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Rational Bubbles and The Spirit of Capitalism

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Abstract

This paper explores the existence of rational bubbles in the pricing of an asset that pays no dividend. I find that when "the spirit of capitalism" is introduced into a growth model, rational bubbles do exist provided that the marginal benefit from holding wealth is nontrivial relative to the marginal utility of consumption as time goes to infinity. I use phase diagrams to discuss the property of the bubbly equilibrium and I use two examples to describe the bubbly equilibrium trajectory explicitly and more intuitively. Moreover, I show that a stochastic bubble, which bursts with an exogenous probability, could exist. This could provide a simple theoretical foundation to explore economic implications of the collapse of bubbles.

Keywords: bubbles, the spirit of capitalism, growth
JEL Classification: E2, E44

1 Introduction

Most observers believe that bubbles exist in financial markets. However, it is difficult to model rational bubbles in a general equilibrium framework with a finite number of rational individuals who live infinitely. Tirole (1982) argued that bubbles cannot exist in this type of infinite-horizon model unless asset

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traders do not have fully rational expectation. Therefore, the early endeavors to model bubbles in general equilibrium focus on overlapping generations models. For example, Tirole (1985) disscusses necessary and sufficient conditions for the existence of bubbles in the Diamond model (1965) and Weil (1987) complements Tirole's (1985) analysis on deterministic bubbles by examining stochastic bubbles.

Kocherlakota (1992) pointed out that Tirole's (1982) analysis does not consider no-Ponzi-game conditions (debt accumulation constraints), namely, individuals are allowed to borrow and lend freely in all periods regardless of their debt position. When debt accumulatioin constraints are imposed, either in the form of a wealth constraint, or in the form of an exogenous short sale restriction, rational bubbles could exist in infinite-horizon models. Intuitively, this is because these constraints limit the ability of individuals to undertake arbitrage trades that rule out the bubbles. Technically, these constraints on debt accumulation prevent transversality conditions from eliminating the bubbles. The transversality conditions are believed by Kamihigashi (2008, 2009) as the hurdle for the existence of rational bubbles in standard models. By putting the wealth directly into the individuals' preferences to capture the idea of the "spirit of capitalism" and "status seeking", Kamihigashi shows that bubbles cannot be ruled out by the transversality conditions: bubbles could exist when the marginal benefit of wealth does not decline to zero as wealth goes to infinity.

This paper shows that Kamihigashi's condition on the marginal utility of wealth is sufficient, but not necessary for the existence of bubbles. Rational bubbles could arise in an infinite-horizon model provided that the marginal benefit of wealth is nontrivial relative to the marginal utility of consumption as time goes to infinity. Whether the marginal benefit of wealth approaches zero as the wealth goes to infinity is not essential.

I use the infinite-horizon model to study the impact of rational bubbles on the real economy. One nice feature of an infinite-horizon model is that it eliminates the concern of incomplete markets that arises naturally in the overlapping generations framework. Also, the infinite-horizon model is the basis for a vast literature on asset pricing.

With special functional forms of utility and production functions, the bubbly equilibrium can be illustrated explicitly. The set of examples I construct will sharpen our intuition. I find that economies with a lower rate of time discount, stronger "spirit of capitalism", or higher technology level, would allow for greater sizes of bubbles. By comparing the bubbly economy with the bubbleless economy, one could see that the bubbles stimulate consumption, crowd out investment and slow down economic growth. This effect on economic growth is similar to the findings in Saint-Paul (1992), Yanagawa and Grossman (1992), and King and Ferguson (1993), all of which are based on Tirole's overlapping generations framework.

Besides the deterministic rational bubbles discussed above, the setup here makes the stochastic bubbles possible. These stochastic bubbles burst with an exogenous and constant probability. In an example with explicit solution, we see that the bubbly equilibrium exhibits a positive relationship over time between the size of the stochastic bubbles and the physical capital stock, a feature that is consistent with the results in Weil (1987). From this example, it is clear that a higher probability for the bubble to collapse reduces the size of the bubble, lowers consumption, and raises economic growth. Intuitively, with a higher likelihood for the bubble to burst, individuals will speculate less and invest more in physical capital. In fact, there are multiple stochastic bubbly equilibria, each with a distinct bursting probability.

Putting wealth into the utility function might seem to be similar to putting money in the utility function (MIU). Thus, the existence of bubbles on the pricing of a non-dividend bearing asset may seem identical to the positive valuation of money that is intrinsically useless. This observation however is not accurate. In typical MIU models, money enters the utility function independently of the physical capital stock; in our model, wealth equals the bubble plus the capital stock, making the marginal benefit of the bubble dependent on the capital stock. Wheras money could be neutral as in Sidrauski (1967), the bubbles in my model have an impact on the real economy.

The rest of the paper is organized as follows. Section 2 surveys the methodologies developed so far on rational bubbles. Section 3 sets up the general model. Section 4 studies the existence of rational bubbles using phase diagrams. The property of bubbly equilibria is discussed. Section 5 gives two examples with explicit solutions for the bubbly equilibrium. Section 6 extends the discussion to a stochastic bubbly economy. Section 7 concludes.

2 A Review of Methodology

This section provides a brief survey of the methodologies used in deriving rational bubbles in general equilibrium. Since the methodology of introducing bubbles in this paper is similar to that of introducing money by the assumption of "money in utility function", the literature on goods price bubbles in MIU models is also reviewed here.

2.1 Literature on Asset Price Bubbles

The literature on the different methodologies used to introduce rational bubbles into a general equilibrium framework can be broadly classified into three types. Each of these is briefly reviewed below.

Largely influenced by the non-existence result presented in Tirole (1982) in the infinite horizon framework, the early research turned to finite-horizon models, particularly the overlapping generations models. This branch starts from Tirole (1985). He introduces intrinsically useless paper into the Diamond model and argues that the existence of bubbles is dependent on the inefficiency of the bubbleless equilibrium, i.e, the real interest rate is less than the growth rate of output. The inefficiency condition can also be interpreted to imply that bubbles exist only if the present value of aggregate income (or aggregate consumption) is infinite. Weil (1987) complements Tirole's (1985) analysis by studying a stochastic bubble, which is believed to collapse with a constant probability. When the probability of the persistence of a bubble is larger than a threshold level, the so called "minimum rate of confidence", this type of stochastic bubbles will exist. It is also proved that this minimum rate of confidence depends on the degree of inefficiency of the bubbleless economy: the more inefficient the bubbleless economy, the lower the value of the minimum rate of confidence.

The intuition behind the above findings is simple. The existence of either deterministic bubbles or stochastic bubbles crowds out productive investment which in turn decreases the capital level, and raises the real interest rate. Given the fact that bubbles grow at the same rate as the real interest rate, if the bubbleless economy is already efficient, then bubbles should grow at a higher speed, which cannot be supported by economic growth. Thus, bubbles will be ruled out by real resource constraint.

Within the setting of endogenous growth, this framework also can be used to explore the relationship between bubbles and economic growth. Saint-Paul (1992), Yanagawa and Grossman (1992), and King and Ferguson (1993) reported that bubbles would retard growth when endogenous growth was introduced by externality in capital accumulations. On the other hand, Olivier (2000) argued that bubbles on equity would encourage the creation of firms and promote economic growth if endogenous growth was due to research and development (R&D).

Recently, under the assumption of imperfect financial markets, Tirole's framework has been used to explain a number of issues. For example, Caballero and Krishnamurthy (2006) explores emerging market crisis resulting from the bursting of bubbles; Caballero, Farhi, and Hammour (2006) provides a framework for understanding the "speculative growth" episodes in the U.S.; Farhi and Tirole (2010) analyzes the relationship between bubbles and liquidity; and Martin and Ventura (2010) revisits economic growth with bubbles.

Another branch of this literature emphasizes the importance of no-Ponzi-game conditions (constraints on debt accumulation) for the existence of bubbles in infinite-horizon models. Kocherlakota (1992) first pointed this out by showing that an individual cannnot reduce his asset position permanently when facing constraints on debt accumulation. Technically, these constraints help to guarantee transversality conditions not to be violated when asset price has a bubble term. If this constraint is a wealth constraint, the sufficient and necessary conditions for the existence of a bubble is zero net supply of the asset. On the other hand, if this constraint is an exogenous short sales constraint, bubbles can arise if and only if the growth rate of individual's income is not less than the real interest rate. As Kocherlakota (2008) stressed, with short sales constraints, bubbles can arise even if the present value of aggregate consumption is finite.

Based on this finding, Kocherlakota (2009) modeled a stochastic bubble in the price of collateral, which is intrinsically worthless, by introducing borrowing constraints faced by infinitely-lived entrepreneurs. The effects of bursting bubbles and the discussions of policies after the collapse of bubbles are provided. Pengfei Wang and Yi Wen (2009) took this analysis a step further by studying bubbles that may arise on assets with positive intrinsic values.

