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Production as behavior toward risk

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INTRODUCTION

Recent contributions to the theory of the firm have recognized the impact of uncertainty on both the firm's behavior and its equilibrium position. Models were developed by Baron (1970, 1971), Sandmo (1971) and Leland (1972) that explicitly incorporate a random output price as well as the firm's attitude toward the risk associated with production under price uncertainty. These models assume that, in the absence of complete foresight, the firm seeks to maximize the expected utility of its profits, in contrast to the traditional profit-maximizing firm operating under certainty. It is then shown that a firm which displays aversion toward risky activities will produce a smaller output under price uncertainty than under certainty, and that, contrary to the firm operating in a certain world, the risk averse firm may alter its optimal level of production in response to a change in its fixed costs.

These and other comparative-statics results derived by Baron, Sandmo and Leland are based on the general principle of expected utility maximization, first developed by Arrow (1965) in his work on the theory of risk aversion. An alternative approach is based on the mean-standard deviation framework (M-SD hereafter) developed by Markowitz (1952, 1959) and extended by Tobin (1958, 1965).

The former approach directly maximizes expected utility without specifying either the distribution of future profits or the particular shape of the firm's utility function. It is therefore very powerful and can be easily subjected to a comparative-statics analysis once the Arrow-Pratt risk aversion function is introduced into the analysis. In contrast, the M-SD approach, based on only the first two moments of the distribution of wealth, suffers from well known limitations. It requires the use of either a quadratic utility function of profits or a normal distribution of profits. Despite its drawbacks, the M-SD approach remains very popular, partly because of its simplicity and its geometric treatment of the optimization problem.

This geometric treatment was first applied to the theory of the firm under uncertainty by Hawawini (1978). However, this approach leads to serious difficulties when one attempts to introduce the Arrow-Pratt risk aversion function in an operationally usable form within the M-SD framework. In this paper we treat the problem analytically in order to remove the ambiguities in the early geometrical treatment. This treatment does not provide proofs of the conclusions which are reached by a graphical method. The analytical approach in this article furnishes an unambiguous framework for the geometrical examination of the behavior of the firm under uncertainty. In particular, the comparative statics of the firm, within the M-SD framework, becomes a simple and rigorous exercise.

The remaining part of this article is divided into two sections. The first presents our model of the risk averse firm under price uncertainty and examines the impact of uncertainty on the firm's production decision. The second section develops a geometric method for comparative-statics analysis within the M-SD framework.

I. A MODEL OF THE RISK AVERSE FIRM IN THE M-SD FRAMEWORK

1. Equilibrium in the M-SD framework

In a world of certainty and regardless of the structure of the output market, the firm's equilibrium position is found at the level of production for which the firm's marginal revenue (MR) equals its marginal cost (MC). Barring risk neutrality, the MR = MC rule does not hold under uncertainty where the firm is assumed to maximize its expected utility of profits. For example, in the case of risk aversion, MR > MC.

A new, general marginal rule, based on the M-SD framework, can be devised; this rule determines the firm's equilibrium position under price uncertainty as well as under certainty.

It states that the firm's equilibrium position is found at the level of production for which the firm's Marginal Rate of Substitution (MRS) in taste for expected profits versus risk equals the firm's Marginal Expected Profits (MEP). Instead of the traditional MR = MC rule, we now have the MRS = MEP rule.

This new rule can be explained as follows. First note that under uncertainty profits are a random variable and risk, represented by the variability of profits, will be shown to increase with output. In other words, more output means more risk. Now, a risk averse firm dislikes risk in the sense that it will accept to produce a higher level of output (take on more risk) only if it is compensated with additional expected profit and we can define the firm's MRS as the expected profits-risk trade off that leaves the firm equally satisfied (at the same level of expected utility). Define the firm's MEP as the additional expected profits the firm can physically secure if it bears one more unit of risk (produce more), then optimal output is attained where MRS = MEP. If MRS $<$ MEP, the firm can increase its satisfaction (expected utility) by raising output. Likewise, if MRS $>$ MEP, the firm can increase its satisfaction by reducing output.

