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Abstract

A cake is a metaphor for a heterogeneous, divisible good. When two players divide such a good, there is always a perfect division—one that is efficient (Pareto-optimal), envy-free, and equitable—which can be effected with a finite number of cuts under certain mild conditions; this is not always the case when there are more than two players (Brams, Jones, and Klamler, 2011b). We not only establish the existence of such a division but also provide an algorithm for determining where and how many cuts must be made, relating it to an algorithm, "Adjusted Winner" (Brams and Taylor, 1996, 1999), that yields a perfect division of multiple homogenous goods.

1. Introduction

There has been a substantial literature on cake-cutting over the past fifteen years; books giving overviews of both existence results and algorithms for physically cutting a cake include Brams and Taylor (1996), Robertson and Webb (1998), Barbanel (2005), and Brams (2008). Review articles of the fair-division literature that discuss cake-cutting include Brams (2006) and Klamler (2010).

Recent algorithms, involving both discrete and continuous (moving-knife) procedures, are analyzed in Barbanel and Brams (2004, 2011) and Brams, Jones, and Klamler (2006, 2011a, 2011b). There is also a growing literature on pie-cutting (Brams, Jones, and Klamler, 2008; Barbanel, Brams, and Stromquist, 2009; Barbanel and Brams, 2011), in which radial cuts are made from the center of a pie rather than parallel cuts being made along the edge of a cake.

A key question that has *not* been addressed in most of this literature is how many cuts are required to give what we call a *perfect division* of a cake (or pie)—one that satisfies the following three properties that encapsulate the idea of fair division:

1. *Efficiency (Pareto-optimality)*: There is no other division that gives each player a portion that he or she values at least as much and that one player values strictly more.
2. *Envy-freeness*: Each player values his or her portion at least as much as that of every other player and, consequently, does not envy any other player.

[1] We gratefully acknowledge the valuable comments of Peter Landweber and Ariel D. Procaccia on an earlier version of this paper.

3. *Equitability*: Each player values his or her portion exactly the same as other players value their own portions—that is, each player thinks that his or her portion is the same fraction of his or her perceived value of the entire cake.

If there are $n > 2$ players, there may be no perfect division of a cake (Brams, Jones, and Klamler, 2011b). But if $n = 2$, we prove there is such a division; how many cuts are required depends on the players' preferences for different parts of the cake.

It is well known that if $n = 2$ and only one cut is allowed, there is a division that satisfies properties (2) and (3) (Jones, 2002)—but not necessarily property (1)—though no algorithm is known for obtaining it (Brams, Jones, and Klamler, 2006). But if more than one cut is allowed, efficiency can generally be obtained, as we will show, at least if the players have the help of a referee who has complete information about their preferences.[2]

To illustrate the problem of limiting two players to one cut, think of a cake in which player 1 most likes the parts near the edges, whereas player 2 most likes the part around the center. Then an efficient division would be to give player 1 the two parts near the edges, and player 2 the remainder around the center, which requires two cuts. This efficient division can be made equitable with appropriate cuts—giving each player, say, ¾ of the cake as each values it—which is also envy-free since both players receive more than half the value of the cake.

We give conditions under which there exists an *optimal number of cuts*, by which we mean a number such that fewer cuts are not sufficient to give a perfect division and more do not help. Beyond demonstrating the existence of a perfect division for two players, we provide an algorithm for (i) determining the optimal number of cuts (if such a number exists) and (ii) specifying where they must be made. Not only are our results constructive, but we also show that our algorithm is applicable to pie-cutting and related to a discrete algorithm, "Adjusted Winner" (Brams and Taylor, 1996, 1999; Jones, 2002; Brams, 2008), for dividing multiple homogenous goods.

A cake, by contrast, is a single heterogeneous good, which the players may value different parts of differently. While the cake may be a swirl of flavors and toppings, we assume that the players can attach a value to it at every point along an edge. (We will be more specific about these values in the next section, when we introduce probability density functions.) Physically, the cake may be thought of as a rectangle, valued along an edge, but it could be any shape and valued along a straight line that passes through it.

In section 2, we state the measure-theoretic assumptions we make about a cake as a mathematical object when valued along a line segment. We then briefly describe our algorithm for dividing it, based on the information that the players provide to a referee

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[2] Consequences of this assumption for pie-cutting are discussed in Barbanel and Brams (2011). Suffice it to say here that it is easier for a referee to compute an equitable division of a cake between two players than it is of a pie, which requires a continuous comparison of the values of the different players' pieces as "moving knives" are rotated.

about their subjective valuations of the cake, but we reserve for section 3 the details of its implementation.

In section 3, we introduce a geometric framework for viewing and analyzing our algorithm, and we illustrate its use with several examples, showing how it succeeds in finding a perfect division under different conditions. We complement this analysis in section 4 by showing, for certain classes of situations, how to calculate the number of cuts needed to obtain a perfect division; only in extreme cases is an infinite number of cuts required.

In section 5, we show that Adjusted Winner can be viewed as a special case of our algorithm. Adjusted Winner was first analyzed in Brams and Taylor (1996), popularized in Brams and Taylor (1999), and patented by New York University in 1999. NYU licensed the patent rights to a Boston law firm in 2007; the firm formed a company, Fair Outcomes, Inc., which markets it and other patented fair-division algorithms. In section 6, we offer some concluding remarks, emphasizing the applicability of our algorithm to real-life problems of fair division, such as the division of land.

