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Brams, Steven J. and Jones, Michael A. and Klamler, Christian

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N-Person Cake-Cutting: There May Be No Perfect Division

Steven J. Brams* Michael A. Jones† Christian Klamler‡

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Abstract

A cake is a metaphor for a heterogeneous, divisible good, such as land. A perfect division of cake is efficient (also called Pareto-optimal), envy-free, and equitable. We give an example of a cake in which it is impossible to divide it among three players such that these three properties are satisfied, however many cuts are made. It turns out that two of the three properties can be satisfied by a 3-cut and a 4-cut division, which raises the question of whether the 3-cut division, which is not efficient, or the 4-cut division, which is not envy-free, is more desirable (a 2-cut division can at best satisfy either envy-freeness or equitability but not both). We prove that no perfect division exists for an extension of the example for three or more players.

1 Introduction

Over the past fifteen years, a substantial literature on cake-cutting has sprung up. Books giving overviews of both existence results and algorithms for physically cutting a cake include Brams and Taylor [11], Robertson and Webb [17], Barbanel [1], and Brams [7]. Review articles of fair division that discuss cake-cutting include Brams [6] and Klamler [16].

Recent algorithms, involving both discrete and continuous (moving-knife) procedures, are analyzed in Barbanel and Brams [2, 4], Brams, Jones, and Klamler [8, 10], Caragiannis, Lai, and Procaccia [12], and Cohler et al. [13]. There is also a growing literature on pie-cutting (Brams, Jones, and Klamler [9]; Barbanel, Brams, and Stromquist [5]; Barbanel and Brams [3]), in which radial cuts are made from the center of a pie rather than parallel cuts being made along the edge of a cake.

A key question that is not addressed in any of this literature is whether there always exists a *perfect division* of a cake (or pie)—one that satisfies the following three properties, which capture important features of fair division:

*Department of Politics, New York University, New York, NY 10012 (steven.brams@nyu.edu).

†Mathematical Reviews, 416 Fourth Street, Ann Arbor, MI 48103 (maj@ams.org).

‡Institute of Public Economics, University of Graz, A-8010 Graz, Austria (christian.klamler@uni-graz.at). Klamler was partially supported by Austrian Science Fund (FWF): P 23724-G11 "Fairness and Voting in Discrete Optimization".

1. *Efficiency* (Pareto-optimality): There is no other division that gives players portions that they value at least as much and gives at least one player strictly more.

2. *Envy-freeness*: Each player values its portion at least as much as that of every other player and, consequently, does not envy any other player.

3. *Equitability*: Each player values its portion exactly the same as everybody else values its portion—that is, each player thinks that its portion is the same fraction of its perceived value of the entire cake.

It turns out, surprisingly, that there may be no perfect division, which we demonstrate with an extended example after introducing some notation and definitions.

2 Terminology and an Overview

If there are only two players and a cake must be divided with one cut, then there is a perfect division (Jones [15]), but it may be efficient only with respect to one cut, which we call 1-*efficient*. More than one cut may enable both players to receive more than what one cut gives each player. More generally, a division is *k-efficient* if it is efficient with respect to k cuts. A division that is efficient—independent of the number of cuts—is *o-efficient* (for overall efficiency). Henceforth, o-efficiency is what we mean by "efficiency" in the definition of a perfect division.

To illustrate o-efficiency, consider a rectangular cake, which is chocolate at its two ends—that, together, constitute half the length of the cake—and vanilla around its middle (the other half in length). Assume that player A likes chocolate twice as much as vanilla, and player B has exactly the opposite preference. Then an o-efficient division would be to give A the two chocolate parts, which require two cuts, and B the vanilla part. This allocation is equitable—giving each player $\frac{2}{3}$ of the cake as each values it—which is also envy-free because neither player would prefer to receive the other player's piece.

By contrast, using one cut to divide this cake at the center into two identical halves—and giving one of the halves to each player—is envy-free and equitable. However, it is not o-efficient, because each player receives more ($\frac{2}{3}$ instead of $\frac{1}{2}$) when the aforementioned two cuts are made. Barbanel and Brams [4] give a 2-player algorithm for finding an envy-free and equitable allocation that is o-efficient, which specifies (i) how many cuts are required and (ii) where they must be made to produce a perfect division of a cake.

Other algorithms, one approximate and the other exact, produce a perfect 2-player division of a pie into two wedge-shaped pieces using two cuts (Barbanel and Brams [3]). However, there may be a division using more than two cuts that *Pareto-dominates*—is at least as good for one player and better for the other player—the 2-efficient division, but no algorithm is known for producing such an o-efficient division of a pie. (We show later, however, that our results for a cake also apply to a pie.)

