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Abstract

We assume that a voter's judgment about a proposal depends on (i) the *proposal's probability* of being right (or good or just) and (ii) the *voter's probability* of making a correct judgment about its rightness (or wrongness). Initially, the state of a proposal (right or wrong), and the correctness of a voter's judgment about it, are assumed to be independent. If the *average probability* that voters are correct in their judgments is greater than ½, then the proposal with the greatest probability of being right will, in expectation, receive the greatest number of approval votes. This result holds, as well, when the voters' probabilities of being correct depend on the state of the proposal; when the average probability that voters judge a proposal correctly is functionally related to the probability that it is right, provided that the function satisfies certain conditions; and when all voters follow a leader with an above-average probability of correctly judging proposals. However, it is possible that voters may more frequently select the proposal with the greatest probability of being right by reporting their independent judgments—as assumed by the Condorcet Jury Theorem—rather than by following any leader. Applications of these results to different kinds of voting situations are discussed.

When Does Approval Voting Make the "Right Choices"?

1. Introduction

In a recent paper, Mahendra Prasad (2011) proposes an extension of the Condorcet jury theorem (CJT) to voting on multiple proposals, wherein each proposal has a probability of being right, and each voter believes that every proposal is either right or wrong.[1] He argues that the proposal most likely to be right will be chosen by approval voting (AV). His paper includes both an overview of the classical political theory literature relating to normative social choice—especially the writings of Condorcet and Rousseau—and a survey of modern social choice theory that extends the CJT to multiple proposals.

In this paper, we assume that there are multiple proposals on a ballot; as in a referendum with several propositions that voters can support or oppose, more than one proposal can be approved. Although our world is black and white—a proposal is either right or wrong, and a voter's judgment about it is either correct or incorrect—we embed it in a probabilistic framework, wherein each proposal has a probability of being right, and each voter has a probability of correctly judging its state.[2] A proposal's state, and a voter's judgment about it, are assumed, initially, to be independent.

The paper proceeds as follows. In section 2, we prove that AV in expectation chooses those proposals mostly likely to be right if and only if the *average probability* that a voter is correct about the state of a proposal is greater than ½ (Theorem 1).

[1] For "right" one can substitute desirable characteristics such as "just" or "good."
[2] In contradistinction to Prasad (2011), who assumes that proposals are either right or wrong with certainty, we assume their rightness is probabilistic, making Prasad's deterministic assumption a special case.

In section 3, we assume that the probability that a voter is correct depends on a proposal's state—whether it is right or wrong. We then show that AV chooses those proposals most likely to be right if and only if the *sum of the average probabilities* that a voter is correct about right and wrong proposals is greater than 1 (Theorem 2), which is a refinement of Theorem 1.

In section 4, we prove a negative result: AV does *not* always choose the proposal most likely to be right when the probability that a voter is correct depends on the proposal (Theorem 3). But in section 5 we show that if the average probability that a voter is correct, and the probability that a proposal is right, are functionally related in a certain way, the proposals that receive the most votes are most likely to be right (Theorem 4), echoing Theorems 1 and 2.

In section 6, we ask the following question: If all voters follow a leader who has an above-average probability of correctly judging whether a proposal is right, is their aggregated judgment better than when they vote independently? It turns out that it is—in the sense that AV better distinguishes right from wrong proposals—if the probability that proposals are right is never less than ½ (Theorem 5). Surprisingly, however, voters who make independent judgments may have a greater probability of selecting the right proposals than following a leader, showing that different measures of the "rightness" of decisions can diverge (Theorem 6).

In section 7, we discuss applications of our results to different kinds of elections, pointing out that the deliberations of committees—including the one that debated US options in the 1962 Cuban missile crisis (EXCOM)—probably best approximate the use of AV. AV is also applicable to referendums with multiple propositions, wherein voters may approve of more than one.

In section 8, we relate our results to the CJT. The CJT concerns a *single proposal*, and states that if (i) each voter has the same probability, greater than ½, of being correct, and (ii) voters' judgments of correctness are independent, then the probability that a majority of voters is correct approaches 1 as the number of jurors approaches infinity.[3]

Unlike the CJT, Theorems 1-6 do not posit a quota, such as a simple majority, but instead answer the question of which, among *multiple proposals*, are most likely to be right. We show under what conditions a proposal's AV total can be interpreted as a measure of its probability of being right. We also consider the possibility of strategic voting.

In section 9, we summarize our results for juries that must weigh multiple charges or counts, legislatures that must decide among multiple bills or amendments to bills, and elections with multiple candidates. In these very different settings, the most approved choices tend to be those with the highest probabilities of being right.

2. Judging Multiple Proposals

We begin by computing the expected number of approval votes for a proposal when voters vote for all proposals that they judge to be right. Let $p(i)$ be the probability that proposal i is right, $i = 1, 2, \ldots, m$, and let $q(j)$ be the probability that voter j judges a proposal correctly, $j = 1, 2, \ldots n$. We assume initially that the probability that a proposal is right, and that a voter judges it correctly, are independent events, so $p(i)$ and $q(j)$ are

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[3] For background and references on the CJT, see "Condorcet Jury Theorem," Wikipedia (2011); additional references will be given in section 8.

unconditional probabilities. We also assume that these probabilities fall strictly between 0 and 1,

$$0 < p(i), q(j) < 1 \text{ for all } i \text{ and } j,$$

so no proposal is certain to be right or wrong, and no voter's judgment is always correct or incorrect.

