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2009

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October 23, 2008

Abstract

In this paper a three-dimensional environmental defensive expenditures model with delay is considered. The model is based on the interactions among visitors V, quality of ecosystem goods E, and capital K, intended as accommodation and entertainment facilities, in Protected Areas (PAs). The tourism user fees ($TUFs$) are used partly as a defensive expenditure and partly to increase the capital stock. The stability and existence of Hopf bifurcation are investigated. It is that stability switches and Hopf bifurcation occurs when the delay t passes through a sequence of critical values, τ_0. It has been that the introduction of a delay is a destabilizing process, in the sense that increasing the delay could cause the bio-economics to fluctuate. Formulas about the stability of bifurcating periodic solution and the direction of Hopf bifurcation are exhibited by applying the normal form theory and the center manifold theorem. Numerical simulations are given to illustrate the results.

1 Introduction

The number of people visiting protected natural areas has increased considerably over the last three decades; this trend is expected to continue. Research suggests that tourists visiting natural areas are, at least in part, motivated by a desire to experience some degree of solitude away from the crowds associated with typical mass tourism attractions (see e.g. Eagles (2002)). As the demand for nature-based tourism is growing, resource managers are facing the problem of accommodating an ever increasing number of tourists while preserving the very qualities of the natural site appreciated by tourists (and others). A number of relatively simple, market-based mechanisms commonly known as tourism

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user fees ($TUFs$) can gather significant revenues from tourism-based activities, which can then be directed toward supporting Protected Areas (PAs) and other conservation efforts. The fees partially reflect the cost of supplying recreational services, the demand for natural resources, and the value the visitors place on their experience at the site. $TUFs$ can be structured around many activities. For example: PAs entrance fees, restaurant and lodging concession fees, fees/permits for hiking, scuba diving, fishing, etc. This revenue is used for: operation and maintenance for PA, upgrading of facilities and conservation programs. In this paper we consider that the TUF is dependent linearly on the number of visitors and that a part of it (η) is used for conservation programs (defensive expenditures) and the other part $(1-\eta)$ (defined as reinvestment rate) for upgrading facilities and maintenance of PA growth of capital stock. Therefore, the parameter h reflects the decision of the managers of PAs on the distribution of the investments between the defense of the environmental (defensive expenditures) and the increase of the capital stock (reinvestment rate). In this work, we formulate a simple model, with an unique equilibrium (when it exists) between the variable of state considered; moreover this equilibrium is always stable. Therefore variations of the parameter η does not alter the stability of the system. The aim of this work is to analyze how the stability of the equilibrium changes when a delay τ is introduced. In fact we think that dynamics of the environmental resource and capital stock at time t depend on the number of visitors in the past. In this model, we can see how the stability changes giving rise to a Hopf bifurcation when the delay t passes through a sequence of critical values, to. Hopf bifurcation allowed us to find the existence of a region of instability in the neighborhood of a fixed point where the managers of PAs can stabilize the system if the delay is sufficiently short, but the system becomes unstable when the delay is too long. The equilibrium can be brought back to the stability (in the sense that it is without periodic oscillations) in various ways, for example reducing the delay, increasing the defensive expenditures or improving the adopted technology in order to defend the environmental resource. This paper is organized as follows: in Section 2 the model is presented; in Section 3 and Section 4 fixed point, stability analysis and the existence of Hopf bifurcation, respectively, with $\tau = 0$ and $\tau > 0$ are studied; in Section 5, the direction of Hopf bifurcation and the stability and the period of bifurcating periodic solutions on the center manifold are determined; in Section 6 numerical simulations are presented.

2 The Model

The model, refer a generic Protected Area (PA) and have three variables: the visitors $V(t)$ in PA at time t, the environmental resource stock $E(t)$ and the capital stock $K(t)$ intended as structures, into the PA, for visitors activities.

Dynamic of the visitors V

Visitors are attracted by the infrastructures and services included in the variable K and by the natural resource E. Both stocks are combined by means of an additive [1] function, which assumes a degree of substitution between the environmental resource and the capital, in the sense that destinations with low capital stock can receive the same number of visitors as those with better infrastructures if they have a large natural resource stock.
Dynamics of the number of visitors is

$$V\dot{(t)} = m_1 E + m_2 K - aV^2(t) \tag{1}$$

where the parameter $a > 0$ represents the crowding effect. This means that the PA becomes less attractive when the number of tourists visiting the protected area increases, and this gives rise to a decrease in the number of visitors.