We focus in this paper on n-person cake-cutting. In section 3, we state the measure-theoretic assumptions we make about a cake as a mathematical object when valued along a line segment. We also give a 3-player example from an earlier paper (Brams, Jones, and Klamler [8]) that demonstrates, using two cuts, that there is no equitable division that is also envy-free, ruling out a perfect division of a cake that uses a minimal number of cuts ($n - 1$ if there are n players).

In section 4 we show that increasing the number of cuts beyond two can produce an equitable division in our example that is also o-efficient, but it is not envy-free. Likewise, there is an equitable division that is envy-free, but it is not o-efficient. We summarize in a table our results about this example, pointing out that an o-efficient, equitable division (with 4 cuts) is not envy-free; an envy-free, equitable division (with 3 cuts) is not o-efficient; and an equitable division (with 2 cuts) is neither o-efficient nor envy-free. Furthermore, 5 or more cuts do not satisfy all three properties.

In short, two of our three properties of a perfect division are satisfied, but all three are unattainable in our example, even when there is no restriction on the number of cuts. Surprisingly, the division that maximizes the sum of player valuations, which we call the *maxsum division*, uses a minimal number of cuts (two) and, while 2-efficient, is *neither* envy-free nor equitable. We then prove our main result: There is a cake (the 3-player example) in which there is *no* perfect division, however many cuts are made. We show this is true not just for three players but for any number.

This impossibility result complements an earlier finding for pies, which demonstrated that no envy-free division may be efficient (Barbanel, Brams, and Stromquist [5]), much less equitable. However, this was proved only when a minimal number of cuts was allowed (n if there are n players).

In section 5, we mention some examples in which it would be useful to find o-efficient, equitable divisions, even if they are not perfect. We conclude by posing three open questions.

3 Assumptions about Cake-Cutting and Our Original Example

A cake is a heterogeneous good, which the players may value different parts of differently. While the cake may be a swirl of flavors and toppings, we assume that each player can attach a value to it at every point along an edge. (Physically, the cake may be thought of as a rectangle, valued along an edge, but it could be any shape and valued along any straight line that passes through it.)

Formally, we make the following assumptions about a cake and the players' valuations of it:

1. The cake is defined by the $[0, 1]$-interval, and a division of the cake is a partition in which players receive disjoint subsets and the union of these subsets is the whole cake.

2. Each player i has a probability density function (pdf), called a value function, $v_i : [0, 1] \to \mathbb{R}^+ \cup \{0\}$ with $\int_0^1 v_i(t)dt = 1$, for which player i's value of the piece $[a, b]$ is given by $\int_a^b v_i(t)dt$.

3. Let $\mathcal{W}$ be the collection of all subsets of $[0, 1]$. Player i's preferences for pieces of cake are represented by the probability measure $\mu_i : \mathcal{W} \to [0, 1]$, where $\mu_i(S) = \int_S v_i(t)dt$.

Because the value functions are pdfs, their probability measures μ_i satisfy (a) and (b) below; in addition, we assume (c) to ensure that all players i attach positive value to all portions of the cake:

(a) μ_i is *countably additive*, i.e., if $T_1, T_2, ...$ is a countable collection of pairwise disjoint elements of $\mathcal{W}$, then $\mu_i(\bigcup_{j=1}^{\infty} T_j) = \sum_{j=1}^{\infty} \mu_i(T_j)$;

(b) μ_i is *nonatomic*, i.e., for any $S_k \in \mathcal{W}$, if $\mu(S_k) > 0$, then for some $S_l \subseteq S_k$, it follows that $S_l \in \mathcal{W}$ and $\mu_i(S_k) > \mu_i(S_l) > 0$; and

(c) μ_i is *mutually absolutely continuous*, i.e., if for any $S_k \in \mathcal{W}$ and for some j, $\mu_j(S_k) = 0$, then $\mu_i(S_k) = 0$ for all i.

Hence, μ_i assigns a value to player i's share of the cake according to the total area under i's pdf v_i for all the different pieces assigned to player i (a). In addition, nonatomicity (b) says that any single point has no value, and mutual absolute continuity (assumption c) states that if a portion of a cake is of zero value to one player, it is of zero value to all players.