The third method of modeling rational bubbles is by assuming that wealth has a direct effect on the preference function. This is modeled in the same way as the "spirit of capitalism" models.

Kamihigashi (2008) first introduced rational bubbles on assets by this method. In this paper, he argued that bubbles may exist if "the marginal utility of wealth does not decline to zero as wealth goes to infinity". The relationship between bubbles and output, or, capital stock, depends on the property of the production function. For a production function with decreasing returns to scale, this relationship is negative. On the other hand, it might be positive for a production function with increasing returns to scale. However, all of these analyses are under the restrictive assumption of linear utility in consumption.

Kamihigashi (2009) discussed the existence of asset price bubbles in an exchange economy with status seeking. When status is modeled by the ratio of individual wealth to aggregate wealth, bubbles are ruled out by the transversality condition. This is because the marginal utility of individual wealth converges to zero along with a growing price path. This means that the effect of status seeking disappears. However, if the status is formulated by the difference of individual wealth and aggregate wealth, then bubbles might exist since the marginal utility of wealth remains as a positive constant.

2.2 Literature on Goods Price Bubbles

Due to the similarity in methodologies between the third method of introducing bubbles as described above and including money into the utility function, the literature on price level bubbles is also briefly reviewed below. However, this literature focuses on conditions that would rule out this type of bubbles.

Obstfeld and Rogoff (1983) reexamined the conditions to rule out speculative hyperinflations and hyperdefaltions as discussed in Brock (1974) and Brock (1975). It argued that the restriction on preference was infeasible to rule out speculative hyperinflations in a pure fiat monetary system and that the government's fractional backing for money is needed. Speculative hyperinfaltions imply that as the price level goes to infinity, the real balance will converge to zero. Conditions to rule out the non-monetary economy (with speculative hyperinflations), can guarantee that the saddle path is the unique equilibrium in dynamic monetary models.

Obstfeld and Rogoff (1986) revisited the conditions to ruling out these price level bubbles in an environment with extrinsic uncertainty. It found that a fractional backing regime also can rule out the explosive price level bubbles. In addition, they proved a new necessary condition for equilibrium to strengthen the existing arguments for ruling out implosive price bubbles.

Buiter and Sibert (2007) pointed out that there is confusion on the correct specification of the transversality conditions in these monetray models. Therefore, the existence of deflationary bubbles and the terminal conditons to rule them out, are revisited.

3 The Model

This section describes an economy where "the spirit of capitalism" is necessary for the existence of rational asset bubbles.

Time is continuous. An infinite number of identical individuals, who live forever, are continuously and evenly distributed in [0,1]. Every individual can rent his physical capital to firms that are owned by all of the individuals, receives the lump-sum transfer of the firms' profit, Π, and a rental at the rate of r. This rate is also the real interest rate. The capital stock is denoted by k. Each individual is also able to invest in financial assets. For convenience, I suppose that there is only one kind of zero-dividend asset in this economy. Based on the standard definition, the fundmental value of this asset should be zero. Therefore, once the price of this asset, which is denoted by q, is positive, we say that an asset bubble exists. The total supply of this asset is normalized by 1. The amount of this asset held by the individual is denoted by s.

Each individual wishes to maximize the sum of time discounted utility values

$$\int_0^\infty e^{-\rho t} U(c,a)dt, \quad \rho > 0,$$

facing his budget constraint given by

$$\dot{a} = rk - c + \dot{q}s + \Pi,$$

where ρ is the rate of time preference, $U(c,a)$ is the utility function, which is continuous, differentiable, strictly increasing and concave in all of its arguments. Here, c is the amount of consumption, and $a \equiv qs + k$ is the amount of wealth, which is equal to the sum of values of asset and physical capital. The aim of setting the wealth term directly into utility function is to model "the spirit of capitalism". This follows the methodology of Mordecai Kurz (1968) and Hengfu Zou (1991).

The Hamiltonian of the representative agent's optimal problem can be written as

$$\begin{aligned} \mathcal{H} &= U(c,a) + \lambda(a - qs - k) \\ &\quad + \mu(rk - c + \dot{q}s + \Pi). \end{aligned}$$

The first order conditions are given by the Euler equation

$$\frac{\dot{\mu}}{\mu} = \rho - \frac{U_a'}{\mu} - r, \tag{1}$$

where $\mu = U_c'$, and the non-arbitrage condition

$$\frac{\dot{q}}{q} = r. \tag{2}$$

The transversality conditions can be written as

$$\lim_{t\to\infty} e^{-\rho t}\mu k = 0, \tag{3}$$

$$\lim_{t\to\infty} e^{-\rho t}\mu q s = 0. \tag{4}$$

There are infinite number of homogeneous firms exist in this economy. Each of them wishes to maximize its current profit

$$\Pi \equiv f(k) - \delta k - rk.$$

From the first order condition, the rate of rental (also the real interest rate) is given by

$$r = f'(k) - \delta. \tag{5}$$

At equilibrium, the goods market clearing condition is given by

$$\dot{k} = f(k) - \delta k - c, \tag{6}$$

and the asset market clearing condition is

$$s = 1.$$

Combining the Euler equation (1) with equation (2), we can obtain that

$$\frac{\dot{\mu}}{\mu} + \frac{\dot{q}}{q} = \rho - \frac{U_a'}{\mu},$$

which means that the product of μ and q is always growing at the rate of $\rho - \frac{U_a'}{U_c'}$.

Here, I use the fact that $\mu = U_c'$. Therefore, as long as

$$\lim_{t\to\infty} \frac{U_a'}{U_c'} > 0, \tag{7}$$

the product of μ and q will eventually grow at a rate less than ρ. This guarantees that the transversality condition will never rule bubbles out. This necessary condition (7) actually requires that the marginal benefit from holding wealth is nontrivial relative to the marginal uitility of consumption as time approaches to infinity. Thus, bubbles in the price of asset, as a part of wealth, is not worthless at the end. However, if the spirit of capitalism does not exist, then $\frac{U_a'}{U_c'}$ would always be zero. Thus, bubbles will be ruled out by the transversality condition. This reveals the necessity of the "spirit of capitalism" for the existence of asset bubbles in this economy.

It also should be highlighted that any further restriction on the form of preference function is not necessary at all. Thus, Kamihigashi's requirement for $\lim_{a\to\infty} U_a' > 0$ should be just a sufficient condition for the existence of bubbles. When $\lim_{a\to\infty} U_a' = 0$, as long as the margnial utility of consumption, U_c', also converges to zero, the condition (7) may also hold. Therefore, it is still possible that bubbles arise in this case. The following sections of this paper verify this point.

4 Bubbly Equilibrium

This section analyzes the bubbly equilibrim in the above economy by phase diagrams. This analysis also verifies the existence of rational bubbles in an economy with "spirit of capitalism", even if Kamihigashi's condition is not satisfied.

Before the analysis of the bubbly equilibrium, we first need to examine the system of equations that determines the dynamic system of the economy modeled above. In order to keep this analysis simple, I consider two-dimensional dynamic system. From the point of view of the policy function, the representative agent's cosumption must be a function of state variables, such as his wealth level, a, the lump sum transfer of profit, Π, and the real interest rate, r. At equilibrium, all of these state variables are functions of physical capital stock, k, and asset price, q; and, the consumption of the representative individual is equal to aggregate consumption. This implies that aggregate consumption would also be a function of physical capital and asset price, i.e., $c(k, q)$. Thus, the goods market clearing condition (6) can be rewritten as

$$\dot{k} = f(k) - \delta k - c(k, q). \tag{8}$$

From the non-arbitrage condition (2) and equation (5), we can obtain that

$$\frac{\dot{q}}{q} = f'(k) - \delta. \tag{9}$$

The above pair of differential equations (8) and (9) describes a two-dimensional dynamic system, which determines the time paths of the asset bubble and the stock of physical capital.

In order to make a feasible analysis through the above dynamic system, however, it is also necessary to verify the property of the function $c(k, q)$. In an economy in which the production function exhibits decreasing return to physical capital, an increase in the aggregate capital stock would decrease the real interest rate. This lessens the individual's incentive to save and also his disposable income. On the other hand, this increase would also raise the individual's disposable income by improving firms' profits. However, at equilibrium, the aggregate disposable income is equal to the aggregate output, $f(k)$, which must be monotonically increasing in the aggregate capital stock. Thus, the income effect on aggregate consumption due to an increase of aggregate capital stock, must be positive. At the same time, through the wealth effect, the individual's optimal consumption should be monotonically increasing both in his stock of physical capital and in the value of his asset. At equilibrium, this means that aggregate consumption must also be a monotonically increasing function of aggregate capital stock and asset price, i.e.,

$$\frac{\partial c}{\partial k} > 0, \frac{\partial c}{\partial q} > 0.$$

Now, the analysis on bubbly equilibrium can be given through a phase diagram. The $\frac{\dot{q}}{q} = 0$ locus is a vertical line at the corresponding capital level, where the real interest rate is zero, and the $\dot{k} = 0$ locus might be a hump-shaped curve, whose peak is to the left of the above vertical line. As shown in Figure 1, there is a saddle path converging to the bubbly steady state. The points on this path imply an one-to-one relationship between the asset bubble and physical capital stock at this bubbly equilibrium. It also shows that the bubble is positively related to physical capital stock. Trajectories below the saddle path eventually converge to the bubbleless steady state. Trajectories above the saddle path are not economic equilibria since capital will always be through the negative side.