2. Assumptions and the Cake-Cutting Algorithm

We make the following assumptions about the cake to be divided and the players' valuations of it:

1. The cake C comprises the unit interval [0,1]. There are two players, player 1 and player 2, whom we will refer to as "he" and "she," respectively.
2. A *piece* of cake refers to a connected subset of the cake. A *portion* of cake is a finite collection of pieces.
3. Player 1 and player 2 have *probability density functions* (*pdfs*) f_1 and f_2, respectively, whose domain is the cake (i.e., the interval [0,1]), with each taking on only positive values.[3] The total area under the graphs of each of these functions is 1, as required for a pdf.
4. We define functions μ_1 and μ_2, called *measures*, whose domain consists of the set of all portions of cake, and whose values for a given portion of cake are given by the area under the parts of the graphs of f_1 and f_2, respectively, that correspond to that portion of cake.

Intuitively, we may view the pdfs f_1 and f_2 as giving a value for the worth of every point of cake to each of the players. To be sure, any single point has value zero to each player, but areas corresponding to intervals and collections of intervals of cake—given by μ_1 and μ_2—have positive value. If we are given the measures μ_1 and μ_2, we can obtain the pdfs f_1 and f_2 from these measures.

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[3] This is a necessary technical assumption. Because we will later define a new function that is the ratio of f_1 and f_2, we require this assumption to preclude dividing by 0. It says, in effect, that neither of the two players ever considers any part of the cake worthless.

In brief, the rules of our algorithm for dividing a cake are as follows (details, with examples, follow later):[4]

1. Two players, player 1 and player 2, independently submit their pdfs to a referee, who could be a computer that accepts this information. (The referee makes only "objective" decisions based on this information.)
2. The referee places marks on the cake at all points where the pdfs cross (either by intersecting at a point, or by discontinuously jumping over each other). Initially, the referee awards each subinterval, defined by two adjacent marks, to the player whose pdf is greater in that subinterval.
3. If the players' pdfs do not cross at a point but are equal over some subinterval, then the referee places marks at the endpoints of this subinterval and awards it to player 1 initially.
4. If the sum of the areas that the two players receive from their subintervals are equal, the algorithm terminates. The players receive the subintervals that they were awarded initially.
5. If the areas are not equal, the player who receives the larger sum (say, player 1) gives subintervals, or portions thereof (i.e., sub-subintervals), to player 2 according to the ratio of player 1's pdf to player 2's pdf, starting with the subinterval in which this ratio is the smallest (possibly 1 if the players pdfs are equal on some subinterval) and letting this ratio increase.
6. This giveback process may entail awarding sub-subintervals from different subintervals that player 1 was awarded initially. When the areas of the two players are equal, the algorithm terminates.

In an earlier article (Barbanel and Brams, 2011), but in a different context (pie-cutting), we asked the question of whether a player would be truthful in submitting his or her pdf to the referee. After all, if player 1 has some information about the pdf that player 2 will submit, might player 1 be able to submit a false pdf that increases the value of his portion of the cake, compared with what he would receive if he were truthful?

The answer is "yes," but unless player 1's information is complete (e.g., obtained from a spy who received advance information on the pdf that player 2 would submit to the referee), this is a risky strategy to pursue. If player 1 is off by the slightest amount, he could end up actually doing worse than if he were truthful, which guarantees him a minimum of 50 percent of his value of the cake, and generally more. We will return to this point in section 5, where we compare our algorithm with Adjusted Winner.

[4] Independently, Cohler, Lai, Parkes, and Procaccia (2011) give an algorithm (Algorithm 2) that is very similar to ours. Whereas we focus on efficient allocations that are equitable, they focus on those which maximize the total valuation of the players, which may be different. They restrict their algorithm to piecewise linear pdfs, whereas we consider other functions; also, we relate our algorithm to Adjusted Winner, which assumes, in effect, piecewise constant pdfs. In footnote 5 we mention other differences in our approach and theirs in the aforementioned paper and a related paper with overlapping authors, which reflects a computer-science perspective that we regard as complementary to our approach.

3. Finding Perfect Divisions

In general, many divisions of a cake are efficient (e.g., giving all the cake to one player), and at least one is equitable. Obtaining envy-freeness is trivial because, as we will see, it follows from the satisfaction of efficiency and equitability.

The key to finding a perfect division, and determining the number of cuts that it requires, is the *relative* worth of each point of the cake—which we shall refer to as a *bit* of cake—to the two players. We assume that f_1 and f_2 are each *piecewise continuous* (i.e., each comprises a finite number of continuous pieces) as well as being positive over the domain [0,1].

For each bit k of cake, let $f(k) = \frac{f_1(k)}{f_2(k)}$. Then f is a function with domain [0,1] that takes on only positive values. When player 1 values a bit more than player 2, f is greater than 1; otherwise, f is less than or equal to 1 (when player 2 values a bit at least as much as player 1).

Let G be the graph of f. G need not be connected; because f_1 and f_2 are each piecewise continuous, however, so is f, and hence G comprises a finite number of connected pieces. Since f provides a bijection between the cake C and the graph G, we can identify C with G. Any division of the graph G into portions corresponds to a division of the cake C into portions. We exploit this identification between C and G in what follows by sometimes referring to G as the cake itself.

