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Abstract

We characterize weak continuity of an interval order \( \preceq \) on a topological space \( (X, \tau) \) by using the concept of a scale in a topological space.

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1 Introduction

An interval order \( \preceq \) on a set \( X \) is in some sense the simplest kind of binary relation with nontransitive indifference since, under certain conditions, it can be represented by a pair \( (u, v) \) of real-valued functions on \( X \) (this means that, for all \( x, y \in X \), \( x \preceq y \) if and only if \( u(x) \leq v(y) \)). If in addition \( X \) is endowed with a topology \( \tau \), then one may look for a pair \( (u, v) \) of continuous real-valued functions representing an interval order \( \preceq \) on \( (X, \tau) \) (see e.g. Bosi, Candeal and Induráin [2] and Bosi, Candeal, Campión and Induráin [3]).

With a view to possible general conditions guaranteeing the existence of such a continuous representation, Bosi [1] introduced the concept of a weakly continuous interval order. In this paper, we characterize weak continuity of an interval order by using the concept of a scale in a topological space.
2 Notation and preliminaries

We first recall that an interval order \( \preceq \) on an arbitrary nonempty set \( X \) is a binary relation on \( X \) which is reflexive and in addition verifies the following condition for all \( x, y, z, w \in X \):

\[
(x \preceq z) \text{ and } (y \preceq w) \Rightarrow (x \preceq w) \text{ or } (y \preceq z).
\]

The irreflexive part of an interval order \( \preceq \) will be denoted by \( \prec \) (i.e., for all \( x, y \in X \), \( x \prec y \) if and only if \( x \not\preceq y \) and not \( y \not\preceq x \)).

Fishburn [6] showed that if \( \preceq \) is an interval order on a set \( X \), then each of the following two binary relations \( \preceq^* \) and \( \preceq^{**} \) on \( X \) is a total preorder (i.e., a total and transitive binary relation):

\[
x \preceq^* y \iff (z \preceq x \Rightarrow z \preceq y) \text{ for all } z \in X,
x \preceq^{**} y \iff (y \preceq z \Rightarrow x \preceq z) \text{ for all } z \in X.
\]

The irreflexive parts of \( \preceq^* \) and \( \preceq^{**} \) will be denoted by \( \prec^* \) and \( \prec^{**} \).

If \( \preceq \) is an interval order on a set \( X \), then denote by \( L_\prec(x) \) (\( U_\prec(x) \)) the strict lower (upper) section of any element \( x \in X \) (i.e., for every \( x \in X \), \( L_\prec(x) = \{y \in X : y < x\} \) and \( U_\prec(x) = \{y \in X : x < y\} \)).

A pair \((u, v)\) of real-valued functions on \( X \) is said to represent an interval order \( \preceq \) on \( X \) if, for all \( x, y \in X \),

\[
x \preceq y \iff u(x) \leq v(y).
\]

We say that a pair \((u, v)\) of real-valued functions on \( X \) almost represents an interval order \( \preceq \) on \( X \) if, for all \( x, y \in X \),

\[
(x \preceq y \Rightarrow u(x) \leq v(y)) \text{ and } (x \prec y \Rightarrow v(x) \leq u(y)).
\]

The following proposition holds which illustrates the importance of the concept of a pair of continuous real-valued functions almost representing an interval order in connection with the problem concerning the existence of a representation by means of a pair of continuous real-valued functions.

**Proposition 2.1** An interval order \( \preceq \) on a topological space \((X, \tau)\) is representable by means of a pair \((u, v)\) of continuous real-valued functions with values in \([0, 1]\) if and only if there exists a countable family \(\{(u_n, v_n)\}_{n \in \mathbb{N} \setminus \{0\}}\) of pairs of continuous real-valued functions on \((X, \tau)\) with values in \([0, 1]\) almost representing \( \preceq \) such that for every \( x, y \in X \) with \( x \prec y \) there exists \( n \in \mathbb{N} \setminus \{0\} \) with \( v_n(x) < u_n(y) \).
Weakly continuous preferences

Proof. The “only if” part is clear. Hence, assume that there exists a countable family \( \{(u_n, v_n)\}_{n \in \mathbb{N} \setminus \{0\}} \) of pairs of continuous real-valued functions on \((X, \tau)\) with values in \([0, 1]\) almost representing \( \preceq \) such that for every \( x, y \in X \) with \( x \prec y \) there exists \( n \in \mathbb{N} \setminus \{0\} \) with \( v_n(x) < u_n(y) \). Define functions \( u \) and \( v \) on \( X \) as follows:

\[
    u(x) = \sum_{n=1}^{\infty} 2^{-n} u_n(x), \quad v(x) = \sum_{n=1}^{\infty} 2^{-n} v_n(x) \quad (x \in X)
\]

in order to immediately verify that \((u, v)\) is a continuous representation of the interval order \( \preceq \) on the topological space \((X, \tau)\).

An interval order \( \preceq \) on a topological space \((X, \tau)\) is said to be continuous if \( L_\prec(x) \) and \( U_\prec(x) \) are both open subsets of \( X \) for every \( x \in X \). Further, we say that it is strongly continuous if it is continuous and in addition the associated total preorders \( \preceq^* \) and \( \preceq^{**} \) are both continuous.

We now recall the definition of a weakly continuous interval order presented by Bosi [1].

**Definition 2.2 (weakly continuous interval order)** We say that an interval order \( \preceq \) on a topological space \((X, \tau)\) is weakly continuous if for every \( x, y \in X \) such that \( x \prec y \) there exists a pair \((u_{xy}, v_{xy})\) of continuous real-valued functions on \((X, \tau)\) satisfying the following conditions:

(i) \((u_{xy}, v_{xy})\) almost represents \( \preceq \);
(ii) \( v_{xy}(x) < u_{xy}(y) \).