Assume that a voter approves of all proposals that he or she judges to be right and none that he or she judges to be wrong. Then there are two ways that voter j can decide to vote for proposal i: either (i) proposal i is right, and voter j judges it correctly, which has probability $p(i)q(j)$; or (ii) proposal i is wrong, and voter j judges it incorrectly, which has probability $[1 - p(i)]\,[1 - q(j)]$. We wish to calculate the *expected number of approval votes* for proposal i, $AV(i)$, and *the average per voter of this expected number*, $av(i) = AV(i)/n$.

The theorem that follows depends on the *average probability that a voter is correct*, $\bar{q} = \sum_{j=1}^{n} q(j)/n$, about a proposal (this probability, for now, is assumed to be the same for all proposals i). As we show next, the value of this average can guarantee that proposals that are more likely to be good receive, in expectation, more approval votes.

Theorem 1. *For any two proposals, i_1 and i_2, the statement that*

$$av(i_1) > av(i_2) \textit{ if and only if } p(i_1) > p(i_2)$$

is true if and only if $\bar{q} > ½$.

Proof. As noted above, the probability that voter j votes for proposal i is $p(i)q(j) + [1 - p(i)][1 - q(j)]$. It follows that the expected number of approval votes by voter j for proposal i is also $p(i)q(j) + [1 - p(i)][1 - q(j)]$. Summing this expectation over all voters yields the expected number of approval votes received by proposal i:

$$AV(i) = \sum_{j=1}^{n} \{p(i)q(j) + [1 - p(i)][1 - q(j)]\}. \tag{1}$$

Multiplying out and summing terms of (1) yields

$$AV(i) = \sum_{j=1}^{n} [1 - p(i) - q(j) + 2p(i)q(j)]$$

$$= n - np(i) - \sum_{j=1}^{n} q(j) + 2p(i)\sum_{j=1}^{n} q(j)$$

$$= n - \sum_{j=1}^{n} q(j) + p(i)[\sum_{j=1}^{n} 2q(j) - n]. \tag{2}$$

Dividing (2) by the number of voters, n, yields

$$av(i) = 1 - \bar{q} + p(i)[2\bar{q} - 1]. \tag{3}$$

It follows from (3) that $av(i_1) - av(i_2) = [p(i_1) - p(i_2)]\ [2\bar{q} - 1]$. Therefore, the signs of $av(i_1) - av(i_2)$ and $p(i_1) - p(i_2)$ are the same if and only if $2\bar{q} - 1 > 1$, which is equivalent to $\bar{q} > ½$. Moreover, (3) shows that if $\bar{q} = ½$, then $av(i_1) = av(i_2)$ without regard to the values of $p(i_1)$ and $p(i_2)$, and that if $\bar{q} < ½$, the sign of $av(i_1) - av(i_2)$ is opposite to that of $p(i_1) - p(i_2)$. ■

As an illustration of Theorem 1, assume that jurors in a criminal trial vote on multiple proposals (i.e., charges against a defendant). Effectively they are using AV to provide a measure of support for each charge. Because, as shown by (3), $av(i)$ is a strictly increasing function of $p(i)$ if $\bar{q} >$ ½, the charge that receives the greatest support is the one most likely to be right.

Note that $AV(i) = n[av(i)]$ is the sum of n independent Bernoulli random variables (i.e., binomial random variables with one trial). Consequently, even though $p(i_1) > p(i_2)$ and, as Theorem 1 guarantees, $av(i_1) > av(i_2)$, there is some probability that an unlikely event occurs and, for example, the actual vote for i_2 exceeds the actual vote for i_1. However, by the law of large numbers, the probability that such a reversal occurs approaches zero as the number of voters, n, approaches infinity. Moreover, this reversal probability diminishes as the gap between $av(i_1)$ and $av(i_2)$ increases, helping to ensure that the proposal most likely to be right is chosen by AV.

3. State Dependence

Assume next that the probability that voter j is correct about a proposal is not a constant, $q(j)$, but, instead, depends on whether the proposal is right or wrong:

- If proposal i is right, then voter j will judge it correctly with probability $q_r(j)$;
- If proposal i is wrong, then voter j will judge it correctly with probability $q_w(j)$.

Define $\overline{q_r} = \sum q_r(j)/n$ and $\overline{q_w} = \sum_{j=1}^{n} q_w(j)/n$ to be the *average probabilities that a voter makes a correct judgment about, respectively, right and wrong proposals*. This assumption of state dependence produces a refinement in Theorem 1:

Theorem 2. *For two proposals i_1 and i_2, the statement that*

$$av(i_1) > av(i_2) \textit{ if and only if } p(i_1) > p(i_2)$$

is true if and only if $\overline{q_r} + \overline{q_w} > 1$.

Proof. We rewrite (1), replacing $q(j)$ by $q_r(j)$ if the proposal i, when evaluated by voter j, is right, and with $q_w(j)$ if this proposal is wrong:

$$AV(i) = \sum_{j=1}^{n} \left\{ p(i)q_r(j) + [1 - p(i)][1 - q_w(j)] \right\}.$$

Multiplying and summing terms gives

$$AV(i) = \sum_{j=1}^{n} \left\{ 1 - p(i) - q_w(j) + p(i)[q_r(j) + q_w(j)] \right\}$$

$$= n - \sum_{j=1}^{n} q_w(j) + p(i)[\sum_{j=1}^{n} q_r(j) + \sum_{j=1}^{n} q_r(j) - n]$$

$$= n - n\overline{q_w} + p(i)[n\overline{q_r} + n\overline{q_w} - n]. \tag{4}$$

Dividing (4) by the number of voters, n, gives

$$av(i) = 1 - q_r + p(i)[q_r + q_w - 1]. \tag{5}$$

Note that $av(i)$ is increasing in $p(i)$ if and only if $\overline{q_r} + \overline{q_w} > 1$. The remainder of the proof is analogous to that of Theorem 1.■

Thus, the proposal that is most likely to be right receives, in expectation, the greatest number of approval votes, given that the sum of the average probabilities of being correct exceeds 1. Theorem 2 constitutes a generalization of Theorem 1, to which it is equivalent in the special case that $\overline{q_r} = \overline{q_w} = \overline{q}$, or when the average probability that a voter's judgment is correct does not depend on the state of the proposal.