The three properties that determine a perfect division can now formally be defined. Let $N = \{1, 2, \ldots, n\}$ be the set of players and $S = (S_1, ..., S_n)$ be a partition of $[0, 1]$. Then S is

- *envy-free* if and only if for all $i, j \in N$, $\mu_i(S_i) \geq \mu_i(S_j)$;
- *equitable* if and only if for all $i, j \in N$, $\mu_i(S_i) = \mu_j(S_j)$; and
- *efficient* if and only if there is no partition $S' \neq S$ such that for all $i \in N$, $\mu_i(S'_i) \geq \mu_i(S_i)$ and $\mu_j(S'_j) > \mu_j(S_j)$ for some $j \in N$.

We do not count as cuts those that subdivide a player's portion into adjacent subpieces. For example, if we use two cuts to divide a cake between two players, and give one player two adjacent pieces, we consider this a 1-cut division, because it is equivalent to dividing the cake with just one cut. On the other hand, if we give one player the two (disjoint) endpieces, and the other player the middle piece, this is a 2-cut division.

To show that it is not always possible to divide a cake among three players into envy-free and equitable portions using two cuts, assume that two players, A and B, have piecewise linear value functions that are symmetric and V-shaped (Brams, Jones, and Klamler [8]; this example was later used for a different purpose in Brams, Jones, and Klamler [10]). The value functions are:

$$v_A(t) = \begin{cases} -4t + 2 & \text{for } t \in [0, 1/2] \\ 4t - 2 & \text{for } t \in (1/2, 1] \end{cases} \text{ and } v_B(t) = \begin{cases} -2t + 3/2 & \text{for } t \in [0, 1/2] \\ 2t - 1/2 & \text{for } t \in (1/2, 1] \end{cases}.$$

Whereas both value functions have maxima at $t = 0$ and $t = 1$ and a minimum at $t = \frac{1}{2}$, A's function is steeper (higher maximum, lower minimum) than B's, as illustrated in Figure 1, giving it a greater amplitude. In addition, suppose that a third player, C, has a uniform value function, $v_C(t) = 1$, for $t \in [0, 1]$.

In this example, every envy-free allocation of the cake will be one in which A gets the portion to the left of x, B the portion to the right of $1 - x$ (A and B could be interchanged),

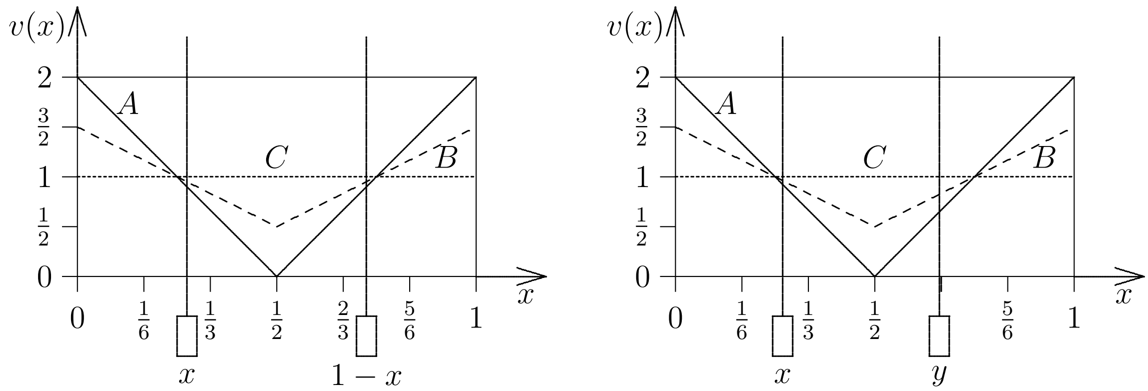


Figure 1: A 2-cut envy-free solution (left) cannot be equitable. A 2-cut equitable solution (right) cannot be envy-free.

and C the portion in the middle (we will later give the bounds on x that ensure envy-freeness); see Figure 1 (left). If the horizontal lengths of A's and B's portions are not the same (i.e., x), the player whose portion is shorter in length will envy the player whose portion is longer.

Unfortunately, no such envy-free allocation is equitable, because if A and B receive equal-length endpieces—ensuring that each values each piece equally, which precludes envy—A will receive a larger portion in its eyes than B receives in its eyes, precluding equitability. Thus, such a 2-cut envy-free allocation is not equitable in this example, nor is a 2-cut equitable allocation envy-free, though both these allocations are 2-efficient.[1]

It is instructive in this example to calculate the equitable division in which A gets the left piece defined by the interval $[0, x]$, C gets the middle piece defined by the interval $(x, y]$—where $y < 1 - x$—and B gets the right piece defined by the interval $(y, 1]$, as in Figure 1 (right). The players' values will be equal—and, therefore, the allocation to the three players will be equitable—when the areas (i.e., cake portions) of A, B, and C over their respective intervals are equal:

$$\int_0^x (-4t + 2)dt = \int_x^y dt = \int_y^1 (2t - \frac{1}{2})dt.$$

After integration and evaluation of the integrals, we have two quadratic equations in two unknowns,

$$-2x^2 + 3x - y = 0$$
$$-2x + 2y^2 + y - 1 = 0,$$

whose four solutions include one feasible solution: $x \approx 0.269$ and $y \approx 0.662$. Players A, B, and C each value their pieces at 0.393, so each thinks it receives nearly 40 percent of the value of the cake. This equitable allocation is 2-efficient, but, as we will show later, it is not o-efficient.