Dynamic of the environmental resource E

Following Becker (1982) and Cazzavillan and Musu (2001), the environmental resource stock is defined as the difference between the maximum tolerable pollution stock $\overline{P}$ and the current pollution stock $0 \leq P(t) \leq \overline{P}$

$$E(t) = \overline{P} - P(t)$$

Differentiating with respect to time we obtain the law of evolution of the environmental stock

$$\dot{E}(t) = \dot{P}(t) \tag{2}$$

We then assume that a constant proportion $0 < r < 1$ of the pollution stock is assimilated at each time t. Moreover, we assume that the asset E decreases proportionally with the level of tourist entries. When no resources can be devoted

[1] The results do not change if we use a moltiplicative function such as mEK

to abatement expenditures, residents can influence the pollution stock only by controlling tourist entries V(t)

$$\dot{P}(t) = bV(t) - r(\overline{P} - E(t)) \tag{3}$$

where $b > 0$. Combining (2) and (3) we finally get

$$\dot{E}(t) = r(\overline{P} - E(t)) - bV(t) \tag{4}$$

Visitors impact negatively on the environmental resource, but environment and infrastructures are attractive for visitors . Therefore the manager of PA uses a share η of total revenues generated by $TUfs$ to defend the environmental resource in the PA (environmental defensive expenditures). This expenditures are proportional to the number of visitors. Therefore the dynamics of the environmental resource is

$$\dot{E(t)} = r(\overline{P} - E) - bV(t) + c\eta V(t) \tag{5}$$

The parameter $c > 0$ is a constant parameter determining how much an additional unit of defensive expenditure rises the environmental resource. In other words, it indicates the technology that it comes adopted in order to defend the environmental quality.

Dynamic of the capital stock K

The other share $(1 - \eta)$ of the total revenue is used to increase the capital stock

$$\dot{K(t)} = (1 - \eta)V(t) - \delta K(t) \tag{6}$$

Capital stock is assumed to depreciate at the rate $\delta > 0$. As stated in the introduction, considering the delay $\tau > 0$ we formulate the model as follows:

$$\begin{aligned} \dot{V(t)} =& m_1 E(t) + m_2 K(t) - aV^2(t) \\ \dot{E(t)} =& r(\overline{P} - E(t)) - bV(t - \tau) + c\eta V(t - \tau) \\ \dot{K(t)} =& (1 - \eta)V(t - \tau) - \delta K(t) \end{aligned} \tag{7}$$

where m_1, m_2, a b,c, η, δ, $\overline{P}$, τ strictly positive constants, while $0 \leq \eta < 1$ and $0 < r < 1$.

3 Fixed point and stability analysis with $\tau = 0$

By a simple computation, it is straightforward to obtain the following results

Proposition 1 *A fixed point $S = (V_\infty, E_\infty, K_\infty)$ exist if only if one of the following conditions holds:*

a) $b - c\eta < 0$

b) $b - c\eta > 0$ *and* $\overline{P}a\delta r - m_2(1-\eta)(b-c\eta) > 0$

where

$$V_\infty = \frac{\rho}{2} + \frac{1}{2}\sqrt{\rho^2 + 4\frac{m_1\overline{P}}{a}}, \qquad E_\infty = \overline{P} - \frac{b-c\eta}{r}V_\infty, \qquad K_\infty = \frac{1-\eta}{\delta}V_\infty \quad (8)$$

with $\rho = \frac{1}{a}(\frac{m_2(1-\eta)}{\delta} - \frac{m_1(b-c\eta)}{r})$

Proof. Being $K_\infty = \frac{(1-\eta)}{\delta}V_\infty$, V_∞ and E_∞ are determined by intersections of the functions $F(V,E) = m_1E + m_2\frac{1-\eta}{\delta}V - aV^2 = 0$ and $G(E,V) = r(\overline{P} - E) - (b - c\eta)V = 0$, therefore V_∞ is solution of the equation $V^2 - \frac{1}{a}(\frac{m_2(1-\eta)}{\delta} - \frac{m_1(b-c\eta)}{r}) - \frac{m_1\overline{P}}{a} = 0$ ∎

Proposition 2 *The fixed point $S = (V_\infty, E_\infty, K_\infty)$ is always attractive*

Proof. Straightforward calculation enable the definition of the Jacobian matrix J calculated in $S = (V_\infty, E_\infty, K_\infty)$

$$J(S) = \begin{pmatrix} -2aV_\infty & m_1 & m_2 \\ -b + c\eta & -r & 0 \\ 1-\eta & 0 & -\delta \end{pmatrix} \quad (9)$$

