When worlds collide: Different comparative static predictions of continuous and discrete agent models with land

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Abstract

This paper presents a difference in the comparative statics of general equilibrium models with land when there are finitely many agents, and when there is a continuum of agents. Restricting attention to quasi-linear and Cobb-Douglas utility, it is shown that with finitely many agents, an increase in the (marginal) commuting cost increases land rent per unit (that is, land rent averaged over the consumer's equilibrium parcel) paid by each consumer. In contrast, with a continuum of agents, average land rent goes up close to the central business district, is constant at some intermediate distance, and decreases for consumers farther away. Therefore, there is a qualitative difference between the two types of models, and this difference is potentially testable.

1 Introduction

Models with a continuum of consumers are often employed for reasons of mathematical convenience or simplicity. Moreover, they can make precise the notion of perfect competition. As the number of agents in the world is finite, models with an infinite number of agents are not realistic unless they are close to models with a finite number of agents, in terms of equilibria and comparative statics. The scattered literature on general equilibrium models with land has tried to investigate the similarity or dissimilarity between the equilibria of these two types of models. This line of inquiry has met with limited success only; see Berliant (1985, 1991), Asami *et al* (1991), Kamecke (1993), Papageorgiou and Pines (1990), and Berliant and ten Raa (1991). The intuition for the dissimilarity between the models is that any partition of a σ-finite measure space, such as a Euclidean space, can have only countably many elements of positive measure. So except for a negligible set of consumers out of a continuum, all must consume or even be endowed with a set of measure zero. A corollary is that economies with a finite number of consumers approximating these continuum economies must have land consumption or endowments tending to zero almost surely.

This paper presents a dissimilarity in the comparative statics of these two types of models in the cases of quasi-linear and Cobb-Douglas utility functions. The comparative static of interest here is the effect of a change in marginal commuting cost on the per unit land rent paid, averaged over a consumer's equilibrium parcel, for each consumer. The model considered here is the standard closed city model (with an exogenous Central Business District (CBD) and an endogenous city boundary). It is shown that when the number of agents is finite, an increase in the (marginal) commuting cost increases average land rent paid by each consumer. In contrast, the canonical result when there is a continuum of agents is that average land rent goes up close to the CBD, is constant at some intermediate distance, and decreases for consumers farther away; see, for example, Fujita (1989, p. 81, Proposition 3.14, part (iii)).

This is important for both urban economic theory and empirical work. On the theoretical front, this result shows that models with a finite number of consumers are qualitatively different from models with a continuum of consumers, and therefore, in general, it is impossible to conclude that their equilibria are similar. On the empirical front, this result provides a potentially testable prediction. We shall discuss the empirical implications in the conclusions below.

Recent literature on city formation, for example Lucas and Rossi-Hansberg (2002), or the new economic geography, for example Fujita and Thisse (2002), generally employ a continuum of consumers and ordinarily have land as a commodity at least implicitly. We have postulated in our work an exogenously given CBD. In most models of city formation, the CBD or location of firms is endogenous, and there is an agglomeration externality used to determine these locations. However, these models all have embedded in them a model of consumer location and commuting, making our analysis relevant. For example, conditional on the spatial distribution of firms, one might want to consider the consumer location problem.

In the next section we introduce the notation and present the comparative static in the case of quasi-linear and Cobb-Douglas utility for the model with a finite number of consumers. This is essentially the model of Berliant and Fujita (1992) but with an endogenous city boundary that is determined using an exogenous agricultural land rent. The last section presents our conclusions. An appendix contains a complementary theorem on existence of equilibrium for the closed city model where the extent of the city is endogenous and determined by agricultural land rent.[1]

2 Increasing Rents per Unit Parcel

2.1 Notation

Consider the standard general equilibrium model with a linear city (the CBD at 0) and endogenous city size, l. Suppose there are $N \geq 2$ agents. Each agent's utility function is the same and is given by $u(s, z)$, where s is the size of their land parcel and z is consumption of a composite numéraire. Land is assumed to be a normal good. In addition to (s, z), an agent chooses the location of their lot at distance x from the CBD. Each agent has the same endowment w of the numéraire. In order to consume z, an agent has to commute to the CBD to earn income. The exogenous cost of commuting is $t > 0$ per unit distance to the CBD, and is measured in terms of the numéraire. Land price per unit is given by a density p. Agricultural rent outside the city is given by $\xi > 0$.