[1]Not all equitable divisions need be 2-efficient. If C were given an endpiece, and A or B the middle piece and the other endpiece in our example, cutpoints can be found such that all the players receive, in their own eyes, the same value. However, this value will be less than the 2-efficient, equitable value we calculate next in the text. By contrast, an envy-free allocation that uses $n - 1$ cuts if there are n players is always $(n - 1)$-efficient (Gale [14]; Brams and Taylor [11], pp. 150-151).

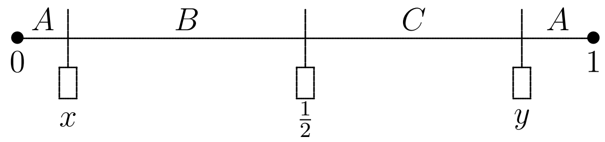
4 More Cuts Do Not Give Perfection

What if we allow more cuts? We next show that doing so does not lead to a perfect division.

Three Cuts

There are two ways of effecting an equitable division such that A receives the two end-pieces that it values most:

1. *Make middle cutpoint at* $\frac{1}{2}$. With three cuts, we can give two narrow pieces near the endpoints to A and split the middle portion of the cake at $\frac{1}{2}$ between B and C so that the three players value each of the pieces they receive exactly the same:



To ensure equitability, B's piece must be greater in length than C's piece—and A's right piece greater in length than A's left piece—which means that C will envy B.

Assume the left cutpoint is x and the right cutpoint is y, with the middle cutpoint $\frac{1}{2}$. The players' values will be equal—and, therefore, the allocation to the three players will be equitable—when

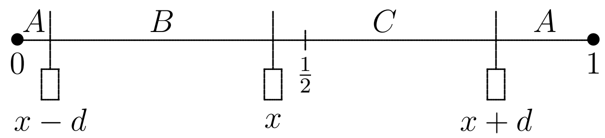
$$\int_0^x (-4t+2)dt + \int_y^1 (4t-2)dt = \int_x^{\frac{1}{2}} (-2t+\frac{3}{2})dt = \int_{\frac{1}{2}}^y dt.$$

After integration and evaluation of the integrals, we have

$$\begin{aligned} 6x^2 - 7x + 4y^2 - 4y + 1 &= 0 \\ 2x^2 - 3x - 2y + 2 &= 0, \end{aligned}$$

whose four solutions include one feasible solution: $x \approx 0.088$ and $y \approx 0.876$. Players A, B, and C each value their allocations at 0.376, which is less than the 0.393 that the earlier 2-efficient allocation gives each player.

2. *Make middle cutpoint at* $x < \frac{1}{2}$. All three players can do better with a different equitable 3-cut division than the equitable division given in (1) above. Moreover, this alternative equitable division eliminates envy by placing the middle cutpoint to the left of $\frac{1}{2}$ and giving B a piece no greater in length than C's piece. In fact, assume that B's and C's pieces are the same distance, d, from x—and so the same length; hence, C will not envy B (nor will B envy C, because C's piece overlaps $\frac{1}{2}$):



Thus, the left cutpoint will be $x - d$, the middle cutpoint will be x, and the right cutpoint will be $x + d$. The players' values will be equal—and, therefore, the allocation to the three players will be equitable—when

$$\int_0^{x-d} (-4t + 2)dt + \int_{x+d}^{1} (4t - 2)dt = \int_{x-d}^{x} (-2t + \frac{3}{2})dt = \int_{x}^{x+d} dt.$$

After integration and evaluation of the integrals, we have

$$8x^2 - 8x + 10d^2 + 3d - 4xd = 0$$
$$2d^2 + d - 4xd = 0,$$

whose four solutions include one feasible solution: $x \approx 0.444$ and $d \approx 0.387$.