This finding, similar as what Tirole (1985) reported, can be summarized by the following proposition.

Proposition 1 *In an economy with "spirit of capitalism", when the production function exhibits decreasing returns, given any initial capital stock, k_0, there is a unique initial value of bubble, $q^*(k_0)$, to guarantee that this economy would converge to the bubbly steady state. The first derivative of $q^*(k_0)$ is positive. If the initial bubble, q_0, is positive but less than $q^*(k_0)$, then this economy converges to the bubbleless steady state. The initial bubble, q_0, cannot be larger than $q^*(k_0)$.*

However, when production function does not exhibit decreasing returns, the analysis by phase diagrams is not feasible because the asset bubble would diverge at the same rate as the real interest rate, which is always positive. However, with some standard specifications, the existence and charaterization of the bubbly equilibrium in this environment are still possible.

To highlight that bubbles still exist in the case of $\lim_{a\to\infty} U'_a = 0$, the preference function is specified as follows.

$$U(c, a) \equiv \frac{c^{1-\sigma} - 1}{1-\sigma} + \eta \frac{a^{1-\gamma} - 1}{1-\gamma}, \sigma \geq 1, \gamma \geq 1, \eta > 0. \tag{10}$$

Here, η measures the "spirit of capitalism". In order to easily compare this with Yanagawa and Grossman (1992), and following Romer (1986) and Xie (1991), a similar production function with positive externality is adopted. That is,

$$f(k) \equiv Ak^{\alpha}\bar{k}^{1-\alpha}, 0 < \alpha < 1, \tag{11}$$

where A is the technology level, $\bar{k}$ is the average capital stock. At equilibrium, $k = \bar{k}$. Thus, this economy can be described by the following system of equations

$$\dot{q} = (\alpha A - \delta)q, \tag{12}$$

$$-\sigma\frac{\dot{c}}{c} = \rho - \frac{\eta c^{\sigma}}{(q+k)^{\gamma}} - (\alpha A - \delta), \tag{13}$$

$$\dot{k} = (A - \delta)k - c, \tag{14}$$

together with transversality conditions (3) and (4).

As mentioned in the above section, the condition of

$$\lim_{t\to\infty} \frac{\eta c^{\sigma}}{(q+k)^{\gamma}} > 0 \tag{15}$$

prevents transversality condition (4) from ruling out the bubbles. Given the real resource constraint, we can obtain that

$$\frac{\dot{c}}{c} \leq \frac{\dot{k}}{k} \leq \frac{\dot{a}}{a}.$$

In order to make sure that condition (15) holds, it is necessary that parameter σ is not less than parameter γ, i.e.,

$$\sigma \geq \gamma.$$

With this restriction on the parameters, the existence of a bubbly balanced growth path and quasi-bubbleless balanced growth path can be verified[1]. On the former path, the growth rate of the real economy is not larger than that of the asset bubble, which is equal to the real interest rate, $\alpha A - \delta$. On the latter path, the situation is just the opposite. There could be some doubt as to whether the individual's problem is well-defined or not in the case when the real interest rate is not larger than the growth rate of the real economy. Appendix 8.3 dispels this concern by proving that the solution determined by the first order conditions and tranversality conditions is the optimal choice for individuals.

For the convenience of analysis using phase diagrams, we can define that

$$\begin{aligned} \tilde{c} &\equiv ce^{-rt}, \\ \tilde{k} &\equiv ke^{-rt}, \\ \tilde{q} &\equiv qe^{-rt}. \end{aligned}$$

Here, $r \equiv \alpha A - \delta$ is the real interest rate. Thus, the above system of equations can be converted into the following new system.

$$\tilde{q} \equiv q_0,$$

[1] See more details in the appendix 8.

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{1}{\sigma}[\frac{\eta \tilde{c}^{\sigma} e^{(\sigma-\gamma)rt}}{(q_0 + \tilde{k})^{\gamma}} - \rho - (\sigma - 1)(\alpha A - \delta)],$$

$$\dot{\tilde{k}} = (1 - \alpha)A\tilde{k} - \tilde{c}.$$

In this new system, $\tilde{c}$ and $\tilde{k}$ both eventually converge to some non-negative constants[2] on the bubbly balanced growth path. However, on the quasi-bubbleless balanced growth path, both of them will diverge.

Suppose that the initial value of bubble q_0 is positive. The phase diagram for the case where $\sigma > \gamma$ is given in Figure 2. Note that the $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ locus shifts downward with time. The bold curve in this figure describes the $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ locus at the initial time, i.e., $t = 0$. Through this phase diagram, we can clearly see that to the left of the vertical dot-dash line of $\tilde{k}_0^*(q_0)$, there is at least one trajectory that eventually converges to the bubbly balanced growth path where $\tilde{c} = 0$ and $\tilde{k} = 0$; and, to the right of this dot-dash line, there also exists at least one trajectory that converges to the bubbleless balanced growth path where $\tilde{c}$ and $\tilde{k}$ diverge. Here, the value of $\tilde{k}_0^*(q_0)$ is given by the positive solution of the following equation

$$\frac{\eta[(1 - \alpha)A]^{\sigma}}{\rho + (\sigma - 1)(\alpha A - \delta)}\tilde{k}_0^{\sigma} = (q_0 + \tilde{k}_0)^{\gamma}. \tag{16}$$

Since $\tilde{k}_0 \equiv k_0$, by the inverse function of $\tilde{k}_0^*(q_0)$, we can also obtain a threshold value of bubble $q_0^*(k_0)$ that is a function of the positive initial capital stock, k_0. Given a positive initial capital stock, i.e., $k_0 > 0$, if the initial value of the bubble q_0 is positive but less than $q_0^*(k_0)$, then, in the space of $\tilde{c}$ and $\tilde{k}$, the initial point of economy, is located to the right of the vertical line of $\tilde{k}_0^*(q_0)$. Thus, an appropriate initial value of consumption $c_0 = \tilde{c}_0$ would be chosen to make sure that the economy is on the trajectory converging to the quasi-bubbleless balanced growth path. If the the initial value of bubble q_0 is larger than $q_0^*(k_0)$, then the initial point of the economy is located to the left of the vertical line of $\tilde{k}_0^*(q_0)$. Thus, the economy would be approaching the bubbly balanced growth path.

The phase diagram for the case where $\sigma = \gamma$ is shown in Figure 3. This case is simpler than the one discussed above, given the fact that $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ locus is stable. We can easily obtain the form of $\tilde{k}_0^*(q_0)$ by solving equation (16). Given

[2] When $\sigma > \gamma$, these constants are zeros; when $\sigma = \gamma$, these constants are increasing in the initial value of bubble, q_0.

any $k_0 > 0$, the threshold value of bubble $q_0^*(k_0)$ is given by

$$\{[(1-\alpha)A][\frac{\eta}{\rho+(\sigma-1)(\alpha A-\delta)}]^{\frac{1}{\sigma}}-1\}k_0.$$

Here, we need to assume that

$$[\frac{\eta}{\rho+(\sigma-1)(\alpha A-\delta)}]^{\frac{1}{\sigma}}\geq 1.$$

If the initial value of bubble q_0 is equal to $q_0^*(k_0)$, then the economy should be located at the intersection of the $\frac{\dot{\tilde{c}}}{\tilde{c}}=0$ locus and the $\dot{\tilde{k}}=0$ locus, which corresponds to the bubbly balanced growth path, and stay there forever. If the initial value of bubble q_0 is positive but less than $q_0^*(k_0)$, then the economy starts at the right of the vertical line of $\tilde{k}_0^*(q_0)$. Thus, the economy would eventually converge to the quasi-bubbleless balanced growth path.

The proposition below summarizes these findings.

Proposition 2 *In an economy with "spirit of capitalism", the preference function and production function are given by (10) and (11), respectively. Suppose that $\sigma \geq \gamma$. Given any initial capital stock, k_0, there is a unique threshold value of bubble, $q_0^*(k_0)$. If the initial bubble, q_0, is positive but less than $q_0^*(k_0)$, then this economy converges to the quasi-bubbleless balanced growth path. When $\sigma > \gamma$, if the initial bubble, q_0, is larger than $q_0^*(k_0)$, then this economy converges to the bubbly balanced growth path. When $\sigma = \gamma$, if the initial bubble, q_0, is equal to $q_0^*(k_0)$, then this economy stays on the bubbly balanced growth path forever.*

5 Explicit Examples of Bubbly Equilibria

This section provides two examples of bubbly equilibria that can be solved explicitly. These explicit examples help us obtain more intuitions about the bubbly economy.