Efficiency

To obtain an efficient division of G, consider giving each bit of cake to the player who values it more, and arbitrarily giving out any bits of cake that are valued equally by the two players. (We must be cautious with this intuition, however, because, as noted earlier, any one bit of cake has value zero to each player.) Any division so obtained—and there may be more than one if the players value some interval equally—is efficient, as we will show. However, something much more general is also true.

Fix any $\alpha > 0$, and draw the line $y = \alpha$ in the same figure as the graph G. This horizontal line divides G. Give all of G that is above the line $y = \alpha$ to player 1, and all of G below the line $y = \alpha$ to player 2. We call such a division $\langle A,B \rangle$ of the cake, whereby player 1 receives portion A and player 2 receives portion B, an *α-division*.

More formally, <*A*,*B*> is an α-division if and only if for any $k \in A$, $\frac{f_1(k)}{f_2(k)} \geq \alpha$, and for any $k \in B$, $\frac{f_1(k)}{f_2(k)} \leq \alpha$. Because we have not specified what happens to points on *G* which coincide with the line *y* = α, where $\alpha > 0$, there may be more than one α-division for a fixed such α (about which we will say more later). Notice that the algorithm we presented in section 2 yields an initial division that is a 1-division; the adjustments (in step 6 of the algorithm) correspond to changing the value of α from α = 1.

Claim. *A division <A,B> of the cake is efficient if and only if for some* $\alpha > 0$*, it is an* α*-division*.

Before proving the claim, we illustrate our analysis with four examples of pairs of pdfs. Their corresponding graphs *G* are shown in Figure 1, in which f_1 is the small-block curve and f_2 is the large-block curve. (For simplicity, these examples involve only linear functions.)

Figure 1 about here

Example *a*:

$$f_1(x) = \begin{cases} 1.5 \text{ for } 0 \leq x < 0.6 \\ 0.25 \text{ for } 0.6 \leq x \leq 1 \end{cases}$$

$$f_2(x) = \begin{cases} 0.625 \text{ for } 0 \leq x < 0.4 \\ 1.25 \text{ for } 0.4 \leq x \leq 1 \end{cases}$$

Example *b*:

$$f_1(x) = 2x \text{ for } 0 \leq x \leq 1$$

$$f_2(x) = 1 \text{ for } 0 \leq x \leq 1$$

Example *c*:

$$f_1(x) = \begin{cases} 0.8 \text{ for } 0 \leq x < 0.25,\ 0.5 \leq x < 0.75 \\ 1.2 \text{ for } 0.25 \leq x < 0.5,\ 0.75 \leq x \leq 1 \end{cases}$$

$$f_2(x) = \begin{cases} 1.2 \text{ for } 0 \leq x < 0.25,\ 0.5 \leq x < 0.75 \\ 0.8 \text{ for } 0.25 \leq x < 0.5,\ 0.75 \leq x \leq 1 \end{cases}$$

Example *d*:

$$f_1(x) = \begin{cases} 0.8 \text{ for } 0 \leq x < 0.25,\ 0.5 \leq x < 0.75 \\ 1.2 \text{ for } 0.25 \leq x < 0.5,\ 0.75 \leq x \leq 1 \end{cases}$$

$$f_2(x) = 1 \text{ for } 0 \leq x \leq 1$$

These functions are all *piecewise linear* (i.e., each comprises a finite number of linear pieces). The functions in examples 1*a*, 1*c*, and 1*d*, are also *piecewise constant* (i.e., each comprises a finite number of constant pieces). We also note that, as required, the area under each pdf is 1.

In Figure 2, we show the graphs G of $f = f_1/f_2$ corresponding to each of the examples in Figure 1. In examples 1*b* and 1*d*, the graph G is the same as the graph of f_1, player 1's pdf, because f_2 has a constant value of 1 (i.e., it is a uniform pdf) and, hence, for any bit of cake k in each of these two examples, $f(k) = \frac{f_1(k)}{f_2(k)} = \frac{f_1(k)}{1} = f_1(k)$.

Figure 2 about here

Next, we use G to find efficient divisions. Intuitively, efficient divisions should give pieces of G having big y-coordinates to player 1 (since player 1 puts high value on these pieces), and pieces of G having small y-coordinates to player 2 (since player 2 puts high value on these pieces).

In Figure 3, G is the darker graph in each example. To illustrate the calculation of an equitable allocation, we have drawn two horizontal lines in Figure 3*a* at $y = 2$ and $y = 1.2$. First consider the line $y = 2$. Making this the dividing line for allocating cake to players 1 and 2, we give all of G above this line to player 1, and all of G below this line to player 2. Thus, player 1's portion of the original cake (i.e., the line interval [0,1]) is [0,0.4), and player 2's portion is [0.4,1]. Using the players' pdfs, we find that player 1's valuation of his portion is (0.4)(1.5) = 0.6, and player 2's valuation of her portion is (0.6)(1.25) = 0.75. Clearly, this is not an equitable division, but by our claim it is efficient.

Figure 3 about here

Next, consider the other horizontal line in Figure 3a, $y = 1.2$. Using this line, we give interval [0,0.4) to player 1, and interval [0.6,1] to player 2. Because the part of G corresponding to interval [0.4,0.6) is on the line $y = 1.2$, *any* division of this piece of cake between the players yields an efficient division. Thus, while there is just one efficient division arising from the line $y = 2$, there are many arising from the line $y = 1.2$.