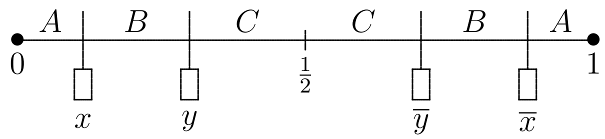
This makes the left cutpoint $x - d \approx 0.056$, the middle cupoint $x \approx 0.444$, and the right cutpoint $x + d \approx 0.831$. Players A, B, and C each value their pieces at 0.387, which is also how much A values B's piece—so A does not envy B, rendering the equitable allocation envy-free.[2] But 0.387 is less than the equitable (but not envy-free) allocation of 0.393 we found for the 2-efficient allocation. Therefore, this 3-cut allocation is not *o*-efficient.

In fact, this 3-cut allocation is not 3-efficient. Recall that the 2-efficient, equitable allocation gives each player 0.393. By using a third cut to remove a small portion from A's piece and giving it to B, or a small portion from B's piece and giving it to A, the recipient of these pieces will obtain more than 0.393 while the donor will retain less. As long as the donor retains at least 0.387, this 3-cut allocation will Pareto-dominate the 3-player, equitable allocation (with middle cut at $x < \frac{1}{2}$), which is why the latter allocation is not 3-efficient, though it is efficient among 3-cut, equitable allocations.

Four Cuts

With four cuts, we can give each of the three players pieces on both sides of $\frac{1}{2}$, which for A and B will be disconnected, but for C will be connected as one piece at $\frac{1}{2}$. Assume the two cutpoints to the left of $\frac{1}{2}$ are x and y, and the two to the right are $\overline{x} = 1 - x$ and $\overline{y} = 1 - y$, and that we give A and B two pieces each (C's portion is shown as two adjacent pieces around $\frac{1}{2}$, but it is a single connected piece because there is no cut at $\frac{1}{2}$):

[2]Intuitively, the reason that both A and B value the piece defined by $[0.056, 0.444]$ equally is that the midpoint of this interval is 0.25, where the players' pdfs intersect. A prefers the left half of this interval, $[0.056, 0.25]$, by the same amount that B prefers the right half, $[0.25, 0.444]$. The same is true for the less preferred halves, so the full interval is of equal value to A and B.



The players' values will be equal—and, therefore, the allocation to the three players will be equitable—when x and y on the left half satisfy the following equations ($\overline{x}$ and $\overline{y}$ on the right half satisfy analogous equations):[3]

$$\int_0^x (-4t+2)dt = \int_x^y (-2t+\frac{3}{2})dt = \int_y^{\frac{1}{2}} dt.$$

After integration and evaluation of the integrals, we have

$$-6x^2 + 7x + 2y^2 - 3y = 0$$
$$2x^2 - 3x - 2y^2 + 5y - 1 = 0,$$

whose four solutions include one feasible solution: $x \approx 0.114$ and $y \approx 0.299$, which gives A, B, and C pieces they value at 0.201 to the left of $\frac{1}{2}$. The cutpoints on the right, $\overline{y} \approx 0.701$ and $\overline{x} \approx 0.886$, are symmetrical, enabling each player to obtain an equitable allocation of 0.403. This allocation gives each player more than the 2-efficient, equitable allocation (0.393) and the 3-cut equitable and envy-free allocation (0.387), making it the *maximin allocation*—it maximizes the minimum value a player receives.

However, this 4-efficient allocation is not envy-free. A, which obtains the two pieces defined by the intervals $[0, x]$ and $[\overline{x}, 1]$, will envy B, which obtains the two pieces defined by the intervals $[x, y]$ and $[\overline{y}, \overline{x}]$. A values the sum of B's two intervals as 0.435, which is more than the 0.403 it receives from its portion of the equitable 4-cut allocation.

Maxsum

The solution that maximizes the sum of the values to the players is that which gives pieces to those players who value them most. Note that the value functions of A, B, and C in our example intersect at two points, $x = \frac{1}{4}$ and $x = \frac{3}{4}$. The part of the cake defined by the interval $[0, \frac{1}{4}]$ and $[\frac{3}{4}, 1]$ is most valued by A, and the part defined by $[\frac{1}{4}, \frac{3}{4}]$ is most valued by C. The combined value to A of its two endpieces is $\frac{3}{4}$, whereas the value to C of its middle piece is $\frac{1}{2}$, so the total value that these two players attach to their pieces is $\frac{5}{4} = 1.25$, making the allocation *o*-efficient.

But it is blatantly unfair to B, who receives nothing. If B, instead of A, were awarded the right endpiece, the total value to the three players would be 1.1875, because A values the interval defined by $[0, \frac{1}{4}]$ at 0.375, B values the interval defined by $[\frac{3}{4}, 1]$ at 0.3125, and C values the interval defined by $[\frac{1}{4}, \frac{3}{4}]$ at 0.5. This division, which requires only two cuts, is not envy-free (B envies C for what it thinks is a piece worth 0.375), equitable, or *o*-efficient, so it fails to satisfy all three properties.