5.1 Example 1

Suppose the production function is given by

$$f(k) = Ak^{\alpha}, 0 < \alpha < 1,$$

and the preference function is given by (10). Following proposition presents an explicit solution of the bubbly equilibrium.

Proposition 3 *When* $\alpha\sigma = 1, \gamma = 1, \frac{\eta[(1-\alpha)A]^{\frac{1}{\alpha}}}{\rho} > 1,$ *and* $q_0 = \{\frac{\eta[(1-\alpha)A]^{\frac{1}{\alpha}}}{\rho} - 1\}k_0$*, the solution of the representative individual's optimization problem is given by the policy function* $c = (1-\alpha)Ak^{\alpha}$*. A bubble exists in this economy and its process can be described by* $q = \{\frac{\eta[(1-\alpha)A]^{\frac{1}{\alpha}}}{\rho} - 1\}k$*. Finally, this economy converges to the bubbly steady state.*

Proof. First, we conjecture $c = (1-\alpha)Ak^{\alpha}$, then we verify that this is optimal. Substituting our conjecture into the goods market clearing condition (8), we obtain that

$$\frac{\dot{k}}{k} = \alpha A k^{\alpha-1} - \delta. \tag{17}$$

Comparing the above result with the non-arbitrage condition (9), we find that the bubble grows at the same rate as the accumulation of capital,

$$\frac{\dot{q}}{q} = \frac{\dot{k}}{k} = \alpha A k^{\alpha-1} - \delta.$$

This means that the size of bubble is always proportional to the capital stock. Since $a = q+k$ at equilibrium, the Euler equation (1) can be rewritten as follows given our conjecture.

$$\frac{U_a'}{U_c'} = \frac{\eta[(1-\alpha)A]^{\sigma}k}{q+k} = \rho \tag{18}$$

If $q_0 = \{\frac{\eta[(1-\alpha)A]^{\frac{1}{\alpha}}}{\rho} - 1\}k_0$, it is easy to obtain

$$q = \{\frac{\eta[(1-\alpha)A]^{\frac{1}{\alpha}}}{\rho} - 1\}k,$$

which guarantees the second equality of (18) to hold. $\frac{U_a'}{U_c'} = \rho > 0$ ensures that the bubble cannot be ruled out by the transversality condition. Determined by equation (17), capital stock k will eventually converge to a fixed point: $k^* = (\frac{\delta}{\alpha A})^{\frac{1}{\alpha-1}}$. The bubble eventually converges to $\{\frac{\eta[(1-\alpha)A]^{\frac{1}{\alpha}}}{\rho} - 1\}k^*$, which is equal to $\{\frac{\eta[(1-\alpha)A]^{\frac{1}{\alpha}}}{\rho} - 1\}(\frac{\delta}{\alpha A})^{\frac{1}{\alpha-1}}$. Finally, consumption is equal to $(1-\alpha)A(k^*)^{\alpha}$, which is $(1-\alpha)A(\frac{\delta}{\alpha A})^{\frac{\alpha}{\alpha-1}}$. This is exactly the steady state when the size of the bubble stays constant. ■

By solving equation (17), we can obtain the law of motion for the capital stock k as

$$k = [\frac{\alpha A}{\delta}(1 - e^{-(1-\alpha)\delta t}) + k_0^{1-\alpha} e^{-(1-\alpha)\delta t}]^{\frac{1}{1-\alpha}}.$$

Together with the policy function $c = (1-\alpha)Ak^{\alpha}$, and, $q = \{\frac{\eta[(1-\alpha)A]^{\frac{1}{\alpha}}}{\rho} - 1\}k$, the explicit solution of bubbly equilibrium is given.

From this explicit example, the effect of the determinants of bubble can be directly illustrated as follows. A richer atmosphere of spirit of capitalism, η, or, a higher technology level, A, allows larger sizes of bubbles; while, the lower patience, a higher value of ρ, reduce the size of the bubble. These relationships obtained from this simple example, are consistent with conventional wisdom.

Initially, there is a common consensus that bubbles boom during the prosperity of an economy and burst during a recession. In standard business cycle theory, fluctuations of the economy are usually believed to be driven by shocks on the total-factor productivity (TFP), which can be measured by parameter A. The co-movement between business cycles and flutuations in the size of the bubbles, stemming from changs of TFP, is presented clearly by this explicit solution.

Secondly, the patience of asset traders affects the size of bubbles. Intuitively, more impatient traders in financial market generally means that the bubbles have less probabilty to last, since impatient people prefer to consume immediately rather than wait for possible returns in the future. the rate of time preference, ρ, measures the degree of impatience. A higher value of ρ means more impatience. Therefore, the inverse relationship between the size of a bubble and the rate of time preference, generated by this example, fits for this intuition.

Finnally, speculation is usually believed to be more rampant in an economy with a richer culture of "spirit of capitalism". A higher value of η means a more entrenched culture of the capitalism. Thus, η's positive effect on the size of the bubble, given by this explicit solution, is consistent with this intuition.

5.2 Example 2

Let the form of preference function be

$$\log c + \eta \log a, \eta > 0, \tag{19}$$

i.e., $\sigma = \gamma = 1$, and the production function is given by (11). Based on these specifications, the following two propositions provide explicit solutions of bubbly equilibrium and bubbleless equilibrium, respectively.

Proposition 4 *When* $\frac{\eta(1-\alpha)A}{\rho} > 1$, *and initial value of this bubble,* q_0, *is equal to* $[\frac{\eta(1-\alpha)A}{\rho} - 1]k_0$, *the solution of the representative individual's optimization problem is given by the policy function* $c = (1-\alpha)Ak$. *A bubble exists in this economy and its process can be described by* $q = [\frac{\eta(1-\alpha)A}{\rho} - 1]k$. *This bubbly economy is on a balanced growth path whose growth rate is given by the real interest rate,* $\alpha A - \delta$.

Proof. At first, we conjecture an explicit solution to be given by

$$c = \varphi a, \varphi > 0,$$

$$q = \theta k, \theta > 0.$$

From the above conjecture, we obtain that

$$\frac{\dot{c}}{c} = \frac{\dot{a}}{a},$$

$$\frac{\dot{q}}{q} = \frac{\dot{k}}{k}.$$

Given the fact that at equilibrium

$$a = q + k,$$

we obtain that all variables grow at the same rate, i.e.,

$$\frac{\dot{a}}{a} = \frac{\dot{q}}{q} = \frac{\dot{k}}{k} = \frac{\dot{c}}{c} = \alpha A - \delta.$$

From equation (12) and (13), it is easy to see that

$$\varphi = \frac{\rho}{\eta}.$$

This also ensures that the transversality condition holds. From equation (12) and (14), we obtain

$$c = (1-\alpha)Ak.$$

Since

$$\frac{c}{a} = \frac{c}{(\theta + 1)k},$$

we find that

$$\theta = \frac{\eta(1-\alpha)A}{\rho} - 1.$$

This requires that

$$q_0 = [\frac{\eta(1-\alpha)A}{\rho} - 1]k_0.$$

To guarantee that $\theta > 0$, the parameter restriction,

$$\frac{\eta(1-\alpha)A}{\rho} > 1,$$

is needed. ■

Proposition 5 *Under the same parameters restriction* $\frac{\eta(1-\alpha)A}{\rho} > 1$, *when the initial value of this bubble,* q_0, *is equal to* 0, *the solution of the representative individual's optimization problem is given by the policy function* $c = \frac{(1-\alpha)A+\rho}{\eta+1}k$. *No bubble exists in this economy. This bubbleless economy is on the balanced growth path. Its growth rate is given by* $A - \delta - \frac{(1-\alpha)A+\rho}{\eta+1}$, *which is larger than the real interest rate,* $\alpha A - \delta$.

