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Abstract

This paper examines the effect of the timing of childbirth on capital accumulation and welfare in a simple overlapping generations model, where each agent lives for four periods and works for two periods. We show that delayed childbearing not only reduces population, but also generates fluctuations in the age composition of workers in the labor force. This causes the aggregate saving rate to fluctuate, which leads to cycles in the capital–labor ratio. When all agents delay childbearing, we analytically show that both the capital–labor ratio and the welfare of all agents can fall in the long run, despite the population decline. When a fraction of agents delay childbearing, it has differential welfare effects on agents depending on their positions in the demographic cycles. The effects of lower lifetime fertility and technological progress are also examined.

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Keywords: Economic growth, Overlapping generations, Cycles, Population, Delayed childbearing.

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1 Introduction

Consider a fall in population induced by a decline in the number of births in the economy, taking as given mortality and migration. It is well known that a lower population growth raises the capital–labor ratio in the Solow–Swan growth model. The same property holds in Diamond's (1965) overlapping generations model, and it enhances welfare as long as the economy is dynamically efficient; i.e., when the interest rate exceeds the population growth rate. Of course, the declining birth rate can cause welfare problems when the population size has some positive externality, or when social security systems are explicitly considered.[1] Apart from these issues, it has been generally perceived that the population decline is favorable to economic welfare.[2]

This paper considers an overlapping generations model without external effects or a social security system. Nonetheless, we show that a population decline can worsen the welfare of agents if it is caused by a change in the timing of childbirth or, more specifically, when many people decide to delay childbearing to older ages.

Delayed childbearing has been broadly observed in developed countries. Between 1975 and 2005, the fraction of Japanese children who were born to mothers in their 20s decreased from 75% to 45%, whereas those born to mothers in their 30s increased from 20% to 52%. A similar trend is observed in the United States and advanced European countries (Gustafsson and Kalwij 2006), and also in Canada, Australia, and New Zealand (Sardon 2006). Interestingly, as pointed out by Bongaarts and Feeney (1998), even when the cohort's lifetime fertility rate (the number of children a mother has in her lifetime) does not fall, the delayed childbearing alone leads to a decline in the number of childbirths, measured by the total period fertility rates (TPFRs). Ogawa and Retherford (1993), Kohler et al. (2002), and Sobotka (2004) confirmed that, to a certain extent, the delay of marriage and motherhood is

[1] To support a pay-as-you-go pension system, the economy must have enough children. For the relationship between endogenous fertility and optimal social security, see Zhang and Zhang (2007) and Yew and Zhang (2009).

[2] A notable exception is d'Albis (2007), which showed that when agents are uncertain about the length of their life and there is a perfect annuity market, the capital–labor ratio may respond nonmonotonically to the population growth rate.

responsible for the observed period fertility rate decline (now known as the "tempo effect").

The seminal studies that incorporated the tempo effect into economic theory are Happel et al. (1984) and Cigno and Ermisch (1989). These studies constructed models where women endogenously choose the timing of childbearing considering the fact that childbearing interrupts their work experience for a certain period, which affects their lifetime income profiles through their career paths or the accumulation of human capital.[3] Incorporating this idea into the theory of economic growth, Iyigun (2000), Blackburn and Cipriani (2002), Mullin and Wang (2002), and d'Albis et al. (2010) constructed dynamic general equilibrium models where the timing of childbirth is endogenous.[4]

Complementary to these preceding studies, this paper focuses on the aspect that delayed childbearing changes the age structure of the labor force. When a considerable fraction of mothers begin to delay childbearing, it causes a temporary baby bust in the economy, and the echoes of the initial baby bust create long-lasting demographic cycles. We construct an overlapping generations model where agents work for more than one period so that the demographic cycles are translated into fluctuations in the age structure of the labor force. As the variation in the age composition of workers affects the distribution of income among different cohorts (see Berger 1989), demographic cycles lead to cycles in the aggregate saving rate, which drive fluctuations in the capital–labor ratio. We will show that the fluctuations in the capital–labor ratio have differential welfare effects on agents depending on their positions in the demographic cycles. This point was not found by earlier studies. For instance, Iyigun (2000) considered a small open economy with a fixed capital–labor ratio, savings were not allowed in Blackburn and Cipriani (2002), and Mullin and Wang (2002) and d'Albis et al.

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[3]For empirical studies on this issue, see Buckles (2008), which employed the National Longitudinal Survey of Youth to investigate the return to delayed childbearing in the US. Using Japanese panel data, Higuchi (2001) investigated the effects of labor market changes on the timing of marriage, childbirth, and employment.

[4]Iyigun (2000) built a growth model where women face a tradeoff between childbearing and human capital accumulation when young, and derived multiple steady state equilibria. Blackburn and Cipriani (2002) illustrated the mechanism where an increase in longevity delays the timing of childbearing. Mullin and Wang (2002) constructed an endogenous childbirth timing model where the solution is obtained as a closed form. d'Albis et al. (2010) proved the existence of a monetary equilibrium in a model where the age of childbearing is determined endogenously.

(2010) assumed a linear technology where one unit of effective labor produces a fixed amount of output. The remainder of the paper is structured as follows. Section 2 introduces the theoretical model. Section 3 analytically examines the impact of delayed childbearing on capital accumulation and welfare. Section 4 numerically examines the general case where only a fraction of agents delay their childbearing. Section 5 considers extensions of the model with a lower lifetime fertility rate and technological progress. Section 6 concludes the paper. Appendices A and B provide the proofs of the lemmas.

2 Model

2.1 Demographic Structure

Let us consider an overlapping generations model where each agent lives for four periods, referred to as child, young, middle-aged, and old. A group of young agents in period t (i.e., those who are born in period $t-1$) is called generation t, and its cohort size is denoted by N_t. Each agent has one child during her lifetime (the gender of the agents is not considered), and she is able to bear a child either in her youth or middle age. In this paper, we say that an agent "delays childbearing" if she bears her (only) child in her middle age.

Let us denote by $\lambda_t \in [0,1]$ the fraction of agents among generation-t agents who delay childbearing. This means that among the generation-t agents with population N_t, the fraction λ_t bear their children in their middle age (period $t+1$), and the remaining fraction $1-\lambda_t$ bear their children in their youth (period t). The cohort size of generation $t+1$, born in period t, is thus determined by:

$$N_{t+1} = (1-\lambda_t)\, N_t + \lambda_{t-1} N_{t-1}. \tag{1}$$

To highlight the effect of age distribution on capital accumulation and welfare as simply as possible, the timing of childbearing is assumed to be exogenous throughout the analysis.[5]

[5] Doepke (2005) showed that the timing of childbirth is affected by the child mortality rate in a sequential fertility choice model. A decline in child mortality also reduces the uncertainty about the number of surviving children, which lowers the fertility rate and raises educational investment, causing the demographic transition (see Kalemli-Ozcan 2002, 2003; Tamura 2006).

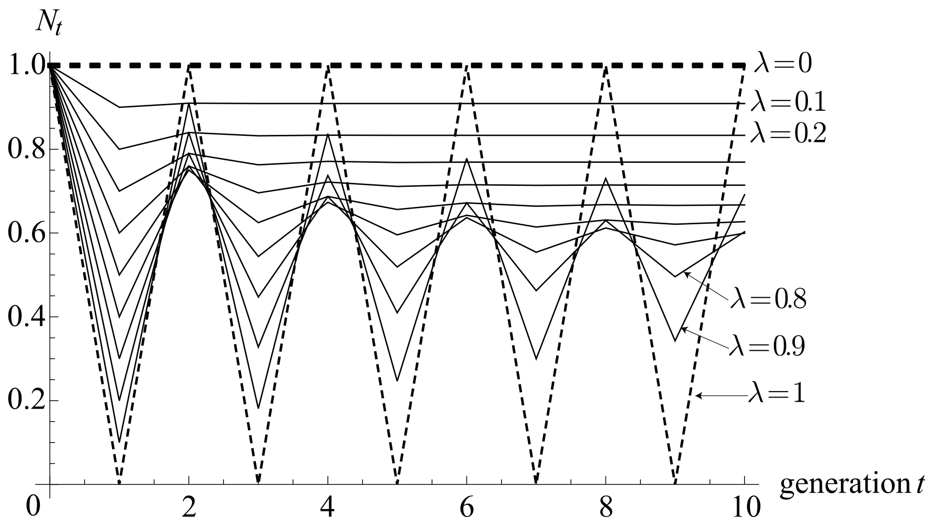

Figure 1: Fluctuations in Cohort Size N_t over Generations

We consider the situation where all agents until generation -1 bear their children when they are young, and from generation 0 a constant fraction λ of agents bear their children when they are middle-aged, i.e.:

$$\lambda_t = \begin{cases} 0, & t < 0, \\ \lambda, & t \geq 0. \end{cases} \tag{2}$$

We normalize the cohort size so that $N_0 = 1$ holds. As equations (1) and (2) imply that the cohort size is constant until period 0, $N_t = 1$ holds for all $t \leq 0$. When delayed childbearing begins, the period fertility rate temporarily falls. In period 0, only fraction $1-\lambda$ of generation-0 young agents bear children, while the generation-(-1) middle-aged agents do not bear children because they completed childbearing in the previous period (i.e., $\lambda_{-1} = 0$). Thus, the cohort size of generation 1, who are born in period 0, is given by:

$$N_1 = 1 - \lambda. \tag{3}$$

From period 1 on, not only a fraction $1 - \lambda$ of young agents, but also a fraction λ of middle-aged agents bear children. Hence, the period fertility rate recovers to some extent, which is consistent with Bongaarts and Feeney (1998). Substituting $N_0 = 1$, $N_1 = 1-\lambda$, and equation (2) into (1), the cohort size after generation 0 is solved as $N_t = \frac{1}{1+\lambda}\left(1 + (-1)^t \lambda^{t+1}\right)$ for $t \geq 0$. Figure 1 depicts the sequences of N_t for various levels of λ. It shows that the

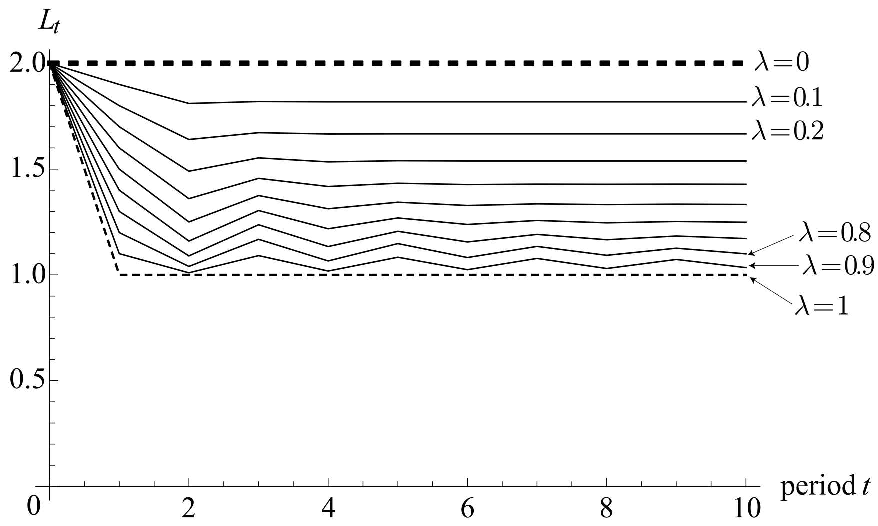

Figure 2: Dynamics of Labor Force L_t

cohort size N_t fluctuates after delayed childbearing begins (i.e., after period 0), and that the amplitude of oscillation is larger when λ is higher. This indicates that the initial fluctuation of age structure (i.e., the fall in the fertility rate in period 0 and a recovery in period 1) has recurrent "echo effects" over many generations. If $\lambda \in (0, 1)$, the fluctuation decays and N_t converges to a stationary level at $\lim_{t\to\infty} N_t = 1/(1+\lambda)$,[6] although N_t fluctuates forever in the polar case of $\lambda = 1$.