In the following subsections we derive analytically the shape of the firm's MEP curve (under imperfect and perfect competitions) and the firm's MRS curve (under risk aversion), both drawn in the $(dE(\pi) \,/\, d\,\sigma(\pi), \sigma(\pi))$ — plane as illustrated in figure 1. We will prove that

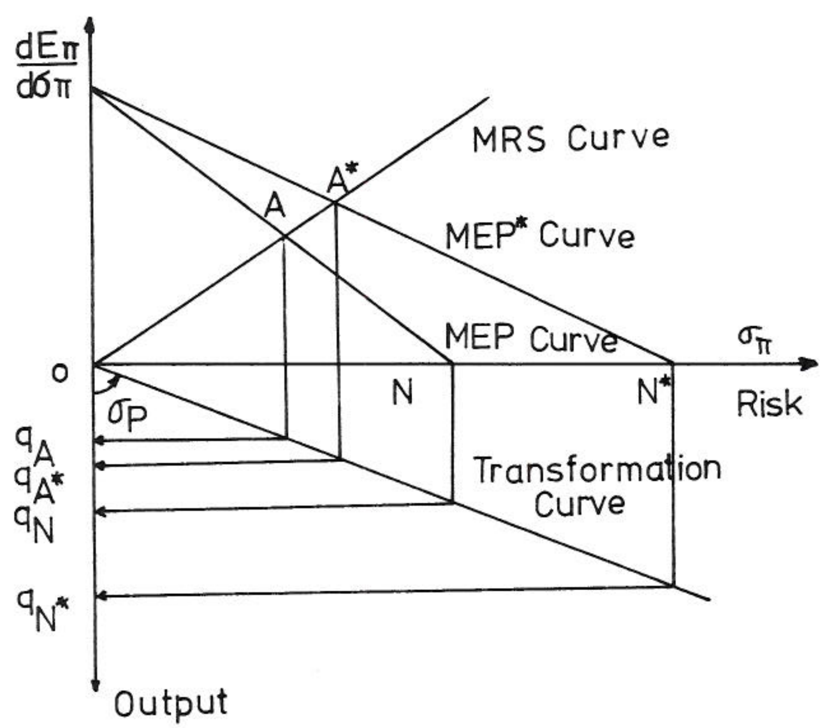

Figure 1

$$\frac{d\,\text{MEP}}{d\,\sigma(\pi)} < 0 \text{ and } \frac{d\,\text{MRS}}{d\,\sigma(\pi)} > 0$$

where $\sigma(\pi)$, the standard deviation of profits, is the risk variable. The MEP curve is therefore downward sloping, the MRS curve is upward sloping, and equilibrium is determined geometrically at their intersection. This simple diagrammatical model will be shown to yield considerable information on the behavior of the risk averse firm facing an uncertain demand for its product.

2. The Marginal Expected Profits curves

Assume the firm is an imperfect competitor facing a downward sloping random demand function such as

$$p = G(q, e) = g(q) + e \tag{1}$$

in which the random disturbance e is additive and normally distributed with zero mean and standard deviation $\sigma(e)$. The output q is under the firm's control and the only source of uncertainty is with respect to the price p, assumed unknown to the firm at the time production decisions are made. From equation (1) it follows that the price is a normally distributed random variable[1] with mean $g(q)$ and standard deviation $\sigma(p) = \sigma(e)$.

The firm's profits function $\pi(q)$ can be written as

$$\pi(q) = pq - V(q) - F = g(q)q + qe - V(q) - F \tag{2}$$

where $V(q)$ is the variable costs function such as $V(0) = 0$ and F is the fixed costs. From equation (2) it follows that profits are a normally distributed random variable with expected value (mean) $E(\pi)$ and standard deviation $\sigma(\pi)$ given by

$$E(\pi) = g(q)q - V(q) - F \tag{3}$$

$$\sigma(\pi) = \sigma(p)q. \tag{4}$$

[1] Theoretical support for the normality assumption is found in the work of Sengupta (1967) who proves that when many firms and many consumers interact in the market, the resulting price will be closely approximated by a normally distributed random variable. It is clear, however, that under imperfect competition normality may be a strong assumption.

Because profits are normally distributed, the variability of profits around the mean, $\sigma(\pi)$, is the appropriate variable representing the risk faced by the firm producing under price uncertainty. Referring to equation (3) note that, given $\sigma(p)$, the risk $\sigma(\pi)$ is under the firm's control; it can choose the level of risk it is willing to bear simply by varying the level of output it produces. Equation (4) can be represented geometrically in the $(\sigma(\pi), q)$-plane as a straight line passing through the origin and increasing with output q at a constant rate $\sigma(p)$, the slope of the line. This is illustrated in the lower quadrant in figure 1 with output increasing as we move away from the origin, downward. This line will be referred to as the risk-output Transformation Curve.