As usual in this model, an agent's budget constraint is given by $z + \int_x^{x+s} p(s)ds \leq w - tx$. In other words, total spending of an agent on consumption of z units of the numéraire, and a lot of size s at distance x from CBD is less than or equal to the agent's endowment less the commuting cost tx.

[1] Although equilibrium in the quasi-linear utility case is found explicitly, the Cobb-Douglas case yields a model with no known result on existence of equilibrium. Of course, without such a result the comparative static could be vacuous.

An equilibrium is given by a collection $(s_n^*, z_n^*)_{n=1}^N$, and a price density function p such that consumers are optimizing and markets clear. Agent marginal rates of substitution, land price, and agricultural rent determine the city size endogenously, as the sum of individual parcel sizes.

2.2 Equilibrium Parcels and Their Comparative Statics

As in Berliant and Fujita (1992), let $Z(s,u)$ be the level of consumption required to achieve utility u, when lot size is s. That is, $Z(\cdot,u)$ is the equation of the indifference curve for utility u. Let $\zeta(\cdot,\cdot) \equiv -Z(\cdot,\cdot)$. Then, using results from Berliant and Fujita (1992), $\zeta_s > 0$, $\zeta_{ss} < 0$, $\zeta_u < 0$, and $\zeta_{su} > 0$. Notice that $\zeta_s(s,u)$ is the (negative of) slope of an indifference curve, and hence it is a marginal rate of substitution.

Consider an equilibrium allocation $(s_n^*, z_n^*)_{n=1}^N$, and let equilibrium utility levels be $(u_n^*)_{n=1}^N$. Notice that as agents have the same endowments and utility function, equilibrium utility levels are identical: $u_1^* = u_2^* = \cdots = u_N^*$. As usual, we label agents by their distance from the CBD, with agent 1 being closest to the CBD and agent N being farthest away from the CBD. Moreover, as shown in Berliant and Fujita (1992), an equilibrium land price function is one that is monotonically non-increasing over distance from the CBD, and has the following form. Over agent's 1 parcel, the price is constant at that person's equilibrium MRS; over agent 2's parcel, at the front end, the price decreases as the MRS of agent 1 until it hits the level of agent 2's MRS, and then stays at the level of agent 2's equilibrium MRS, over agent 3's parcel, at the front end, the price decreases as the MRS of agent 2 until it hits the level of agent 3's equilibrium MRS, and then stays at the level of agent 3's equilibrium MRS, and so on.

As usual, first order conditions imply that in equilibrium,

$$\begin{aligned} &\zeta_s(s_n^*, u_n^*) = \zeta_s(s_{n+1}^*, u_{n+1}^*) + t, && \text{for } n = 1, \ldots, N-1, \quad \text{and} \\ &\zeta_s(s_n^*, u_n^*) = \xi && \text{for } n = N. \end{aligned}$$

In particular, $\zeta_s(s_N^*, u_N^*) = \xi$ implies that $\zeta_s(s_{N-1}^*, u_{N-1}^*) = \xi + t$, and proceeding inductively, $\zeta_s(s_{N-k}^*, u_{N-k}^*) = \xi + kt$, for $k = 0, \ldots, N-1$. Changing index yields

$$\zeta_s(s_n^*, u_n^*) = \xi + (N-n)t \quad n = 1, \ldots, N. \tag{1}$$

This provides a relationship between equilibrium marginal rates of substitution in terms of the exogenous parameter of interest t. The relationship is helpful in proving the main result in this paper. Toward that goal, the comparative statics of the equilibrium parcel sizes are computed first, as follows.