2.2 Economic Environment

Agents undertake no economic activity in their childhood, supply one unit of labor inelastically in their youth and middle age, respectively, and retire when old. The total labor force in period t is thus expressed as:

$$L_t = N_t + N_{t-1}, \tag{4}$$

which is depicted in Figure 2 for various levels of λ. This figure shows that the delayed childbearing decreases the labor force permanently even when the lifetime fertility rate is

[6]This is consistent with Lotka's stable population theory, which states that in a closed economy where there is no migration, a long-run age distribution becomes time invariant when age-specific fertility and mortality rates are constant (see Keyfitz and Caswell 2005).

held constant, and the level of L_t is lower when a larger fraction of agents decide to delay childbearing (i.e., when λ is higher).[7] Observe also that the labor force L_t has much smaller oscillations than the cohort size N_t (in fact, there is no oscillation when $\lambda = 1$). We will show that the fluctuations in the age composition of the labor force, rather than in the size of the labor force itself, drive the economic dynamics in this model.

There is a single final good in each period that can be used for either consumption or investment. Consumption takes place when agents are middle-aged and old.[8] The utility of a generation-t agent is given by:

$$U_t = \log c_{m,t+1} + \beta \log c_{o,t+2}, \tag{5}$$

where $c_{m,t+1}$ and $c_{o,t+2}$ represent generation-t consumption in their middle age (period $t+1$) and old age (period $t+2$), respectively.

Let w_t and r_t denote the wage rate and the gross interest rate (i.e., including the principal) in period t. Then, the budget constraint of a generation-t agent is:

$$a_{y,t} = w_t, \tag{6}$$

$$c_{m,t+1} + a_{m,t+1} = w_{t+1} + r_{t+1} a_{y,t}, \tag{7}$$

$$c_{o,t+2} = r_{t+2} a_{m,t+1}, \tag{8}$$

where $a_{y,t}$ and $a_{m,t+1}$ denote the amounts of assets held by a generation-t agent when she is young and middle-aged, respectively. Maximizing (5) subject to (6)-(8) yields:

$$c_{m,t+1} = (1-z)\left(r_{t+1} w_t + w_{t+1}\right), \tag{9}$$

$$a_{m,t+1} = z\left(r_{t+1} w_t + w_{t+1}\right), \tag{10}$$

where $z \equiv \beta/(1+\beta)$ denotes the propensity to save by the middle-aged, which is a key parameter in the following analysis.

[7] Using $N_t = \frac{1}{1+\lambda}\left(1 + (-1)^t \lambda^{t+1}\right)$, the total labor force is expressed as $L_t = \frac{1}{1+\lambda}\left(2 + (-1)^{t-1} \lambda^t (1-\lambda)\right)$ for $t \geq 1$. In period 0, it is given by $2 - \lambda$. It can be shown analytically that L_t is a decreasing function with respect to λ when $0 \leq \lambda \leq 1$. Similarly, the total population, $\sum_{j=-1}^{2} N_{t-j}$, also decreases as λ increases.

[8] For simplicity, we do not explicitly consider consumption in childhood and youth as the main results are not qualitatively affected. We also ignore the utility from and the costs of having children. See, for example, Tamura (2006) for the fertility decision through utility maximization.

Observe that in period t, aggregate savings consist of the asset holdings of young agents, $a_{y,t}N_t$, and the assets held by the middle-aged, $a_{m,t}N_{t-1}$. These aggregate savings, denoted by S_t, become the capital stock in the next period. From (6) and (10), this means that the capital stock in period $t+1$, denoted by K_{t+1}, is determined as:

$$K_{t+1} = S_t = a_{y,t}N_t + a_{m,t}N_{t-1} = w_t N_t + z(r_t w_{t-1} + w_t)N_{t-1}. \tag{11}$$

Goods are produced competitively by a representative firm using labor and the capital stock. The aggregate amount of production is given by a standard Cobb–Douglas production function $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$, where parameter $A > 0$ is total factor productivity, whereas parameter $\alpha \in (0,1)$ represents the share of capital. The production function can be expressed in terms of per-worker values as $y_t = Ak_t^{\alpha}$, where $y_t \equiv Y_t/L_t$ is output per worker and $k_t \equiv K_t/L_t$ is the capital–labor ratio. Denoting the capital depreciation rate by $\delta \in [0,1]$, the profit-maximizing condition for the firm implies that the factor prices in equilibrium are:

$$r_t = A\alpha k_t^{\alpha-1} + 1 - \delta \equiv r\left(k_t\right), \tag{12}$$

$$w_t = A\left(1-\alpha\right)k_t^{\alpha} \equiv w\left(k_t\right). \tag{13}$$

Substituting these factor prices into (11) gives the evolution of per-worker capital over generations:

$$k_{t+1} = A\left(1-\alpha\right)\frac{N_t k_t^{\alpha} + zN_{t-1}\left[\left(A\alpha k_t^{\alpha-1} + 1 - \delta\right)k_{t-1}^{\alpha} + k_t^{\alpha}\right]}{N_{t+1} + N_t}, \tag{14}$$

where we used the fact that $k_{t+1} = K_{t+1}/L_{t+1} = K_{t+1}/(N_{t+1}+N_t)$. Recalling that the timing of childbirth λ_t for all t is given by (2), equation (1) and initial condition $N_0 = 1$ determines the demographic dynamics N_t for all t. Then, given the path of N_t and the initial two values of capital, k_0 and k_{-1}, (14) determines the dynamic path of the capital–labor ratio k_t for all t.

3 Dynamic Effects of Delayed Childbearing

In the following, we investigate the dynamic effects of delayed childbearing on capital accumulation and welfare. Throughout this section, we focus on the polar case of $\lambda = 1$, where

	period -1	period 0	period 1	even periods $t = 2, 4, \dots$	odd periods $t = 3, 5, \dots$
Old	1	1	1	1	0
Middle-Aged (worker)	*1*	*1*	*1*	*0*	*1*
Young (worker)	*1*	*1*	*0*	*1*	*0*
Child	1	0	1	0	1

Table 1: Evolution of Demographic Structure when $\lambda = 1$. Numbers in italic indicate the cohorts in the labor force

all agents beginning from generation 0 bear children in middle age. Although this case is not very plausible, it allows us to analytically explain the effect of delaying childbearing in a simple way. We also assume full capital deprecation ($\delta = 1$) in this section. The general case with $\lambda,\ \delta \in (0, 1)$ will be numerically investigated in the next section.

3.1 Equilibrium Path When All Agents Delay Childbearing ($\lambda = 1$)

When $\lambda = 1$, the demographic dynamics (1) simplify to $N_{t+1} = N_{t-1}$ for all $t \geq 1$. Substituting $N_0 = 1$ and $N_1 = 0$ from (3) into this equation, it turns out that $N_t = 1$ for all even t and $N_t = 0$ for all odd t. Table 1 describes the implied demographic structure at each point in time. Note that the whole labor force consists only of young workers in even periods, and only of middle-aged workers in odd periods.[9]

With the path of N_t, we can derive the equilibrium path of the capital–labor ratio k_t, given the initial k_0 and k_{-1} values. Substituting $N_0 = N_{-1} = 1$ and $N_1 = 0$ into (14) for

[9]Of course, this is an extreme possibility: young and middle-aged workers would coexist if $\lambda \in (0, 1)$. However, the important point is that the composition of young and middle-aged workers in the labor force fluctuates, which is still true for $\lambda \in (0, 1)$. Observe from the demographic dynamics illustrated by Figure 1 that the young workers are the majority (i.e., $N_t > N_{t-1}$) in the even periods, whereas the middle-aged workers are the majority (i.e., $N_t < N_{t-1}$) in the odd periods.

$t = 0$, we obtain the capital–labor ratio in period 1:[10]

$$k_1 = A(1-\alpha)\left[(1+z)k_0^\alpha + zA\alpha k_0^{\alpha-1}k_{-1}^\alpha\right]. \tag{15}$$

For k_2 and onwards, substituting $\{N_0, N_1, N_2, N_3, \cdots\} = \{1, 0, 1, 0, \cdots\}$ into (14) gives:

$$\text{for } t \geq 1, \quad k_{t+1} = \begin{cases} A\,(1-\alpha)\,k_t^\alpha & \text{if } t \text{ is even,} \\ A\,(1-\alpha)\,z\left\{k_t^\alpha + A\alpha k_{t-1}^\alpha k_t^{\alpha-1}\right\} & \text{if } t \text{ is odd.} \end{cases} \tag{16}$$

This pattern of dynamics can be intuitively interpreted in terms of the aggregate saving rate (adjusted for labor force growth), defined by:

$$v_t \equiv \frac{S_t}{Y_t} \Bigg/ \frac{L_{t+1}}{L_t} = \frac{K_{t+1}/L_{t+1}}{Y_t/L_t} = \frac{k_{t+1}}{Ak_t^\alpha}. \tag{17}$$

As labor force L_t is constant at 1 for all $t \geq 1$ (see Figure 2),[11] v_t simply represents the aggregate saving rate for $t \geq 1$.

Using this definition, the first line of equation (16) can be restated as $v_t = 1-\alpha$. In even periods, young agents are the sole workers, and thus they earn the labor share of output, $(1-\alpha)Y_t$. At the same time, they are also the sole savers in even periods, and because they save their income entirely, aggregate savings coincide with their income, $(1-\alpha)Y_t$. Therefore, in even periods, the aggregate saving rate v_t is determined by the labor share of the production, $1-\alpha$.

For odd periods, the second line of equation (16) can be restated as $v_t = (1-\alpha)\,z\,(1+\alpha/v_{t-1})$. Note that v_{t-1} in this equation refers to the aggregate saving rate in even periods, which is $1-\alpha$ as shown above. By substituting $v_{t-1} = 1-\alpha$ into the above equation, it simplifies to $v_t = z$. In odd periods ($t \geq 3$), the middle-aged are the only workers. In addition, the capital used in odd periods is owned solely by the middle-aged, because they are the only savers in the previous period (when they were young in even periods). Therefore, they earn the entire output Y_t. The middle-aged are the sole savers in odd periods, and they save fraction z of

[10] Recall also that we have assumed $\delta = 1$ for this section.

[11] When $\lambda \in (0, 1)$, labor force L_t does fluctuate in transition dynamics. However, comparing Figure 2 and Figure 1 shows that the fluctuations in labor force L_t are much smaller than in the demographic dynamics of N_t.

their income. Therefore, the aggregate saving rate v_t coincides with their saving propensity, z.

To summarize, the aggregate saving rate v_t exhibits a two-period cycle after period 2:[12]

$$\text{for } t \geq 2, \quad v_t = \begin{cases} 1-\alpha & \text{if } t \text{ is even,} \\ z & \text{if } t \text{ is odd.} \end{cases} \tag{18}$$

Note that either the saving rate in even periods $1-\alpha$, or that in odd periods z, could be larger. On one hand, young workers have a high saving propensity (unity), but they save only out of labor income (w_t). On the other hand, middle-aged workers earn both labor and capital income ($w_t + r_t w_{t-1}$), but their saving propensity is lower ($z < 1$).[13]

Using the values of v_t in (18), we can derive the sequence of k_t. Note that (17) implies a simple relationship between the aggregate saving rate v_t and the evolution of the capital–labor ratio k_t:

$$k_{t+1} = v_t A k_t^{\alpha}. \tag{19}$$

Taking the logs of (19) and applying this recursively, we obtain:

$$\log k_t = \left(\sum_{j=0}^{t-3} \alpha^j \right) \log A + \left(\sum_{j=0}^{t-3} \alpha^j v_{t-1-j} \right) + \alpha^{t-2} \log k_2 \quad \text{for } t \geq 3, \tag{20}$$

where $k_2 = A(1-\alpha)z \left\{ k_1^{\alpha} + A\alpha k_0^{\alpha} k_1^{\alpha-1} \right\}$ from (16), k_1 is given by (15), and k_0 (and k_{-1}) is given as the initial value. This equilibrium path has the following property.