In Figure 3*b*, we show the horizontal line $y = \sqrt{5} - 1$, which is not an arbitrary choice, as we will see shortly. The x-coordinate of the point of intersection of this line with G is $\frac{\sqrt{5}-1}{2}$. Hence, in the associated efficient division of the original cake, player 1's portion is the interval $\left[\frac{\sqrt{5}-1}{2}, 1\right]$, and player 2's portion is the interval $\left[0, \frac{\sqrt{5}-1}{2}\right)$. (We have assigned the bit of cake at $x = \frac{\sqrt{5}-1}{2}$ arbitrarily, since a single bit has no value to either player.) We use the players' pdfs to compute each player's valuation of his or her own portion as follows:

Player 1: {Area under the line $y = 2x$ from $x = \frac{\sqrt{5}-1}{2}$ to $x = 1$} =

area of top triangle + area of bottom rectangle =

$\left(\frac{1}{2}\right)$(base)(height) + (base)(height) =

$$\left(\frac{1}{2}\right)\left[1-\left(\frac{\sqrt{5}-1}{2}\right)\right]\left[2-\left(\sqrt{5}-1\right)\right]+\left[1-\left(\frac{\sqrt{5}-1}{2}\right)\right]\left(\sqrt{5}-1\right)=\frac{\sqrt{5}-1}{2}$$

Player 2: {Area of rectangle from $x = 0$ to $x = \frac{\sqrt{5}-1}{2}$} =

(base)(height) =

$$\left(\frac{\sqrt{5}-1}{2}\right)(1)=\frac{\sqrt{5}-1}{2}$$

Thus, we see that besides being efficient, this division is equitable, which is why we chose the dividing line $y = \sqrt{5}-1$. (We will have more to say about this shortly.)

Proof of Claim

To establish the forward direction of the claim, assume that for no $\alpha > 0$ is $<A,B>$ an α-division. We must show that $<A,B>$ is not efficient. Our assumption implies that no horizontal line can be drawn so that all of A lies on or above this line, and all of B lies on or below this line. It follows that there are pieces of cake $A' \subseteq A$ and $B' \subseteq B$ with all of A' below all of B' (i.e., each point of A' has a y-coordinate that is less than the y-coordinate of each point of B').

We may assume (by shrinking either A' or B', if necessary) that $\mu_1(A') = \mu_1(B')$. Because all of A' is below all of B', we know that for any $k_a \in A'$ and $k_b \in B'$, $\frac{f_1(k_a)}{f_2(k_a)} < \frac{f_1(k_b)}{f_2(k_b)}$. This implies that $\frac{\mu_1(A')}{\mu_2(A')} < \frac{\mu_1(B')}{\mu_2(B')}$ and, hence, that $\mu_2(A') > \mu_2(B')$. Finally, we compare the division $<A,B>$ with the division $<(A\backslash A') \cup B',(B\backslash B') \cup A'>$, which we can view as being obtained from $<A,B>$ by having the players trade the pieces A' and B'. Player 1 equally values these two divisions (since $\mu_1(A') = \mu_1(B')$), but player 2 values the division $<(A\backslash A') \cup B',(B\backslash B') \cup A'>$ more (since $\mu_2(A') > \mu_2(B')$). This implies that $<A,B>$ is not efficient.

To establish the reverse direction of the claim, fix some $\alpha > 0$ such that $<A,B>$ is an α-division. We must show that $<A,B>$ is efficient.

For any $k \in A$, $\frac{f_1(k)}{f_2(k)} \geq \alpha$, and for any $k \in B$, $\frac{f_1(k)}{f_2(k)} \leq \alpha$. This tells us that for any $A' \subseteq A$ and $B' \subseteq B$, $\frac{\mu_1(A')}{\mu_2(A')} \geq \alpha$, $\frac{\mu_1(B')}{\mu_2(B')} \leq \alpha$ and, hence, $\mu_1(A') \geq \alpha\mu_2(A')$ and $\mu_1(B') \leq \alpha\mu_2(B')$.

Changing from a division of the cake <*A*,*B*> to some other division can be viewed as a trade between players 1 and 2 of some $A' \subseteq A$ and $B' \subseteq B$. Suppose that player 1 values his new portion more than he values A, his original portion. This means that $\mu_1(B') > \mu_1(A')$, and, hence, $\alpha\mu_2(A') \leq \mu_1(A') < \mu_1(B') \leq \alpha\mu_2(B')$. Thus, $\mu_2(A') < \mu_2(B')$, which tells us that player 2 values her new portion less than she valued B, her original portion. This establishes that any trade that increases one player's valuation must decrease the other player's valuation. Consequently, <*A*,*B*> is efficient, which establishes the claim.

Equitability

The claim tells us that there is at least one efficient division associated with any horizontal line. We are now ready to consider divisions that are both efficient and equitable. The method for obtaining equitability is straightforward. Suppose that $\alpha_1, \alpha_2 > 0$, with α_1 near zero and α_2 a large positive number. Then for any division associated with the line $y = \alpha_1$, player 1 will get what he considers to be a large portion, whereas player 2 will get what she considers to be a small portion. By contrast, in any division associated with the line $y = \alpha_2$, the situation will be reversed.

Recall that an equitable division is one in which the two players value their portions equally. The lines $y = \alpha_1$ and $y = \alpha_2$ give efficient divisions that are on the two extremes of being inequitable, the first favoring player 1 and the second player 2. We wish to adjust α so that (one of) the division(s) associated with the line $y = \alpha$ gives the two players portions that they value equally. We imagine moving a horizontal line from just above the x-axis gradually upwards, stopping at the point where some division associated with this line gives equal values (in each player's own measure) to the two players.