The above relationship implies that for $n = 1, \dots, N$,

$$N - n = \frac{\partial}{\partial t}\zeta_s(s_n^*, u_n^*) = \zeta_{ss}(s_n^*, u_n^*)\frac{\partial s_n^*}{\partial t} + \zeta_{su}(s_n^*, u_n^*)\frac{\partial u_n^*}{\partial t}.$$

Equation (1) helps determine how equilibrium parcel size changes with respect to t, as follows. As $\zeta_{ss} < 0$, and $\zeta_{su} > 0$, it follows that for $n = 1, \dots, N$,

$$\frac{\partial u_n^*}{\partial t} < 0 \quad \Rightarrow \quad \frac{\partial s_n^*}{\partial t} < 0.$$

Recall that $u_1^* = u_2^* = \cdots = u_N^*$. Moreover, as land is a normal good, for agent 1 (closest to the CBD), $\frac{\partial u_1^*}{\partial t} < 0$. These observations imply that for $n = 1, \dots, N$,

$$\frac{(N-n) - \zeta_{su}(s_n^*, u_n^*)\frac{\partial u_n^*}{\partial t}}{\zeta_{ss}(s_n^*, u_n^*)} \quad = \quad \frac{\partial s_n^*}{\partial t} \quad < \quad 0.$$

Thus, for each agent, as commuting cost increases, the equilibrium parcel size, (and therefore, city size) decreases.

2.3 Equilibrium Prices and Rents

For notational convenience, write $u_1^* = u_2^* = \cdots = u_N^* \equiv u$, and write s_n^* as s_n. With this convention, as is well-known, the equilibrium price density is as follows.

$$p(s) = \begin{cases} \zeta_s(s_1, u) & \text{on } [0, s_1] \\ \zeta_s(s - \sum_{k=1}^{n-1} s_k, u) & \text{on } \left[\sum_{k=1}^{n} s_k, \sum_{k=1}^{n-1} s_k + s_{n+1}\right] \qquad n = 1, \dots, N-1, \\ \zeta_s(s_{n+1}, u) & \text{on } \left[\sum_{k=1}^{n-1} s_k + s_{n+1}, \sum_{k=1}^{n+1} s_k\right] \qquad n = 1, \dots, N-1, \\ & \text{where, for } \; n = 1, \; \sum_{k=1}^{n-1} s_k \equiv 0. \end{cases}$$

Define the total land rent paid by consumer n to be

$$\text{rent}_n = \int_{\sum_{k=1}^{n-1} s_k}^{\sum_{k=1}^{n} s_k} p(s)ds$$

2.4 Comparative Statics of Rent per Unit Parcel

This subsection presents the main comparative statics result; how average land rent, or rent per unit, changes with respect to transport cost; that is, $\frac{\partial}{\partial t}\left(\frac{\text{rent}_n}{s_n}\right)$.

Theorem 1: *If utility is quasi-linear, $u(s,z) = v(s) + z$ (where v is increasing and concave), or if utility is Cobb-Douglas, $u(s,z) = s^{\alpha} z^{1-\alpha}$ (where $\alpha \in (0,1)$), then for $n = 1, ..., N$, $\frac{\partial}{\partial t}(\frac{\text{rent}_n}{s_n}) > 0$.*

Proof: Notice that $\text{rent}_1 = \zeta_s(s_1, u)s_1$, and therefore,

$$\frac{\partial}{\partial t}\left(\frac{\text{rent}_1}{s_1}\right) \;=\; \frac{\partial}{\partial t}\zeta_s(s_1,u) \;=\; \frac{\partial}{\partial t}(\xi + (N-1)t) \;=\; N-1 \;>\; 0.$$

Therefore, for the first agent, rent per unit increases with commuting cost. Moreover, for $n = 1, \ldots, N-1$,

$$\begin{aligned}
\text{rent}_{n+1} &= \textstyle\int_{\sum_{k=1}^{n} s_k}^{\sum_{k=1}^{n-1} s_k + s_{n+1}} \zeta_s(s - \sum_{k=1}^{n-1} s_k, u) ds + \int_{\sum_{k=1}^{n-1} s_k + s_{n+1}}^{\sum_{k=1}^{n+1} s_k} \zeta_s(s_{n+1}, u) ds \\
&= \zeta(s_{n+1}, u) - \zeta(s_n, u) + \zeta_s(s_{n+1}, u)s_n \\
&= \zeta(s_{n+1}, u) - \zeta(s_n, u) + (\xi + (N-n-1)t)s_n.
\end{aligned}$$