Using equations (3) and (4) we can easily derive the MEP. Recall that the MEP is the additional expected profits the firm can secure if it is willing to accept a small increase in risk, we can therefore write

$$\text{MEP} = \frac{d\,\text{E}(\pi)}{d\,\sigma(\pi)} = \frac{d\,\text{E}(\pi)}{d\,q} \Big/ \frac{d\,\sigma(\pi)}{d\,q} .$$

Differentiating equation (3) and equation (4) with respect to output we get

$$\text{MEP} = \frac{g'(q)q + g(q) - \text{V}'(q)}{\sigma(p)} \tag{5}$$

where primes indicate derivatives with respect to q.

Equation (5) is the MEP under imperfect competition. In the case of perfect competition, the demand is perfectly elastic and we have $g'(q) = 0$, $g(q) = \alpha$, where α is the expected price of output. The competitive MEP* can be written

$$\text{MEP*} = \frac{\alpha - \text{V}'(q)}{\sigma(p)}\ , \ \alpha > 0 \tag{6}$$

We can identify the conditions under which the MEP curve is a decreasing function of risk (downward sloping MEP curve). From equation (5) we can determine the slope of the MEP curve as

$$\frac{d\,\text{MEP}}{d\,\sigma(\pi)} = \frac{d\,\text{MEP}}{dq} \Big/ \frac{d\sigma(\pi)}{dq} = \frac{1}{\sigma^2(p)}\left(g''(q)q + 2\,g'(q) - \text{V}''(q)\right).$$

The first two terms in parentheses are equal to the derivative of the firm's marginal revenue and the third one is the derivative of the firm's marginal cost. If the demand curve is downward sloping so is the marginal revenue curve and $g''(q)q + 2g'(q) < 0$. Assuming increasing marginal costs, it follows that $d\,\text{MEP}/d\sigma(\pi) < 0$.

For the sake of exposition and without loss of generality, assume that the demand function is linear in q and that the variable costs function is quadratic in q, that is, $g'(.) < 0$, $g''(.) = 0$ and $V''(.) > 0$. We can write

$$g(q) = \alpha - \beta q \; ; \alpha > 0, \beta > 0 \tag{7}$$

$$V(q) = aq + bq^2; 0 < a < \alpha \, , b > 0 \, . \tag{8}$$

Taking account of equation (4) and substituting $g(q)$ and $V(q)$ in the MEP expressed in equation (5), we obtain

$$\text{MEP} = \left(\frac{\alpha - a}{\sigma(p)}\right) - 2\left(\frac{\beta + b}{\sigma^2(p)}\right)\sigma(\pi) \, . \tag{9}$$

Equation (9) is the equation of the MEP curve for the firm under imperfect competition. The MEP curve under perfect competition is a special case of equation (9) for which α equals the expected price and β equals zero, that is, $g(q) = \alpha$ and the firm faces a perfectly elastic demand for its product. The equation of the MEP curve under perfect competition is, therefore, given by

$$\text{MEP}^* = \left(\frac{\alpha - a}{\sigma(p)}\right) - \left(\frac{2b}{\sigma^2(p)}\right)\sigma(\pi) \, . \tag{10}$$

The MEP curves under imperfect and perfect competitions are drawn in the upper quadrant in figure 1. They have the same intercept but different negative slopes. Since the slope of the MEP cuive under imperfect competition is smaller than that under perfect competition, the former curve will be to the left of the latter.

3. The Marginal Rate of Substitution curves

Assume that the firm's attitude toward risk can be summarized by a von Neumann-Morgenstern (1947) utility-of-profits function, $U(\pi)$, such as

$$\frac{d\,U(\pi)}{d\pi} = U'(\pi) > 0 \text{ and } \frac{d^2\,U(\pi)}{d\pi^2} = U''(\pi) < 0 \, . \tag{11}$$

These conditions can be shown to imply that the firm is risk averse[2]. It prefers more profits to less ($U'(\pi) > 0$) with a declining preference for incremental units of profits ($U''(\pi) < 0$). Profits π are normally distributed with expected value $E(\pi)$ and standard deviation $\sigma(\pi)$. Thus, we can write

$$\pi = E(\pi) + \sigma(\pi) . Z \tag{12}$$

where Z is normally distributed with zero mean and unit standard deviation.