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[12]There is no cycle in the knife-edge case of $1-\alpha = z$. For completeness, v_0 is obtained by $v_0 = k_1 / \left(A k_0^{\alpha}\right)$, where k_0 is a part of the initial condition and k_1 is given by (16). The level of v_1 is then obtained by $v_1 = (1-\alpha)\, z \,(1 + \alpha / v_0)$.

[13]From the Family Income and Expenditure Survey for wage-earning households with two or more persons in Japan, we confirmed that the average saving rate (1− the average propensity to consume) tends to fall with the age of the household head, from 32.0% (thirties) to 28.8% (forties) to 25.4% (fifties) and then to 11.3% (sixties) using 2000–2010 data. While some other reports find flat or rising age-saving profiles (even after the retirement age), Jappelli and Modigliani (2005) pointed out that these are because contributions to pension funds (including employers' contribution) are not regarded as savings, and also because pension incomes are treated as income although they should be regarded as dissavings. They estimated the effects of social security on the age-saving profile in Italy, which showed that actual savings are highest when the household head is in his/her late thirties and then falls to zero around age 60.

Proposition 1 ***(Limit cycles when all agents delay childbearing):***
In the equilibrium with $\lambda = 1$, $\{k_t\}_{t=0}^{\infty}$ converges to a two-period limit cycle regardless of the initial values. Define $k^*_{even} \equiv \lim_{s\to+\infty} k_{2s}$ and $k^*_{odd} \equiv \lim_{s\to+\infty} k_{2s+1}$, where s is an integer. Depending on $z \equiv \beta/(1+\beta)$, the relative magnitude of k^*_{even} and k^*_{odd} is:
(Case I) if $z < 1-\alpha$, $k^*_{even} < k^*_{odd}$ holds.
(Case II) if $z > 1-\alpha$, $k^*_{even} > k^*_{odd}$ holds.[14]

Proof: As $t \to \infty$, the first term of (20) converges to $(1-\alpha)^{-1}\log A$, whereas the third term vanishes because $\alpha \in (0,1)$. When t is even (i.e., when $t = 2s$ for some integer s), from (18), the second term is expanded as $\log z + \alpha \log(1-\alpha) + \alpha^3 \log z + \alpha^4 \log(1-\alpha) + \cdots$, which converges to $(1-\alpha)^{-1}\log V_{even}(z)$, where:

$$V_{even}(z) \equiv \left[(1-\alpha)^{\alpha} z\right]^{\frac{1}{1+\alpha}} \tag{21}$$

is a geometric weighted average of the aggregate saving rate v_t.[15] Similarly, when t is odd (i.e., when $t = 2s+1$), the second term is expanded as $\log(1-\alpha) + \alpha \log z + \alpha^3 \log(1-\alpha) + \alpha^4 \log z + \cdots$, which converges to $(1-\alpha)^{-1}\log V_{odd}(z)$, where:

$$V_{odd}(z) \equiv \left[(1-\alpha) z^{\alpha}\right]^{\frac{1}{1+\alpha}}. \tag{22}$$

From these, we conclude that the values of k_t in even and odd periods, respectively, converge to:

$$\lim_{s\to\infty} \log k_{2s} = \log k^*_{even} = \frac{1}{1-\alpha}\left[\log V_{even}(z) + \log A\right], \tag{23}$$

$$\lim_{s\to\infty} \log k_{2s+1} = \log k^*_{odd} = \frac{1}{1-\alpha}\left[\log V_{odd}(z) + \log A\right]. \tag{24}$$

Note that $V_{even}(z) < V_{odd}(z)$ holds if $z < 1-\alpha$ (Case I), whereas the opposite holds if $z > 1-\alpha$ (Case II). Therefore, $k^*_{even} < k^*_{odd}$ holds if and only if $z < 1-\alpha$. ■

[14] Although the condition for Case II ($z > 1-\alpha$) might seem unlikely to hold, this is only because of the simplifying assumption of complete capital depreciation ($\delta = 1$). In Section 4, we show that a lower δ makes Case II more likely, and it is shown that both cases happen within a reasonable parameter range (see also footnote 24).

[15] Observe that $V_{even}(z)$ in (21) puts a higher weight on z than on $(1-\alpha)$ because in even periods the most recent aggregate saving rate is $v_{2s-1} = z$. Similarly, $V_{odd}(z)$ in (22) puts a higher weight on $(1-\alpha)$.

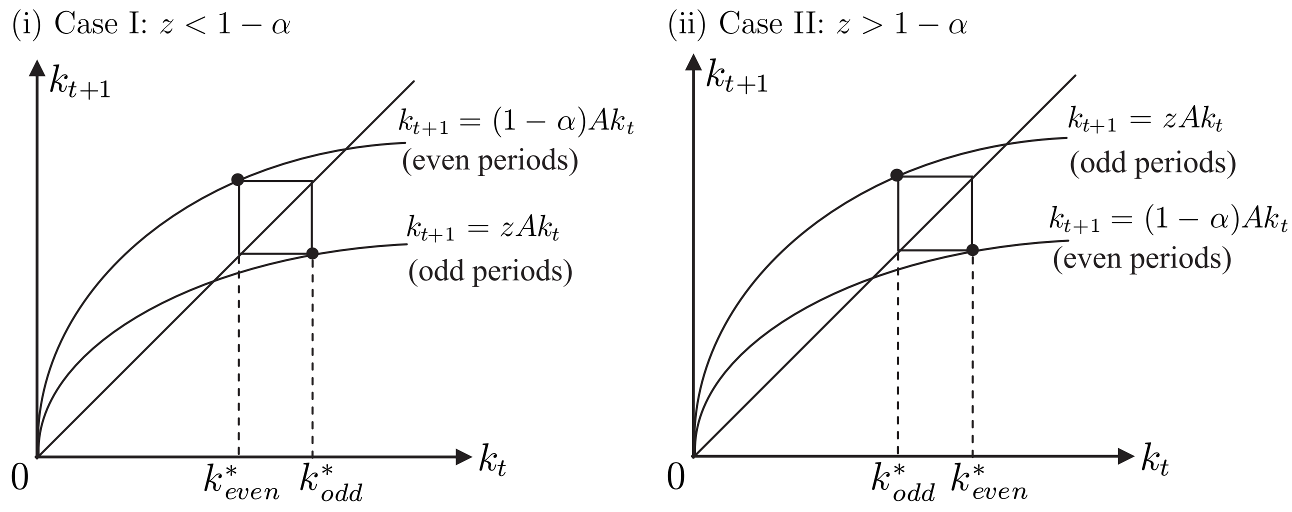

Figure 3: Limit Cycles in k_t with Alternating Aggregate Saving Rate v_t

Proposition 1 states that if all agents from period 0 delay childbearing, the capital–labor ratio k_t eventually converges to a two-period limit cycle. This fluctuation is driven not by the size of the labor force (which is constant), but by the age distribution within it, through the fluctuations in the aggregate saving rate v_t. Figure 3 illustrates the limit cycles for the two cases, where the two loci are drawn by substituting $1 - \alpha$ (even periods) and z (odd periods) for v_t in the capital accumulation equation (19). Panel (i) shows that in Case I ($z < 1 - \alpha$), the aggregate saving rate is higher in even periods, which results in a higher capital stock in odd periods. Conversely, panel (ii) depicts that the higher saving rate in odd periods results in the higher capital stock in even periods in Case II ($z > 1 - \alpha$).

3.2 Effects on Capital Accumulation

As we have seen in Figure 2, delayed childbearing lowers the labor force permanently. This subsection examines how this affects capital accumulation in the economy by comparing the capital–labor ratio in the limit cycles to the economy without delayed childbearing.

Note that without delayed childbearing (i.e., when $\lambda = 0$), $N_t = 1$ holds for all t from (1) and the initial condition $N_0 = 1$. By substituting $N_{t-1} = N_t = N_{t+1} = 1$ and $\delta = 1$ into the capital accumulation equation (14), and rewriting the resulting equation using $v_t \equiv k_{t+1}/(Ak_t^\alpha)$, we obtain the evolution of the aggregate saving rate v_t for the case of

$\lambda = 0$. From it, we find that the steady state level of v_t is a (positive) solution to a quadratic equation $\xi(v) \equiv 2v^2 - (1-\alpha)(1+z)v - \alpha(1-\alpha)z = 0$, which we obtain as:

$$v^*(z) \equiv \frac{1}{4}\left\{(1-\alpha)(1+z) + \sqrt{(1-\alpha)^2(1+z)^2 + 8\alpha(1-\alpha)z}\right\}. \tag{25}$$

As (20) holds for any λ, we obtain the steady state capital–labor ratio k^* for $\lambda = 0$ by substituting (25) into (20):

$$\log k^* = \frac{1}{1-\alpha}\left[\log v^*(z) + \log A\right]. \tag{26}$$

It is apparent from (23), (24) and (26) that the relative magnitudes of the capital–labor ratios, k^*_{odd}, k^*_{even}, and k^*, can be obtained by comparing $V_{odd}(z)$, $V_{even}(z)$, and $v^*(z)$. To focus on the relevant situation, we assume that the share of capital is not too high: $\alpha < \left(\sqrt{5}-1\right)/2 \approx 0.618$. With this assumption, we can show the following property.

Lemma 1 ***(Comparison of*** $V_{odd}(z)$ ***and*** $V_{even}(z)$ ***to*** $v^*(z)$***):***
(i) At $z = 1-\alpha$, $V_{odd}(z) = V_{even}(z) = v^*(z) = 1-\alpha$ *holds.*
(ii) There exist $\widehat{z} \in (0, 1-\alpha)$ *such that* $V_{odd}(\widehat{z}) = v^*(\widehat{z})$ *holds.* $V_{odd}(z) > v^*(z)$ *holds if and only if* $z \in (\widehat{z}, 1-\alpha)$.
(iii) $V_{even}(z) > v^*(z)$ *holds if and only if* $z > 1-\alpha$.

Proof: Property (i) is immediately confirmed by comparing (21), (22), and (25) at $z = 1-\alpha$. The proofs of (ii) and (iii) are given in Appendix A. ■

As summarized in Table 2, Proposition 1 and Lemma 1 imply three possibilities regarding the relative magnitudes of k^*_{odd}, k^*_{even}, and k^*:[16]

Proposition 2 ***(Comparison of capital–labor ratios between the limit cycle at*** $\lambda = 1$ ***and the steady state at*** $\lambda = 0$***):***
(Case Ia) If $z < (0, \widehat{z})$, $k^*_{even} < k^*_{odd} < k^*$ *holds.*
(Case Ib) If $z < (\widehat{z}, 1-\alpha)$, $k^*_{even} < k^* < k^*_{odd}$ *holds.*
(Case II) If $z > 1-\alpha$, $k^*_{odd} < k^* < k^*_{even}$ *holds.*

[16] It can also be shown that if $z = 1-\alpha$ (i.e., when capital does not fluctuate), $k^*_{even} = k^*_{odd} = k^*$ holds. In addition, if $z = \widehat{z}$, $k^*_{even} < k^*_{odd} = k^*$ holds. We ignore these border cases because they do not occur except for a (measure 0) coincidence.