The characteristic polynomial of $J(S)$ is

$$P(\lambda) = \lambda^3 - (tr(J))\lambda^2 + M_2\lambda - det(J) \quad (10)$$

where

$$tr(J) = -r - \delta - 2aV_\infty \tag{11}$$

$$M2 = 2aV_\infty r + 2aV_\infty \delta - m_2(1-\eta) + m_1(b - c\eta) + \delta r \tag{12}$$

and it is easily computed that

$$det(J) = -a\delta r\sqrt{q^2 + \frac{4m_1\overline{P}}{a}} \tag{13}$$

From Routh-Hurwitz theorem the cubic polynomial (10) has all negative real parts roots, if and only if

$$tr(J) < 0 \tag{14}$$

$$M_2 > 0 \tag{15}$$

$$det(J) < 0 \tag{16}$$

$$H = |tr(J)||M_2| - |det(J)| > 0 \tag{17}$$

The determinant and the trace of J are always negative. Applying the implicit function theorem to the function $F(V, E) = 0$, we obtain $f_V = -\dfrac{F_V}{F_E} = \dfrac{2aV_\infty - \frac{m_2(1-\eta)}{\delta}}{m_1} > 0$, therefore the coefficient M_2 is always positive. From easy calculations one obtains that H is always positive. ■

4 Stability analysis and Hopf bifurcation with $\tau > 0$

By the linear transform

$$\begin{cases} x_1(t) = V(t) - V_\infty \\ x_2(t) = E(t) - E_\infty \\ x_3(t) = K(t) - K_\infty \end{cases} \tag{18}$$

system (7) becomes

$$\begin{aligned}\dot{x_1} =& m_1x_2(t) + m_2x_3 - ax_1^2(t) - 2aV_\infty x_1(t)\\ \dot{x_2} =& -rx_2(t) - (b - c\eta)x_1(t - \tau)\\ \dot{x_3} =& (1 - \eta)x_1(t - \tau) - \delta x_3(t)\end{aligned} \tag{19}$$

The associated characteristic equation of system (7) is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_1\lambda + b_0)e^{-\lambda\tau} = 0 \tag{20}$$

where

$$\begin{aligned}a_2 &= 2aV_\infty + \delta + r\\ a_1 &= (r + \delta)2aV_\infty + \delta r\\ a_0 &= 2aV_\infty\delta r\\ b_1 &= -m_2(1 - \eta) + m_1(b - c\eta)\\ b_0 &= m_1\delta(b - c\eta) - m_2r(1 - \eta)\end{aligned} \tag{21}$$

Follow the result proved in Ruan and Wei (2003) by using Rouche's theorem, we can say that as τ vary, the sum of the order of the zeros of exponential polynomial (20) on the open right half plane can charge only if a zero appears on or crosses the imaginary axis.
Therefore, if $i\omega(\omega > 0)$ is a root of (20) then ω satisfies

$$s(z) = z^3 + pz^2 + qz + n \tag{22}$$

where $z = \omega^2$, $p = a_2^2 - 2a_1$, $q = a_1^2 - 2a_0a_2 - b_1^2$, $n = a_0^2 - b_0^2$
In order to find a signs of the roots of the third degree polynomial, we introduce the following proposition

Proposition 3 *For the third degree polynomial $s(z)$, we have the following results*

(i) if $r \leq 0$, then equation (22)has at least one positive root;

(ii) if $r \geq 0$ and $\Delta = p^2 - 3q \leq 0$, the equation (22) has no positive roots;

(iii) if $r \geq 0$ and $\Delta > 0$, the equation (22) has positive roots iff $z_1^* = \dfrac{-p + \sqrt{\Delta}}{3} > 0$ and $s(z_1^*) \leq 0$.

Therefore, if there is al least a positive $\omega_0 = \sqrt{z_0}$ satisfy the equation (22), then the characteristic equation (20) has a pair purely imaginary roots of the form $\pm i\omega$ with

$$\tau_0 = \frac{1}{\omega_0}\{cos^{-1}(\frac{b1\omega_0^4 + (a_2b_0 - a_1b_1)\omega_0^2 - a_0b_0}{(-b_0)^2 + b_1^2\omega_0^2}) + 2j\pi\} \tag{23}$$

where $j = 0, 1...$
Thus, we obtain the following proposition

Proposition 4 *For the equation (20), we have*

(i) if $n \geq 0$ and $\Delta = p^2 - 3q \leq 0$, then, for all $\tau \geq 0$, all roots with positive real part of (20) has the same sum to those of the polynomial (20) for $\tau = 0$;

(ii) if either $n < 0$ or $n \geq 0$ and $\Delta > 0$, $z_1^* > 0$ and $s(z_1^*) \leq 0$, then for $\tau \in [0, \tau_0]$, all roots with positive real parts of (20) has the same to those of the polynomial of (20) for $\tau = 0$.

