} - \frac{1}{2} \\ (Vu)(x) &= x u'(x). \end{aligned}$$

$$\begin{cases} D(V) = \{u \in L^2 : u \in AC(L^2) \text{ and } u' \in L^2\} \\ \\ D(B) = D(V) \\ \\ D(A) = \{u \in D(B) : \; u' \in AC(L^2)\} \end{cases}$$

One can verify that $A = B^2 - \left(\gamma^2 + \frac{2r}{b^2}\right) I$.

Proposition 2. *V is the infinitesimal generator of a C_0-group, defined by $(G_0(t)f)(x) = f(e^t x)$.*

Proof. First we prove that the operator $R : L^2(0,\infty) \to L^2(0,\infty)$ given by

$$(Rf)(x) = x\int_x^\infty \frac{f(u)}{u^2}du$$

is well defined and bounded. Indeed

$$\begin{aligned}|(Rf)(x)|^2 &= \left(x\int_x^\infty \frac{f(u)}{u^2}du\right)^2 \le x^2\left(\int_x^\infty \frac{|f(u)|}{\sqrt[4]{u}}\frac{\sqrt[4]{u}}{u^2}du\right)^2\\ &\le x^2\int_x^\infty \frac{|f(u)|^2}{\sqrt{u}}du\int_x^\infty \frac{\sqrt{u}}{u^4}du\\ &= x^2\int_x^\infty \frac{|f(u)|^2}{\sqrt{u}}du\int_x^\infty u^{-\frac{7}{2}}du = \frac{2}{5}\int_x^\infty \frac{|f(u)|^2}{\sqrt{ux}}du\end{aligned}$$

and so

$$\begin{aligned}\|Rf\|_2^2 &= \int_0^\infty |(Rf)(x)|^2dx \le \frac{2}{5}\int_0^\infty\int_x^\infty \frac{|f(u)|^2}{\sqrt{ux}}dudx\\ &= \frac{2}{5}\int_0^\infty\int_0^u \frac{|f(u)|^2}{\sqrt{u}\sqrt{x}}dudx = \frac{2}{5}\int_0^\infty \frac{|f(u)|^2}{\sqrt{u}}2\sqrt{u}du = \frac{4}{5}\|f\|_2^2\end{aligned}$$

It follows that R is well-defined and bounded $\|R\| \le \frac{2}{\sqrt{5}}$. One can easily see that $Rf \in D(V)$, for all $f \in L^2(0,\infty)$,

$$(VRf)(x) = x(Rf)'(x) = -f(x) + (Rf)(x)$$

for all $f \in L^2(0,\infty)$, and

$$(RVf)(x) = -f(x) + (Rf)(x)$$

for all $f \in D(V)$. It results that $I - V$ is invertible ($1 \in \rho(V)$) and $R(1,V) = (I-V)^{-1} = R$. Also we have that

$$\|G_0(t)f\|_2^2 = \int_0^\infty |f(e^t s)|^2ds = \int_0^\infty |f(v)|^2e^{-t}dv = e^{-t}\|f\|_2^2,$$

for all $t \ge 0$, $f \in L^2(0,\infty)$ which implies that $\omega_0(G_0) = -\frac{1}{2}$. □

Remark 2. *B is the infinitesimal generator of the C_0-group G defined by $G(t) = e^{\gamma t}G_0(t)$, for all $t \in \mathbb{R}$.*

Remark 3. *The following formula*

$$\frac{e^{-a|y|}}{2a} = \int_0^\infty e^{-at} \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}} dt, \quad \text{for all } a > 0,\ y \in I\!R. \tag{10}$$

is always valid for all $a > 0$ and $y \in I\!R$.