$\widehat{z}$ $\quad$ $1-\alpha$ $\quad$ z

<table>
<tr><td colspan="2">Case I: $k^*_{even} < k^*_{odd}$</td><td>Case II: $k^*_{odd} < k^*_{even}$</td></tr>
<tr><td>$k^*_{odd} < k^*$</td><td>$k^* < k^*_{odd}$</td><td>$k^*_{odd} < k^*$</td></tr>
<tr><td colspan="2">$k^*_{even} < k^*$</td><td>$k^* < k^*_{even}$</td></tr>
<tr><td>Case Ia: $k^*_{even} < k^*_{odd} < k^*$</td><td>Case Ib: $k^*_{even} < k^* < k^*_{odd}$</td><td>Case II: $k^*_{odd} < k^* < k^*_{even}$</td></tr>
</table>

$k^*_{even} = k^*_{odd} = k^*$

Table 2: Derivation of Proposition 2 (shown in the bottom row). The first row is from Proposition 1. The second and third rows are from Lemma 1(ii) and (iii).

Observe that the lower end of the limit cycle ($\min\{k^*_{odd}, k^*_{even}\}$) is always smaller than the steady state level k^* in the economy without delayed childbearing (which we call the benchmark economy). In addition, if z is sufficiently small ($z < \widehat{z}$), the upper end of the limit cycle can also be smaller than k^*. This means that the long-term levels of the capital–labor ratio k_t in the delayed childbearing economy are always smaller than the steady state capital–labor ratio k^* in the benchmark economy. This might seem paradoxical, given that in the delayed childbearing economy, the labor force remains low compared with the benchmark economy (compare $\lambda = 1$ to $\lambda = 0$ in Figure 2). This paradoxical result can be explained by the alternating age composition in the labor force. Recall from (18) that the aggregate saving rate alternates between $1 - \alpha$ and z. In Case Ia, the saving propensity of the middle-aged agents, $z \equiv \beta/(1+\beta)$, is small. Thus, in odd periods, when the labor force is entirely composed of middle-aged agents, the aggregate saving rate $v_t = z$ is low. This makes capital per worker in the next period (k^*_{even}) considerably smaller than in the benchmark (k^*), and therefore also the wage rate. As a result, workers in even periods, who are composed of young agents, receive substantially lower incomes than in the benchmark economy. Thus, even though the aggregate saving rate in even periods $v_t = 1 - \alpha$ is higher than that in the benchmark economy ($v^*(z)$), the amount of aggregate savings can be lower, which explains the possibility of $k^*_{even} < k^*_{odd} < k^*$.[17]

[17] One may then wonder why $k^* < k^*_{odd} < k^*_{even}$ never occurs in Case II. As we assumed α to be lower than

3.3 Welfare Effects

We now examine how the cycles in the capital–labor ratio in the delayed childbearing economy affect the welfare of agents. Note that, by substituting (8), (9), and (10) into (5), the utility of generation-t agents (those who are born in period $t-1$) is written as:

$$U_t = \beta \log z + \log(1-z) + (1+\beta)\log\left(r_{t+1}w_t + w_{t+1}\right) + \beta \log r_{t+2}. \tag{27}$$

Let us call those agents born in odd periods and thus young in even periods the "even-period generations." In the delayed childbearing economy ($\lambda = 1$), the whole population is composed only of the even-period generations ($N_t = 0$ for all odd t). Therefore, the long-term welfare of agents in the limit cycle can be measured by $U^*_{even} \equiv \lim_{s\to+\infty} U_{2s}$. Using the limit-cycle values of the capital–labor ratio, we can write long-term welfare with $\lambda = 1$ as:

$$U^*_{even} = (1+\beta)\log[A\alpha\left(k^*_{odd}\right)^{\alpha-1}\left(k^*_{even}\right)^{\alpha} + \left(k^*_{odd}\right)^{\alpha}] - \beta\left(1-\alpha\right)\log k^*_{even} + C, \tag{28}$$

where C is a constant term defined as $C \equiv \beta\log\beta - (1+\beta)\log(1+\beta) + \beta\log A\alpha + (1+\beta)\log A\left(1-\alpha\right)$. Similarly, long-term welfare in the benchmark economy ($\lambda = 0$) can be written as:

$$U^* = (1+\beta)\log[A\alpha\left(k^*\right)^{2\alpha-1} + \left(k^*\right)^{\alpha}] - \beta\left(1-\alpha\right)\log k^* + C. \tag{29}$$

Comparing (28) with (29), we have the following property.

Lemma 2 ***(Difference between*** U^*_{even} ***and*** U^****):***
(i) U^*_{even} *is lower than* U^* *if and only if* $\Omega(z) < 0$, *where function* $\Omega(z)$ *is defined by:*

$$\Omega\left(z\right) \equiv -\log\left(1-\alpha\right)\left[\frac{\alpha}{v^*\left(z\right)} + 1\right] - \frac{\alpha}{1-\alpha}\log\frac{v^*\left(z\right)}{V_{odd}\left(z\right)} + z\log\frac{v^*\left(z\right)}{V_{even}\left(z\right)}. \tag{30}$$

(ii) $\lim_{z\to 0}\Omega\left(z\right) = -\infty$ *and* $\Omega(1-\alpha) = 0$ *hold.*

$\left(\sqrt{5}-1\right)/2$, the wage equation $w_t = A(1-\alpha)k_t^{\alpha}$ has a certain degree of concavity with respect to k_t. This concavity implies that, while a negative deviation k^*_{even} from k^* in Case I results in a substantial drop in the wage income, a positive deviation of k^*_{even} from k^* in Case II results in a relatively small wage increase. Therefore, k^*_{odd} does not exceed k^* in Case II.

Proof: Given in Appendix B. ■

As function $\Omega(z)$ is continuous, Lemma 2 implies that when z is sufficiently close to 0, $\Omega(z)$ must be negative, and hence $U^*_{even} < U^*$ holds. The next proposition states this result.

Proposition 3 ***(Welfare comparison between*** $\lambda = 1$ ***and*** $\lambda = 0$***):***
There exists a value $\widetilde{z} \in (0, 1-\alpha]$ *such that* $U^*_{even} < U^*$ *holds whenever* $z < \widetilde{z}$*.*

As long as the saving propensity of the middle-aged, $z \equiv \beta/(1+\beta)$, is sufficiently small, or equivalently when the agents discount the future significantly (i.e., β is small), the delayed childbearing ($\lambda = 1$) causes the long-run welfare of agents to fall compared with the case where delayed childbearing does not occur ($\lambda = 0$). This again seems paradoxical, because when the population falls from the initial level, it is usually anticipated that each agent enjoys a higher per-worker capital and hence higher consumption. This does not hold true in this case, similar to the discussion in the previous subsection, because of the fluctuations in the age composition of workers.

4 Numerical Analysis

This section considers a general case where only a fraction of agents delay childbearing. When $\lambda \in (0, 1)$, the fluctuations in N_t gradually settle to a long-term value (see Figure 1). Nonetheless, the fluctuations in N_t continue for an extended number of generations, especially when λ is relatively large.[18] This section examines their effects on capital accumulation and welfare in the transitional dynamics. We also relax the assumption of complete capital depreciation.

4.1 Equilibrium Dynamics under $\lambda \in (0, 1)$

For a given value of λ, the path of N_t is readily calculated as depicted in Figure 1 using (1) and (2) along with initial condition $N_0 = 1$. As $N_t = 1$ for all $t \leq 0$, we reasonably assume that the economy has reached the steady state under $N_t = 1$ by period -1, and also

[18] For example, if 80% of agents delay childbearing ($\lambda = 0.8$), it can be seen that substantial fluctuations in N_t remain even after 10 generations (around 200 years).

remains at the same steady state at period 0.[19] We previously calculated this steady state in Subsection 3.2 as the benchmark case, where the steady state level of the capital–labor ratio k^* is given by (26). Thus, we use $k_{-1} = k_{-0} = k^*$ as the initial condition to calculate the path of k_t using (14).

We specify the parameters as follows. As an agent lives for four periods, one period in the model can be considered as approximately 20 years. If agents discount future consumption by 1% per quarter, as is often assumed in the literature, the discount factor β will be $(1+0.01)^{-4\times 20} \approx 0.45$. Therefore, we take $\beta = 0.45$ as the reference value, and also examine the low-beta ($\beta = 0.1$) and the high-beta ($\beta = 0.9$) cases. For the depreciation parameter δ, Nadiri and Prucha (1996) estimated a yearly depreciation rate for the physical capital stock of 5.9%, and 1.2% for the R&D capital stock. The capital stock K_t in our model includes both physical and R&D capital stocks, but these estimates suggest that a good fraction of the aggregate capital stock that remains after 20 years would be R&D capital. Therefore we use a yearly depreciation rate of 2% as a reference (which means $\delta = 0.33$ for a period of 20 years), and also examine the case of a higher depreciation rate of 5% per year ($\delta = 0.64$). The share of capital α is set to 0.4.[20]

Figure 4 shows the equilibrium paths of k_t for $\beta = 0.1$, 0.45, and 0.9, respectively, and also for $\delta = 0.33$ and 0.64. Each panel depicts 10 paths of k_t, where each path corresponds to the cases of $\lambda = 0.1, 0.2, \ldots, 0.9$, and 1. In period 1, the labor force falls from 2 to $2 - \lambda$ because fraction λ of parents in the previous period decided to delay childbearing, and hence there are only $1 - \lambda$ young workers in this period. Note also that the aggregate capital stock is the same as in the initial steady state, because it is determined by the aggregate savings in the previous period.[21] Therefore, the initial response of the capital–labor ratio is always

[19] Note that, even though λ_t jumps up from 0 to $\lambda > 0$ in period 0, the population is not immediately affected, nor is the capital–labor ratio, because the fertility in period 0 determines the amount of labor supplied in period 1 and beyond.

[20] As we do not distinguish between physical and human capital, the share of K_t, α, should be higher than the conventionally measured share of physical capital. Thus, we choose $\alpha = 0.4$, although the value of α does not substantially change the pattern of the dynamics. The scaling parameter A is set to 1.5. Under these parameter values, we confirmed that dynamic efficiency $r_t > 1$ is always satisfied at the steady state.