Consequently, for $n = 1, \ldots, N-1$,

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\left(\tfrac{\text{rent}_{n+1}}{s_{n+1}}\right) \\
&= \tfrac{1}{s_{n+1}^2} s_{n+1}\left[\zeta_s(s_{n+1},u)\tfrac{\partial s_{n+1}}{\partial t} + \zeta_u(s_{n+1},u)\tfrac{\partial u}{\partial t} - \zeta_s(s_n,u)\tfrac{\partial s_n}{\partial t} - \zeta_u(s_n,u)\tfrac{\partial u}{\partial t}\right] \\
&\quad - \tfrac{1}{s_{n+1}^2}\left[\zeta(s_{n+1},u) - \zeta(s_n,u)\right]\tfrac{\partial s_{n+1}}{\partial t} \\
&\quad + \tfrac{1}{s_{n+1}^2} s_{n+1}\left[s_n(N-n-1) + \zeta_s(s_{n+1},u)\tfrac{\partial s_n}{\partial t}\right] \\
&\quad - \tfrac{1}{s_{n+1}^2}\zeta_s(s_{n+1},u)s_n\tfrac{\partial s_{n+1}}{\partial t} \\
&= \tfrac{1}{s_{n+1}^2} s_{n+1}\left[\zeta_u(s_{n+1},u) - \zeta_u(s_n,u)\right]\tfrac{\partial u}{\partial t} \\
&\quad + \tfrac{1}{s_{n+1}^2} s_{n+1}s_n(N-n-1) \\
&\quad + \tfrac{1}{s_{n+1}^2} s_{n+1}\left[\zeta_s(s_{n+1},u) - \zeta_s(s_n,u)\right]\tfrac{\partial s_n}{\partial t} \\
&\quad + \tfrac{1}{s_{n+1}^2}\left[\zeta_s(s_{n+1},u)(s_{n+1}-s_n) - (\zeta(s_{n+1},u) - \zeta(s_n,u))\right]\tfrac{\partial s_{n+1}}{\partial t}.
\end{aligned}$$

As lot sizes are positive, the above relationship implies that for $n = 1, \ldots, N-1$,

$$\begin{aligned}
\tfrac{\partial}{\partial t}\left(\tfrac{\text{rent}_{n+1}}{s_{n+1}}\right) \;>\; 0 \;\Leftrightarrow\; & s_{n+1}s_n(N-n-1) \;+\; s_{n+1}\left[\zeta_u(s_{n+1},u) - \zeta_u(s_n,u)\right]\tfrac{\partial u}{\partial t} \\
&+\; s_{n+1}\left[\zeta_s(s_{n+1},u) - \zeta_s(s_n,u)\right]\tfrac{\partial s_n}{\partial t} \\
&+\; \left[\zeta_s(s_{n+1},u)(s_{n+1}-s_n) - (\zeta(s_{n+1},u) - \zeta(s_n,u))\right]\tfrac{\partial s_{n+1}}{\partial t} \quad > \quad 0.
\end{aligned}$$

Notice that the first term on the right-hand side is non-negative, the third term is positive because $\zeta_{ss} < 0$ implies that $\zeta_s(s_{n+1},u) - \zeta_s(s_n,u) < 0$ and that $\frac{\partial s_n}{\partial t} < 0$, and the fourth term is positive because $\zeta(\cdot,u)$ is concave and $\frac{\partial s_{n+1}}{\partial t} < 0$. In general, the second term is non-positive, because

$\zeta_{su} \geq 0$ implies that ζ_u is (weakly) increasing in s, $s_n < s_{n+1}$, and $\frac{\partial u}{\partial t} < 0$. Thus, in general, it is possible that the expression on the right-hand size is not positive. However, as documented next, the expression on the right-hand size is positive for the frequently-used classes of quasi-linear and Cobb-Douglas utility.

For quasi-linear utility, the second term above equals zero, as follows. Write $u(s,z) = v(s) + z$, where v is increasing and concave, and notice that $\zeta(s,u) = v(s) - u$. Consequently, $\zeta_u = -1$ and $\zeta_{su} = 0$. In particular, ζ_u does not depend on s, and the second term above equals zero. Therefore, in the case of quasi-linear utility, for every n, $\frac{\partial}{\partial t}\left(\frac{\text{rent}_n}{s_n}\right) > 0$.