The concept of weak continuity described in Definition 2.2 is reminiscent of the concept of weak continuity of a preorder on a topological space (see e.g. Bosi and Herden [5]). Every interval order that is representable by means of a pair of continuous functions \((u, v)\) and at same time is such that the associated total preorders \( \preceq^* \) and \( \preceq^{**} \) are not continuous provides an example of a weakly continuous interval order which is continuous but not strongly continuous. For example, this is the case of the interval order \( \preceq \) on \( X = [3, 5] \cup [9, 25] \) defined by \( x \preceq y \iff x \leq y^2 \) (see Bosi, Candeal and Induráin [2, Example 3.2]) when \( X \) is endowed with the induced Euclidean topology on the real line.

3 Weak continuity of interval orders

In the sequel, we shall refer to the well known notion of a scale in a topological space (see e.g. Gillman and Jerison [7]).

**Definition 3.1** If \((X, \tau)\) is a topological space and \( S \) is a dense subset of \([0, 1]\) such that \( 1 \in S \), then a family \( \{G_r\}_{r \in S} \) of open subsets of \( X \) is said to be a scale in \((X, \tau)\) if the following conditions hold:
(i) \( G_1 = X \);
(ii) \( G_{r_1} \subseteq G_{r_2} \) for every \( r_1, r_2 \in S \) such that \( r_1 < r_2 \).

We are now ready to characterize the weak continuity of an interval order on a topological space.

**Proposition 3.2** Let \( \prec \) be an interval order on a topological space \((X, \tau)\). Then the following conditions are equivalent:

(i) \( \prec \) is weakly continuous;

(ii) For every pair \((x, y) \in X \times X\) such that \( x \prec y \) there exist two scales \( \{G^*_r(x, y)\}_{r \in S} \) and \( \{G^{**}_r(x, y)\}_{r \in S} \) in \((X, \tau)\) such that the family \( \{(G^*_r(x, y), G^{**}_r(x, y))\}_{r \in S} \) satisfies the following conditions:

(a) \( z \prec w \) and \( w \in G^*_r(x, y) \) imply \( z \in G^*_r(x, y) \) for every \( z, w \in X \) and \( r \in S \);

(b) \( z \prec w \) and \( w \in G^{**}_r(x, y) \) imply \( z \in G^{**}_r(x, y) \) for every \( z, w \in X \) and \( r \in S \);

(c) \( x \in G^*_r(x, y) \) and \( y \not\in G^{**}_r(x, y) \) for every \( r \in S \setminus \{1\} \).

**Proof.** Consider a pair \((x, y) \in X \times X\) such that \( x \prec y \).

(i) \( \Rightarrow \) (ii). Since \( \prec \) is weakly continuous, there exists a pair \((u_{xy}, v_{xy})\) of continuous real-valued functions on \((X, \tau)\) such that \((u_{xy}, v_{xy})\) almost represents \( \prec \) and in addition \( v_{xy}(x) < u_{xy}(y) \). Without loss of generality, we can assume that both \( u_{xy} \) and \( v_{xy} \) take values in \([0, 1]\) and that \( v_{xy}(x) = 0 \), \( u_{xy}(y) = 1 \). Define \( S = \mathbb{Q} \cap [0, 1] \), \( G^*_r(x, y) = v_{xy}^{-1}([0, r]) \), \( G^{**}_r(x, y) = u_{xy}^{-1}([0, r]) \) for every \( r \in S \), and \( G^{**}_1(x, y) = G^*_1(x, y) = X \) in order to immediately verify that \( \{G^*_r(x, y)\}_{r \in S} \) and \( \{G^{**}_r(x, y)\}_{r \in S} \) are two scales in \((X, \tau)\) such that the family \( \{(G^*_r(x, y), G^{**}_r(x, y))\}_{r \in S} \) satisfies the above conditions (a), (b) and (c).

(ii) \( \Rightarrow \) (i). From the assumptions, there exist two scales \( \{G^*_r(x, y)\}_{r \in S} \) and \( \{G^{**}_r(x, y)\}_{r \in S} \) such that the family \( \{(G^*_r(x, y), G^{**}_r(x, y))\}_{r \in S} \) satisfies the above conditions (a), (b) and (c). Define two functions \( u_{xy}, v_{xy} : X \to [0, 1] \) as follows:

\[
u_{xy}(z) = \inf \{ r \in \mathbb{Q} \cap [0, 1] : z \in G^{**}_r(x, y) \} \quad (x \in X),
\]

\[
u_{xy}(z) = \inf \{ r \in \mathbb{Q} \cap [0, 1] : z \in G^{**}_r(x, y) \} \quad (x \in X).
\]

We have that \( u_{xy} \) and \( v_{xy} \) are both continuous functions on \((X, \tau)\) with values in \([0, 1]\) (see e.g. the proof of the lemma on pages 43-44 in Gillman and Jerison...
We claim that the pair \((u_{xy}, v_{xy})\) almost represents the interval order \(\preceq\) and satisfies the condition \(v_{xy}(x) < u_{xy}(y)\).

From condition (c), we have that \(v_{xy}(x) = 0\) and \(u_{xy}(y) = 1\). It remains to show that the pair \((u_{xy}, v_{xy})\) almost represents the interval order \(\preceq\). First consider any two elements \(z, w \in X\) such that \(z \prec w\). Then, by condition (b), we have that \(v_{xy}(z) \leq u_{xy}(w)\). Finally, observe that if \(z, w \in X\) are any two elements such that \(z \preceq w\), then we have that \(u_{xy}(z) \leq v_{xy}(w)\) by condition (a). This consideration completes the proof.

\[\square\]

**References**