Suppose that $z_0 = \omega_0^2$ and $s^{'}(z_0) \neq 0$, we obtain that $\frac{d(Re\lambda(\tau_0^j))}{d\tau}$ and $s^{'}(z_0)$ have the same sign.
Remembering that the conditions (14)-(17) are always verified we have the following theorem about the stability of the fixed point S of system (7) and Hopf bifurcations.

Theorem 1 *Let τ_0 and ω_0.*

(i) if $n \geq 0$ and $\Delta = p^2 - 3q \leq 0$, the positive equilibrium S of system (7) is asymptotically stable for all $\tau \geq 0$;

(ii) if either $n < 0$ or $n \geq 0$, $\Delta > 0$, $z_1^* > 0$ and $s(z_1^*) \leq 0$, then the positive equilibrium S of system (7) is asymptotically stable for $\tau \in [0, \tau_0)$;

(iii) if the condition of (ii) are satisfied and $s^{'}(z_0) \neq 0$, then system (7) undergoes a Hopf bifurcation at the equilibrium S when $\tau = \tau_0$.

Moreover, we have the following theorem

Theorem 2 *Necessary condition for existence of a Hopf bifurcation is that $b - c\eta > 0$.*

Proof. See Appendix ■

Proposition 2 says that, the system (7) undergoes to Hopf bifurcation at the equilibrium S when $\tau = \tau_0$, when the technology or the defensive expenditures sufficiently is not elevated, regarding the impact negative of the visitors on the environmental. In other words if we not to sufficiently defend the environmental resource them can carry to one destabilization of the fixed point when $\tau > \tau_0$.

5 Direction and stability of the Hopf bifurcation

The following algorithm for computing the period and the stability of the Hopf periodic orbit follows Hassard et. al (1981), Sun et al. (2007) and Liao et al. (2007). Consider the autonomous equation

$$\dot{x}(t) = L_\mu(x_t) + f(\mu, x_t) \tag{24}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3$, and $L_\mu : C \longrightarrow \mathbb{R}$, $f : \mathbb{R} \times C$ are given, respectively, by

$$\begin{aligned} L_\mu(\phi) \quad &= \quad (\tau_0 + \mu) \begin{pmatrix} -2aV_\infty & m_1 & m_2 \\ 0 & -r & 0 \\ 0 & 0 & -\delta \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} \\ &+ (\tau_0 + \mu) \begin{pmatrix} 0 & 0 & 0 \\ c\eta - 2bV_\infty & 0 & 0 \\ 1 - \eta & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix} \end{aligned} \tag{25}$$

and

$$f(\tau, \phi) = (\tau_0 + \mu) \begin{pmatrix} -a\phi_1^2(0) \\ 0 \\ 0 \end{pmatrix} \tag{26}$$

An orbit corresponding to a solution $x(t)$ of (24) is a curve of C traced out by the family of functions $x(\boldsymbol{.})$, $(x_t(t) = x_t(t+\theta))$ as t rangers over $(0, \infty)$; the orbit of a periodic solution is a closed curve in C. The individual periodic orbits will belong to slices C_μ, (μ constant) of C.

By the Riesz representation theorem, there exists a function $\rho(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu \phi = \int_{-1}^{0} d\rho(\theta, 0)\phi(0) \qquad \text{for } \phi \in C \tag{27}$$

in fact, we can choose

$$\begin{aligned} \rho(\theta, \mu) \quad &= \quad (\tau_0 + \mu) \begin{pmatrix} -2aV_\infty & m_1 & m_2 \\ 0 & -r & 0 \\ 0 & 0 & -\delta \end{pmatrix} \delta(\theta) \\ &\quad -(\tau_0 + \mu) \begin{pmatrix} 0 & 0 & 0 \\ c\eta - 2bV_\infty & 0 & 0 \\ 1 - \eta & 0 & 0 \end{pmatrix} \delta(\theta + 1) \end{aligned} \tag{28}$$

where δ is defined by $\delta(\theta) = \begin{cases} 0 & \theta \neq 0 \\ 1 & \theta = 0 \end{cases}$

For $\phi \in C^1([-1, 0], \mathbb{R}^3)$ define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta} & \theta \in [-1, 0), \\ \int_{-1}^{0} d\rho(\mu, s)\phi(s) & \theta = 0 \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0 & \theta \in [-1, 0), \\ f(\mu, \phi) & \theta = 0 \end{cases}$$

Then the system (24) can be written as

$$\dot{x_t} = A(\mu)x_t + R(\mu)x_t \tag{29}$$

where $x_t(\theta) = x(t + \theta)$ for $\theta \in [-1, 0]$.