Proof. By equation (12), the price of asset should always be zero. This means that

$$a = k.$$

We conjecture the explicit solution is given by

$$c = \theta k,$$

which implies that

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k}.$$

From equation (13) and (14), we obtain that

$$\theta = \frac{(1-\alpha)A + \rho}{\eta + 1},$$

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = A - \delta - \frac{(1-\alpha)A + \rho}{\eta + 1}.$$

Given the parameters restriction

$$\frac{\eta(1-\alpha)A}{\rho} > 1,$$

it is easy to find that

$$\eta(1-\alpha)A > \rho.$$

Adding $(1-\alpha)A$ to both sides of this inequality, we obtain that

$$(1-\alpha)A > \frac{(1-\alpha)A + \rho}{\eta + 1}.$$

Therefore,

$$A - \delta - \frac{(1-\alpha)A + \rho}{\eta + 1} > \alpha A - \delta.$$

This means that the growth rate is higher than the real interest rate. ■

From Proposition 4, we find that relationships between bubbles and economic environment are qualitatively consistent with those in example 1. By comparing these explicit solutions given by two propositions above, the effect of bubbles on the real economy can also be clarified, as listed in Table 8.4. In this Table, the social welfare measure only includes the utility from consumption. If the utility from wealth is also included, the result will clearly bias in favor of the bubbly equilibrium. Given the parameters restriction $\eta(1-\alpha)A > \rho$ and the same initial capital level, it is easy to find that consumption in bubbly economy is more than that in the bubbleless economy. Together with the fact that at equilibrium the investment is equal to $Ak - c$, investment in the bubbly economy must be less than that in the bubbleless economy. This implies that bubbles stimulate consumption and crowd out investment. It also explains why the bubbleless economy has a higher growth rate than the bubbly economy. This result is similar to the findings of Saint-Paul (1992), Yanagawa and Grossman (1992), and King and Ferguson (1993), where bubbles exist in Tirole's OLG framework.

6 Stochastic Bubbles

This section discusses a stochastic bubbly economy, where bubbles might burst with an exogenous constant probability. Following the specifications given in example 2 of the above section, an explicit solution can be obtained through the methodology of Rebelo and Xie (1999).

Suppose that the bubble still exists at the current moment, i.e., $q > 0$. The process of the bubble can be described as follows,

$$dq = \left\langle \begin{array}{c} \psi q dt, \text{ with probability } 1 - \varepsilon dt \\ -q, \text{ with probability } \varepsilon dt \end{array} \right. ,$$

where $\psi > 0$, and $\varepsilon > 0$. If the bubble bursts, i.e., $q = 0$, then the price of this asset will always be zero as shown by the above process. This implies that bubble cannot be reborn.

I conjecture the process of average capital $\bar{k}$ as follows,

$$d\bar{k} = \psi \bar{k} dt.$$

Suppose the process of asset volume held by the representative individual, is given by

$$ds = \iota dt,$$

where ι is a choice variable that measures the increment of the assets held by the representative individual, given its price level. Thus, the budget can be written as follows,

$$dk = (Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c - q\iota)dt.$$

The Hamilton-Jacobi-Bellman equation is,

$$\begin{aligned} 0 \;\; = \;\; & \max_{c,\iota}\{U(c, ps + k) + V_1(q, \bar{k}, k, s)\psi q + V_2(q, \bar{k}, k, s)\psi \bar{k} \\ & +V_3(q, \bar{k}, k, s)(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c - q\iota) + V_4(q, \bar{k}, k, s)\iota \\ & -(\varepsilon + \rho)V(q, \bar{k}, k, s) + \varepsilon V(0, \bar{k}, k, s)\} \end{aligned}$$

Guess the form of value function to be

$$V(q, \bar{k}, k, s) \equiv \chi + h \log(qs + k) + b \log \bar{k},$$

it is easy to obtain that

$$V_1(q, \bar{k}, k, s) = \frac{hs}{qs + k},$$

$$V_2(q, \bar{k}, k, s) = \frac{b}{\bar{k}},$$

$$V_3(q, \bar{k}, k, s) = \frac{h}{qs+k},$$

$$V_4(q, \bar{k}, k, s) = \frac{hq}{qs+k}.$$

Thus, the Hamilton-Jacobi-Bellman equation above can be rewritten as

$$\begin{aligned} 0 \quad = \quad & \max_{c,\iota}\{\log c + \eta \log(qs+k) + \frac{hs}{qs+k}\psi q + b\psi \\ & + \frac{h}{qs+k}(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c) \\ & -(\varepsilon+\rho)[\chi + h\log(qs+k) + b\log\bar{k}] + \varepsilon(\chi + h\log k + b\log\bar{k})\}. \end{aligned} \tag{20}$$

The optimal condition for consumption is given by

$$c = \frac{qs+k}{h}. \tag{21}$$

The partial derivatives of equation (20) with respect to $\bar{k}$, s, q, and k, respectively, should all be zero. That is,

$$(1-\alpha)hAk^{\alpha}\bar{k}^{1-\alpha} = \rho b(qs+k), \tag{22}$$

$$\begin{aligned} & \eta(qs+k) + \psi hk \\ = \quad & (\varepsilon+\rho)h(qs+k) + h(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c), \end{aligned} \tag{23}$$

$$\begin{aligned} & \eta(qs+k) + h\psi k \\ = \quad & (\varepsilon+\rho)h(qs+k) + h(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c), \end{aligned} \tag{24}$$

$$\begin{aligned} & \eta(qs+k) + h(\alpha Ak^{\alpha-1}\bar{k}^{1-\alpha} - \delta)(qs+k) + \frac{\varepsilon h}{k}(qs+k)^2 \\ = \quad & hs\psi q + (\varepsilon+\rho)h(qs+k) + h(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c). \end{aligned} \tag{25}$$

By equation (21) and equation (22), we obtain that

$$c = \frac{qs+k}{h} = \frac{(1-\alpha)Ak^{\alpha}\bar{k}^{1-\alpha}}{\rho b}. \tag{26}$$

Substituting above equation and equation (22) into equation (23), we obtain that

$$\frac{\bar{k}}{k} = [\frac{\psi+\delta}{A\{[(\varepsilon+\rho)h-\eta-1]\frac{(1-\alpha)}{\rho b}+1\}}]^{\frac{1}{1-\alpha}}.$$

Given the fact that at equilibrium $k = \bar{k}$, the condition below should hold,

$$\psi = A\{[(\varepsilon+\rho)h-\eta-1]\frac{(1-\alpha)}{\rho b}+1\} - \delta. \tag{27}$$

Thus, individual's capital stock is always equal to the average level, i.e.,

$$k = \bar{k}.$$

By equation (26), we know that

$$[\frac{(1-\alpha)hA}{\rho b} - 1]k = qs.$$

From equation (24) and equation (22), we obtain that

$$(\alpha A k^{\alpha-1}\bar{k}^{1-\alpha} - \delta) + \frac{\varepsilon}{k}(qs+k) = \psi.$$

Substituting equation (22) into the above, we find that

$$\alpha A k^{\alpha-1}\bar{k}^{1-\alpha} - \delta + \frac{\varepsilon(1-\alpha)h}{\rho b} A k^{\alpha-1}\bar{k}^{1-\alpha} = \psi.$$

Together with the fact that $k = \bar{k}$, the following equation can be obtained.

$$\alpha A - \delta + \frac{\varepsilon(1-\alpha)hA}{\rho b} = \psi \tag{28}$$

The budget constraint can be rewritten as

$$dk = [Ak - \delta k - \frac{(1-\alpha)Ak}{\rho b} - q\iota]dt.$$

At equilibrium, ι is equal to zero. Thus,

$$d\bar{k} = [A - \delta - \frac{(1-\alpha)A}{\rho b}]\bar{k}dt.$$

Comparing this with the conjectured process of $\bar{k}$, it is easy to find that

$$\psi = A - \delta - \frac{(1-\alpha)A}{\rho b}. \tag{29}$$

Solving the equations system consisting of (27), (28) and (29), we can obtain values of constants b, h, and ψ as follows.

$$b = \frac{\varepsilon\eta + \rho + \varepsilon}{\rho(\rho + \varepsilon)},$$

$$h = \frac{\eta}{\rho + \varepsilon},$$

$$\psi = A - \delta - \frac{(1-\alpha)A(\rho + \varepsilon)}{\varepsilon\eta + \rho + \varepsilon}.$$

Substituting the results above into equation (20), we obtain that

$$\begin{aligned} 0 \quad = \quad & \log\frac{(1-\alpha)A}{\rho b} - 1 + b\psi - \rho a \\ & + \frac{\psi[(1-\alpha)hA - \rho b]}{(1-\alpha)A} + \frac{(A-\delta)\rho b}{(1-\alpha)A} \\ & + \log k + \varepsilon h \log k - \rho b \log k. \end{aligned}$$

It is easy to check that the sum of coefficients of $\log k$ is zero. When χ takes the following value

$$\frac{1}{\rho}\log\frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta + \rho + \varepsilon} + \frac{\eta + 1}{\rho^2}[A - \delta - \frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta + \rho + \varepsilon}],$$

the sum of the constant terms is also zero.

Based on contents that are derived above, we can find that this stochastic bubble exists only if the probability of the bursting of the bubble, ε, is less than the upper limit,

$$\bar{\varepsilon} \equiv \frac{\eta(1-\alpha)A - \rho}{\eta + 1}. \tag{30}$$

If the bubble lasts, this bubbly economy can be described by following three equations.

$$c = \frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta + \rho + \varepsilon}k, \tag{31}$$

$$q = [\frac{\eta(1-\alpha)A}{\varepsilon\eta + \rho + \varepsilon} - 1]k, \tag{32}$$

$$
\begin{aligned}
g &= \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{q}}{q} \\
&= A - \delta - \frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta+\rho+\varepsilon}
\end{aligned} \tag{33}
$$

Once the bubble bursts, this economy jumps to the bubbleless economy described in Proposition 5.