[21] This result depends on the logarithmic period utility function, which implies that the savings of agents

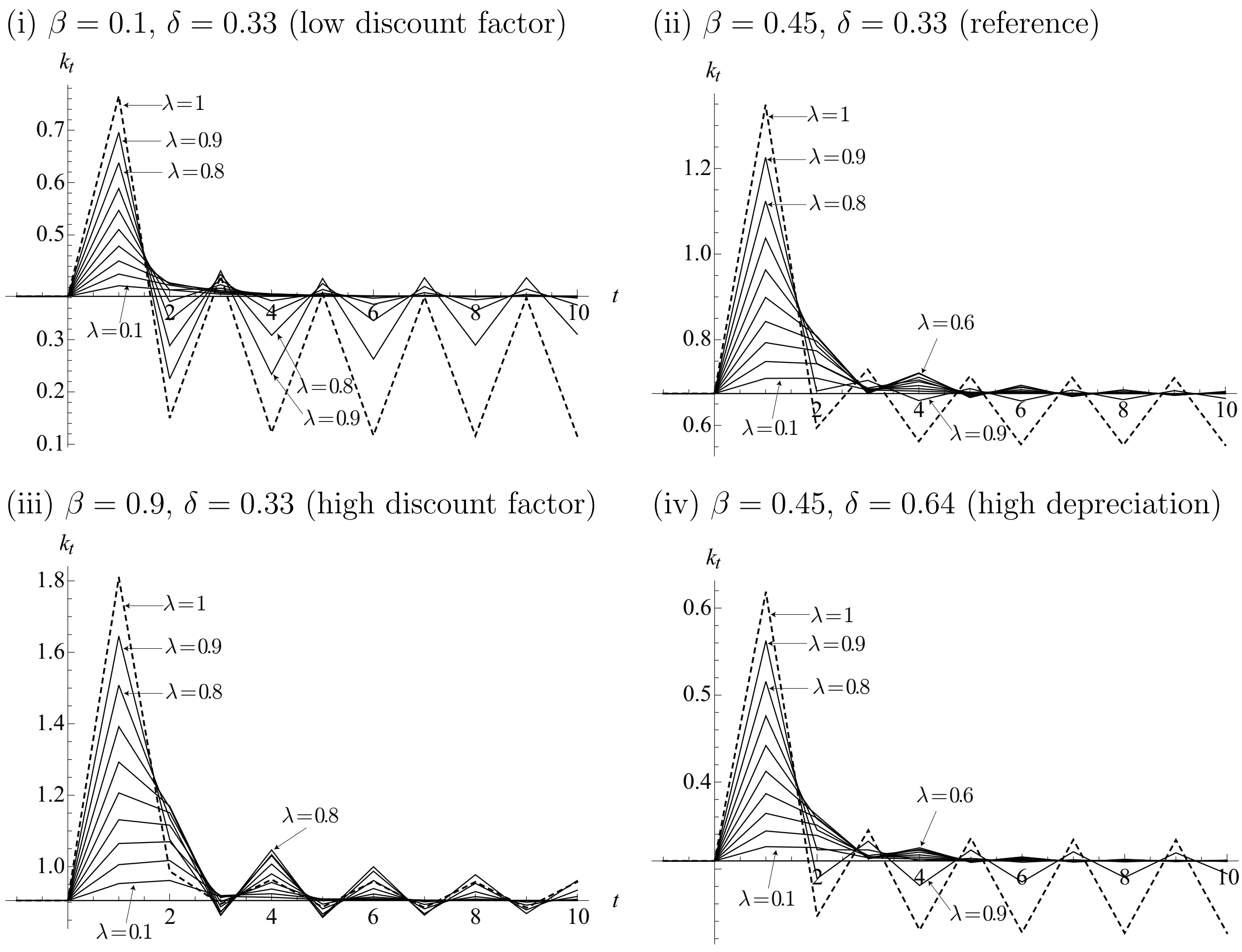

Figure 4: Evolution of Capital–Labor Ratio k_t

positive, and $k_1 = (2/(2-\lambda))\, k^*$ is higher when λ is higher.

Observe from Figure 2 that the labor force falls further in period 2 (except for the case of $\lambda = 1$, where L_t falls to the bottom only in one period). At the same time, however, the aggregate capital stock is also lower than the initial steady state, because there were fewer young workers in the previous period ($N_1 = 1 - \lambda$) who contributed to aggregate savings. Figure 4 shows that the second effect dominates, and the size of the fall in k_t at $t = 2$ is

do not depend on the interest rate. If the agents are more risk averse (i.e., if the intertemporal elasticity of substitution is lower than unity), the middle-aged agents in period 0 would somewhat increase savings, because they know that the interest rate in period 1 will be lower because of the reduced labor supply, and would want to supplement old-age consumption by saving more. Therefore, the magnitude of the initial fluctuations will be larger than shown in this paper.

larger when λ is larger. In addition, when β is small (i.e., when the saving propensity of the middle-aged $z = \beta/(1+\beta)$ is small), the major portion of the aggregate savings depends on the savings by the young workers. Therefore, with large λ and small β, the fall in aggregate savings in $t = 2$ is so large that k_2 falls below (or "overshoots") the initial capital–labor ratio k^*.[22]

The pattern of dynamics after period 3 depends both on λ and β. When only a small fraction of parents delay childbearing, the fluctuations in cohort size N_t disappear in a relatively short period of time. Therefore, with small λ, k_t settles to the steady state value k^* relatively quickly, without cycles. If λ is relatively large, two-period cycles in k_t are present, which last for many generations. The pattern of the cycles is comparable to the results we obtained in Proposition 2. Figure 4(i) shows that when β is small, the capital–labor ratio k_t is smaller in even periods than in odd periods, which corresponds to Case I (a and b) in Proposition 2. In particular, the values of k_t in even periods are far below the steady state value k^*, whereas in odd periods they are barely above k^* (except for the case of $\lambda = 1$, where k_t in odd periods is also smaller than k^*, as we mentioned in Case Ia in Proposition 2). This asymmetry can be understood in terms of the reason why both k^*_{even} and k^*_{odd} can be lower than k^* when z is small, which we discussed in Subsection 3.2.

Figure 4(iii) shows that the pattern of the cycle is opposite when β is large. The capital–labor ratio is larger in even periods than in odd periods, similar to Case II in Proposition 2. When β is at an intermediate value ($\beta = 0.45$), Figure 4(ii) suggests that the pattern is similar to Case I if λ is large, whereas it is similar to Case II if λ is intermediate (and no cycle if λ is small). Finally, Figure 4(iv) illustrates that a higher δ shifts the entire path of the capital–labor ratio k_t downwards, but the effect of δ on the pattern of the fluctuations is not clear from this figure.

To show the dependence of the pattern of cycles on parameter values more explicitly, we experimented with 40000 combinations of λ and β by varying each of them from 0.005 to 1.00 in 200 steps, and we repeated this for two values of δ. We calculated the dynamic path of k_t for each combination of parameters until period 10, and then classified the result

[22] In Figure 4, it can be observed that $k_2 < k^*$ occurs when $\beta = 0.1$ and $\lambda \geq 0.6$, and also when $\beta = 0.45$ and $\lambda \geq 0.9$.

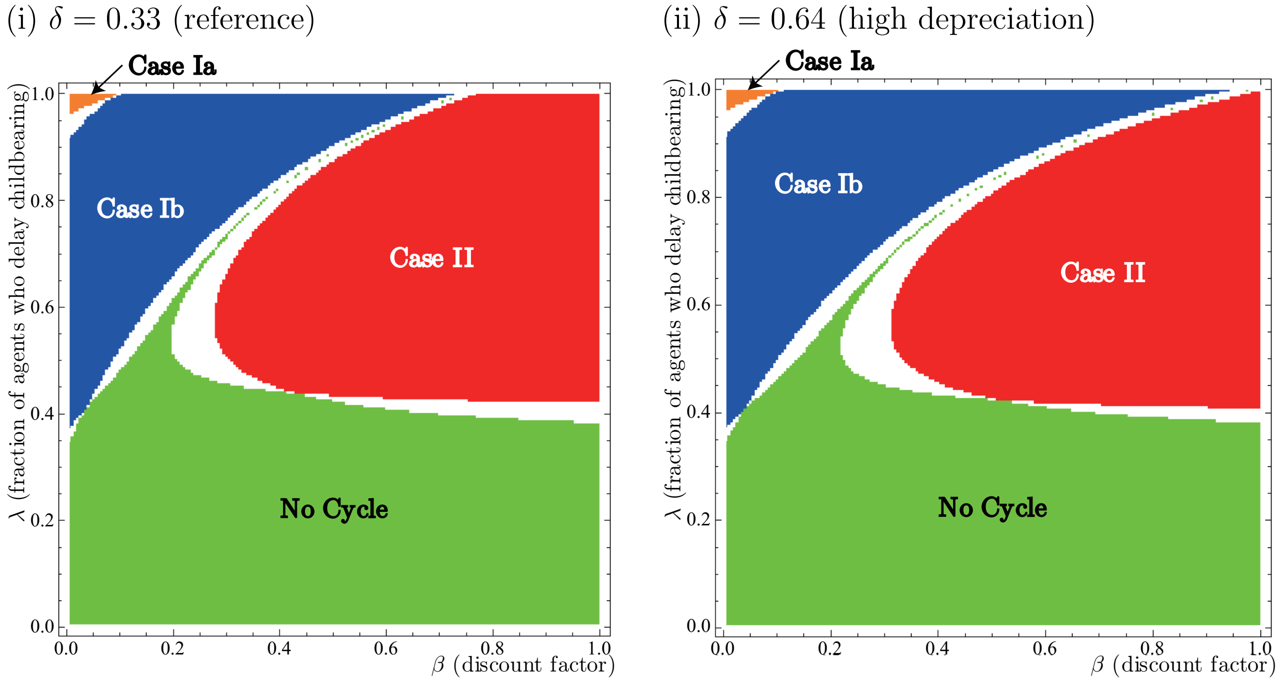

Figure 5: Pattern of Cycles in k_t

according to the pattern of movements, based on that in Proposition 2. The phase diagrams depicted in Figure 5 summarize the result. When the combination of λ and β belongs to the area labeled as Case Ib, we find [k_t in even periods] $<$ k^* $<$ [k_t in odd periods] holds for all $t > 3$, whereas we find [k_t in even periods] $<$ [k_t in odd periods] $<$ k^* in the small area labeled as Case Ia.[23] Similarly, in the area labeled as Case II, [k_t in odd periods] $<$ k^* $<$ [k_t in even periods] holds. In the area "No cycle," $k_t > k^*$ holds for all $t > 3$. The remaining white areas correspond to the border cases where the movements of k_t do not fit exactly any of the above patterns (e.g., when cycles are present until a certain period but disappear before period $t = 10$).

Figure 5 confirms that cycles in the capital–labor ratio emerge when a certain fraction

[23] As explained in the text, we classify the pattern of the dynamics according to the level of k_t relative to the steady state value k^*. An alternative method of classification is to focus on the first difference of the capital–labor ratio, $k_t - k_{t-1}$, and examine if it is greater (or less) than zero. This calculation shows that the resulting phase diagram is almost identical to Figure 5. The sign of $k_t - k_{t-1}$ is positive only in odd periods in the area labeled as Case Ia and Ib, and the opposite holds in Case II. The sign of $k_t - k_{t-1}$ is negative for all $t > 3$ in the "No cycle" area because k_t monotonically falls to the steady state level.

(around 0.4) of agents delay childbearing. When cycles emerge, the capital–labor ratio is higher in odd periods if the discount factor β (or equivalently the propensity to save z) is small, and vice versa. Observe also that the border between Case Ib and Case II bends toward the right as λ increases. Thus, for a given intermediate β, the pattern of cycles can be reversed depending on the fraction of agents who delay childbearing (λ). In addition, comparing panels (i) and (ii) in Figure 5 shows that a higher depreciation rate δ shifts the border to the right. Intuitively, when δ is higher, the gross interest rate falls, which reduces the income of the middle-aged agents. This lowers aggregate savings in odd periods (when the middle-aged workers are the majority in the labor force), and in turn reduces the capital stock in even periods, making Case Ib more likely.[24] Finally, observe that Case Ia is obtained under a reasonable depreciation rate, although it occurs only when β is very small (i.e., when agents discount the future quite significantly) and λ is close to one (i.e., when almost everyone delays childbearing).