For $\psi \in C^1([0, 1], (\mathbb{R}^3)^*)$, the adjoint operator A^* is defined by

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds} & s \in [0, 1), \\ \int_{-1}^{0} d\rho^T(t, 0)\psi(-t) & s = 0 \end{cases}$$

To construct coordinates to describe the center manifold C_0 near $\mathbf{0} \in \mathbb{R}^3$, we need of a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta) d\rho(\theta) \phi(\xi) d\xi, \tag{30}$$

where $\rho(\theta) = \rho(\theta, 0)$. By the discussion in Section 4, we know that $\pm\omega\tau_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We first need to compute the eigenvector of $A(0)$ and A^* corresponding to $i\omega\tau_0$ and $-i\omega\tau_0$, respectively.

Suppose that $q(\theta) = (1, \beta, \gamma)^T e^{i\theta\omega\tau_0}$ is the eigenvector of $A(0)$ corresponding to $i\omega\tau_0$. Then $A(0)q(\theta) = i\omega\tau_0 q(\theta)$. If follows from the definition of $A(0)$ and (27) and (28) that

$$\tau_0 \begin{pmatrix} i\omega + 2aV_\infty & -m_1 & -m_2 \\ -(c\eta - 2V_\infty b)e^{-i\omega\tau_0} & i\omega + r & 0 \\ -(1-\eta)e^{-i\omega\tau_0} & 0 & i\omega + \delta \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{31}$$

Thus, we can easily obtain

$$q(0) = (1, \beta, \gamma)^T = (1, \frac{(-b + c\eta)e^{-i\omega\tau}}{i\omega + r}, \frac{(1-\eta)e^{-i\omega\tau}}{i\omega + \delta})^T \tag{32}$$

On the other hand, suppose that $q^*(s) = D(1, \beta^*, \gamma^*)e^{is\omega\tau_0}$ is the eigenvectors of A^* corresponding to $-i\omega\tau_0$. By the definition of A^* and (27) and (28), we have

$$\tau_0 \begin{pmatrix} -i\omega + 2aV_\infty & -(c\eta - 2bV_\infty)e^{-i\omega\tau_0} & -(1-\eta)e^{-i\omega\tau_0} \\ -m_1 & -i\omega + r & 0 \\ -m_2 & 0 & -i\omega + \delta \end{pmatrix} (q^*(0))^T = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{33}$$

which means

$$q^*(0) = D(1, \beta^*, \gamma^*) = D(1, \frac{m_1}{-i\omega + r}, \frac{m_2}{\delta - i\omega}) \tag{34}$$

In order to assume $< q^*(s), q(\theta) >= 1$, we need to determine the value of D. From (30), we have

$$
\begin{aligned}
< q^*(s), q(0) > &= \overline{D}(1, \overline{\beta^*}, \overline{\gamma^*})(1, \beta, \gamma)^T \\
&- \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{D}(1, \overline{\beta^*}, \overline{\gamma^*}) e^{-i(\xi-\theta)\omega\tau_0} d\rho(\theta)(1, \beta, \gamma)^T e^{i\psi\omega\tau_0} d\xi \\
&= \overline{D}\{1 + \beta\overline{\beta^*} + \gamma\overline{\gamma^*} - \int_{-1}^{0} (1, \overline{\beta^*}, \overline{\gamma^*})\theta e^{i\theta\omega\tau_0} d\rho(\theta)(1, \beta, \gamma)^T\} \\
&= \overline{D}\{1 + \gamma\overline{\gamma^*} + \beta\overline{\beta^*} - \tau_0 e^{-i\omega\tau_0}(\overline{\beta^*}(c\eta - 2bV_\infty) + (1-\eta)\overline{\gamma^*})
\end{aligned}
\tag{35}
$$

In the remainder of this section, we will follow the ideas and use the same notations as Hassard et. al (1981).
Let x_t be the solution of (29) when $\mu = 0$. Define

$$
z(t) =< q^*, x_t >, \quad W(t, \theta) = x_t - 2Re\{\ z(t)q(\theta)\} \tag{36}
$$