4. Proposal Dependence

In section 2, we assumed that the probability that voter *j* correctly judges proposal *i* is $q(j)$; this probability depends only on the voter and not on the proposal. In section 3, we assumed that this probability depends on the proposal, *i*, but only insofar as it depends on the true state, right (*r*) or wrong (*w*). Thereby we replaced $q(j)$ with two probabilities, $q_r(j)$ and $q_w(j)$, that were in general different. If, for example, voter *j* is better at correctly judging proposals that are right than those that are wrong, $q_r(j) > q_w(j)$.

In this section, we assume that voter *j*'s ability to judge a proposal may be different for *every* proposal, even those in the same state of rightness or wrongness. Thereby the probability that voter *j*'s judgment is correct about proposal *i* is $q(i,j)$, a function of *i* as well as *j*.

Let $\sum_{j=1}^{n} q(i,j) = Q(i)$. Then $\overline{q(i)} = Q(i)/n$ is *the average probability that a voter is correct about proposal* i. Thus we tie the correctness of a voter's judgment to the proposal being considered—contra Theorems 1 and 2—which leads to a negative result:

Theorem 3. *If voters' probabilities of correct judgment are proposal dependent, then, even if* $\overline{q(i)} > \frac{1}{2}$ *for all values of i, it is possible for two proposals,* i_1 *and* i_2*, to satisfy* $av(i_1) < av(i_2)$ *and* $p(i_1) > p(i_2)$.

Proof. We replace the $q(j)$ in (1) with $q(i,j)$ to obtain

$$AV(i) = \sum_{j=1}^{n} \{p(i)q(i,j) + [1 - p(i)][1 - q(i,j)]\}$$

$$= \sum_{j=1}^{n} [1 - p(i) - q(i,j) + 2p(i)q(i,j)]$$

$$= n - np(i) - \sum_{j=1}^{n} q(i,j) + 2p(i)\sum_{j=1}^{n} q(i,j)$$

$$= n - q(i) + p(i)[-n + 2q(i)]. \qquad (6)$$

Dividing (6) by the number of voters, n, gives

$$av(i) = 1 - \overline{q(i)} + p(i)[-1 + 2\overline{q(i)}]. \qquad (7)$$

We next show that the tie-in of the expected average number of approval votes of a proposal and its probability of being right that we found in Theorems 1 and 2 does not necessarily hold when there is proposal dependence. Consider two proposals, $i = 1$ and 2, where $p(1) = 0.8$, $\overline{q(1)} = 0.55$, and $p(2) = 0.7$, $\overline{q(2)} = 0.75$. From (7),

$av(1) = 1 - 0.55 + (0.8)[-1 + 2(0.55)] = 0.45 + (0.8)(0.1) = 0.53.$

$av(2) = 1 - 0.75 + (0.7)[-1 + 2(0.75)] = 0.25 + (0.7)(0.5) = 0.60.$

Hence, even though $p(1) > p(2)$, $av(1) < av(2)$. ■

Theorem 3 shows that a proposal more likely to be right (i_1) may receive, in expectation, fewer approval votes than a proposal less likely to be right (i_2). Thus, unlike in Theorems 1 and 2, Theorem 3 demonstrates that the expected number of approval votes for a proposal need *not* increase in its probability of being right. The reason is that both the second and the third terms on the right side of (7) depend on i; in particular, the second term, $\overline{q(i)}$, is not a constant, as were the analogous terms, $\bar{q}$ and $\bar{q}_r$, in (3) and (5), respectively.[4]

In addition, whereas the bracketed term in (7) causes $av(i)$ to increase as $p(i)$ increases if $\overline{q(i)} > ½$, the other appearance of $\overline{q(i)}$ in (7) has the opposite effect because of its negative sign. Hence, $av(i)$ may not increase when $p(i)$ increases, as the preceding example demonstrates.

Of course, if the average probability that a voter is correct does not depend on the proposal, that is, if $\overline{q(i)} = \bar{q}$ for all i, then Theorem 1 applies. When this is the case, as we showed in section 2, there is a positive association between the approval votes for a proposal and the probability that it is right. Are there other kinds of dependence in which this positive association holds?

5. Other Kinds of Dependence

As we show now, there can be a positive association between the approval votes for a proposal and its probability of being right, even when the voters' probabilities of judging correctly depend on the proposal. We now assume any proposal has a probability p of being right and an average probability $\bar{q}$ or being judged correctly by the voters. We will characterize such a proposal $(p, \bar{q})$.

For illustration, assume that all proposals satisfy $\bar{q} = p$.[5] In words, the probability that a proposal is judged correctly by all voters equals the probability that it is right. Then (7) becomes

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[4] While differentiation of (3) and (5) with respect to $p(i)$ would cause these terms to vanish, this is not the case for $\overline{q(i)}$ in (7) unless it is known that there is no functional relation between $\overline{q(i)}$ and $p(i)$.

$$av = 1 - p + p[-1 + 2p]$$
$$= 2p^2 - 2p + 1, \quad (8)$$

which is the average approval vote of a proposal when $p = \bar{q}$. Differentiating (8) with respect to p yields

$$\frac{dav}{dp} = 4p - 2 = 0, \quad (9)$$

which is positive if and only if $p > ½$, a condition that is analogous to $\bar{q} > ½$ in Theorem 1, wherein $\bar{q}$ was assumed constant.