4.2 Welfare Analysis under $\lambda \in (0, 1)$

While we examined U_t only for even-period generations in Subsection 3.3, here we examine U_t for both even- and odd-period generations because $\lambda \in (0, 1)$ implies that $N_t > 0$ for all generations t. By substituting the path of k_t into (12) and (13), we obtain factor prices, r_t and w_t, on the equilibrium path. Then, substituting these into (27) gives the welfare U_t for all generations. Similar to Figure 5, we calculated 80000 paths of U_t by varying β, λ, and δ, and classified the pattern of evolution of U_t according to when U_t is above (or below) the welfare of agents in the initial steady state, U^*, as given by (29). Figure 6 shows that the resulting phase diagrams are basically similar to Figure 5.[25]

[24] In Proposition 2, we have shown that the border between Cases Ib and Case II is at $z = 1 - \alpha$, given $\lambda = 1$ and $\delta = 1$. As $z \equiv \beta/(1 + \beta)$, this implies that the border would be at $\beta = (1 - \alpha)/\alpha$, which is 1.5 if $\alpha = 0.4$. Therefore, it is almost impossible to obtain Case II under the assumption of $\lambda = 1$ and $\delta = 1$ (see footnote 14). However, the discussion in the text suggests that this is only because the highest combination of λ and δ pushes the border too far away in the direction of the higher β. Under realistic values of δ, Figure 5 shows that both Case I and Case II are possible with a plausible range of β.

[25] Strictly speaking, there is a slight difference in the upper-right corner of Figure 6(ii), where the pattern becomes ambiguous. Note that β is close to 1 in this region, which means that the agents do not care about

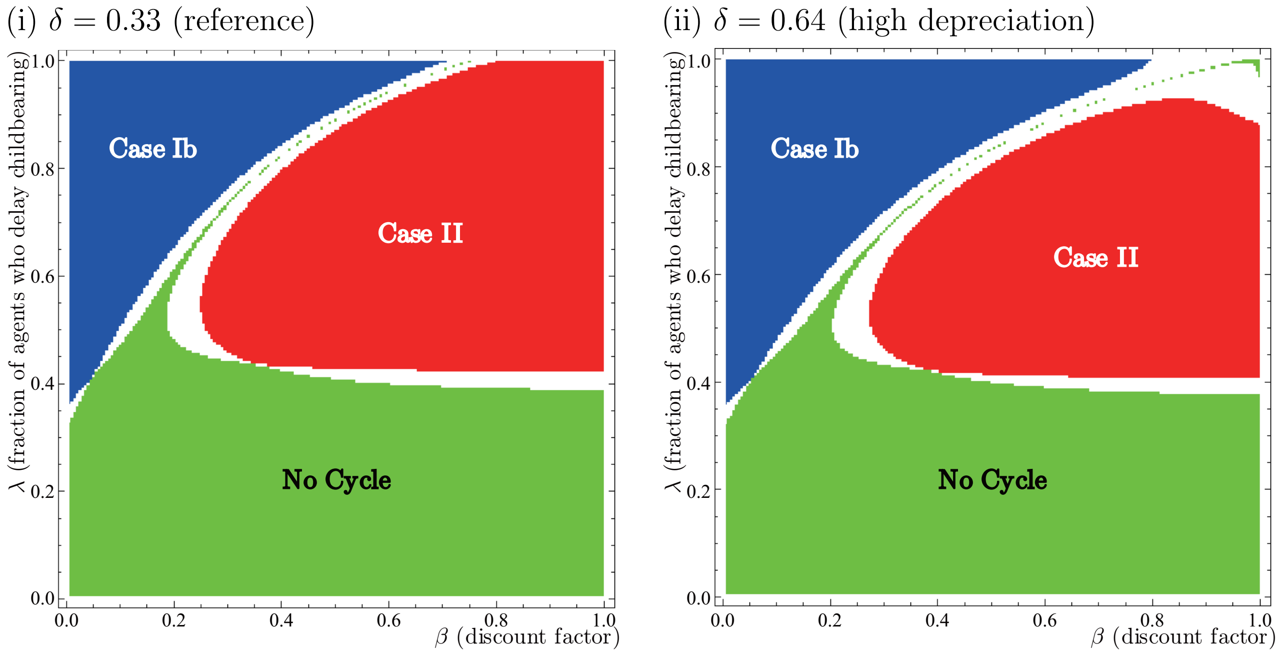

Figure 6: Pattern of Cycles in U_t

Types of income	wage at young	interest at middle	wage at middle	interest at old
Odd-period generations (smaller population)	higher w_t ($\because k_t > k^*$)	higher r_{t+1} ($\because k_{t+1} < k^*$)	lower w_{t+1} ($\because k_{t+1} < k^*$)	lower r_{t+2} ($\because k_{t+2} > k^*$)
Even-period generations (larger population)	lower w_t ($\because k_t < k^*$)	lower r_{t+1} ($\because k_{t+1} > k^*$)	higher w_{t+1} ($\because k_{t+1} > k^*$)	higher r_{t+2} ($\because k_{t+2} < k^*$)

Table 3: Effects of Delayed Childbearing on Income Profile (Case I)

Types of income	wage at young	interest at middle	wage at middle	interest at old
Odd-period generations (smaller population)	lower w_t ($\because k_t < k^*$)	lower r_{t+1} ($\because k_{t+1} > k^*$)	higher w_{t+1} ($\because k_{t+1} > k^*$)	higher r_{t+2} ($\because k_{t+2} < k^*$)
Even-period generations (larger population)	higher w_t ($\because k_t > k^*$)	higher r_{t+1} ($\because k_{t+1} < k^*$)	lower w_{t+1} ($\because k_{t+1} < k^*$)	lower r_{t+2} ($\because k_{t+2} > k^*$)

Table 4: Effects of Delayed Childbearing on Income Profile (Case II)

It is intuitive that when k_t converges monotonically to k^*, the welfare of generations U_t also converges to the steady state value U^*. Therefore, the region of "No cycle" in Figure 5 naturally corresponds to the same region in Figure 6. The correspondence of the other regions can be understood in terms of the incomes that agents earn throughout their lives. Consider the case where the combination of β and λ belongs to the "Case Ib" region of Figure 5. This means that, after the initial response, the capital–labor ratio k_t is higher than the steady state value k^* in odd periods, whereas $k_t < k^*$ in even periods. This pattern of movement in k_t affects the income profiles of agents differently depending on whether they belong to odd- or even-period generations. The odd-period generations (i.e., those who are young in an odd period t) enjoy high wage incomes in their youth because $k_t > k^*$, and also high interest incomes in their middle age because $k_{t+1} < k^*$. Although they suffer from low wage incomes in their middle age (because $k_{t+1} < k^*$) and low interest incomes in their old age (because $k_{t+2} > k^*$), the high incomes in the earlier part of their life affect their welfare more significantly because of discounting, and hence U_t tends to be higher than the steady state level, U^*. On the contrary, as summarized by the bottom row in Table 3, even-period generations (i.e., those who are young in an even period) lose income in the earlier part of their lives. Thus, their lifetime welfare U_t tends to be lower than U^*. As a result, [the welfare of the even-period generations] $< U^* <$ [the welfare of odd-period generations] holds in the region labeled "Case I" in Figure 6.

Recall from Figure 2 that even-period generations have larger cohort sizes than odd-period generations, and the difference is more significant when λ is higher. Therefore, the result in Table 3 suggests that the majority of agents in the economy suffer from welfare loss when the economy lies in Case I (i.e., when β, or equivalently z, is small). This can be viewed as a generalized result of Proposition 3, which has shown that the welfare of all agents falls if z is sufficiently small in the case where only even-period generations exist ($\lambda = 1$).

When β and λ belong to "Case II", the effect of delayed childbearing on the incomes of the odd- and even-period generations, respectively, are summarized in Table 4. In this case,

the timing of consumption. We guess that this is one reason why cycles in U_t are less evident in this region (see Tables 3 and 4). Another slight difference is that there is no "Case Ia" region in Figure 6(i), while there was a small region of "Case Ia" in Figure 5(i).

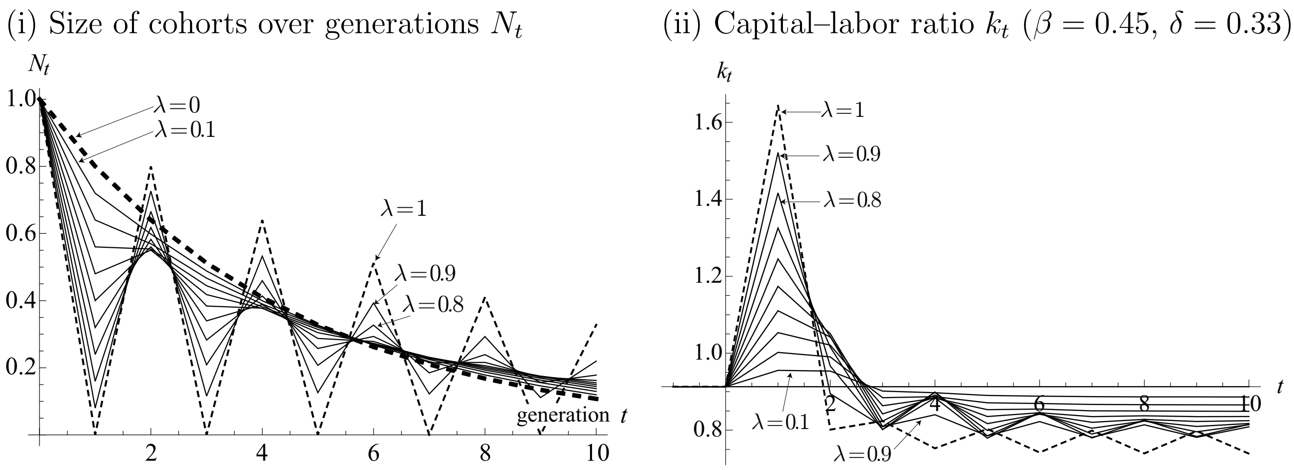

Figure 7: Demographic and Equilibrium Dynamics under Declining Population ($n = 0.8$)

$U_t < U^*$ holds for odd-period generations and $U_t > U^*$ for even-period generations. This implies that the majority of the population will benefit from delayed childbearing, while those born in-between the big cohorts experience a fall in their lifetime utility.

5 Extensions and Robustness

5.1 Declining Population

Prior to the previous section, we examined the effect of delayed childbearing by assuming that each agent has exactly one child in her lifetime. This is equivalent to assuming that the lifetime fertility rate (LFR) is exactly at the replacement level. However, in most developed countries where delayed childbirth is observed, the lifetime fertility rate is far below the replacement level (with a possible exception of the United States, where the LFR is around the replacement level). This means that the population is declining in the long run, even without delayed childbearing. Here, we briefly examine the effect of delayed childbearing in the economy where each agent has, on average, less than one child in her lifetime.

Suppose that each agent has, on average, $n \in (0, 1)$ children in her lifetime, and also that the number of children does not correlate with the timing of childbearing. Recall that the fraction λ_t of the generation-t agents delay childbearing. This means that from generation-t

agents with population N_t, $n(1-\lambda_t)N_t$ children are born in period t (i.e., when parents are young), and $n\lambda_t N_t$ children are born in period $t+1$ (i.e., when parents are middle-aged). The cohort size of generation $t+1$, born in period t, is thus determined by:

$$N_{t+1} = n\left(1-\lambda_t\right)N_t + n\lambda_{t-1}N_{t-1}. \tag{31}$$

Combining (31) with (2), we obtain the pattern of evolution of N_t.[26] Figure 7(i) depicts the path of N_t for the case of $n = 0.8$, which roughly corresponds to the lifetime fertility rate of $1.68 = 2.1(\text{replacement rate}) \times 0.8$. When $\lambda > 0$, the initial fall in the cohort population ($N_1 = n(1-\lambda) < N_0 = 1$) is more significant than the benchmark case ($\lambda = 0$), not only because each agent has fewer children in their life, but also because a fraction λ of young agents in period 0 postpone childbearing until the next period. However, in the long run, the delay of motherhood slows the pace of depopulation compared with the case of $\lambda = 0$. As a result, for larger t, the population is actually higher when a larger fraction of agents delay childbearing.