We would like to prove that the MRS curves of the risk averse firms are increasing positive functions of risk that pass through the origin as shown in the upper quadrant in figure 1[3]. This proposition can be proved as follows. Consider a constant level of expected utility, that is, $E[U(\pi)]$ = constant, and differentiate this expression with respect to risk $\sigma(\pi)$. We obtain

$$\frac{d\,E[U(\pi)]}{d\,\sigma(\pi)} = E\left[\frac{d\,U(\pi)}{d\,\sigma(\pi)}\right] = E\left[U'(\pi) . \frac{d\,\pi}{d\,\sigma(\pi)}\right] = 0 . \tag{13}$$

Using equation (12), equation (13) can be rewritten as

$$E\left[U'(\pi) . \left(\frac{d\,E(\pi)}{d\,\sigma(\pi)} + Z\right)\right] = 0 . \tag{14}$$

In equation (14), $d\,E(\pi)/d\,\sigma(\pi)$ is the firm's marginal rate of substitution. It is the expected profits-risk trade off that keeps the firm on the same level of expected utility. Solving equation (14) for the MRS, we get

$$\text{MRS} = \frac{d\,E(\pi)}{d\,\sigma(\pi)} = \frac{-E[U'(\pi) . Z]}{E[U'(\pi)]} > 0 . \tag{15}$$

The denominator in equation (15) is positive since $U'(\pi) > 0$ and the sign of the MRS is, therefore, determined by that of the numerator. Noting that $E(AB) = \text{Cov}(A, B) + E(A)\,E(B)$, the numerator can be rewritten as

$$\begin{aligned} -E[U'(\pi) . Z] &= -\text{cov}(U'(\pi), Z) - E[U'(\pi)]\,E(Z) \\ &= -\text{cov}(U'(\pi), Z) \end{aligned}$$

[2] We implicitly assume that the transitivity axiom is satisfied and that $U(\pi)$ represents group preferences if decision-making involves more than one individual.

[3] We have drawn linear MRS curves for simplicity and without loss of generality. The MRS curve may be either concave or convex without affecting our conclusions.

since E (Z) = 0. The covariance has the sign of the derivative $dU'(\pi)/dZ = U''(\pi)\,\sigma(\pi)$, which is negative since $U''(\pi) < 0$. This implies that the MRS is positive for the risk averse firm.

Evaluating the value of the MRS at the origin where $\sigma(\pi) = 0$, we get

$$\text{MRS}(0) = \frac{-E[U'(E(\pi) + 0.Z).Z]}{E[U'(E(\pi) + 0.Z)]} = \frac{-E[U'(E(\pi)).Z]}{E[U'(E(\pi))]}.$$

Since $E(\pi)$ is not a random variable, then

$$\text{MRS}(0) = -E(Z) = 0$$

since E (Z) = 0. It follows that the MRS curves pass through the origin regardless of the firm's attitude toward risk.

Finally, differentiating equation (14) with respect to risk we have

$$\frac{d\,\text{MRS}}{d\sigma(\pi)} = \frac{d^2 E(\pi)}{d\sigma(\pi)^2} = \frac{-E[U''(\pi).(\text{MRS} + Z)^2]}{E[U'(\pi)]} > 0. \qquad (16)$$

Since $U''(\pi) < 0$, the sign of the derivative of the MRS with respect to risk is clearly positive. Thus the MRS curves are an increasing function of risk for the risk averse firm. For the risk neutral firm the marginal utility of profits is constant and positive, $U'(\pi) > 0$, $U''(\pi) = 0$. This implies that for the risk neutral firm the MRS is equal to zero. See equation (15) and the related discussion. The MRS curves of the risk neutral firm coincide with the risk axis in figure 1.

4. Comparative production in the $(dE(\pi)/d\,\sigma(\pi),\ \sigma(\pi))$ - plane

We can now compare the level of output produced by the risk averse firm under certainty to the level of output produced under price uncertainty, as well as the level of output produced, under risk aversion, by the competitive firm to that produced by the imperfect competitor. The optimal output is obtained at the point where MEP = MRS. At this point the firm's expected profits-risk trade off in opportunities (MEP) equals the firm's expected profits-risk trade off in taste (MRS).