On the center manifold $\mathbf{C}_0$ we have $W(t, \theta) = W(z(t), \overline{z}(t), \theta)$,
where

$$
W(z, \overline{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + ..., \tag{37}
$$

z and $\overline{z}$ are local coordinates for center manifold $\mathbf{C}_0$ in the direction of q^* and $\overline{q^*}$. Note that W is real if x_t is real. We consider only real solutions. For the solution $x_t \in \mathbf{C}_0$ of (24), since $\mu = 0$, we have

$$
\begin{aligned}
\dot{z}(t) &= i\tau_0\omega z+ < \overline{q}^*(0), f(0, W(z, \overline{z}, \theta) + 2Re\{\ zq(\theta)\}\ > \\
&= i\tau_0\omega z + \overline{q}^*(0)f(0, W(z, \overline{z}, \theta) + 2Re\{\ zq(\theta)\} \\
&= i\tau_0\omega z + \overline{q}^*(0)f_0(z, \overline{z})
\end{aligned}
\tag{38}
$$

which we rewrite in abbreviated form as $\dot{z}(t) = i\tau_0\omega z(t) + g(z, \overline{z})$
with

$$
\begin{aligned}
g(z, \overline{z}) &= \overline{q}^*(0)f_0(z, \overline{z}) \\
&= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{20}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + ...
\end{aligned}
\tag{39}
$$

Noticing that $x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta)) = W(t,\theta) + zq(\theta) + \overline{zq}(\theta)$ and $q(\theta) = (1, \beta, \gamma)^T e^{i\theta\omega\tau_0}$, we have

$$
\begin{aligned}
x_{1t}(0) =& z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3),\\
x_{2t}(0) =& \beta z + \overline{\beta}\overline{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3),\\
x_{3t}(0) =& \gamma z + \overline{\gamma z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\overline{z} + W_{02}^{(3)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3),\\
x_{1t}(-1) =& ze^{-i\omega\tau_0} + \overline{z}e^{i\omega\tau_0} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\overline{z}\\
&+ W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3).
\end{aligned}
\tag{40}
$$

Thus, from (39), we have

$$
\begin{aligned}
g(z,\overline{z}) =&\\
&= \overline{q}^*(0)f_0(z,\overline{z}) = \tau_0\overline{D}(1,\overline{\beta}^*,\overline{\gamma}^*)\begin{pmatrix} -ax_{1t}^2(0)\\ 0\\ 0\end{pmatrix}\\
&= -\tau_0\overline{D}a[z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3)]^2
\end{aligned}
$$

Comparing the coefficients with (39), we get

$$
\begin{aligned}
g_{20} &= -2\tau_0 a\overline{D}\\
g_{11} &= -2\tau_0 a\overline{D}\\
g_{02} &= -2\tau_0 a\overline{D}\\
g_{21} &= -\frac{1}{2}\tau_0 a\overline{D}(4W_{20}^{(1)}(0) + 8W_{11}^{(1)}(0))
\end{aligned}
\tag{41}
$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them. From (29) and (36) we have

$$\dot{W} = \dot{x}_t - \overline{z}q - \dot{\overline{zq}}$$
$$\begin{cases} AW - 2Re\{\ \overline{q}^*(0)f_0 q(0)\} & ,\theta \in [-1,0], \\ AW - 2Re\{\ \overline{q}^*(0)f_0 q(0)\}\ + f_0 & ,\theta = 0, \end{cases}$$
$$= AW + H(z,\overline{z},\theta). \tag{42}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + ... \tag{43}$$

Expanding the above series and comparing the corresponding coefficients, we obtain

$$(A - 2i\omega\tau_0)W_{20}(\theta) = -H_{20}(\theta), \qquad AW_{11}(\theta) = -H_{11}(\theta)... \tag{44}$$

from (42), we know that for $\theta \in [-1,0)$,

$$\begin{aligned} H(z,\overline{z},\theta) &= -\overline{q}^*(0)f_0 q(\theta) - q^*(0)\overline{f}_0\overline{q}(\theta) \\ &= -gq(\theta) - \overline{gq}(\theta). \end{aligned}$$

Comparing the coefficients with (43) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta) \tag{45}$$

and

$$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{\theta}. \tag{46}$$

From (44), (45) and the definition of A, it follows that

$$\dot{W}_{20}(\theta) = 2i\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta).$$