In a similar way to that in Subsection 4.1, substituting the path of N_t into (14) gives the equilibrium dynamics for k_t, as shown by Figure 7(ii). When compared with Figure 4(ii), we observe that, although the pattern of the fluctuations are similar, the long-run capital–labor ratio k_t is lower than in the initial steady state, and the difference is larger when λ is higher. Intuitively, delayed childbearing in this economy (with $n < 1$) raises the long-run rate of population growth, which naturally leads to a lower capital–labor ratio through a capital-dilution effect.[27]

As the capital-dilution effect has already been well studied, we examine whether there are cycles in the paths of k_t and U_t after removing this effect.[28] The results are shown in

[26]With the initial condition of $N_0 = 1$ and $N_1 = n(1-\lambda)$, equation $N_{t+1} = n\left(1-\lambda\right)N_t + n\lambda N_{t-1}$ for $t \geq 1$ can be solved as $N_t = c_1\sigma_1^t + c_2\sigma_2^t$, where $\sigma_1 = (n/2)\left\{1-\lambda+\sqrt{(1-\lambda)^2+(4\lambda/n)}\right\} > n$ and $\sigma_2 = n(1-\lambda)-\sigma_1 < 0$, given $\lambda,\ n \in (0,1)$. As $|\sigma_2| < |\sigma_1| < 1$, the evolution of N_t in the long run is dominated by the $c_1\sigma_1^t$ term, which means that delayed childbearing increases the long-term rate of population growth from n to $\sigma_1 > n$.

[27]See Blanchet (1988) and Brander and Dowrick (1994) for more discussion on the capital-dilution effect by demographic growth.

[28]Using the long-term rate of population growth with delayed childbearing $\sigma_1 =$

(i) Cycles in k_t (ii) Cycles in U_t

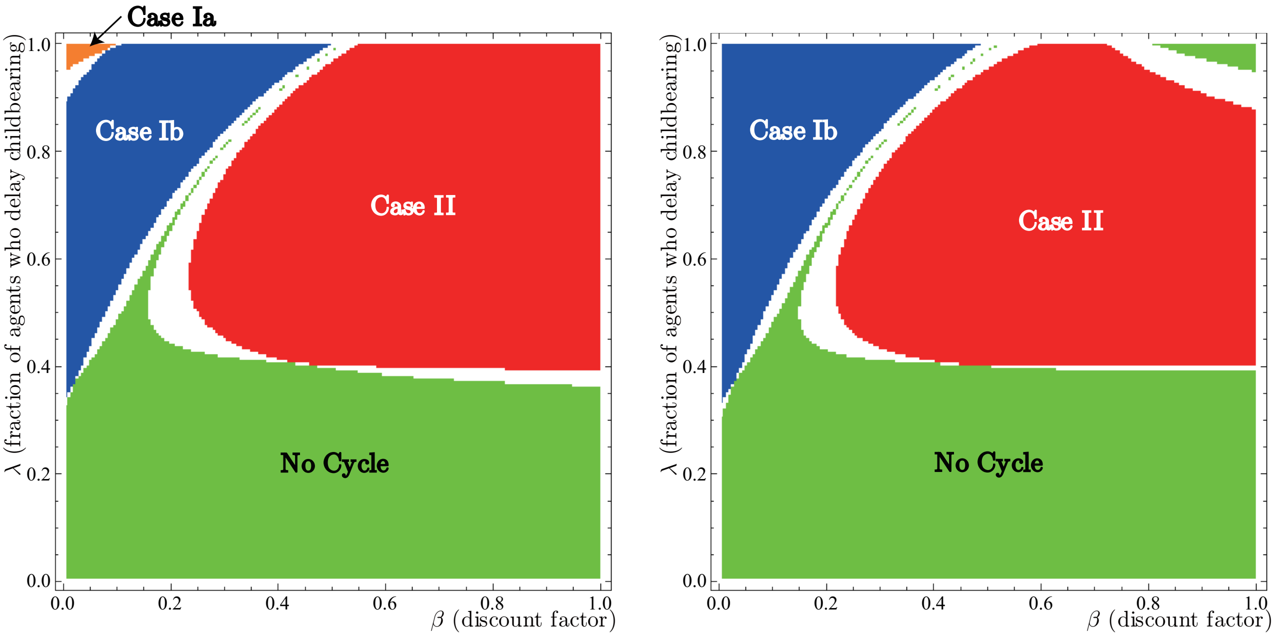

Figure 8: Pattern of Cycles with Declining Population ($n = 0.8,\ \delta = 0.33$)

Figure 8. By comparing Figure 8(i) with Figure 5(i), we observe that the border between Case Ib and Case II shifts to the left because of a lower n. In addition to the effect of overall population decline, a lower n also has an effect on the composition of the labor force: if agents have fewer children, the fraction of younger workers *ceteris paribus* will fall compared with older (middle-aged) workers. This increases the aggregate savings in odd periods (when the middle-aged workers are the majority in the labor force), and in turn raises the capital stock in even periods, making Case II more likely.

The pattern of cycles in U_t, shown in Figure 6(ii), generally matches the pattern in k_t, although in the upper-right corner we find that the welfare is higher than the long-term level both for the odd- and even-period generations, at least until $t = 10$. However, note that this gain in welfare exists only after controlling for the capital-dilution effect. The overall effect of delayed childbearing on the capital–labor ratio and welfare is certainly more negative than

$(n/2)\left\{1 - \lambda + \sqrt{(1-\lambda)^2 + (4\lambda/n)}\right\}$, we calculate the long-term levels of k_t and U_t, which depend on λ because of the capital-dilution effect (see footnote 26). Then, we examine if there are cycles in the paths of k_t (and U_t) relative to their respective long-term levels.

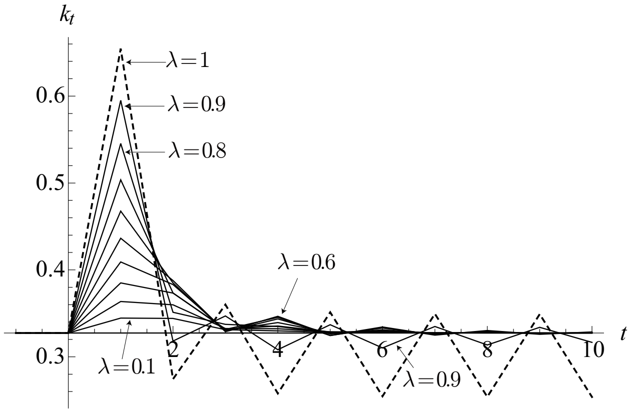

Figure 9: Equilibrium Dynamics with Technological Progress ($\gamma = 1.49$, $\beta = 0.45$, $\delta = 0.33$)

analyzed in the previous section because of the capital-dilution effect that shifts the entire paths of k_t and U_t downward.

5.2 Technological Progress

To ensure the robustness of the results obtained so far, here we confirm that the inclusion of technological progress does not significantly change the pattern of cycles induced by delayed childbearing. Assume that in every period there is exogenous technological progress that increases labor productivity by a factor of $\gamma > 1$. When labor productivity at period 0 is normalized to unity, production per worker can be represented as $y_t = A\gamma^t k_t^\alpha$, where $k_t \equiv K_t/(\gamma^t L_t)$ now represents the amount of capital per efficiency unit of labor. Note that the amount of labor income for each worker (not efficiency unit) should be modified from (13) to $w_t = A(1-\alpha)\gamma^t k_t^\alpha$, whereas the expression for r_t is the same as (12). Then, instead of (14), we obtain the evolution of $k_t \equiv K_t/(\gamma^t L_t)$ as:

$$k_{t+1} = \frac{A\,(1-\alpha)}{\gamma} \frac{N_t k_t^\alpha + z N_{t-1}\left[\left(A\alpha k_t^{\alpha-1} + 1 - \delta\right) k_{t-1}^\alpha/\gamma + k_t^\alpha\right]}{N_{t+1} + N_t}. \tag{32}$$

Figure 9 shows the path of k_t in the presence of yearly labor productivity growth of 2%, i.e., when labor productivity is multiplied by $\gamma = 1.49 \approx (1+0.02)^{20}$ in each period. It looks almost the same as the reference case of Figure 4(ii), but the level of the whole path is lower than the equilibrium without technological progress. This is because technological progress

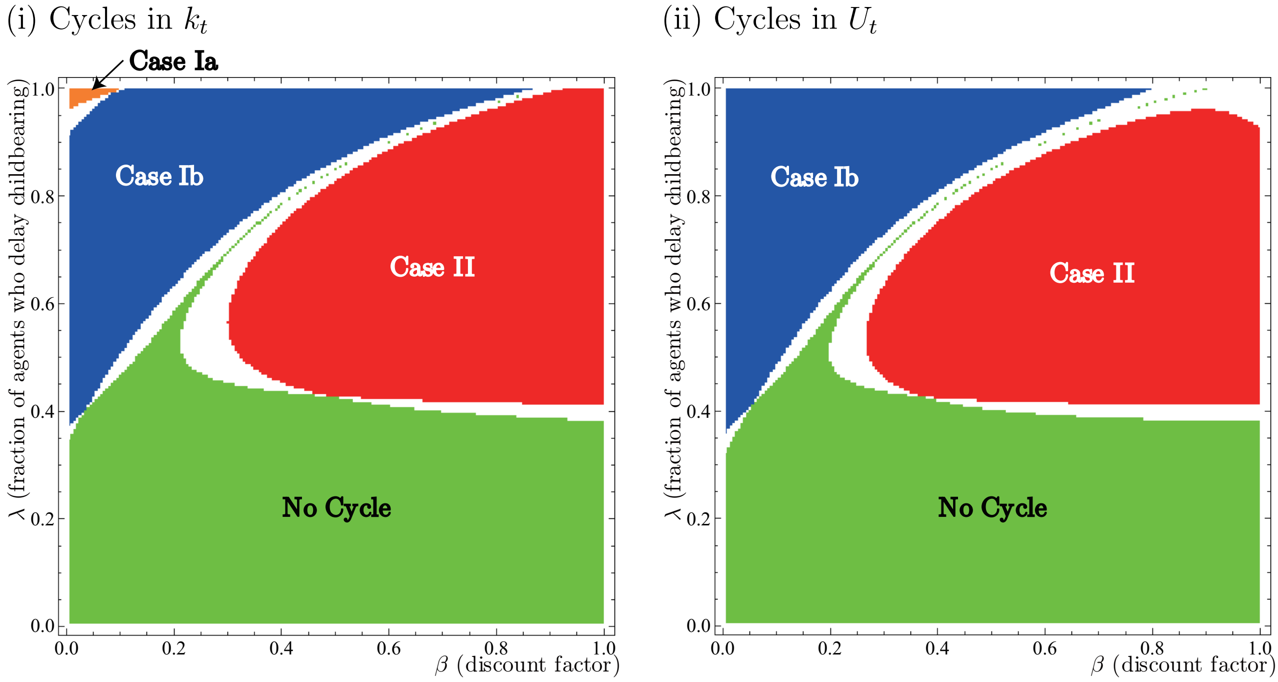

Figure 10: Pattern of Cycles with Technological Progress ($\gamma = 1.49$, $\delta = 0.33$)

expands the labor force measured in efficiency units, and thus dilutes capital per efficiency unit of labor.