For Cobb-Douglas utility, the entire expression above is positive, as follows. For notational convenience, the argument u in the functions ζ, ζ_s, ζ_{ss}, ζ_u, and ζ_{su} is suppressed for now. Notice that for $n = 1,\ldots,N-1$, $s_{n+1}s_n(N-n-1) \geq 0$ implies that in order to conclude that for $n = 1,\ldots,N-1$, $\frac{\partial}{\partial t}\left(\frac{\text{rent}_{n+1}}{s_{n+1}}\right) > 0$, it is sufficient to show that

$$\begin{aligned}
&[\zeta_s(s_{n+1})(s_{n+1}-s_n) - (\zeta(s_{n+1}) - \zeta(s_n))]\, \tfrac{\partial s_{n+1}}{\partial t} \\
&+ \quad s_{n+1}\left[\zeta_s(s_{n+1}) - \zeta_s(s_n)\right] \tfrac{\partial s_n}{\partial t} \\
&+ \quad s_{n+1}\left[\zeta_u(s_{n+1}) - \zeta_u(s_n)\right] \tfrac{\partial u}{\partial t} \quad > \quad 0.
\end{aligned}$$

Using the equilibrium relationship that for $n = 1,\ldots,N$,

$$\frac{\partial s_n}{\partial t} = \frac{(N-n) - \zeta_{su}(s_n)\frac{\partial u}{\partial t}}{\zeta_{ss}(s_n)},$$

the expression on the left-hand side of the above inequality can be written as follows.

$$\begin{aligned}
&[\zeta_s(s_{n+1})(s_{n+1}-s_n) - (\zeta(s_{n+1}) - \zeta(s_n))]\, \tfrac{(N-n-1)-\zeta_{su}(s_{n+1})\frac{\partial u}{\partial t}}{\zeta_{ss}(s_{n+1})} \\
&\quad + \quad s_{n+1}\left[\zeta_s(s_{n+1}) - \zeta_s(s_n)\right] \tfrac{(N-n)-\zeta_{su}(s_n)\frac{\partial u}{\partial t}}{\zeta_{ss}(s_n)} \\
&\quad + \quad s_{n+1}\left[\zeta_u(s_{n+1}) - \zeta_u(s_n)\right] \tfrac{\partial u}{\partial t} \\
&= [\zeta_s(s_{n+1})(s_{n+1}-s_n) - (\zeta(s_{n+1}) - \zeta(s_n))]\, \tfrac{(N-n-1)}{\zeta_{ss}(s_{n+1})} \\
&\quad + \quad s_{n+1}\left[\zeta_s(s_{n+1}) - \zeta_s(s_n)\right] \tfrac{(N-n)}{\zeta_{ss}(s_n)} \\
&\quad + \quad [\zeta_s(s_{n+1})(s_{n+1}-s_n) - (\zeta(s_{n+1}) - \zeta(s_n))]\, \tfrac{-\zeta_{su}(s_{n+1})}{\zeta_{ss}(s_{n+1})}\tfrac{\partial u}{\partial t} \\
&\quad + \quad s_{n+1}\left[\zeta_s(s_{n+1}) - \zeta_s(s_n)\right] \tfrac{-\zeta_{su}(s_n)}{\zeta_{ss}(s_n)}\tfrac{\partial u}{\partial t} \\
&\quad + \quad s_{n+1}\left[\zeta_u(s_{n+1}) - \zeta_u(s_n)\right] \tfrac{\partial u}{\partial t}.
\end{aligned}$$

For this last expression, notice that the first term is non-negative, and the second term is positive. In the case of Cobb-Douglas utility, the sum of the remaining three terms equals zero, as shown below.

Consider the utility function, $u(s,z) = s^{\alpha} z^{1-\alpha}$. Then

$$\zeta(s,u) = -\frac{u^{\frac{1}{1-\alpha}}}{s^{\frac{\alpha}{1-\alpha}}} \qquad \zeta_s(s,u) = \frac{\alpha}{1-\alpha}\frac{u^{\frac{1}{1-\alpha}}}{s^{\frac{1}{1-\alpha}}} \qquad \zeta_{ss}(s,u) = -\frac{\alpha}{(1-\alpha)^2}\frac{u^{\frac{1}{1-\alpha}}}{s^{\frac{1}{1-\alpha}+1}}$$
$$\zeta_u(s,u) = -\frac{1}{1-\alpha}\frac{u^{\frac{\alpha}{1-\alpha}}}{s^{\frac{\alpha}{1-\alpha}}} \quad \zeta_{su}(s,u) = \frac{\alpha}{(1-\alpha)^2}\frac{u^{\frac{\alpha}{1-\alpha}}}{s^{\frac{1}{1-\alpha}}} \quad \frac{\zeta_{su}}{\zeta_{ss}}(s,u) = -\frac{s}{u}.$$