Thus, if $\bar{q} = p$, approval votes track the probability that proposals are right when $p > ½$, illustrating how the conclusions of Theorems 1 and 2 can hold even when the correctness of voters' judgments is proposal-dependent. From (9) we know that an infinitesimal increase in p will produce an infinitesimal increase in av. More specifically, within the interval $½ < p \leq 1$, any two proposals can be compared, and the one with the greater p will, in expectation, receive more approval votes.

We now generalize the forgoing example to all proposals that have a probability p of being correct. We assume that the probabilities are selected from the interval of real numbers, [0,1], and for every proposal, the average probability that the voters judge it correctly, $\bar{q}$, is related to p by a differentiable function, which we denote $\bar{q}(p)$.[6] In Theorem 1, this function $\bar{q}(p)$ was constant (i.e., did not depend on p); in the example

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[5] Because $p(i)$ and $\overline{q(i)}$ are assumed equal and, therefore, do not depend on i (their values depend only on each other), we can eliminate i as an argument.

[6] This assumption is in fact rather restrictive. It implies, for example, that any two proposals with the same value of p also have the same value of $\bar{q}$.

just discussed, $\bar{q}(p) = p$. Now we do not assume a particular form for $\bar{q}(p)$ but, rather, derive the conditions that it must satisfy to make the expected number of approval votes a strictly increasing function of *p*.

Theorem 4. *Suppose that for all proposals* $(p, \bar{q})$, $\bar{q}$ *is functionally related to p and, moreover,* $\bar{q}(p)$ *is differentiable. If there is a subinterval of values of p where*

$$\frac{d\bar{q}}{dp} > \frac{1-2\bar{q}}{2p-1} \text{ whenever } p > 1/2;\ \bar{q} > ½ \text{ if } p = ½; \text{ and } \frac{d\bar{q}}{dp} < \frac{2\bar{q}-1}{1-2p} \text{ whenever } p < 1/2,$$

then, provided $p_1 > p_2$ *and* p_1 *and* p_2 *both lie in this subinterval, AV chooses* p_1 *over* p_2 *in expectation.*

Proof. Writing (7) without dependence on *i*, we differentiate *av* with respect to *p*:

$$\frac{dav}{dp} = -\frac{d\bar{q}}{dp} + p\left[2\frac{d\bar{q}}{dp}\right] + [-1 + 2\bar{q}]$$

$$= \frac{d\bar{q}}{dp}[-1 + 2p] + [2\bar{q} - 1]. \tag{10}$$

Note that if $p = ½$, the right side of (10) will be positive if and only if $\bar{q} > ½$. Similarly, if $p > ½$, then $-1 + 2p > 0$, in which case the right side of (10) will be positive if and only if $\frac{d\bar{q}}{dp} > \frac{1-2\bar{q}}{2p-1}$. The remaining condition, $p < ½$, is analogous to $p < ½$. ■

It is easy to link Theorem 4 with Theorem 1: When $\bar{q}(p) = \bar{q}$ is constant, all three conditions of Theorem 4 are satisfied if and only if $\bar{q} > ½$, in which case the subinterval is [0,1]. In the case of $\bar{q}(p) = p$ discussed earlier, the conditions of Theorem 4 are satisfied for the subinterval (½,1]—that is, if and only if $p > ½$.

A parallel example to that of $\bar{q}(p) = p$ is $\bar{q}(p) = 1 - p$, in which case the subinterval is [0,½), and the conditions of Theorem 4 are satisfied if and only if $p < ½$. Many other examples could be constructed. The most realistic, we think, are those in which $\bar{q}(p)$ is monotonically increasing in p, but not necessarily linearly. For example, $\bar{q}(p)$ may increase slowly near $p = ½$, but then rapidly as p approaches 1, if the proposals most likely to be right are much more likely to be judged correctly.

To summarize, when voters' probabilities of being correct depend on the proposal being considered, AV does not necessarily single out the proposals most likely to be right (Theorem 3). However, if the average voter's probability of being correct is a differentiable function of the probability of the proposal's being right, then Theorem 4 provides conditions on this function that ensure that the expected number of approval votes of a proposal reflects the probability that that proposal is right.

6. Follow-the-Leader

For convenience, we henceforth assume that a voter's judgment is equally good—on any proposal, whether it is right or wrong—rendering Theorem 1 applicable. In expectation, therefore, the proposal that receives the most approval votes is the one with the greatest probability of being right if and only if $\bar{q} > ½$.

We next ask whether voters might improve the chance that a proposal most likely to be right is selected if they all follow the advice of some leader, $j = L$. We denote by

$av_L(i)$ the average number of approval votes received by proposal i when all voters follow L.

One might expect that follow-the-leader would be an especially good strategy for selecting the proposal most likely to be right when $q(L) > \bar{q}$, or L has an above-average probability of judging proposals correctly. The next theorem shows that this is indeed true—follow-the-leader surpasses the independent judgments of the voters in distinguishing candidates with the greatest probabilities of being right, based on their AV totals. However, this result is complicated by an issue that we will discuss shortly.

Theorem 5. *If $q(L) > ½$, then for two proposals i_1 and i_2, $av_L(i_1) > av_L(i_2)$ if and only if $p(i_1) > p(i_2)$. Moreover, $av_L(i_1) - av_L(i_2) > av(i_1) - av(i_2)$ if and only if $q(L) > \bar{q}$.*

Proof. Replacing $\bar{q}$ by $q(L)$ in equation (3)—because the average q for L is simply $q(L)$ when all voters follow his or her advice—gives the average number of approval votes that proposal i receives from a voter who votes according to L's choice:

$$av_L(i) = 1 - q(L) + p(i)[2q(L) - 1]. \tag{11}$$

It follows from (11) that

$$av_L(i_1) - av_L(i_2) = [p(i_1) - p(i_2)]\ [2q(L) - 1].$$

Thus, if $q(L) > ½$, then the proposal that is more likely to be right (i.e., i_1) receives, in expectation, a greater number of approval votes if all voters follow the leader, L.