It is worth pointing out that there is no *o*-efficient division, even for two players, if their value functions cross an infinite number of times (Barbanel and Brams [4]). The reason is

[3]If the allocations to the three players on the left and right halves were different, the division would not be 4-efficient.

that while adding another cut can always bring one closer to the maxsum allocation, it will never be achieved—and so give o-efficiency—with a finite number of cuts.

Envy-Freeness

So far we have found that the 2-efficient and 4-efficient equitable allocations are not envy-free, whereas one 3-cut equitable allocation is. However, the latter division is not o-efficient. Thus, none of the equitable divisions, using 2, 3, or 4 cuts, is perfect, because they satisfy at most two of the three properties of a perfect division (the 2-efficient equitable division is neither envy-free nor o-efficient).

In general, there will be a plethora of envy-free divisions. For the example, every division that gives player A a piece that begins at 0 and ends between $x \approx 0.271$ and $x \approx 0.333$, B the same-length piece on the right, and C the middle piece, is envy-free and o-efficient, but it is not equitable and, hence, not perfect.

Before stating the theorem, we summarize in Table 1—for the cake defined in section 3 and illustrated in Figure 1—our previous findings for equitable divisions, which show the impossibility of a perfect division of up to 4 cuts. Specifically, the 4-efficient, equitable division is o-efficient, but it is not envy-free; one 3-cut, equitable division is envy-free, but it is not o-efficient; the 2-efficient, equitable division is neither envy-free nor o-efficient.

	2 cuts	**3 cuts**	**4 cuts**
Value to each player of equitable division	0.393	0.376/0.387	0.403
Is equitable division envy-free?	No	Yes: 0.387	No
Is equitable division o-efficient?	No	No	Yes

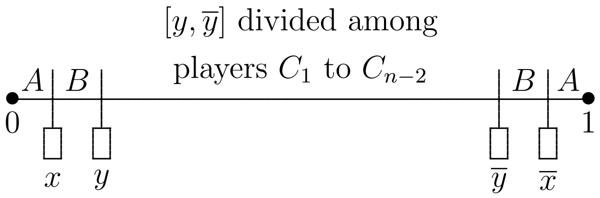
Table 1: Equitable allocations for the 3-player example illustrated in Figure 1.

The following theorem generalizes this impossibility result, proving that there is no perfect division in our example for not just up to 4 cuts but for any number, and for any number of players as well:

Theorem. *There exists a cake for which there is no perfect division, however many players (greater than two) there are and however many cuts are made.*

Proof. We have already shown that there is no perfect division of our cake for three players, making up to 4 cuts. Now, we introduce additional players to those considered in section 3 and illustrated in Figure 1. In particular, let players A and B have preferences defined as before and players C_i for $i = 1$ to $n-2$ have value functions $v_{C_i}(t) = 1$ for $t \in [0,1]$. We prove that there is no perfect division by showing that there exists an equitable, o-efficient allocation that is not envy-free.

Like the 3-player, 4-cut allocation, the n-player example exploits symmetry, where player A receives $[0,x] \cup [1-x,1]$, player B receives $[x,y] \cup [1-y,1-x]$, and players C_1 through C_{n-2} divide the remaining cake $[y,1-y]$ into $n-2$ pieces of the same length $\frac{1-2y}{n-2}$. The interval $[y,1-y]$ is divided by cuts at $y+\frac{i(1-2y)}{n-2}$ for $i=1$ to $n-3$. Denote such an allocation by $S = (S_A, S_B, S_{C_1}, \ldots, S_{C_{n-2}})$. In the figure, $\overline{x} = 1-x$ and $\overline{y} = 1-y$.



The cutpoints x and y are determined to ensure equitability, so that

$$2\int_0^x(-4t+2)dt = 2\int_x^y(-2t+\frac{3}{2})dt = \frac{1-2y}{n-2}.$$

After integration and evaluation of the integrals, we have

$$\begin{aligned} 4x-4x^2 &= \frac{1-2y}{n-2} \\ 2(x^2-y^2)+3(y-x) &= \frac{1-2y}{n-2}. \end{aligned}$$

Solving for y in terms of x and n in the first equation, and then eliminating y from the second equation, yields a 4th degree polynomial in x. Of the four roots, one is $x = 1$, two are complex, and the fourth determines the cutpoints x and y as decreasing functions of n (naturally, as n increases both A and B receive less of the cake). Oddly, we don't have to explicitly calculate x and y to prove that the allocation is *o*-efficient and that A envies B.