From equation (32), we can find that the relationship between the size of bubble and the capital stock in this stochastic economy is similar as what Weil (1987) reported, which is based on Tirole's overlapping generations framework. However, explicit solutions in this example help us analyze other economic issues more intuitively.

By comparing the explicit solutions of the deterministic bubbly economy, the deterministic bubbleless economy, and this stochastic bubbly economy, it is easy to find the real effects of uncertainty in this economy. Given the fact that

$$
\frac{(1-\alpha)A+\rho}{\eta+1} < \frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta+\rho+\varepsilon} < (1-\alpha)A,
$$

we can say that consumption in this stochastic environment is larger than that in bubbleless case but less than that in the case where the bubble is certain. Since the aggregate investment is equal to $Ak - c$, the investment in this stochastic case is less than that in the deterministic bubbleless economy but larger than that in the deterministic bubbly economy. Thus, stochastic bubbles stimulate consumption and crowd out investment. This is similar to what we obtain in the deterministic case, except to a weaker degree. This also suggests that the growth rate in this stochastic economy is higher than that in the bubbly economy without uncertainty and lower than that in the deterministic bubbleless case.

From this explicit example, we can also see directly how the probability of the bursting of the bubbles, ε, affects the real economy. The fact that

$$
\frac{\partial(c/k)}{\partial\varepsilon} < 0
$$

illustrates the negative relationship between the probability of the bursting of the bubble and consumption. Given that the growth rate of the real economy is equal to $A - \delta - \frac{c}{k}$, there is a positive relationship between this probability ε and economic growth rate. Thus, a smaller probability of the bursting of the bubble means more consumption and a lower growth rate, while the situation is just the opposite with a higher probability of bursting.

The size of the bubble is also affected by this probability of bursting. A higher probability would reduce the size of bubble; and, a smaller possibility to burst allows for a larger size of the bubble. As the probability of the bursting of the bubbles, ε, converges to zero, this stochastic bubbly economy approaches the deterministic bubbly economy; while, as this probability ε converges to its upper limit, $\bar{\varepsilon}$, this stochastic bubbly economy converges to the deterministic bubbleless economy.

The relationship above obtained from this example is consistent with our intuition. When a bubble has a higher possibility of bursting, the expected wealth decreases. By the wealth effect, consumption will also be reduced. Investors would adjust their portfolios and put more weight on physical capital. Thus, the value of financial assets would be even lower. Since the fundmental value of this financial asset must not be lower, the size of bubble would be reduced. At the same time, higher investment stimulates economic growth.

Furthermore, how the upper limit of the probability of the bursting of the bubble, $\bar{\varepsilon}$, is affected by economic environment, gives us a hint on where this type of stochastic bubbles usually arise. From equation (30), it is easy to see that higher values of η and A, and lower value of ρ, will raise the upper limit, $\bar{\varepsilon}$. This higher upper bar of the probability of the bursting means a higher probability of the existence of stochastic bubbles. Thus, this type of stochastic bubbles more frequently appear in an environment where people care more about their wealth, or, the technology level is higher, or, the society is more patient. This is consistent with our thinking.

In addition, this explicit example of stochastic bubble also verifies the existence of a series of stochastic bubbly equilibria. From equation(32), we can obtain that

$$\varepsilon = \frac{\frac{\eta(1-\alpha)A}{\frac{q}{k}+1} - \rho}{\eta + 1}.$$

To satisfy the following restriction

$$0 \leq \varepsilon < \bar{\varepsilon},$$

together with equation (30), we need that

$$0 < q \leq [\frac{\eta(1-\alpha)A}{\rho} - 1]k.$$

Since the growth rate of the bubble is equal to the growth rate of the real economy at this stochastic bubbly equilibrium, the inequality above will hold at any time. Thus, the following proposition about stochastic bubbly equilibrium can be naturally obtained.

Proposition 6 *In an economy with "spirit of capitalism", the preference function and production function are given by (19) and (11), respectively. For any positive innitial value of bubble,* q_0*, which is not larger than the threshold value of* $[\frac{\eta(1-\alpha)A}{\rho}-1]k_0$*, there exists an exogenous stochastic bubbly equilibrium, where*

$$dq = \langle \begin{array}{c} \psi q dt, \ \textit{with probability } 1-\varepsilon dt \\ -q, \ \textit{with probability } \varepsilon dt \end{array},$$

and the real economy grows at the rate of ψ*. Here, the probability of the bursting of the bubble* ε *is given by*

$$\frac{\frac{\eta(1-\alpha)A}{q_0/k_0+1}-\rho}{\eta+1},$$

which implies that the stochastic bubbly equilibrium with a higher ratio of initial value of bubble to initial capital stock, q_0/k_0*, requires a smaller probability of bursting,* ε*, and*

$$\psi \equiv A-\delta-\frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta+\rho+\varepsilon}.$$

7 Conclusion

This paper focuses on rational bubbles of zero-dividend assets in an infinite-horizon model. In the environment with "spirit of capitalism", as long as, eventually, the marginal benefit from holding wealth is not trivial relative to the marginal utility of consumption, rational bubbles do exist. The analysis of phase diagram suggests similar results as in Tirole (1985). However, my infinite-horizon model eliminates the concern of incomplete market generated by the structure of overlapping generations. Since infinite-horizon framework is the common base for a vast literature on asset pricing and macroeconomics, many economic issues about bubbles can be discussed on the basis of my framework.

In addition, this paper also discusses an economy where a bubble might burst with an exogenous probability. It gives a simple theoretical foundation to explore economic implications of the collapse of a bubble. As an interesting further direction, issues about financial crisis can be explored by introducing a banking sector into this framework.

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8 Appendix

8.1 Bubbly Balanced Growth Path

This appendix shows how to obtain the bubbly balanced growth path under specifications given in section 4.

At a bubbly balanced growth path, the growth rate of the bubble should not be less than the growth rate of the real economy. Otherwise, the value of the bubble would be of trivial relevance to the real economy. This implies that

$$\frac{\dot{q}}{q} \geq \frac{\dot{k}}{k}.$$

Given the parameter restriction

$$\sigma \geq \gamma,$$

to ensure that the condition (15) holds, the growth rate of consumption should not be larger than the growth rate of the bubble. By the real resource constraint, we can obtain that

$$\frac{\dot{c}}{c} \leq \frac{\dot{k}}{k}.$$

However, the growth rate of consumption should not be less than the growth rate of capital. Otherwise, by equation (14), the growth rate of capital eventually converges to $A-\delta$, which is larger than the growth rate of bubble, $\alpha A-\delta$. Thus, at this bubbly balanced growth path, consumption and physical capital have the same growth rate, which is not larger than the growth rate of the bubble, i.e.,

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} \leq \frac{\dot{q}}{q}.$$

Since the term of $\frac{\eta c^{\sigma}}{(q+k)^{\gamma}}$ eventually converges to some positive constant, we find that eventually

$$\sigma\frac{\dot{c}}{c} = \gamma\frac{\dot{q}}{q},$$

which means that

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\gamma}{\sigma}(\alpha A - \delta).$$

From equation (13) and (14), respectively, we obtain that

$$\frac{\eta c^{\sigma}}{(q+k)^{\gamma}} = (\gamma - 1)(\alpha A - \delta) + \rho,$$

$$\frac{c}{k} = A - \delta - \frac{\gamma}{\sigma}(\alpha A - \delta).$$

8.2 Quasi-Bubbleless Balanced Growth Path

This appendix verifies the existence of quasi-bubbleless balanced growth path under specifications given in section 4.

At this balanced growth path, the value of the bubble might not be zero at all. But, since the growth rate of the bubble is less than the growth rate of the real economy, the value of the bubble will finally be trivial. By the real resource constraint, we obtain that

$$\frac{\dot{c}}{c} \leq \frac{\dot{k}}{k}.$$

This implies that at least the growth rate of capital is larger than the growth rate of the bubble, i.e.,

$$\frac{\dot{k}}{k} > \frac{\dot{p}}{p} = \alpha A - \delta.$$

When $\sigma > \gamma \geq 1$, if the growth rate of consumption is equal to the growth rate of capital, then the term $\frac{\eta c^{\sigma}}{(q+k)^{\gamma}}$ would eventually converges to positive infinity. By equation (13), we obtain that

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} \to +\infty.$$

This is contradictory with equation (14). Therefore, the growth rate of consumption should be less than the growth rate of capital, i.e.,

$$\frac{\dot{c}}{c} < \frac{\dot{k}}{k}.$$

From equation (14), it is clear that the growth rate of capital eventually converges to $A - \delta$. By equation (13), we find that the growth rate of consumption eventually converges to $\frac{\gamma}{\sigma}(A - \delta)$ and the term $\frac{\eta c^{\sigma}}{(q+k)^{\gamma}}$ converges to $\gamma(A - \delta) + \rho - (\alpha A - \delta)$.