But recall from (3) that

$$av(i_1) - av(i_2) = [p(i_1) - p(i_2)]\ [2\bar{q} - 1].$$

Then

$$\frac{av_L(i_1) - av_L(i_2)}{av(i_1) - av(i_2)} = \frac{2q(L) - 1}{2\bar{q} - 1}. \tag{12}$$

It is easy to verify that the fraction on the right side is greater than 1 if and only if $q(L) > \bar{q}$. ■

Thus, L must indeed be above average in his or her ability to judge proposals correctly to make it rational for voters to follow his or her advice rather than relying on their own independent judgments. In doing so, voters increase the chances that AV will choose the proposal(s) most likely to be right—by widening the difference between the approval votes of better and worse proposals—compared with what independent judgments produce.

More specifically, from (12) the gap between $av_L(i_1)$ and $av_L(i_2)$ is an increasing linear function of $q(L) - \bar{q}$. This implies that it is beneficial to choose the best leader—that is, the person with the highest value of $q(L)$—assuming this information is known.

But there is another issue, which we illustrate with a simple example with two voters, L and F (for follower), where $q(L) = 0.8$ and $q(F) = 0.6$ so that $\bar{q} = 0.7$. Assume there are two proposals, where $p(1) = 0.9$ and $p(2) = 0.8$.

If the voters exercise their judgments independently, then according to (3),

$$av(1) = 1 - 0.7 + (0.9)[2(0.7) - 1] = 0.3 - (0.9)(0.4) = 0.66$$

$$av(2) = 1 - 0.7 + (0.8)[2(0.7) - 1] = 0.3 - (0.8)(0.4) = 0.62.$$

On the other hand, if the voters follow the leader L, then according to (11),

$av_L(1) = 1 - 0.8 + (0.9)[2(0.8) - 1] = 0.2 - (0.9)(0.6) = 0.74$

$av_L(2) = 1 - 0.8 + (0.8)[2(0.8) - 1] = 0.2 - (0.8)(0.6) = 0.68.$

Notice that proposal 1 garners, in expectation, more approval votes than proposal 2, regardless of whether the voters make their own independent judgments or follow *L*. However, as guaranteed by Theorem 5, follow-the-leader provides a bigger "spread" between the proposals 1 and 2 (0.74 – 0.68 = 0.06) than if *L* and *F* record their independent judgments (0.66 – 0.62 = 0.04). Thus, if the procedure were repeated many times, we would expect that the average vote for proposal 1 would be greater under follow–the-leader.

Surprisingly, it is not true that follow-the-leader *more surely* chooses the proposal with the greater probability of being right. As we show next, the probability that follow-the-leader favors proposal 1 over proposal 2 is 0.237. On the other hand, the probability that independent judgments favors proposal 1 is 0.331. Hence, the probability that follow-the-leader gives the correct decision is actually less than independent judgments.

How can this be? We prove this result, using the forgoing example, next.

Theorem 6. *The proposal that is most likely to be right can be chosen less frequently under follow-the-leader than under independent judgments.*

Proof. Under follow-the-leader, proposal 1 beats proposal 2 if and only if *L* approves of proposal 1 and does not approve of proposal 2, judgments we indicate by (1,0). This is because if *L* approves of proposal 1 and disapproves of proposal 2, so will *F*; hence, proposal 1 will defeat proposal 2 by 2-0.

The leader, *L*, approves of proposal 1 when

1. *L* judges proposal 1 correctly (with probability 0.8), and proposal 1 is right (with probability 0.9), which has a joint probability of 0.72.

2. *L* judges proposal 1 incorrectly (with probability 0.2), and proposal 1 is wrong (with probability 0.1), which has a joint probability of 0.02.

These probabilities sum to 0.74. Similarly, *L* disapproves of proposal 2 when

1. *L* judges proposal 2 correctly (with probability 0.8), and proposal 2 is wrong (with probability 0.2), which has a joint probability of 0.16.

2. *L* judges proposal 2 incorrectly (with probability 0.2), and proposal 2 is right (with probability 0.8), which has a joint probability of 0.16.

These probabilities sum to 0.32.

It follows that the probability that *L*'s judgment is (1,0) is the product, $(0.74)(0.32) \approx 0.237$. Of course, in the follow-the-leader model, *F* follows *L*, so proposal 1 defeats proposal 2 under follow-the-leader with probability 0.237. By comparison, L's judgment is (1,1) with probability 0.503, (0,1) with probability 0.177, and (0,0) with probability 0.083. In the latter three cases, of course, proposal 1 does not defeat proposal 2 under follow-the-leader.

These possibilities exhaust the approval/disapproval choices of *L*, so their probabilities necessarily sum to 1. Note that the most likely event is (1,1), occurring more than half the time (0.053), in which *L (*and *F)* approve of both proposals and thereby create a tie between them.