To show that A envies B, we need to show that A values $[x, y]$ more than $[0, x]$. However, because A's value of $[0, x]$ is the same as B's value of $[x, y]$, A envies B if the difference between A's and B's values of $[x, y]$ is positive. On $[0, 1/2]$, $v_A(t) - v_B(t) = -2t + 1/2$ and

$$\int_x^y -2t+\frac{1}{2}dt = -y^2+\frac{y}{2}+x^2-\frac{x}{2} = (x-y)(x+y)+\frac{y-x}{2} = (y-x)\left[\frac{1}{2}-(x+y)\right].$$

Because $y > x$, this expression is positive when $1/2-(x+y) > 0$. We see that this holds for the 3-player, 4-cut example—in which the cutpoints are $x \approx 0.114$ and $y \approx 0.299$—so their sum is less than $1/2$. Because x and y are decreasing in n, the inequality $1/2 - (x + y) > 0$ holds for all $n \geq 3$. Hence, for this equitable allocation, A envies B.

It remains to show that the allocation is *o*-efficient. S is *o*-efficient if there does not exist another division $T = (T_A, T_B, T_{C_1}, \ldots, T_{C_{n-2}})$ such that for all $i \in \{A, B, C_1, \ldots, C_{n-2}\}$, $\mu_i(T_i) \geq \mu_i(S_i)$; and for some $j \in \{A, B, C_1, \ldots, C_{n-2}\}$, $\mu_j(T_j) > \mu_j(S_j)$. Denote the sum of lengths of the intervals defining S_i by $L(S_i)$. Assume that there exists a division T that Pareto-dominates division S. Let $S_C = S_{C_1} \cup S_{C_2} \cup \cdots \cup S_{C_{n-2}}$ be the share of all players C_i. Define T_C similarly. A necessary condition for T to Pareto-dominate S is that $L(T_C) \geq L(S_C)$.

In particular, suppose $L(T_C) = L(S_C)$. If $T_C \neq S_C$ (i.e., T_C is *not* a single piece centered around $t = \frac{1}{2}$), then the value of the remaining cake to each of players A and B will be less than its value if T_C is centered around $t = \frac{1}{2}$. Therefore, however the remaining cake is allocated between A and B, they together benefit most when T_C is a single piece centered around $t = \frac{1}{2}$.

We next ask whether there is an allocation of the remaining cake to players A and B—different from that given by S—such that one player (e.g., A) does at least as well, and the other (B) does better, when $T_C = S_C$. We restrict attention to the case in which A and B exchange portions of their S shares in the interval $[0, \frac{1}{2}]$, holding C's share fixed, which comprises the shares of all the C_i's, constant in S and T.

At all points $t \in [0, \frac{1}{2}]$, the value functions $v_A(t)$ and $v_B(t)$ are decreasing. Because the ratio, $\frac{v_A}{v_B}$, is also decreasing in t,

$$\frac{d}{dt}\left(\frac{v_A}{v_B}\right) = \frac{-2}{(\frac{3}{2} - 2t)^2} < 0,$$

$v_A(t)$ is decreasing faster in t than $v_B(t)$. Now S_A will differ from T_A by $X \equiv S_A \cap T_B$, and S_B will differ from T_B by $Y \equiv S_B \cap T_A$.

In effect, X is what A loses when the allocation changes from S to T (because it goes to B), and Y is what A gains (because it comes from B); the opposite is true for B. Because of the decreasing value functions and the decreasing ratio of value functions, if $\mu_A(Y) \geq \mu_A(X)$, then $\mu_B(X) < \mu_B(Y)$, so T does not Pareto-dominate S.

We need not assume that $L(T_C) = L(S_C)$. If $L(T_C) > L(S_C)$, players A and B will have less cake to share. Hence, however they divide it, both cannot do at least as well as they did under S, so again T does not Pareto-dominate S. Although the foregoing result applies only to the interval $[0, \frac{1}{2}]$, it can be extended to the interval $[\frac{1}{2}, 1]$ because of the symmetry of the pdfs about $t = \frac{1}{2}$. Also, by this symmetry, there are no Pareto-improving exchanges of pieces between the left and right halves.

