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Abstract

We characterize weak continuity of an interval order \( \succcurlyeq \) on a topological space \((X, \tau)\) by using the concept of a scale in a topological space.

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1 Introduction

An interval order \( \succcurlyeq \) on a set \( X \) is in some sense the simplest kind of binary relation with nontransitive indifference since, under certain conditions, it can be represented by a pair \((u, v)\) of real-valued functions on \( X \) (this means that, for all \( x, y \in X, x \succcurlyeq y \) if and only if \( u(x) \leq v(y) \)). If in addition \( X \) is endowed with a topology \( \tau \), then one may look for a pair \((u, v)\) of continuous real-valued functions representing an interval order \( \succcurlyeq \) on \((X, \tau)\) (see e.g. Bosi, Candeal and Induráin [2] and Bosi, Candeal, Campión and Induráin [3]).

With a view to possible general conditions guaranteeing the existence of such a continuous representation, Bosi [1] introduced the concept of a weakly continuous interval order. In this paper, we characterize weak continuity of an interval order by using the concept of a scale in a topological space.
2 Notation and preliminaries

We first recall that an interval order $\succeq$ on an arbitrary nonempty set $X$ is a binary relation on $X$ which is reflexive and in addition verifies the following condition for all $x, y, z, w \in X$:

$(x \succeq z) \text{ and } (y \succeq w) \Rightarrow (x \succeq w) \text{ or } (y \succeq z)$.

The irreflexive part of an interval order $\succeq$ will be denoted by $\prec$ (i.e., for all $x, y \in X$, $x \prec y$ if and only if $(x \succeq y)$ and not($y \succeq x$)).

Fishburn [6] showed that if $\succeq$ is an interval order on a set $X$, then each of the following two binary relations $\succeq^*$ and $\succeq^{**}$ on $X$ is a total preorder (i.e., a total and transitive binary relation):

$x \succeq^* y \Leftrightarrow (z \succeq x \Rightarrow z \succeq y)$ for all $z \in X$,

$x \succeq^{**} y \Leftrightarrow (y \succeq z \Rightarrow x \succeq z)$ for all $z \in X$.

The irreflexive parts of $\succeq^*$ and $\succeq^{**}$ will be denoted by $\prec^*$ and $\prec^{**}$.

If $\succeq$ is an interval order on a set $X$, then denote by $L_\prec(x)$ ($U_\prec(x)$) the strict lower (upper) section of any element $x \in X$ (i.e., for every $x \in X$, $L_\prec(x) = \{y \in X : y < x\}$ and $U_\prec(x) = \{y \in X : x < y\}$).

A pair $(u, v)$ of real-valued functions on $X$ is said to represent an interval order $\succeq$ on $X$ if, for all $x, y \in X$,

$x \succeq y \Leftrightarrow u(x) \leq v(y)$.

We say that a pair $(u, v)$ of real-valued functions on $X$ almost represents an interval order $\succeq$ on $X$ if, for all $x, y \in X$,

$(x \succeq y \Rightarrow u(x) \leq v(y)) \text{ and } (x \prec y \Rightarrow v(x) \leq u(y))$.

The following proposition holds which illustrates the importance of the concept of a pair of continuous real-valued functions almost representing an interval order in connection with the problem concerning the existence of a representation by means of a pair of continuous real-valued functions.

**Proposition 2.1** An interval order $\succeq$ on a topological space $(X, \tau)$ is representable by means of a pair $(u, v)$ of continuous real-valued functions with values in $[0, 1]$ if and only if there exists a countable family $\{(u_n, v_n)\}_{n \in \mathbb{N} \setminus \{0\}}$ of pairs of continuous real-valued functions on $(X, \tau)$ with values in $[0, 1]$ almost representing $\succeq$ such that for every $x, y \in X$ with $x < y$ there exists $n \in \mathbb{N} \setminus \{0\}$ with $v_n(x) < u_n(y)$. 