We next show, though without giving full details, that independent judgments give a higher probability that proposal 1 will defeat proposal 2 than does follow-the-leader, even though *L* is more likely to be right than *F* (0.8 vs. 0.6) and, therefore, a better-than-average judge. Proposal 1 can defeat proposal 2 in three different ways, shown in the first column of the following table:

Vote Combinations and Probabilities of Being Right under Independent Judgments

Vote Combinations	**Leader (L)**	**Follower (F)**	**Probability**
(2,1)	(1,1)	(1,0)	0.128
	(1,0)	(1,1)	0.077
(2,0)	(1,0)	(1,0)	0.069
(1,0)	(1,0)	(0,0)	0.044
	(0,1)	(1,0)	0.021
Sum			0.331

The leader (*L*) and follower (*F*) columns show the approvals of proposals 1 and 2, by each voter, that yield the vote combinations to the left. For example, (2,1) can occur if *L* approves of proposals 1 and 2 but only one of *L* or *F* approves of proposal 2. We have calculated the probabilities, in a manner analogous to that for *L* in the case of (1,0) under follow-the-leader, in the fourth column for independent judgments. They sum to 0.331, which is 40 percent higher than the probability of 0.237 that we found for follow-the-leader.[7]

[7] The divergence between the probability that a proposal is right and its expected approval vote is mirrored in other realms. For example, the probability of a net gain from a sequence of bets, on the one hand, and the expected amount from that gain on the other, may be at odds. Davis (2001, pp. 23-29) gives the example of a seemingly profitable bet in which you win twice your bet with probability ½ and lose just your bet with probability ½. If you start with $100 and always bet 75 percent of your capital, after only two bets your expected winnings are $189.06, but 75 percent of the time you end up with less than $100. To maximize the rate of growth of your capital, it turns out, you should bet a fraction chosen not to maximize your expected winnings, but rather their logarithm, which is known as Kelly betting. This is a so-called Markov betting system, because the outcome depends only on your present bankroll and your probability of winning (Epstein, 1977, pp. 60-62).

Because in our example there are only two proposals and two voters—each with a high probability of correctly judging the proposals—there is a substantial probability of ties (more than 50 percent). Although we focused on the situation in which the proposal more likely to be right (proposal 1) receives strictly more support than the other proposal (proposal 2)—making proposal 1 the "winner"—in situations such as referendums with multiple propositions on the ballot, more than one proposition might win (e.g., with majority approval).

If the number of voters is large, ties become highly unlikely. While follow-the-leader and independent judgments will give a similar, if not the same, ranking of proposals, there is an important difference in how they determine rankings.

In comparing two proposals, follow-the-leader makes an error whenever the leader makes an error, which clearly depends on $q(L)$; this error rate does not decrease as the number of voters decreases. But the effective error rate does decrease in the case of independent judgments, which in general yields a number of approval votes very close to the expected number if the number of voters is large.

As long as there is some difference in the expected number of approval votes that different proposals receive, a large enough electorate will reliably distinguish better from worse proposals under both follow-the-leader and independent judgments. Whereas follow-the-leader is superior at drawing this distinction, its dependence on all-or-nothing votes gives it a higher error probability in selecting the right proposal(s). As with the Condorcet Jury Theorem (CJT)—to which we will later compare our results—having a large number of voters tends to produce the right outcome, however votes are aggregated or whatever the decision rule is for choosing a winner.

7. Applications to Politics

At the beginning of their deliberations in a case, jurors are often divided on a verdict. But, typically, they move toward a decision—unanimous or near-unanimous, as specified by the rules that govern the case—making hung juries relatively rare. On average, about 10 percent of all cases result in hung juries (Hannaford-Agor, Hans, Mott, Munsterman, 2002).

During their deliberations, jurors will often be persuaded by the juror who offers the most persuasive arguments, whom we assume has an above-average probability of being correct. Assume that this juror is L, and for definiteness assume that $q(L) = \max\{q(j): j = 1, 2, \ldots, n\}$—that is, L has the highest $q(j)$.

If all jurors follow L, we showed in section 6 that, provided $q(L) > \bar{q} > ½$, the proposals with the greatest $p(i)$'s—the ones most likely to be right—will in expectation garner the most approval votes. However, if the jurors exercise their own independent judgments, the probability of this event's occurring may actually be greater.

This argument for following the lead of L is contrary to that made against "groupthink" (Janis, 1972), in which independent thinking is suppressed in favor of achieving a group consensus, often leading to poor decisions. But if we assume that the average juror is *persuaded* by L, where $q(L) > \bar{q} > ½$, independent thinking will not be suppressed but, instead, be replaced by the superior thinking of L, based on the more persuasive arguments L offers compared with those offered by other jurors.

Of course, if L's arguments persuade jurors to support proposals that are more likely to be wrong than right (i.e., $p(i) < ½$), then follow-the-leader will have a perverse effect. But this will not be true if $q(L) > \bar{q} > ½$, in which case follow-the-leader will draw a sharper distinction than independent judgments between better and worse proposals, though independent judgments may maximize the probability that the better proposal will be chosen.

Our model is applicable to groups other than juries. As a case in point, consider the deliberations of EXCOM, the executive committee of high-level government and other officials who debated options that the United States might choose during the Cuban missile crisis of October 1962 (Brams, 2011, pp. 226-240, and references therein). Although EXCOM members at the outset leaned toward an air strike against the Soviet missiles in Cuba, most of its members were persuaded in the end to recommend to President John Kennedy the less aggressive action of a naval blockade (called, euphemistically, a "quarantine" at the time) and only consider more aggressive action if the blockade failed to induce the Soviets to withdraw their missiles from Cuba.

In the deliberations of EXCOM, Robert Kennedy, the attorney general and brother of President Kennedy, seems to have fulfilled the role of L. He warned that an air strike would be seen as "a Pearl Harbor in reverse, and it would blacken the name of the United States in the pages of history" (Sorensen, 1965, p. 684). To be sure, the fact that Robert Kennedy and his supporters were successful in persuading other EXCOM members to support a blockade cannot be taken as conclusive evidence that follow-the-leader will always succeed, but it does illustrate one instance in which persuasion seems to have abated a major political-military crisis, leading to its peaceful resolution.