So far we have shown that swaps of pieces between players A and B cannot be Pareto-improving over S if $L(T_C) \geq L(S_C)$. But what if some player C_i also swaps with players A and B? By the reasoning given above for exchanges between players A and B, A and B will value less per unit length the pieces they obtain from C_i, whereas C_i will not value them more because C_i's value function is constant. Thus, exchanges that involve any player C_i as well as players A and B will be Pareto-diminishing, rather than Pareto-improving, over S. Hence, the allocation is *o*-efficient and equitable, but it fails to be envy-free. Therefore, there is no perfect division in the n-player example, however many cuts are made. □

In the absence of a perfect division, one might ask what is the next-best solution. If envy-freeness and equitability cannot both be satisfied, as in our 3-player example for 2 and 4 cuts, a possible criterion for deciding between them is the allocation that maximizes the total value to the players—that is, the one that comes closest to being maxsum (algorithms given in Cohler et al. [13] determine this for an envy-free allocation if the players have piecewise constant or piecewise linear valuations of a cake).

In our 3-player example, we know from Table 1 that the 4-cut equitable allocation gives the players a total value of $3(0.403) = 1.209$. It is not hard to show that the envy-free allocation that maximizes the players' total value also uses 4 cuts. Like the equitable allocation, it awards player A two disconnected endpieces, player B two disconnected pieces adjacent to A's, and player C a middle piece.

To prevent A from being envious of B in the 4-cut equitable allocation, we must "adjust" the cutpoints of the equitable allocation so as to maximize the total value to the players while precluding envy. This is a tedious but straightforward calculation, giving intervals of $[0, 0.121) \cup (0.879, 1]$ to A, $[0.121, 0.272) \cup [0.728, 0.879]$ to B, and $[0.272, 0.728)$ to C. Compared with the equitable allocation that gives 0.403 to each player, A receives slightly more value (0.425), B substantially less (0.334), and C the biggest boost (0.456).

These values sum to 1.216, which slightly edges out the equitable one (1.209) in total value. It is not hard to show, however, that there are examples in which the equitable allocation outperforms the envy-free one in maximizing total value, establishing that neither envy-freeness nor equitability always wins in a contest for maximizing the total value if there is no perfect division.

It is worth pointing out that our results are applicable to pie-cutting (Brams, Jones, and Klamler [9]; Barbanel, Brams, and Stromquist [5]; Barbanel and Brams [3]), in which the endpoints of a cake are connected to form a pie. If we do this in our 3-player example, then a minimum of 3 radial cuts are needed to cut the pie into three pieces instead of the 2 cuts that a cake requires (note that the endpoints of the cake constitute one cut).

For the 3-player example, the 4 cuts of our cake that yield an o-efficient, equitable allocation, and the different 4 cuts that yield an o-efficient, envy-free allocation, do the same for a pie. In the case of a pie, however, player A can receive a single connected piece, because the endpoints 0 and 1 in the interval $[0, 1]$ are identified—the two pieces that A receives in the 4-cut division of a cake become one piece in the 4-cut division of a pie. Thus, the impossibility of a perfect division of a cake applies as well to a pie.

5 Conclusions

In the fair division of indivisible goods, there are a number of impossibility results (Brams [7]), so it is perhaps not surprising that there is not always a perfect division of a heterogeneous divisible good like cake. But the fact that more cuts do not necessarily increase the equitable portions that players receive (in going from 2 to 3 cuts in our example), or may preclude a property like envy-freeness (in going from 3 to 4 cuts), indicates that efficiency and envy-freeness may not be enhanced, or even preserved, as the number of cuts increases.

It would be desirable to understand better (i) the conditions under which perfect divisions are possible, and when they are not, and (ii) under what conditions different sets of properties can be satisfied. In the fair division of a divisible good like land, players will often seek connected pieces, for which algorithms that use a minimal number of cuts suffice (Barbanel and Brams [2]). In other situations, however, in which one might seek different parts of a divisible good (e.g., time slots in a day that fit into one's schedule, different sections of a beach that an investor wants to develop), disconnectedness is not a problem. In such situations, algorithms that allow for additional cuts are appealing, especially if they produce divisions closer to being perfect.[4]

But a perfect division may simply not be in the cards, however many cuts are made.

[4]Although our proof of the theorem established that the 4-efficient division is o-efficient for the 3-player example, we know of no algorithm for determining when a k-efficient division is o-efficient if there are more than two players (Barbanel and Brams [4] give an algorithm for two players). Furthermore, even determining when an equitable division is k-efficient is not always straightforward, as we showed in calculating two 3-cut equitable divisions for the 3-player example.

In that case, one can select as the next-best solution the division—either envy-free or equitable—that maximizes the total value of the players.

We conclude with three open questions:

1. What are necessary and sufficient conditions that make perfection possible?

2. More subjectively, when perfection is not possible, which of the three properties is most dispensable?

3. Whether perfection is possible or not, can an envy-causing division Pareto-dominate all envy-free divisions, however many cuts are made?

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