Therefore,

$$\begin{aligned}
& s_{n+1}\left[\zeta_u(s_{n+1}) - \zeta_u(s_n)\right] \quad - \quad s_{n+1}\left[\zeta_s(s_{n+1}) - \zeta_s(s_n)\right]\frac{\zeta_{su}(s_n)}{\zeta_{ss}(s_n)} \\
& \quad - \quad \left[\zeta_s(s_{n+1})(s_{n+1}-s_n) - (\zeta(s_{n+1}) - \zeta(s_n))\right]\frac{\zeta_{su}(s_{n+1})}{\zeta_{ss}(s_{n+1})} \\
&= -\frac{1}{1-\alpha}u^{\frac{\alpha}{1-\alpha}} s_{n+1}\left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}}\right) + \frac{\alpha}{1-\alpha}u^{\frac{1}{1-\alpha}}\left(\frac{1}{s_{n+1}^{\frac{1}{1-\alpha}}} - \frac{1}{s_n^{\frac{1}{1-\alpha}}}\right)\frac{s_{n+1}s_n}{u} \\
& \quad + \left[\frac{\alpha}{1-\alpha}(s_{n+1}-s_n)\frac{u^{\frac{1}{1-\alpha}}}{s_{n+1}^{\frac{1}{1-\alpha}}} + u^{\frac{1}{1-\alpha}}\left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}}\right)\right]\frac{s_{n+1}}{u} \\
&= -\frac{u^{\frac{\alpha}{1-\alpha}}}{1-\alpha} s_{n+1}\left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}}\right) + \frac{\alpha}{1-\alpha}u^{\frac{\alpha}{1-\alpha}}\frac{s_n s_{n+1}}{s_{n+1}^{\frac{1}{1-\alpha}}} - \frac{\alpha}{1-\alpha}u^{\frac{\alpha}{1-\alpha}}\frac{s_{n+1}}{s_n^{\frac{\alpha}{1-\alpha}}} \\
& \quad + \frac{\alpha}{1-\alpha}u^{\frac{\alpha}{1-\alpha}}\frac{s_{n+1}}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{\alpha}{1-\alpha}u^{\frac{\alpha}{1-\alpha}}\frac{s_n s_{n+1}}{s_{n+1}^{\frac{1}{1-\alpha}}} + u^{\frac{\alpha}{1-\alpha}} s_{n+1}\left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}}\right) \\
&= -\frac{u^{\frac{\alpha}{1-\alpha}}}{1-\alpha} s_{n+1}\left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}}\right) + u^{\frac{\alpha}{1-\alpha}} s_{n+1}\left(-\frac{\alpha}{1-\alpha}\frac{1}{s_n^{\frac{\alpha}{1-\alpha}}} + \frac{\alpha}{1-\alpha}\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} + \frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}}\right) \\
&= u^{\frac{\alpha}{1-\alpha}}\left[-\frac{s_{n+1}}{1-\alpha}\left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}}\right) + \frac{s_{n+1}}{1-\alpha}\left(\frac{1}{s_{n+1}^{\frac{\alpha}{1-\alpha}}} - \frac{1}{s_n^{\frac{\alpha}{1-\alpha}}}\right)\right] \\
&= 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \left[\zeta_s(s_{n+1})(s_{n+1}-s_n) - (\zeta(s_{n+1}) - \zeta(s_n))\right]\frac{-\zeta_{su}(s_{n+1})}{\zeta_{ss}(s_{n+1})}\frac{\partial u}{\partial t} \\
& \quad + \quad s_{n+1}\left[\zeta_s(s_{n+1}) - \zeta_s(s_n)\right]\frac{-\zeta_{su}(s_n)}{\zeta_{ss}(s_n)}\frac{\partial u}{\partial t} \\
& \quad + \quad s_{n+1}\left[\zeta_u(s_{n+1}) - \zeta_u(s_n)\right]\frac{\partial u}{\partial t} \\
&= 0.
\end{aligned}$$

Therefore, in the case of Cobb-Douglas utility, for every n, $\frac{\partial}{\partial t}\left(\frac{\mathrm{rent}_n}{s_n}\right) \ > \ 0$.