In democracies, political parties and their candidates put forward proposals to solve problems and advance their positions; suppose that associated with each proposal is a probability of its being right, or at least providing some remedy. Are the proposals selected (including the status quo) the ones most likely to be right?

Our model is inapplicable to legislatures and other voting bodies wherein proposals come up one at a time and then are voted up or down. Because voting is sequential in these bodies, voters cannot approve or disapprove, simultaneously, of multiple proposals. In such settings, the ordering of proposals (e.g., amendments to a bill) that are voted on can, for strategic reasons, critically affect the support they receive, so their votes do not provide an accurate gauge of their degree of sincere support.

In elections in which there are multiple candidates on a ballot, usually a choice of only one candidate is possible. Even if the voter is permitted to rank the candidates, this ranking does not say where the voter would draw the line between approved and disapproved candidates, though systems have been proposed that would allow this (Brams, 2008, ch. 3; Brams and Sanver, 2009).

Besides juries that consider multiple charges, or committees like EXCOM that deliberate over multiple strategies, referendums with multiple propositions on the ballot come closest to fitting the AV model. The propositions can be considered proposals, and voters can approve of more than one.

Usually a simple majority determines which propositions pass. If, however, two or more propositions contradict each other, and each gets a majority of votes—as can happen—the usual rule is that the proposition with the most votes is enacted. Because this is the proposition most likely to be right according to our model, this rule is consistent with passage of those (noncontradictory) propositions most likely to be right.

In both jury/committee settings and referendums, voters typically follow the leads of different proponents, who may espouse different positions. The question our analysis raises is whether the leader who persuades the most voters to approve of his or her favored proposal helps the one most likely to be right.

Because there is not usually a single L but, instead, multiple leaders who take different positions on proposals, one must be careful how to define "right." Previously, we defined $p(i)$ to be the probability that proposal i is right (or good or just).

But suppose that there are two leaders, one of whom supports proposal i and the other of whom opposes it. Assume that all voters support the positions of one of the two leaders. Then if we interpret $p(i)$ to be the probability that the supporter of proposal i is right, and $1 - p(\mathrm{i})$ to be the probability the opponent is right, then AV will choose the proposal with the higher probability of being right.

While this interpretation of our model certainly applies to multiple propositions in a referendum[8]—in which one can approve or not approve of each—how does it apply to elections with multiple candidates? We suggest that a useful way to think about candidates who take positions on multiple proposals is as *composites of positions*. Under AV, the voter who approves of one or more candidates is saying, in effect, that he or she approves of their composite positions—at least more so than the composite positions of other candidates that fail to receive his or her approval.

[8] In a referendum, there is, of course, a third option—namely, to abstain. If there is no quorum, abstention has no effect, but if a minimum percentage (e.g., 50) of the electorate must participate to allow for the passage of a proposition, then if this minimum is not achieved, it seems reasonable to interpret nonenactment as the right choice, even if the proposition receives majority support. This is because the failure to achieve a quorum can be deemed as insufficient support to make a choice binding on the electorate.

In this interpretation, the $p(i)$'s are associated with each candidate i, who represents a composite of positions on what we earlier called proposals—the issues of the day in an election. But are the candidates who receive the most approval the ones whose composites of positions are the ones most likely to be right?

In the context of elections, "appealing" might be a better word to use than "right," because there is usually no right or wrong position, or composite of positions, as such (unlike the guilt or innocence of a defendant in a criminal trial). But if we associate the appeal or popularity of a candidate with his or her being the right choice, then AV will make the right choice in elections.

To be sure, the "people's choice" in such elections is not what many political philosophers, at least since Plato, would consider the right choice. But if the popular will—even if it does not always mirror the ideal of Rousseau's general will—is the cornerstone of democracy, then it is appropriate to consider it synonymous with the right choice in elections.[9]

8. Relationship to the Condorcet Jury Theorem (CJT)

The CJT assumes that there is a *single* proposal, which is either right or wrong. Unlike our model, it does not have a probability associated with being in one state or the other.

Like our model, however, a proposal's rightness or wrongness is judged by jurors who themselves have probabilities of being correct. The CJT says that if *all* jurors have the same probability, greater than ½, of being correct, then the proposal's probability of being judged correctly by a majority of jurors approaches 1 as the number of jurors

[9] Miller (1986) makes this argument in applying the Condorcet jury theorem to elections, but with the qualifying phrase that all voters, even if they have conflicting interests, are "fully informed."

approaches infinity.[10] By contrast, Theorem 1 does not assume that *every* voter j has a probability of being correct, or $q(j) > ½$, but, instead, only that the mean of all voters' probabilities, $\bar{q}$, exceeds ½, in order that the approval votes for proposals mirror their probabilities of being right.[11]

These proposals might be different versions of a bill before a legislature. Similarly, each charge in a criminal trial may be true, or each candidate in an election may be a good choice, with a certain probability. If this is the case, then Theorem 1 says that the bill, charge, or candidate that is most approved is the one that is most likely to be right, true, or good.

Unlike the CJT, majority rule has no special significance: Except when there is proposal dependence (Theorems 3), the proposal that receives the most approval votes, which may or may not be a majority, is the one most likely to be right. But as with CJT, as the number of voters increases, the probability of making the right choice increases, provided that the added voters do not reduce the value of $\bar{q}$.[12]

Might another voting system choose the proposal or proposals most likely to be right? Consider plurality voting, in which each voter is restricted to casting one vote. In order for him or her to select the proposal most likely to be right, voters would have to be able to identify the degrees of rightness of each proposal i, as given by $p(i)$, in order to choose the one most likely to be right. But we assume that the $p(i)$'s are unknown to the

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[10] Nitzan (2010, pp. 205-207) shows under rather general conditions when simple-majority rule gives a higher probability that a proposal is judged correctly than the "expert rule," which is follow-the-leader when the leader is the voter with the greatest probability of being correct.