Referring to figure 1, observe that under imperfect competition equilibrium is found at point A, under price uncertainty and risk aversion, to which corresponds an optimal production of q_A. For the risk neutral firm, under imperfect competition, equilibrium is at point N to which corresponds the optimal production q_N. But the risk neutral firm produces the same level of output as that produced by the firm operating in a world of certainty[4]. It follows that $q_A < q_N = q_C$, where q_C is the certainty output. The same analysis can be carried out for the competitive firm, that is, $q_A^* < q_N^* = q_C^*$. This proves that *risk aversion is a necessary and sufficient condition for a firm's optimal output under price uncertainty to be smaller than its optimal certainty output.*

Again, referring to figure 1, observe that, if we assume that the perfectly competitive firm and imperfectly competitive firm have the same attitude toward risk, then $q_A^* > q_A$. Thus *the risk averse firm produces more under perfect competition than under imperfect competition.* This proposition also holds when the firm is risk neutral or operates under certainty since $(q_N^* = q_C^*) > (q_N = q_C)$.

We have shown that the MRS, which equals $dE(\pi)/d\,\sigma(\pi)$, is always positive. This implies that, at equilibrium, expected profits must be positive since risk is positive. It follows that, *under price uncertainty and risk aversion, equilibrium requires the existence of positive profits.* Note that under certainty equilibrium can exist with negative profits.

Finally, we can examine if, under price uncertainty and risk aversion, equilibrium exists with either constant or decreasing marginal costs. We know that under certainty there is no optimal level of production in these cases. The firm maximizes output in order to maximize profits. It is not so under uncertainty. With constant marginal costs, the MEP curve for the competitive firm is parallel to the risk axis[5] and therefore equilibrium exists under risk aversion but does not exist under either risk neutrality or certainty, because in the former case the MRS curve intersects the MEP curve but in the latter case these two curves are parallel. With decreasing marginal costs, the MEP curve for the competitive firm is upward sloping[6] and will intersect the MRS curve under risk aversion only if the slope of the MEP curve is smaller than the slope of the MRS curve. In the case of imperfect competition and constant marginal costs the MEP curve is flatter than in the case of increasing marginal costs but not horizontal[7], and therefore equilibrium exists under

[4] Being indifferent to risk, the risk neutral firm maximizes expected profits. The optimal output for this firm is, therefore, determined at the point where $dE(\pi)/dq = 0$. But $dE(\pi)/dq = E(d\pi/dq)$ and thus $d\pi/dq = 0$, the certainty optimum, at the same point as for the risk neutral firm.

[5] Under constant marginal costs, $b = 0$ and $MEP^* = (\alpha - a)/\sigma(p)$ (see equation (10)).

[6] Under decreasing marginal costs, $b < 0$ and the slope of the MEP curve for the competitive firm is positive (see equation (10)).

[7] Under constant marginal costs, $b = 0$ and the slope of the MEP curve under imperfect competition is negative (see equation (9)).

risk aversion. With decreasing marginal costs, equilibrium may exist depending upon the slope of the MEP curve compared to that of the MRS curve[8] as in the case of perfect competition. Therefore, *risk aversion is a necessary and sufficient condition for the existence of a firm equilibrium under constant marginal costs. Equilibrium may not exist under decreasing marginal costs.*

II. A GEOMETRIC METHOD FOR COMPARATIVE-STATICS ANALYSIS

1. Comparative statics

The model developed in the previous section can be subjected to a comparative-statics analysis within the M-SD framework. Comparative statics is an exercise whose purpose is to examine the effect on the firm's equilibrium level of output caused by a change in one of the model's parameters, that is, fixed and variable costs, expected price and the variability of the distribution of prices, all of which reflect given market conditions beyond the firm's control. Starting from an initial equilibrium position, a change in one of the model's parameters will create a shift in the firm's MEP curve leading to a new equilibrium position. By comparing the initial and final equilibria we can derive some important comparative-statics results describing the firm's reaction to changes in its environment, including the shape of its output supply curve and its input demand curve.

These comparative-statics results, however, will depend not only on the shifts in the firm's MEP curve but also on the shape of its family of MRS curves. This raises an important question. As the firm is allowed to move between different levels of constant expected utility of profits, what happens to the MRS curves ? The answer to this question is, it depends on the firm's Absolute Risk Aversion function introduced by Arrow (1965) and Pratt (1964).

2. The Absolute Risk Aversion function and the MRS curves

The Absolute Risk Aversion function, $R_A(\pi)$, is defined as follows

$$R_A(\pi) = -\frac{U''(\pi)}{U'(\pi)} \,. \qquad (17)$$

[8] Under decreasing marginal costs, $b < 0$ and the slope of the MEP curve under imperfect competition may be either negative, zero or positive (see equation (9)).

A firm is said to exhibit increasing, constant or decreasing absolute risk aversion according as $R_A(\pi)$ increases, remains constant or decreases with increasing profits.