This adjusting of α is easy to accomplish in example *b* (see Figure 3*b*). By continuously changing α from small to large positive values, the divisions obtained from using the line $y = \alpha$ start by giving large portions to player 1 and small portions to player 2 and then switch, eventually giving large portions to player 2 and small portions to player 1. To calculate the *changeover point* (i.e., the point at which the players value their portions equally), we first compute each player's valuation of his and her portion in terms of α, using that player's pdf:

Player 1: {Area under the line $y = 2x$ from $x = \alpha/2$ to $x = 1$} =

area of top triangle + area of bottom rectangle =

$\left(\frac{1}{2}\right)$(base)(height) + (base)(height) =

$$\left(\frac{1}{2}\right)\left[1-\left(\frac{\alpha}{2}\right)\right][2-\alpha]+\left[1-\left(\frac{\alpha}{2}\right)\right](\alpha)=1-\frac{\alpha^2}{4}$$

Player 2: {Area of rectangle from $x = 0$ to $x = \alpha/2$} =

(base)(height) =

$$\left(\frac{\alpha}{2}\right)(1)=\frac{\alpha}{2}$$

We wish to choose α so that these two valuations are equal. We compute the value of α as follows:

$$1-\frac{\alpha^2}{4}=\frac{\alpha}{2}$$

$$\frac{\alpha^2}{4}+\frac{\alpha}{2}-1=0$$

$$\alpha^2+2\alpha-4=0$$

$$\alpha=\frac{-2\pm\sqrt{20}}{2}=-1\pm\sqrt{5}$$

Clearly, we must have $\alpha > 0$, and thus we set $\alpha=\sqrt{5}-1$. With this choice of α, each player values his or her portion at $\frac{\sqrt{5}-1}{2}$. By the claim, this division is efficient. Thus, we see that in this case a division that is efficient and equitable can be obtained with a single cut of the cake.

The situation in example *a* is rather different (see Figure 3a). Fix any α with $0.2 < \alpha < 1.2$. The division of G given by $y = \alpha$ gives more cake to player 1 than to player 2. In particular, for any such α, player 1 receives interval [0,0.6) and player 2 receives interval [0.6,1]. Thus, using the players' pdfs, we find that player 1's valuation of his piece is (0.6)(1.5) = .9, and player 2's valuation of her piece is (0.4)(1.25) = 0.5. Thus, any such division gives more to player 1 than to player 2.

Next, fix any α with $1.2 < \alpha < 2$. As we saw previously, this choice gives more cake to player 2 than to player 1. In particular, player 1's portion is the interval [0,0.4), which results in a valuation by player 1 of (0.4)(1.5) = 0.6, whereas player 2's portion is the interval [0.4,1), which results in a valuation by player 2 of (0.6)(1.25) = 0.75. Thus, if we want an equitable division, we must set $\alpha = 1.2$.

There are many divisions associated with this horizontal line; we want to find one that divides the interval [0.4,0.6) in such a way as to produce equitability. Let r denote the fraction of this interval that should go to player 1 (and hence 1-r is the fraction that player 2 should receive). Then, since player 1 receives the interval [0,0.4) and player 2 receives the interval [0.6.1], it follows that we must have (1.5)(0.4) + (1.5)(r)(0.2) = (1.25)(1-r)(0.2) + (1.25)(0.4), and hence r = 3/11. Thus, player 1 must receive 3/11 of this segment and player 2 must receive 8/11 of this segment. Because player 1's portion must include segment [0,0.4), and player 2's portion must include interval [0.6,1], we can ensure that the players obtain equitable portions, and also minimize the number of cuts needed, by cutting the interval [0.4,0.6) so that player 1 receives the left 3/11 of this piece and player 2 the right 8/11 of this piece. Thus, the cut should be at x = 0.4 + (3/11)(0.2) = 5/11, giving player 1 the interval [0,5/11), and player 2 the interval [5/11,1]. We verify this by noting that player 1's valuation of his piece is (5/11)(1.5) = 15/22, and player 2's valuation of her piece is [1-(5/11)](1.25) = 15/22. The claim tells us that this division is efficient as well as being equitable.

Next, consider example c (see Figure 3c). Fix any α with $2/3 < \alpha < 3/2$. If we use $y = \alpha$ to divide G, we obtain a division that gives portion [0.25,0.5) ∪ [0.75,1] to player 1 and portion [0,0.25) ∪ [0.5,0.75) to player 2. Then player 1's valuation of his portion, and player 2's value of her portion, are both (0.5)(1.2) = 0.6. Thus, this division is equitable and, by the claim, efficient.

Finally, consider example d (Figure 3d). This may look very similar to example c (indeed, player 1's pdf is the same in these two examples, and the graphs G for these examples certainly look similar), but it is quite different. Fix any α with $0.8 < \alpha < 1.2$. Using $y = \alpha$ to divide G, player 1's portion is [0.25,0.5) ∪ [0.75,1], and player 2's portion is [0,0.25) ∪ [0.5,0.75), exactly as in example c and, also as in example c, player 1's valuation of his portion is (0.5)(1.2) = 0.6. However, player 2's valuation of her portion is (0.5)(1) = 0.5. So while this division is efficient (by the claim), it is not equitable.