3 Main result and some concluding remarks

Proposition 3. *A is the infinitesimal generator of the $\mathscr{C}_0$-semigroup*

$$(T_0(t)f)(x) = \frac{e^{-(\gamma^2+\frac{2r}{b^2})t}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}+\gamma s} f(e^s x) ds. \tag{11}$$

Proof. Applying Theorem 1.1.5. given in Neerven (1996) it follows that $A + \left(\gamma^2 + \frac{2r}{b^2}\right) I = B^2$ generates a holomorphic C_0-semigroup of angle $\frac{\pi}{2}$ and hence A generate a holomorphic C_0-semigroup T_0 of angle $\frac{\pi}{2}$. For $\lambda \in I\!R$, sufficiently large we have that

$$\int_0^\infty e^{-\lambda t} \left[e^{(\gamma^2+\frac{2r}{b^2})t} (T_0(t)f) \right] dt = R(\lambda, B^2)f = -R(\sqrt{\lambda}, B)R(-\sqrt{\lambda}, B)f$$

$$= \frac{1}{2\lambda} \left[R(\sqrt{\lambda}, B)f - R(-\sqrt{\lambda}, B)f \right]$$

$$= \frac{1}{2\sqrt{\lambda}} \left[\int_0^\infty e^{-\sqrt{\lambda}t} G(t) f dt + \int_0^\infty e^{-\sqrt{\lambda}t} G(-t) f dt \right]$$

$$= \frac{1}{2\sqrt{\lambda}} \left[\int_0^\infty e^{-\sqrt{\lambda}t} G(t) f dt + \int_{-\infty}^0 e^{\sqrt{\lambda}t} G(t) f dt \right]$$

$$= \frac{1}{2\sqrt{\lambda}} \int_{-\infty}^\infty e^{-\sqrt{\lambda}|s|} G(s) f ds. \tag{12}$$

Using the Remark 3 we obtain that

$$\int_0^\infty e^{-\lambda t} \left[e^{(\gamma^2+\frac{2r}{b^2})t} T_0(t) f \right] dt = \int_{-\infty}^\infty \left(\int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}} dt \right) G(s) f ds$$

$$= \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{4\pi t}} \left(\int_{-\infty}^\infty e^{-\frac{s^2}{4t}} G(s) f ds \right) dt \tag{13}$$

and from the uniqueness theorem for the Laplace transform it follows that

$$T_0(t)f = \frac{e^{-(\gamma^2+\frac{2r}{b^2})t}}{\sqrt{4\pi t}} \int_{-\infty}^\infty e^{-\frac{s^2}{4t}} G(s) f ds, \quad \text{for all } t > 0. \tag{14}$$

$\square$

Now we consider the Cauchy problem (9). $f_0 \in D(A)$ and consequently (9) has a classical solution given by $f(t) = T(t)f_0$. Indeed

$$(T_0(\tau)f_0)(x) = \frac{\exp\left[-(\gamma^2 + \frac{2r}{b^2})\tau\right]}{\sqrt{4\pi\tau}} \int_{-\infty}^\infty \exp\left[-\frac{s^2}{4\tau} + \gamma s\right] f_0(e^s x) ds$$

$$= \frac{\exp\left[-(\gamma^2 + \frac{2r}{b^2})\tau\right]}{\sqrt{4\pi\tau}} \int_{-\infty}^\infty \exp\left[-\frac{s^2}{4\tau} + \gamma s\right] \delta e^{\alpha s} x^\alpha ds$$

$$= \delta x^\alpha \frac{\exp\left[-(\gamma^2 + \frac{2r}{b^2})\tau\right]}{\sqrt{4\pi\tau}} \int_{-\infty}^\infty \exp\left[-\frac{s^2}{4\tau} + \gamma s + \alpha s\right] ds$$

$$= \delta x^\alpha \frac{\exp\left[-(\gamma^2 + \frac{2r}{b^2})\tau\right]}{\sqrt{4\pi\tau}} \int_{-\infty}^\infty \exp\left[-\frac{1}{4\tau}\left[s^2 - 4\tau(\gamma+\alpha)s\right]\right] ds$$

$$= \delta x^\alpha \exp\left[\left(\alpha^2 + 2\alpha\gamma - \frac{2r}{b^2}\right)\tau\right] \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^\infty \exp\left[-\frac{[s - 2\tau(\gamma+\alpha)]^2}{4\tau}\right] ds$$

Under the following change of variable

$$u = \frac{s - 2\tau(\gamma+\alpha)}{\sqrt{2\tau}}$$

we finally obtain

$$f(x,t) = (T(t)f_0)(x) = \delta x^\alpha e^{-\theta\tau} \tag{15}$$

where

$$\theta = (1-\alpha)\left(\alpha + \frac{2r}{b^2}\right).$$

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