[11] An "extended" CJT ensures that if an *average* juror has a probability of being correct that is greater than ½, the probability that a jury will make the right decision approaches a value less than 1, which is a function of e (Grofman and Owen, 1986).

[12] This is true without regard to the values of the $p(i)$'s, which may happen, for example, if a company will surely fail if it does nothing. However, if there is some less-than-even chance of success if it takes some risky action, then the most approved action, even if it will probably fail, is still better than doing nothing.

voters; in the absence of this knowledge, the aggregation of plurality votes need not single out the proposal most likely to be right, even if $\bar{q}$ is high.

To get plurality voting to choose the proposal most likely to be right, voters' judgments about proposals would need to be conditioned on each proposal's rightness. But even for approval votes, as we showed in Theorem 3, this creates problems. Only when the voters' average probability of judging a proposal correctly is functionally related (in an appropriate manner) to the probability that the proposal is right (Theorem 4) can the approval votes of proposals reflect their probabilities of being right.

Our model assumes that voters respond to a signal—based on $q(j)$ or perhaps $q(L)$—that they receive on each proposal i; this proposal has a probability, $p(i)$, of being right. Might voters do better responding strategically rather than sincerely?

To inquire about strategy presumes that voters have preferences over outcomes, which the $q(j)$'s in our model do not assume. However, if one makes this assumption (see Feddersen and Pesendorfer, 1999, and references therein), jurors can do better by conditioning their decisions on their probabilities of being pivotal, which will depend on both the decision rule and how other jurors vote. Thus, for example, if a verdict requires unanimity in a criminal trial, then a juror will be pivotal if and only if all the other jurors vote to convict, making his or her vote decisive either in convicting or in acquitting the defendant.

But when there are more than a few voters, a voter's pivotalness becomes less meaningful as a basis for making a choice. Indeed, the voter's probability of being decisive becomes negligible as the number of voters becomes larger and larger. Moreover, under AV, the question is less one of making the right choice on a single proposal and more one of where to draw the line between acceptable and unacceptable proposals, as analyzed in Brams and Fishburn (1978, 1983).

In the present model, voters seem well advised to make their own best judgments about proposals, either according to $q(j)$ or by following a leader according to $q(L)$. To deviate from these signals, voters—or the leaders whom they follow—would need to have information, which we do not assume, that there is at least the potential to produce more right choices by ignoring or countermanding their signals. Unless the strategic environment provides voters with the opportunity to obtain this information, it seems reasonable to assume that they will be sincere.

True, if voters follow *different* leaders, then the strategic situation changes—a competitive election is no longer just a search for right choices. For example, a leader may advise a voter not to vote for a candidate for whom the voter receives a favorable signal, lest this candidate beat a candidate preferred by the leader. On the other hand, if multiple proposals can be approved, as in a referendum, then supporting one proposal need not affect the choice of another, in which case strategic voting is not an issue.

9. Conclusions

We have shown that the most approved proposals will be those with the greatest probability of being right if and only if the average probability that the judgment of a voter is correct exceeds ½ (Theorem 1). This necessary and sufficient condition allows some voters to have probabilities of being correct that are less than ½, provided they are counterbalanced by voters who raise the average above ½.

Although Theorem 1 and the subsequent theorems bear some similarity to the CJT, their differences are substantial. First, except for Theorems 3 and 4, the theorems assume that proposals have probabilities of being right independent of the judgments of voters. Second, there is not a single proposal but multiple proposals, all of which may have varying degrees of rightness.

While the most approved proposals under AV will be the ones most likely to be right in most circumstances, this may not true under plurality voting. The reason is that a voter, not knowing the $p(i)$'s, must cast his or her single vote on the basis of his or her $q(j)$ alone, which does not distinguish among proposals. Under AV, however, all voters vote with *some* positive probability for all the proposals; except for Theorem 3, our theorems ensure that the expected number of approval votes is greater for the proposals more likely to be right.

More specifically, this is true not only if proposal probabilities and voter probabilities are based on independent events but also if the probability that a voter makes a correct judgment about a proposal depends on its state (i.e., whether it is right or wrong, as shown in Theorem 2). While this is not generally true if voter probabilities depend upon the proposal being considered (Theorem 3), approval votes track the rightness of proposals if the average probability that a voter is correct, and the probability that a proposal is right, are functionally related in certain ways (Theorem 4).

So does follow-the-leader if the leader has an above-average probability of being correct, which sometimes—but not always —may be preferable to voters' making independent judgments (Theorem 5). This may be one reason why defendants, who think their case is strong, sometimes prefer that their case be heard by a judge with a high $q(j)$ than a jury with a lower $\bar{q}$. However, when the number of voters is small, as in a committee, the independent judgments of its members may more often lead to the right decision (Theorem 6), illustrating the divergence between the probability that a collective choice is right and its expected approval vote.

AV is most applicable to situations in which there are multiple alternatives that voters must choose among, such as criminal charges in a trial, proposals in a committee, or candidates in an election, all of which have some probability of being right (i.e., $p(i) > 0$). We have shown that AV is well suited to finding the best—the most likely to be right, good, or just—among them, although strategic considerations may intercede if there are multiple leaders contesting elections.

We conclude on a note of caution. Our results on selecting the proposals most likely to be right are—except for calculating the probability that a 2-person committee makes the right decision in section 6—based on the expected approval of these proposals, which will not always be realized in practice. Especially if the electorate is small, random variability may occasionally imply that the most approved proposals are not be the ones most likely to be right. As the electorate increases in size, however, the correctness of choices becomes more and more certain under AV—without the need, à la the CJT, to assume that every voter is better than a random coin toss.

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