To obtain an equitable division, consider the line y = 1.2. We must determine how to divide the portion of the graph G that is on y = 1.2 (i.e., [0.25,0.5) ∪ [0.75,1]) between the two players so as to obtain an equitable division. A straightforward calculation (similar to the calculation above for example a) shows that we must give 10/11 of this portion to player 1 and 1/11 of this portion to player 2.

Although there are many ways of dividing [0.25,0.5) ∪ [0.75,1] that will achieve this, we wish to minimize the number of cuts by making one cut in one of the two intervals that make up this portion. Which one we cut is arbitrary. We will cut the first piece in this portion (i.e., [0.25,0.5)) in such a way that player 1 gets the left part and player 2 the right part of this piece. This results in the division that gives to player 1 the portion [0.25,0.25 + 9/44) ∪ [0.75,1] and to player 2 the portion [0,0.25) ∪ [0.25+ 9/44),0.75). We verify below that this efficient division is equitable:

- Player 1's valuation of his portion is (9/44 + 0.25)(1.2) = 6/11.
- Player 2's valuation of her portion = [0.25 + (0.75 - (0.25 + 9/44))](1) = 6/11.

Envy-freeness

Our goal is to obtain a perfect division of *G*, which is not only efficient and equitable but also envy-free. We have shown how to obtain a division satisfying the first two of these properties. We now show that the third property follows from the first two. Suppose <*A*,*B*> is a division that is efficient and equitable. Equitability tells us that $\mu_1(A) = \mu_2(B)$. This common value must be at least 0.5 because, if it were not, then both players would benefit from trading portions, and this would violate efficiency. But if each player values his or her portion to be at least 0.5, then each player values the other player's portion at most .5, and hence neither player is envious of the other player's portion. We note that envy-freeness does not follow from efficiency and equitability if there are more than two players.

Different Entitlements

We close this section with a few comments on the possibility of different entitlements. As an example, suppose that player 1 is entitled to 2/3 of the cake and player 2 is entitled to 1/3 of the cake. Can we find a perfect division in this case?

First, we must consider what it means for a division to be perfect. Whereas efficiency retains its original meaning, we redefine envy-freeness in terms of the entitlements of the players. Thus in our example, player 1 envies player 2 if and only if he values player 2's portion at more than half of his own, and player 2 envies player 1 if and only if she values player 1's portion at more than double her own. As was true with equal entitlements, envy-freeness follows easily from efficiency and equitability.

Equitability in our example means that player 1 values his portion at twice what player 2 values her portion. How do we find an equitable division in the context of unequal entitlements? We start with the line $y = 2$ instead of $y = 1$. Above this line is cake that player 1 values at more than twice player 2's valuation, and below this line is cake that player 2 values at more than half player 1's valuation. We move this line up or down until we find a line such that at least one division associated with this line produces the desired 2:1 ratio, which can readily be accomplished in our four earlier examples.

4. Determining The Maximum Number of Cuts Required

We have shown that a perfect division of a cake always exists and illustrated, in several simple examples, how to find it. We also showed how entitlements can be taken into account in defining perfectness. We next consider how to determine the number of cuts required to produce such a perfect division.

As we observed earlier, the number of cuts in the original cake C corresponds to the number of cuts in the graph G. Focusing on G rather than C, however, we must be careful: What is a cut of G?

Although we continue to use a single horizontal line $y = \alpha$ to cut G, there may be more than one cut. In fact, a cut occurs each time G crosses this line. How can such a crossing occur? There are three possibilities:

i. G "jumps over" the line $y = \alpha$ (i.e., without crossing it at a point).
ii. G crosses the line $y = \alpha$ at a point.
iii. G and the line $y = \alpha$ share a line segment.

These possibilities correspond, respectively, to no point of intersection of the players' pdfs, a single point of intersection, and infinitely many points of intersection. Figure 3*a* (with $y = 2$) and Figure 3*c* illustrate possibility (i); Figure 3*b* illustrates possibility (ii); Figure 3*a* (with $y = 1.2$) and Figure 3*d* illustrate possibility (iii).

We note that these possibilities apply to any single crossing. Two or all three of these possibilities may occur for the same cake and same line $y = \alpha$. In Figure 4, we show a graph G and the line $y = 1$ for which all three possibilities occur.

Figure 4 about here

To count the number of cuts required to achieve perfection, we consider two cases. Then we consider two examples that illustrate the limits of counting cuts.

Case 1. *Both pdfs are piecewise constant*. This is the situation in examples *a*, *c*, and *d* in Figure 1. Clearly, G is piecewise constant. How many constant pieces make up G? Suppose f_1 (player 1's pdf) is made up of m constant pieces, and f_2 (player 2's pdf) is made up of n constant pieces. The ratio of f_1 and f_2 can change, at most, as often as there are endpoints of the intervals on which f_1 and f_2 are constant, not counting the endpoints 0 and 1. Hence, there are at most $(m\text{-}1) + (n\text{-}1) = m\text{+}n\text{-}2$ such changes, which tells us that G is made up of at most $m\text{+}n\text{-}1$ constant pieces. Now consider the line $y = \alpha$ that divides G in an efficient and equitable manner. Either this line does not intersect G, or else it intersects G in one or more line segments. (Note that possibility (ii) above cannot occur in this case.)

Suppose first that $y = \alpha$ does not intersect G (see Figure 3*c*). It is possible that every constant piece that makes up G is on the opposite side of $y = \alpha$ from each constant piece adjacent to it. Then there will be a cut between every pair of adjacent pieces. It follows that we can obtain a perfect division with at most $m\text{+}n\text{-}2$ cuts.