When $\sigma = \gamma \geq 1$, suppose the initial value of the bubble is positive, if the growth rate of consumption is less than the growth rate of capital, i.e.,

$$\frac{\dot{c}}{c} < \frac{\dot{k}}{k},$$

then the term $\frac{\eta c^{\sigma}}{(q+k)^{\gamma}}$ will converge to zero. Thus, the transversality condition

(4) is violated. Therefore, for the case of $q_0 > 0$, it should hold at the quasi-bubbleless balanced growth path that

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} > \frac{\dot{q}}{q} = \alpha A - \delta.$$

Suppose the term $\frac{\eta c^{\sigma}}{(q+k)^{\gamma}}$ eventually converges to a positive constant θ. Given $\sigma = \gamma$, we also obtain that

$$\frac{c}{k} \to (\frac{\theta}{\eta})^{\frac{1}{\sigma}}.$$

Combining equation (13) with equation (14), we obtain that

$$A - \delta - (\frac{\theta}{\eta})^{\frac{1}{\sigma}} = \frac{\alpha A - \delta + \theta - \rho}{\sigma} > \alpha A - \delta.$$

It is easy to find that there is an unique θ to satisfy the above condition given standard calibrated parameters. Thus, the quasi-bubbleless balanced growth path still exists in this case.

8.3 Sufficiency of FOCs and TVCs for Dynamic Optimization

Individual's optimality question can be given by

$$\max \int_0^{\infty} e^{-\rho t} U(c,a) dt,$$

$$\begin{aligned} s.t. \quad &: \quad a = qs + k, \\ \dot{a} \quad &= \quad f(k) - c + \dot{q}s, \\ a \quad &\geq \quad 0, \\ & \quad\quad a_0 \text{ is given.} \end{aligned}$$

Here, $U(c,a)$ is concave, and $U^{''}_{ca} = U^{''}_{ac} = 0$, $f^{''}(k) \leq 0$. The process of dq is exogenous for any individual.

Suppose that $\{c^*, k^*, s^*, a^*\}$ is the solution which satisfies the FOCs and TVCs, and $\{c, k, s, a\}$ is another possible choice. The difference of utilities evaluated at $\{c^*, k^*, s^*, a^*\}$ and at $\{c, k, s, a\}$ is given below.

$$\begin{aligned} D &\equiv \int_0^\infty e^{-\rho t}\{U(c^*, a^*) + \lambda^*(a^* - qs^* - k^*) + \mu^*[f(k^*) - c^* + \dot{q}s^* - \dot{a}^*] \\ &\quad -U(c, a) - \lambda^*(a - qs - k) - \mu^*[f(k) - c + \dot{q}s - \dot{a}]\}dt, \end{aligned}$$

where λ^* and μ^* are the multipliers that satisfy the FOCs and TVCs.

By the fact that

$$\mu^*\dot{q} = \lambda^* q,$$

we obtain that

$$\begin{aligned} D &= \int_0^\infty e^{-\rho t}\{U(c^*, a^*) - U(c, a) + \mu^*[f(k^*) - f(k)] \\ &\quad +\lambda^*(a^* - a) - \mu^*(c^* - c) - \lambda^*(k^* - k) - \mu^*(\dot{a}^* - \dot{a})\}dt. \end{aligned}$$

From the concavity of $U(c, a)$ and $f^{''}(k) \leq 0$, we can find that

$$\begin{aligned} D &\geq \int_0^\infty e^{-\rho t}\{U_c^{'}(c^*, a^*)(c^* - c) - \mu^*(c^* - c) \\ &\quad +\mu^* f'(k^*)(k^* - k) - \lambda^*(k^* - k) \\ &\quad +U_a^{'}(c^*, a^*)(a^* - a) + \lambda^*(a^* - a) - \mu^*(\dot{a}^* - \dot{a})\}dt. \end{aligned}$$

By

$$U_c^{'}(c^*, a^*) = \mu^*,$$

and

$$\mu^* f'(k^*) = \lambda^*,$$

we obtain that

$$\begin{aligned} D &= \int_0^\infty e^{-\rho t}[U_a^{'}(c^*, a^*) + \lambda^*](a^* - a)dt - \int_0^\infty e^{-\rho t}\mu^*(\dot{a}^* - \dot{a})dt, \\ &= \int_0^\infty e^{-\rho t}[U_a^{'}(c^*, a^*) + \lambda^* + \dot{\mu}^* - \rho\mu^*](a^* - a)dt - e^{-\rho t}\mu^*(a^* - a)|_0^\infty. \end{aligned}$$

Since

$$U_a^{'}(c^*, a^*) + \lambda^* + \dot{\mu}^* = \rho\mu^*,$$

we obtain that

$$D = -\lim_{t\to\infty} e^{-\rho t}\mu^* a^* + \lim_{t\to\infty} e^{-\rho t}\mu^* a.$$

By the fact of

$$a \geq 0,$$

and the tranversality condition

$$\lim_{t \to \infty} e^{-\rho t} \mu^* a^* = 0,$$

it is easy to find that

$$D \geq 0.$$

This means that the solution that satisfies with the FOCs and TVCs is optimal.

8.4 Comparison between Bubbly BGP and Bubbleless BGP

	Bubbly BGP	v.s.	Bubbleless BGP
Consumption	$(1-\alpha)Ak$	$>$	$\frac{(1-\alpha)A+\rho}{\eta+1}k$
Bubble	$[\frac{\eta(1-\alpha)A}{\rho} - 1]k$	$>$	0
Growth Rate	$\alpha A - \delta$	$<$	$A - \delta - \frac{(1-\alpha)A+\rho}{\eta+1}$
Saving Rate	α	$<$	$[A - \frac{(1-\alpha)A+\rho}{\eta+1}]/A$
Social Welfare (excluding wealth term)	SW^b	$<$	SW^{nb}
Discount rate of $e^{-\rho t}\mu$	$\rho + \alpha A - \delta > g^b$	..	$\alpha A - \delta + \frac{\eta[(1-\alpha)A+\rho]}{\eta+1} > g^{nb}$