Notice that $q(\theta) = (1,\beta,\gamma)^T e^{i\theta\omega\tau_0}$, hence

$$W_{20}(\theta) = \frac{ig_{20}}{\omega\tau_0}q(0)e^{i\theta\omega\tau_0} + \frac{i\overline{g}_{02}}{3\omega\tau_0}\overline{q}(0)e^{-i\theta\omega\tau_0} + E_1 e^{2\theta\omega\tau_0}, \tag{47}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}) \in R^3$ is costant vector.
Similary, from (44) and (46), we can obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega\tau_0}q(0)e^{i\theta\omega\tau_0} + \frac{i\overline{g}_{11}}{\omega\tau_0}\overline{q}(0)e^{-i\theta\omega\tau_0} + E_2 \tag{48}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}) \in R^3$ is costant vector.
In what follows, we shall seek appropriate E_1 and E_2. From the definition of A and (44), we obtain

$$\int_{-1}^{0} d\rho(\theta)W_{20}(\theta) = 2i\tau_0\omega W_{20}(\theta) - H_{20}(0) \tag{49}$$

and

$$\int_{-1}^{0} d\rho(\theta)W_{11}(\theta) = -H_{11}(0) \tag{50}$$

where $\rho(\theta) = \rho(0, \theta)$. From (42), we have

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2\tau_0 \begin{pmatrix} -a \\ 0 \\ 0 \end{pmatrix} \tag{51}$$

and

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + 2\tau_0 \begin{pmatrix} -a \\ 0 \\ 0 \end{pmatrix} \tag{52}$$

Substituing (47) and (51) into (49) and noticing that

$$\left(i\tau_0\omega I - \int_{-1}^{0} e^{i\theta\omega\tau_0} d\eta(\theta) \right) q(0) = 0 \tag{53}$$

and

$$\left(-i\tau_0\omega I - \int_{-1}^{0} e^{-i\theta\omega\tau_0} d\eta(\theta) \right) \overline{q}(0) = 0 \tag{54}$$

we obtain

$$\left(2i\tau_0\omega I - \int_{-1}^{0} e^{2i\theta\omega\tau_0} d\eta(\theta) \right) E_1 = 2\tau_0 \begin{pmatrix} -a \\ 0 \\ 0 \end{pmatrix} \tag{55}$$

It follows that

$$\begin{aligned}
E_1^{(1)} &= -\,2a\frac{(2i\omega+r)(2i\omega+\delta)}{N}\\
E_1^{(2)} &= -\,2a\frac{(2i\omega+\delta)(c\eta-b)e^{-2i\omega\tau}}{N}\\
E_1^{(3)} &= -\,2a\frac{(2i\omega+r)(1-\eta)e^{-2i\omega\tau}}{N}
\end{aligned}$$

where

$$N = -m_2((1-\eta)e^{i\ \omega\tau_0}(i\omega+r)) + (i\omega+\delta)((i\omega+2aV)(i\omega+r) - m_1(c\eta-b)e^{i\omega\tau_0})$$

Similarly, replacing (48) and (52) into (50), we can get

$$\begin{aligned}
E_2^{(1)} &= -\,2a\frac{r\delta}{2ra\delta V_\infty - m_2r(1-\eta) + m_1\delta(b-c\eta)}\\
E_2^{(2)} &= -\,2a\frac{\delta(c\eta-b)}{2ra\delta V_\infty - m_2r(1-\eta) + m_1\delta(b-c\eta)}\\
E_2^{(3)} &= -\,2a\frac{r(1-\eta)}{2ra\delta V_\infty - m_2r(1-\eta) + m_1\delta(b-c\eta)}
\end{aligned}$$

Thus, we can determine $W_{20}(0)$ and $W_{11}(0)$ from (47) and (48). Furthermore, we can determine g_{21}. Therefore, each g_{ij} in (39) is determinate by the parameters and delay.
The coefficient $c_1(0)$ of the Poincaré normal form is given of these term g_{ij} by formula

$$c_1(0) = \frac{i}{2\tau_0\omega}(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{20}|^2}{3}) + \frac{g_{21}}{2} \tag{56}$$

The following formulas give us the value of $\mu 2$, β_2, and T_2

$$\mu_2 = -\frac{Re\{\ c_1(0)\}}{Re\{\ \lambda'(\tau_0\}} \tag{57}$$

$$\beta_2 = 2Re\{\ c_1(0)\} \tag{58}$$

$$T_2 = -\frac{\Im(c_1(0)) + \mu_2\Im(\lambda'(\tau_0)}{\tau_0\omega} \tag{59}$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical values τ_0, i.e., μ_2 determines the directions oh the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_0$ ($\tau < \tau_0$);

β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$), and T_2 determines the periodic of the bifurcating periodic solutions: the period increase (decrease) if $T_2 > 0$ ($T_2 < 0$).