Suppose next that $y = \alpha$ intersects G in one or more line segments (see Figure 3*a* with $y = 1.2$ and Figure 3*d*). Then, as we showed, we may need to cut inside one—but not more than one—of these line segments. We can choose whether to cut so as to give the left part of this piece to player 1 and the right part to player 2, or vice versa, so that no cut between this segment and at least one of the two adjacent segments is needed. Thus, we see that we may have to make one cut *inside* one of the pieces on which G is constant, in which case we definitely do not cut *between* at least one pair of pieces on which G is constant. It follows, as above, that we can obtain a perfect division with at most $m+n-2$ cuts.

Case 2. *Both pdfs are piecewise linear (but not necessarily piecewise constant).* Suppose that player 1's pdf is made up of m linear pieces, and player 2's pdf is made up of n linear pieces. Then, reasoning as in case 1, we see that G is made up of at most $m+n-1$ linear pieces. The horizontal line $y = \alpha$ that produces a perfect division may intersect each of these linear pieces at most once, which means that we can obtain a perfect division with at most $m+n-1$ cuts.

This is illustrated in Figure 5, where the jagged graph is f_1, player 1's pdf, and player 2's pdf, f_2, is constantly 1. Because f_2 has a constant value of 1, graph G is the same as the graph of f_1. (It is easy to see, by symmetry considerations, that the area under f_1 is 1.) In this example, f_1 is made up of 10 linear pieces and, therefore, so is G, and f_2 is a single linear piece. Our analysis above tells us that we will need at most 10+1-1 = 10 cuts for a perfect division. More specifically, we can see from the figure that we will need exactly 10 cuts, because G crosses any horizontal line $y = \alpha$, where $0.5 < \alpha < 1.5$, 10 times.

Figure 5 about here

These two cases establish the truth of the following:

Theorem.

a. *Suppose that both players' pdfs are piecewise constant and that, in particular, player 1's pdf is made up of m constant pieces, and player 2's pdf is made up of n constant pieces. Then, we can obtain a perfect division with at most m+n-2 cuts.*
b. *Suppose that both players' pdfs are piecewise linear and that, in particular, player 1's pdf is made up of m linear pieces, and player 2's pdf is made up of n linear pieces. Then, we can obtain a perfect division with at most m+n-1 cuts.*

It is clear from the example in Figure 5 that there is no uniform finite upper bound that can be placed on the number of cuts necessary to achieve a perfect division. To be sure, for any positive integer t, we could revise this example in an obvious way so that a perfect division requires t cuts.

In concluding this section, we consider two questions:

1. Can "linear" be changed to "monotone" in our analysis above?
2. Is it always the case, even if we drop our piecewise-linear assumption, that it is possible to obtain a perfect division with a finite number of cuts.

The answer to both questions is "no." For question 1, consider Figure 6, in which f_1 is dotted and f_2 is dashed in Figure 6*a* (f_1 is the function $0.5 + x + (.025)\sin(10\pi x)$, and f_2 is the function $x + 0.5$). Standard calculus methods show that the areas under the graphs of each of these functions is 1, and that f_1 is monotone increasing (f_1' is never 0).

Figure 6 about here

The associated graph G is shown in Figure 6*b*. Clearly, G will cross the line $y = \alpha$ that produces a perfect division many times, even though f_1 and f_2 are each (one-piece) monotone functions. By changing f_1, we can make the number of crossings as large as we want. Hence, our results above for piecewise linear functions do not apply if we replace "linear" by "monotone."

For question 2, define f_1 and f_2 as follows:

$$f_1(x) = \begin{cases} 1 + x\sin(1/x) \text{ for } 0 \le x < 0.5 \\ 0.94 \text{ for } .5 \le x \le 1 \end{cases}$$

$$f_2(x) = 1 \text{ for } 0 \le x \le 1$$

Numerical integration shows that the area under the graph of $1 + x\sin(1/x)$ for the interval $0 \le x \le 0.5$ is approximately 0.53. Our choice of 0.94 for the interval $0.5 \le x < 1$ was dictated by the need to have the area under the graph of f_1 be 0.

The graph of f_1 is shown in Figure 7. Because f_2 is constantly 1, the graph of f_1 is the same as the graph of G. Notice that G changes direction infinitely many times, with smaller and smaller periods, as we approach $x = 0$ from the right. Clearly, G must cross any horizontal line $y = \alpha$ that produces a perfect division infinitely many times. Thus, a perfect division in this case requires infinitely many cuts.

Figure 7 about here

We note that there is nothing significant about the fact that this function is not made up of linear pieces. It is easy to imagine "straightening out" G between the high points and low points so as to make it comprise an infinite number of linear pieces, exhibiting the same sort of behavior (in terms of the need for infinitely many cuts to produce a perfect division) as in the present example.

Finally, it is worth pointing out that our algorithm is also applicable to the perfect division of a pie, which can be visualized as a cake whose endpieces are connected. The only difference is that there may or may not be a cut at the point of connection (depending on whether or not the two end pieces go to the same player). However, the initial distribution of pieces—to the player that values each more—is the same, with the giveback process identical until equitability is achieved. While cake-cutting and pie-cutting differ in important ways (Barbanel, Brams, and Stromquist, 2009), when there are just two players, finding a perfect division of a cake and of a pie are essentially the same.