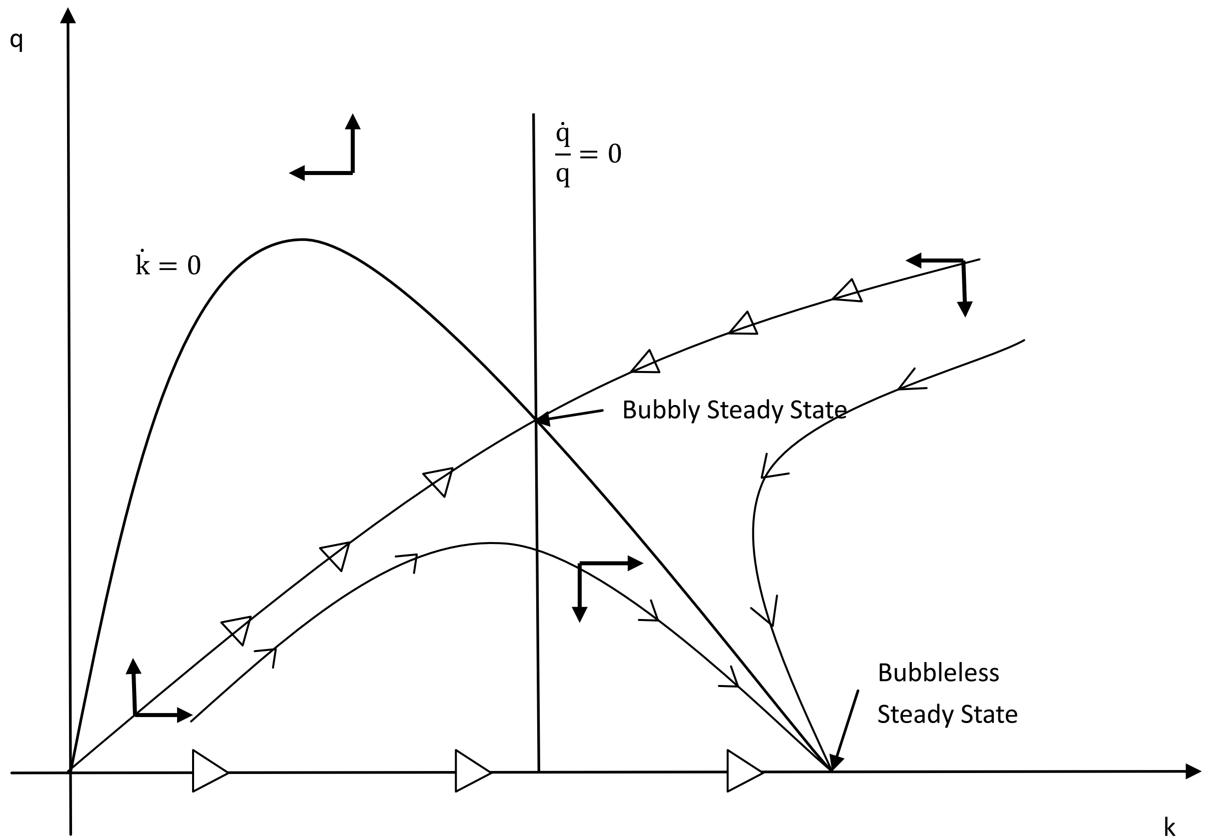

Figure 1: Production of Decreasing Return

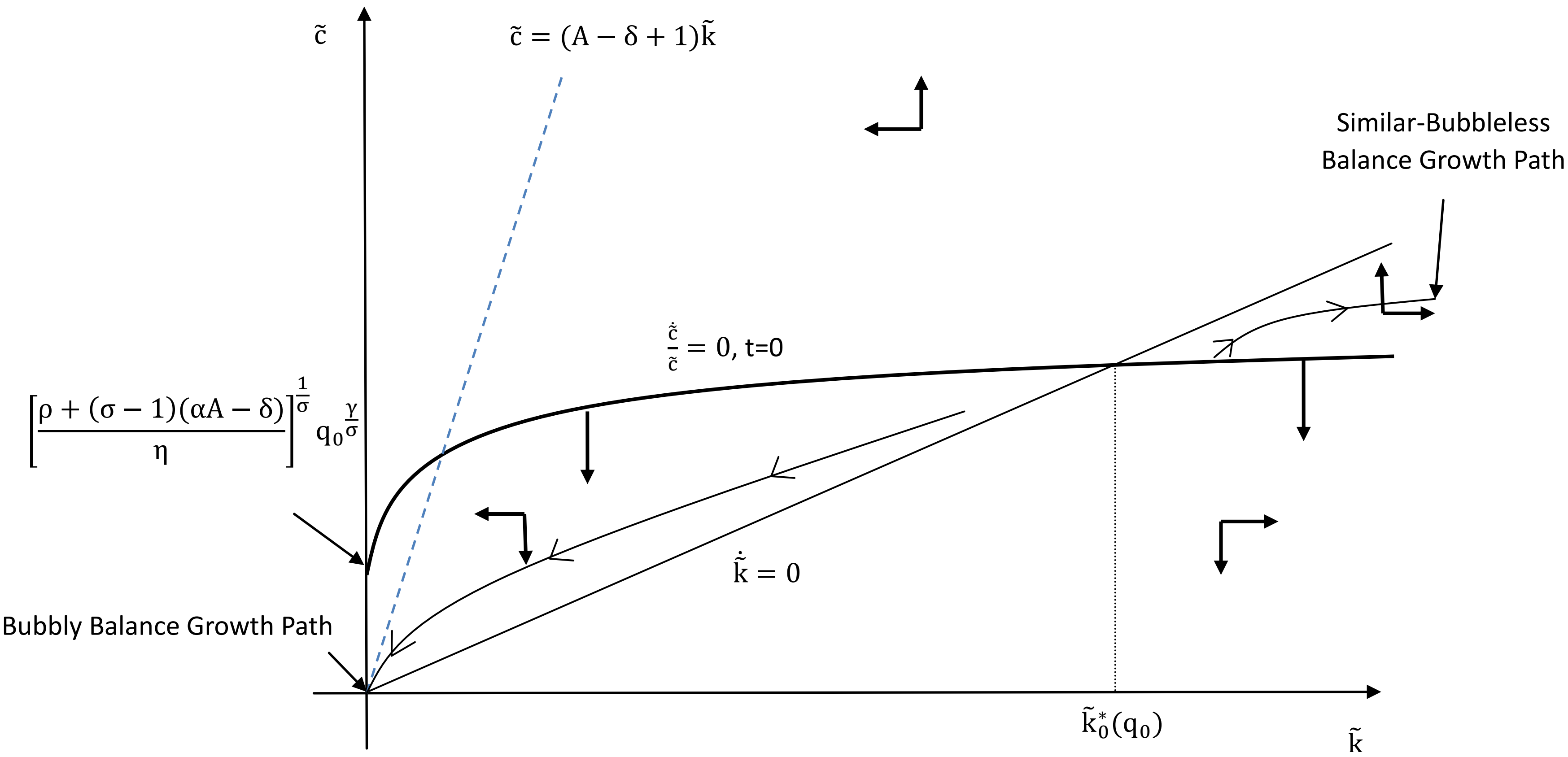

Figure 2: Production of Increasing Return ($\sigma > \gamma$)

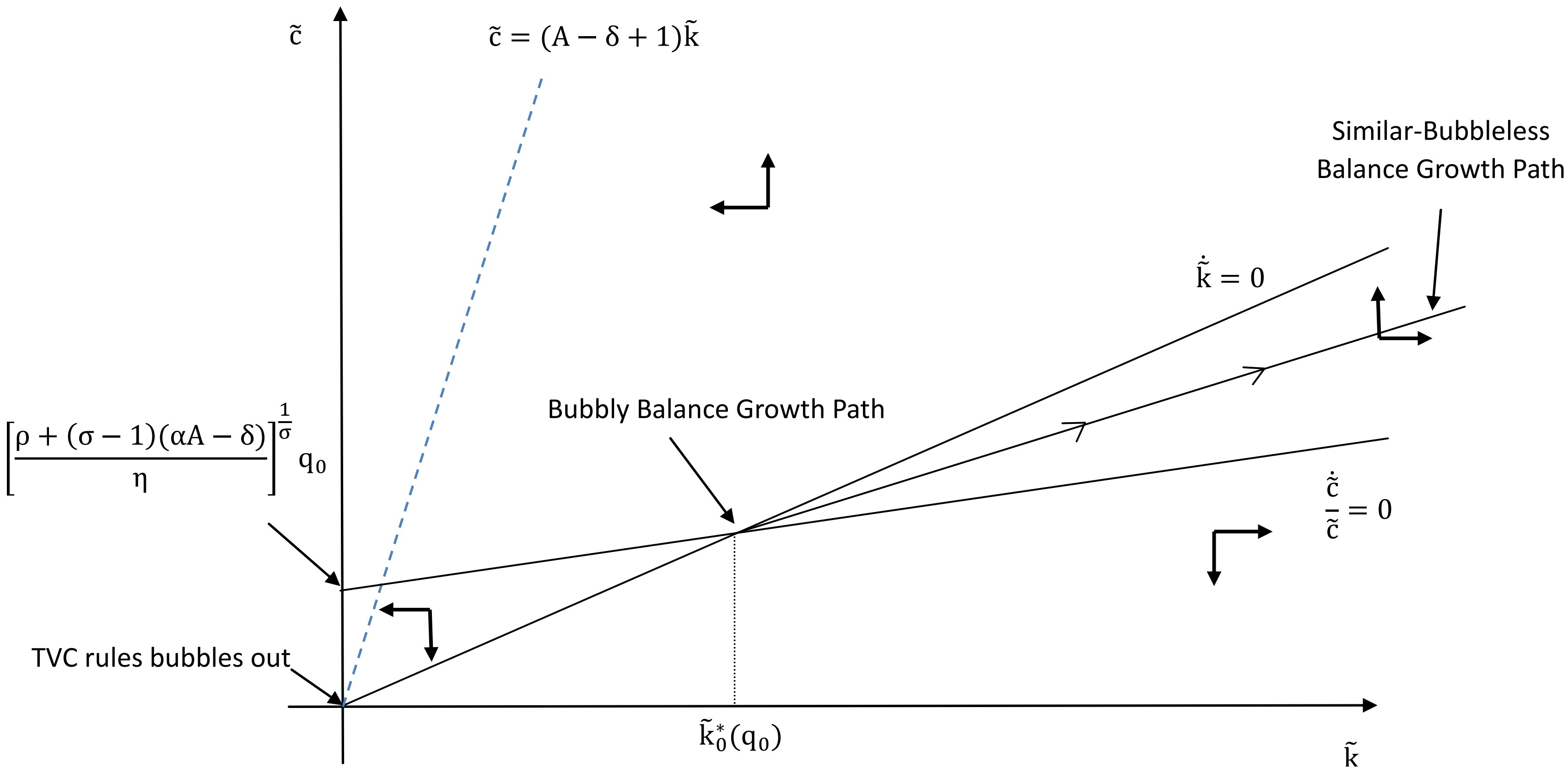

Figure 3: Production of Increasing Return ($\sigma = \gamma$)