Proof. The “only if” part is clear. Hence, assume that there exists a countable family \( \{(u_n, v_n)\}_{n \in \mathbb{N} \setminus \{0\}} \) of pairs of continuous real-valued functions on \((X, \tau)\) with values in \([0, 1]\) almost representing \(\preceq\) such that for every \(x, y \in X\) with \(x \prec y\) there exists \(n \in \mathbb{N} \setminus \{0\}\) with \(v_n(x) < u_n(y)\). Define functions \(u\) and \(v\) on \(X\) as follows:

\[
u(x) = \sum_{n=1}^{\infty} 2^{-n} u_n(x), \quad v(x) = \sum_{n=1}^{\infty} 2^{-n} v_n(x) \quad (x \in X)
\]

in order to immediately verify that \((u, v)\) is a continuous representation of the interval order \(\preceq\) on the topological space \((X, \tau)\).

An interval order \(\preceq\) on a topological space \((X, \tau)\) is said to be continuous if \(L_\prec(x)\) and \(U_\prec(x)\) are both open subsets of \(X\) for every \(x \in X\). Further, we say that it is strongly continuous if it is continuous and in addition the associated total preorders \(\preceq^*\) and \(\preceq^{**}\) are both continuous.

We now recall the definition of a weakly continuous interval order presented by Bosi [1].

**Definition 2.2 (weakly continuous interval order)** We say that an interval order \(\preceq\) on a topological space \((X, \tau)\) is weakly continuous if for every \(x, y \in X\) such that \(x \prec y\) there exists a pair \((u_{xy}, v_{xy})\) of continuous real-valued functions on \((X, \tau)\) satisfying the following conditions:

(i) \((u_{xy}, v_{xy})\) almost represents \(\preceq\);

(ii) \(v_{xy}(x) < u_{xy}(y)\).

The concept of weak continuity described in Definition 2.2 is reminiscent of the concept of weak continuity of a preorder on a topological space (see e.g. Bosi and Herden [5]). Every interval order that is representable by means of a pair of continuous functions \((u, v)\) and at same time is such that the associated total preorders \(\preceq^*\) and \(\preceq^{**}\) are not continuous provides an example of a weakly continuous interval order which is continuous but not strongly continuous. For example, this is the case of the interval order \(\preceq\) on \(X = [3, 5] \cup [9, 25]\) defined by \(x \preceq y \iff x \leq y^2\) (see Bosi, Candeal and Indurain [2, Example 3.2]) when \(X\) is endowed with the induced Euclidean topology on the real line.

### 3 Weak continuity of interval orders

In the sequel, we shall refer to the well known notion of a scale in a topological space (see e.g. Gillman and Jerison [7]).

**Definition 3.1** If \((X, \tau)\) is a topological space and \(S\) is a dense subset of \([0, 1]\) such that \(1 \in S\), then a family \(\{G_r\}_{r \in S}\) of open subsets of \(X\) is said to be a scale in \((X, \tau)\) if the following conditions hold:

\[
sup_{r \in S} G_r = X.
\]
(i) \( G_1 = X \);

(ii) \( G_{r_1} \subseteq G_{r_2} \) for every \( r_1, r_2 \in \mathbb{S} \) such that \( r_1 < r_2 \).

We are now ready to characterize the weak continuity of an interval order on a topological space.

**Proposition 3.2** Let \( \preceq \) be an interval order on a topological space \((X, \tau)\). Then the following conditions are equivalent:

(i) \( \preceq \) is weakly continuous;

(ii) For every pair \((x, y) \in X \times X\) such that \( x < y \) there exist two scales \( \{G_{r}^{(xy)}\}_{r \in \mathbb{S}} \) and \( \{G_{r}^{(xy)}\}_{r \in \mathbb{S}} \) in \((X, \tau)\) such that the family \( \{(G_{r}^{(xy)}, G_{r}^{(xy)})\}_{r \in \mathbb{S}} \) satisfies the following conditions:

(a) \( z \preceq w \) and \( w \in G_{r}^{(xy)} \) imply \( z \in G_{r}^{(xy)} \) for every \( z, w \in X \) and \( r \in \mathbb{S} \);

(b) \( z < w \) and \( w \in G_{r}^{(xy)} \) imply \( z \in G_{r}^{(xy)} \) for every \( z, w \in X \) and \( r \in \mathbb{S} \);

(c) \( x \in G_{1}^{(xy)} \) and \( y \not\in G_{r}^{(xy)} \) for every \( r \in \mathbb{S} \setminus \{1\} \).

**Proof.** Consider a pair \((x, y) \in X \times X\) such that \( x < y \).

(i) \( \Rightarrow \) (ii). Since \( \preceq \) is weakly continuous, there exists a pair \((u_{xy}, v_{xy})\) of continuous real-valued functions on \((X, \tau)\) such that \((u_{xy}, v_{xy})\) almost represents \( \preceq \) and in addition \( v_{xy}(x) < u_{xy}(y) \). Without loss of generality, we can assume that both \( u_{xy} \) and \( v_{xy} \) take values in \([0, 1]\) and that \( v_{xy}(x) = 0 \), \( u_{xy}(y) = 1 \).

Define \( \mathbb{S} = \mathbb{Q} \cap [0, 1] \), \( G_{r}^{(xy)} = v_{xy}^{-1}([0, r]) \), \( G_{r}^{(xy)} = u_{xy}^{-1}([0, r]) \) for every \( r \in \mathbb{S} \), and \( G_{1}^{(xy)} = G_{1}^{(xy)} = X \) in order to immediately verify that \( \{G_{r}^{(xy)}\}_{r \in \mathbb{S}} \) and \( \{G_{r}^{(xy)}\}_{r \in \mathbb{S}} \) are two scales in \((X, \tau)\) such that the family \( \{(G_{r}^{(xy)}, G_{r}^{(xy)})\}_{r \in \mathbb{S}} \) satisfies the above conditions (a), (b) and (c).

(ii) \( \Rightarrow \) (i). From the assumptions, there exist two scales \( \{G_{r}^{(xy)}\}_{r \in \mathbb{S}} \) and \( \{G_{r}^{(xy)}\}_{r \in \mathbb{S}} \) such that the family \( \{(G_{r}^{(xy)}, G_{r}^{(xy)})\}_{r \in \mathbb{S}} \) satisfies the above conditions (a), (b) and (c). Define two functions \( u_{xy}, v_{xy} : X \to [0, 1] \) as follows:

\[
u_{xy}(z) = \inf \{r \in \mathbb{Q} \cap [0, 1] : z \in G_{r}^{(xy)}\} \quad (x \in X),
\]

\[
u_{xy}(z) = \inf \{r \in \mathbb{Q} \cap [0, 1] : z \in G_{r}^{(xy)}\} \quad (x \in X).
\]

We have that \( u_{xy} \) and \( v_{xy} \) are both continuous functions on \((X, \tau)\) with values in \([0, 1]\) (see e.g. the proof of the lemma on pages 43-44 in Gillman and Jerison
We claim that the pair \((u_{xy}, v_{xy})\) almost represents the interval order \(\preceq\) and satisfies the condition \(v_{xy}(x) < u_{xy}(y)\).

From condition (c), we have that \(v_{xy}(x) = 0\) and \(u_{xy}(y) = 1\). It remains to show that the pair \((u_{xy}, v_{xy})\) almost represents the interval order \(\preceq\). First consider any two elements \(z, w \in X\) such that \(z \prec w\). Then, by condition (b), we have that \(v_{xy}(z) \leq u_{xy}(w)\). Finally, observe that if \(z, w \in X\) are any two elements such that \(z \preceq w\), then we have that \(u_{xy}(z) \leq v_{xy}(w)\) by condition (a). This consideration completes the proof.

\[\square\]

References