5. Relationship to Adjusted Winner

Adjusted Winner (AW) is a fair-division procedure in which each of two players distribute a fixed number of points to a set of divisible homogenous goods (Brams and Taylor, 1996, 1999). AW produces a division of goods that is perfect—efficient, envy-free, and equitable—in which at most one of the goods must be divided in a specified ratio between the players; all other goods are distributed in their entirety to one or the other of the two players.

AW can be seen as a special case of the method we have presented. Suppose that there are n goods to be distributed between the two players. Instead of the 100 points that is usually used in AW (of course, the choice of 100 is arbitrary), assume that each of the players has n points to distribute.

Arbitrarily order the goods $g_1, g_2, \ldots, g_n$. Associate the goods with equal-sized pieces of the line segment [0,1]: g_1 is associated with $[0,1/n)$, g_2 with $[1/n,2/n)$, … , g_n with $[(n-1)/n,1]$. Intuitively, we may imagine that each good has size $1/n$, and the goods are lined up on the interval [0,1]. This is our cake (Jones, 1992, uses a somewhat different construction to connect AW to cake-cutting).

Each player's pdf at every point in [0,1] is then given by the number of points that that player associates with the good that includes this point. In this manner, we define the two players' pdfs. We see that each pdf is a piecewise constant function and, by our choice of n points in total, the area under each pdf is 1.

For example, these pdfs might look like those in Figure 1*a*, 1*c*, or 1*d*. The method given by AW can then be seen as a special case of the method described in this paper. We consider the graph G, as in Figures 2*a*, 2*c*, and 2*d*, and start by trying the line $y = 1$ (i.e., giving each good to the player who puts the higher value on it; above the line means player 1, below means player 2). We then adjust the line up or down—depending on which way the transfer must go—transferring first from goods whose associated line segment in G is closest to $y = 1$, and then moving farther away, as needed.

In our general setting, the only cuts needed will be at the endpoints of pieces on which the pdfs are constant, plus at most one additional cut inside of one piece. Of course, there are no actual cuts at endpoints, because the goods are separate; they were artificially put together to form a cake. So the only possible cut is the one inside some interval on which the pdfs are constant, which coincides with AW. As is true of AW, if the players are truthful about their pdfs, they can ensure that they will receive at least half the cake—as they value it—and otherwise not, which gives them a strong incentive not to dissemble, especially if they are risk-averse (Brams and Taylor, 1996, 1999).

6. Conclusions

We proposed a 2-person cake-cutting procedure, in which the players submit their valuations of a cake (i.e., their pdfs) to a referee. The referee initially assigns parts of the cake to the player who values it more (i.e., whose pdf is greater), which will not generally produce an equitable division.

This necessitates a giveback process: The player who values his or her portion more gives parts of this portion back to the other player, starting with those parts in which the ratio of the players' pdfs is closest to 1, until the players value their portions equally. This allocation is not only equitable but also envy-free and efficient—that is, perfect. It can be accomplished with a finite number of cuts if the players' pdfs are piecewise linear, but if they are not (even if they are piecewise continuous), an infinite number of cuts may be required.

In practical applications, we think it reasonable to restrict the players to piecewise linear pdfs. Then our algorithm can be used to parcel out not only cake but also a divisible good like land. Consider, for example, beachfront property co-owned by two developers, who wish to build houses or hotels on the strips that they consider most valuable. Our algorithm could be used to determine who gets what strips (though a potential problem might be that the players end up with strips that they consider too narrow to develop).[5]

We note that the idea of considering ratios of players' pdfs to find efficient divisions of a cake is not new. It was used by Weller (1985) and by Barbanel (2000, 2005), although the geometric structure used there, which makes no continuity assumptions, is rather different from the one we use here. (It was called the "Radon-Nikodym Set" in Barbanel (2000, 2005).) The focus of this approach is on proving the *existence* of efficient and envy-free divisions, not on providing *algorithms* for finding them.

We close by briefly discussing the degree to which we are justified in calling our method an "algorithm." Recall that we assume that each player knows his or her own preferences, represented by a pdf, and communicates this information to a referee, who uses it to decide on the location of the cuts.

If the pdfs are simple functions, such as the constant functions of Adjusted Winner or piecewise linear functions, then they can readily be conveyed to a referee, who

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[5] Caragiannis, Lai, and Procaccia (2011) take into account this possibility in their analysis of "piecewise uniform with minimum length" (PUML) approximate algorithms, which give proportional (for *n* persons) and envy-free (for 2 persons) solutions that require that the players obtain minimal-length portions. The algorithms are not exact and do not incorporate the property of equitability. In Cohler, Lai, Parkes, and Procaccia (2011), approximate 2-person and *n*-person algorithms that give the most efficient of envy-free solutions are analyzed, but these, too, do not incorporate equitability; also, their notion of efficiency—to maximize the total valuation of the players—while it satisfies our notion of efficiency in section 1, is more specific, excluding allocations that ours includes. A focus of both these papers is on algorithms that run in polynomial time, which our exact algorihm trivially does (except when an infinite number of cuts is required).

implements the algorithm. If, however, the pdfs have, for example, infinitely many discontinuities, then it is far from clear how, practically speaking, to communicate this information to the referee, or how the referee can find the cut locations. While in theory such problems may arise, in practice we do not see them as insuperable barriers to dividing land or other divisible goods.

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