We have

$$\frac{dR_A(\pi)}{d\pi} = R_A'(\pi) \begin{matrix} > \\ = \\ < \end{matrix} 0 \text{ implies } \left\{ \begin{matrix} \text{increasing} \\ \text{constant} \\ \text{decreasing} \end{matrix} \right\} \text{ risk aversion, respectively.}$$

Before any meaningful and conclusive comparative statics can be performed, we must relate $R_A(\pi)$, to the firm's MRS curves. Evaluating equation (16), the slope of the MRS curves, at the origin where $\sigma(\pi) = 0$, we get

$$\left[\frac{d\,\text{MRS}}{d\,\sigma(\pi)}\right]_{\sigma(\pi) = o} = \frac{-E\,[U''(E(\pi) + 0)\,(0 + Z)^2]}{E\,[U'(E(\pi) + 0)]}$$

since MRS = 0 at $\sigma(\pi) = 0$. The above can be rewritten as

$$\left[\frac{d\,\text{MRS}}{d\,\sigma(\pi)}\right]_{\sigma(\pi) = o} = -\frac{U''[E(\pi)]}{U'[E(\pi)]}\;E(Z^2).$$

Observing that $E(Z^2)$ is the variance of Z which is equal to one, then

$$\left[\frac{d\,\text{MRS}}{d\,\sigma(\pi)}\right]_{\sigma(\pi) = o} = -\frac{U''[E(\pi)]}{U'[E(\pi)]} = R_A[E(\pi)]. \qquad (18)$$

Equation (18) is crucial. It indicates that the slope of the MRS curves at the origin is equal to the Arrow-Pratt absolute risk aversion function for expected profits. It follows that under Constant Absolute Risk Aversion (CARA) $R_A'[E(\pi)] = 0$ and this slope, which is also the MRS curve under the linearity assumption, is constant regardless of the level of expected utility of profits the firm can attain. Its entire field of expected utility levels is summarized in a unique MRS curve and any movements between levels of expected utility is translated by a movement in the same direction along its unique MRS curve.

Under Increasing Absolute Risk Aversion (IARA) $R_A'[E(\pi)] > 0$ and the slope of the MRS curve (the MRS curve itself under the linearity assumption) will increase with expected profits. A movement to a higher level of expected utility for a given risk will cause the original MRS curve to rotate counterclockwise (steeper slope) around the origin as illustrated in figure 2. The opposite takes place under Decreasing Absolute Risk Aversion (DARA) where a movement to a higher level of expected utility causes a clockwise (flatter slope) rotation around the origin as illustrated in figure 2. Note that contrary to the case of CARA, under both IARA and DARA, to each level of expected utility the firm can attain corresponds a unique MRS curve.

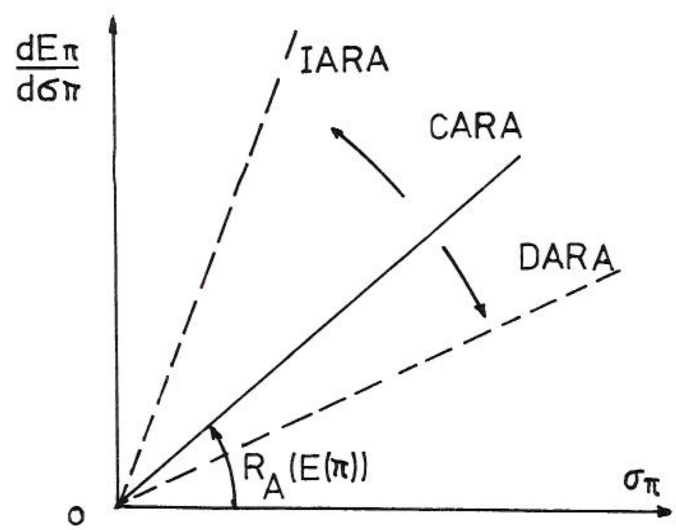

Figure 2

To summarize, we have shown that a shift to a higher level of expected utility of profits for a given risk is represented by a movement along a unique MRS curve under CARA and by a clockwise (counterclockwise) rotation of the MRS curve around the origin under DARA (IARA).

The above results can be easily applied to the model presented earlier in order to determine the behavior of the firm under changing market conditions. The reader will find the comparative-statics propositions, describing the firm's reaction to changes in its environment, within the M-SD framework in our other papers on this subject (see references).

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