Figure 10(i) depicts the pattern of cycles in k_t for various β and λ, under $\gamma = 1.49$ and $\delta = 0.33$. When it is compared with the two panels in Figure 5, this phase diagram matches more closely the high-depreciation case of Figure 5(ii), where $\delta = 0.64$ (5% annum), rather than the reference case with the same depreciation rate ($\delta = 0.33$). This result suggests that technological progress affects the pattern of cycles in k_t in a similar way to a higher depreciation rate. Note that while technological progress in a given period enhances total output Y_t in that period, the amount of remaining capital after depreciation $(1-\delta)K_t$ is unaffected because the latter is determined by the savings in the previous period. Therefore, technological progress reduces $(1-\delta)K_t/Y_t$, and hence lowers the proportion of income received by the middle-aged agents (who have claims on the remaining capital).

We also examined the pattern of cycles in the utility of generations, U_t. Note that, even in the steady state, labor income w_t increases by a factor of γ in each period. By substituting $w_t = A(1-\alpha)\gamma^t k_t^\alpha$ into (27), it can be observed that the utility of generations has a trend term

$(1+\beta)(\log\gamma)t$. Therefore, after calculating the path of U_t for each β and λ by substituting the path of k_t into (27), we removed the trend by subtracting $(1+\beta)(\log\gamma)t$ from it, and then examined the pattern of the cycles in the detrended path of U_t. Figure 10(ii) shows that the result is similar to Figure 6(ii). This confirms that the effects of technological progress on the cycles of k_t and U_t are similar to the effects of a higher depreciation rate.

6 Concluding Remarks

In a simple overlapping generations model, we examined the effects of delayed childbearing on capital accumulation and the welfare of generations. A notable feature of the delayed childbearing economy is that it causes fluctuations in the age composition of workers for a long period of time. As workers at different life stages have different sources of income and also different saving propensities, fluctuations in the age composition affect the aggregate saving rate, causing cycles in the capital–labor ratio. The cycles in the capital–labor ratio cause the lifetime welfare of agents to change generation by generation in an alternating fashion. Depending on the parameters, the majority of agents can experience lower lifetime welfare when the cycles in the capital–labor ratio affect the factor prices in such a way that their income in the early stage of their life falls. We also examined extensions of the model with declining population and technological progress, and confirmed the robustness of our results.

Although our model is very stylized, it gives insights into a possible cause and effects of fluctuations in the age distribution of the labor force, which have been examined in different contexts in the literature. For example, Lee (1997) pointed out that baby booms and busts can cause fluctuations in the age structure. Mankiw and Weil (1989) investigated their effects on the US housing market. Our analysis suggests that delayed childbearing can also generate fluctuations in the age distribution of workers, which have differential welfare effects on cohorts.

This paper attempted to analyze the effects of the age distribution on capital accumulation and economic welfare as intuitively as possible. For this reason, our model treated the timing of childbirth and the number of children as exogenous. However, in analyzing the implications of policies that aim to cope with delayed childbearing and the low fertility rate, it will be necessary to clarify how agents endogenously choose the timing of their childbearing and the number of children. It will also be interesting to investigate the endogenous relationship between delayed childbearing and declining lifetime fertility rate, which in this study we assumed are independent. The exploration of these issues is left for future research.

Appendix A Proof of Lemma 1

Proof of property (ii): From (22) and (25), $V_{odd}(z) \leq v^*(z)$ is equivalent to:

$$1 + z \geq z^{\frac{\alpha}{1+\alpha}} \left\{ 2 (1-\alpha)^{\frac{1-\alpha}{1+\alpha}} - \alpha z^{\frac{1-\alpha}{1+\alpha}} \right\} (1-\alpha)^{-\frac{1}{1+\alpha}} \equiv \rho_{odd}(z). \tag{33}$$

Calculating $\rho'_{odd}(z)$ and $\rho''_{odd}(z)$, we obtain:

$$\rho'_{odd}(z) = \frac{\alpha}{1+\alpha} z^{-\frac{1}{1+\alpha}} \left\{ 2 (1-\alpha)^{\frac{1-\alpha}{1+\alpha}} - z^{\frac{1-\alpha}{1+\alpha}} \right\} (1-\alpha)^{-\frac{1}{1+\alpha}}, \tag{34}$$

$$\rho''_{odd}(z) = -\left(\frac{\alpha}{1+\alpha} \right)^2 z^{-\frac{2+\alpha}{1+\alpha}} \left\{ \frac{2}{\alpha} (1-\alpha)^{\frac{1-\alpha}{1+\alpha}} - z^{\frac{1-\alpha}{1+\alpha}} \right\} (1-\alpha)^{-\frac{1}{1+\alpha}}.$$

Note that $\rho_{odd}(z) = 0$ holds at $z = 0$ and $\left(\frac{2}{\alpha}\right)^{\frac{1+\alpha}{1-\alpha}} (1-\alpha)$. In addition, $\rho_{odd}(z) > 0$ and $\rho''_{odd}(z) < 0$ hold if and only if $z \in \left(0, \left(\frac{2}{\alpha}\right)^{\frac{1+\alpha}{1-\alpha}} (1-\alpha)\right)$. The left-hand side of (33) is linear with respect to z, and the right-hand side, $\rho_{odd}(z)$, has an inverted-U shape while it is positive. From Lemma 1(i), (33) holds as an equality at $z = 1-\alpha$. Moreover, from $\rho'_{odd}(1-\alpha) = \alpha / (1-\alpha^2)$, $\rho'_{odd}(1-\alpha) < 1$ holds because we have assumed $\alpha < \left(\sqrt{5}-1\right)/2$. From these properties, Figure 11 (left) shows that there exist two values of z that satisfy (33) as an equality. One is $z = 1-\alpha$ from Lemma 1(i), and the other solution $\widehat{z}$ satisfies $0 < \widehat{z} < 1-\alpha$. ■

Proof of property (iii): From (21) and (25), $V_{even}(z) \leq v^*(z)$ is equivalent to:

$$1 + z \geq z^{\frac{\alpha}{1+\alpha}} \left\{ 2 z^{\frac{1-\alpha}{1+\alpha}} - \alpha (1-\alpha)^{\frac{1-\alpha}{1+\alpha}} \right\} (1-\alpha)^{-\frac{1}{1+\alpha}} \equiv \rho_{even}(z). \tag{35}$$

Calculating $\rho'_{even}(z)$ and $\rho''_{even}(z)$, we obtain:

$$\rho'_{even}(z) = \frac{1}{1+\alpha} z^{-\frac{1}{1+\alpha}} \left\{ 2 z^{\frac{1-\alpha}{1+\alpha}} - \alpha^2 (1-\alpha)^{\frac{1-\alpha}{1+\alpha}} \right\} (1-\alpha)^{-\frac{1}{1+\alpha}},$$

$$\rho''_{even}(z) = -\left(\frac{\alpha}{1+\alpha} \right)^2 z^{-\frac{2+\alpha}{1+\alpha}} \left\{ \frac{2}{\alpha} z^{\frac{1-\alpha}{1+\alpha}} - (1-\alpha)^{\frac{1-\alpha}{1+\alpha}} \right\} (1-\alpha)^{-\frac{1}{1+\alpha}}.$$

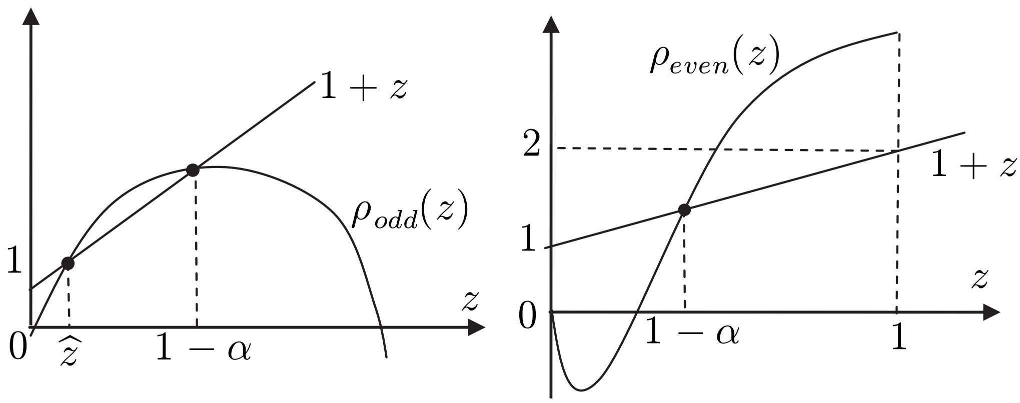

Figure 11: Graphs of Functions $\rho_{odd}(z)$ and $\rho_{even}(z)$

Note that $\rho_{even}(z) = 0$ holds at $z = 0$ and $\left(\frac{\alpha}{2}\right)^{\frac{1+\alpha}{1-\alpha}}(1-\alpha)$. In addition, $\rho_{even}(z) > 0$ and $\rho''_{even}(z) < 0$ hold if and only if $z > \left(\frac{\alpha}{2}\right)^{\frac{1+\alpha}{1-\alpha}}(1-\alpha)$. The left-hand side of (35) is linear with respect to z, and the right-hand side, $\rho_{even}(z)$, is strictly concave when $\rho_{even}(z) > 0$. In addition:

$$\begin{aligned}\rho_{even}(1) &= (1-\alpha)^{-\frac{1}{1+\alpha}}\left[2 - \alpha(1-\alpha)^{\frac{1-\alpha}{1+\alpha}}\right] \\ &> (1-\alpha)^{-\frac{1}{1+\alpha}}(2-\alpha) > (1-\alpha)^{-\frac{1}{2}}(2-\alpha) \geq 2.\end{aligned}$$

Therefore, as depicted in Figure 11 (right), $z = 1-\alpha$ is the unique solution in the range of $z \in (0,1)$. ■

Appendix B Proof of Lemma 2

Proof of property (i): When $\lambda = 1$, the definition of v_t in (17) and equation (18) implies $k^*_{odd}/\left(A\left(k^*_{even}\right)^{\alpha}\right) = 1-\alpha$. Using this equation, (28) is rewritten as:

$$U^*_{even} = (1+\beta)\left[\alpha \log k^*_{odd} - \log(1-\alpha) - z(1-\alpha)\log k^*_{even}\right] + C. \tag{36}$$

Similarly, when $\lambda = 0$, we have $k^*/\left(A\left(k^*\right)^{\alpha}\right) = v^*(z)$. Using this equation, (29) is rewritten as:

$$U^* = (1+\beta)\left[\alpha \log k^* + \log\left[\frac{\alpha}{v^*(z)} + 1\right] - z(1-\alpha)\log k^*\right] + C. \tag{37}$$

Note that, from (23), (24) and (26), $\log k^*_{odd} - \log k^* = \frac{1}{1-\alpha}(\log V_{odd}(z) - \log v^*(z))$ and $\log k^*_{even} - \log k^* = \frac{1}{1-\alpha}(\log V_{even}(z) - \log v^*(z))$ hold. Using these, we eliminate k^*, k^*_{odd} and k^*_{even} from the difference between (36) from (37) to get $U^*_{even} - U^* = (1+\beta)\Omega(z)$. ■

Proof of property (ii): Using (21) and (22), (30) is rewritten as:

$$\begin{aligned} \Omega(z) \equiv & -\log(1-\alpha)\left[\frac{\alpha}{v^*(z)} + 1\right] + \left(z - \frac{\alpha}{1-\alpha}\right)\log v^*(z) \\ & + \frac{\alpha}{1-\alpha^2}\left[\log(1-\alpha) + \alpha\log z\right] - \frac{z}{1-\alpha}\left[\alpha\log(1-\alpha) + \log z\right]. \end{aligned} \tag{38}$$

As $\lim_{z\to 0}\log z = -\infty$, $\lim_{z\to 0} z\log z = 0$ and $v^*(0) = (1-\alpha)/2$, the right-hand side of (38) diverges to minus infinity as $z \to 0$. From Lemma 1(i) and (30), we immediately obtain $\Omega(1-\alpha) = 0$. ■

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