3 Conclusions

We have examined a comparative static in closed city models with an endogenous city boundary both with a continuum and a finite number of consumers, and we have found a difference. For researchers in urban economic theory, the implication is that there are qualitative differences between the models. For empiricists, the possibility of testing the models against one another is real.

Although the quasi-linear and Cobb-Douglas utility cases are sufficient to make our point that the comparative statics in the model with a finite number of consumers and the model with a continuum of consumers can differ,[2] the analogous comparative static for general utility functions seems difficult, or at least algebraically burdensome. But even the result for Cobb-Douglas utility must be backed up by a theorem on existence of equilibrium for the finite model with an endogenous city boundary.[3] We provide this theorem in a brief appendix below.

Which model, finite or continuum of agents, will be verified empirically? Probably this depends on the context. One obvious way to test the models is to look at the per unit cost of land parcels in a city, say Chicago, before and after a change in commuting cost, say the introduction of a new "el" line. The work of McMillen and McDonald (2004) should be useful. Our model does not account for firm relocation and its impact on the comparative static, so this must be controlled for in empirical applications.

It is unclear if the difference in the comparative static presented extends to other comparative statics as well; however, there is another comparative static difference between the models with a finite number of agents and a continuum of agents that applies for all utility functions.

Consider the effect of a change in marginal commuting cost on each consumer's equilibrium marginal rate of substitution of land (for composite consumption commodity), or equilibrium marginal willingness to pay for land.

In the finite model, for each consumer, her equilibrium marginal rate of substitution of land is the same as the price of land at the back of the equilibrium parcel. In this model, a necessary condition for either an equilibrium or Pareto optimum is that for the outermost consumer, the marginal willingness to pay (or rent) at the back end of a consumer's parcel must be equal to agricultural land rent, and this must increase in steps of exactly t as we move inward from the outermost consumer. For example, the second to the outermost consumer must have marginal willingness to pay for land or land rent at the back end of its parcel equal to $\xi + t$; see equation (1). *In summary, the derivative of the marginal willingness to pay for land of each consumer n or the rent at the back end of the equilibrium parcel of consumer n with respect to t is equal to $N - n \geq 0$ for every consumer n.*

[2] It is easy to verify that the result in Fujita (1989) applies to quasi-linear utility.

[3] Notice that existence of equilibrium in the case of quasi-linear utility is not an issue, since we can find the equilibrium explicitly using first order conditions, and thus we have proved that it exists. For more detail and a graphical depiction of the finite model, see Berliant and LaFountain (2006).

In the continuum model, for each consumer, his equilibrium marginal rate of substitution of land is simply equal to the price of land at his equilibrium location. In this model, the average cost of land is equal to the marginal cost, so the comparative static for the continuum model remains the same as the one enumerated above.

This additional comparative static has an impact on some important results. For example, if one extends the model to allow a city developer to maximize land rents by choosing transportation infrastructure t subject to some cost function and subject to the land market equilibrium conditions, we conjecture that the solution will be different in the two models even assuming the same parametric forms, since the derivative of the marginal willingness to pay for land with respect to marginal commuting cost will play a crucial role in the first order conditions for developer optimization.

4 Appendix

Theorem 1 *Under standard regularity assumptions on the utility function (Berliant and Fujita, 1992, Assumption 1) there exists an equilibrium.*

Proof: For $p_1 \equiv \xi + (N-1)t$, construct $x_{n+1}(p_1)$, the sum of consumers' Marshallian demand for land when consumer 1 (the consumer closest to the CBD) faces price p_1, as in Berliant and Fujita (1992, pp. 561-562). Set $l = x_{n+1}(p_1)$. Apply Berliant and Fujita (1992, Proposition 4): under Assumption 1 of their paper, for any fixed $l > 0$, there exists an equilibrium. This equilibrium will have the property that the marginal willingness to pay for land of consumer N, the outermost consumer, is equal to the agricultural land rent.

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