6 Numerical examples

In this section, we present some mumerical result at different values of τ.

6.1 Example 1

We consider system 7, with the following parameters $a = .1$, $m_1 = 5$, $m_2 = 0.001$ $b = 0.85$, $\overline{P} = 25$, $\delta = .1$,$c = 1$, $r = 0.1$, $\eta = 0.75$ the conditions (ii) and (iii) of the Theorem 1 are hold. We choose $\eta = 0.75$, the positive fixed point is $S = (18.30, 6.70, 45.75)$, then we obtain $\omega_0 = 0.092$, $\tau_0 = 25.4$, $Re(c_1(0)) = -0.000745$, $\lambda^{'}(\tau_0) = 0.00048 - i0.00294$,with $\mu_2 > 0$, $\beta_2 < 0$ and period $T = 67$. The Figure 1 show a simulation with $\tau = 20 < \tau_0$. The Figure 2 shows a simulation of a stable periodic orbit with $\tau = 25.6 > \tau_0$.

6.2 Example 2

We consider system 7, with the following parameters $a = .1$, $m_1 = 5$, $m_2 = 1$ $b = 2$, $\overline{P} = 250$, $\delta = .1$,$c = 1$, $r = 0.1$, $\eta = 0.8$ the conditions (ii) and (iii) of the Theorem 1 are hold. We choose $\eta = 0.75$, the positive fixed point is $S = (21, 0.35, 41.6)$, then we obtain $\omega_0 = 1.32$, $\tau_0 = 1.05$, $Re(c_1(0)) = -0.51910^{-5}$, $\lambda^{'}(\tau_0) = 0.467 - i0.732$,with $\mu_2 > 0$, $\beta_2 < 0$ and period $T = 4.7$. The Figure 3 show a simulation with $\tau = 0.5 < \tau_0$. The Figure 4 shows a simulation of a stable periodic orbit with $\tau = 1.1 > \tau_0$.

7 Conclusion

In the present work, starting from a simple model exibiting an always stable equilibrium, we showed that a delay may generate instability (and as a consequence problems in the sustainability of the Protected Areas management) if the condition $b - c\eta > 0$ occurs, that is to say if the difensive expenditures are not sufficiently elevated. Further developments can be identified analyzing a model with differentiated delays, where the negative impact of the visitors on the environmental resource occurs with a delay different from that of the defensive expenditures and the reinvestment on infrastructures.

Appendix

We demonstrate that if $b - c\eta < 0$, n and q are always positive, consequently $z_1^* < 0$, therefore the condition of ii of the Theorem 1 never is verified.
From straightforward calculation we obtain

$$p = 4a^2V^2 + \delta^2 + r^2 > 0 \tag{60}$$
$$n = 4a^2V^2r^2\delta^2 - (-m_2(1-\eta)r + m_1\delta(b - c\eta))^2 \tag{61}$$
$$q = 4a^2V^2\delta^2 + 4a^2V^2r^2 + \delta^2 + r^2 - (m_2(1-\eta) - m_1(b - c\eta))^2 \tag{62}$$

From Proposition 1
$V_\infty = \frac{\rho}{2} + \frac{1}{2}\sqrt{\rho^2 + 4\frac{m_1\overline{P}}{a}}$, where $\rho = \frac{1}{a\delta r}(m_2(1-\eta)r - m_1(b - c\eta)\delta)$

we can write the equation (61) as $n = 4a^2V^2r^2\delta^2 - (-4ar\delta\rho)^2$.
Being $b - c\eta < 0$, then $V_\infty > \frac{\rho}{2}$, and from the equation (60), we find $n > 0$.
Remembering that $z_1^* = \frac{-p + \sqrt{p^2 - 3q}}{3}$, than it is strictly positive if and only if $q < 0$.
Now we demonstrate that q is always positive.
From $n > 0$, we can write
$4a^2V^2r^2 > \frac{m_1^2r^2(1-\eta)^2 + m_1\delta^2|b - c\eta|^2 + 2m_1m_2r\delta|b - c\eta|(1-\eta)}{\delta^2}$
and
$4a^2V^2\delta^2 > \frac{m_1^2r^2(1-\eta)^2 + m_1\delta^2|b - c\eta|^2 + 2m_1m_2r\delta|b - c\eta|(1-\eta)}{r^2}$
replacing this two inequalities in (62) and by straightforward but rather tedious calculations we find $q > 0$.
Note that if $b - c\eta > 0$, a sufficient condition for Hopf bifurcation is that $V_\infty < \frac{|\rho|}{2}$, that is $n < 0$.

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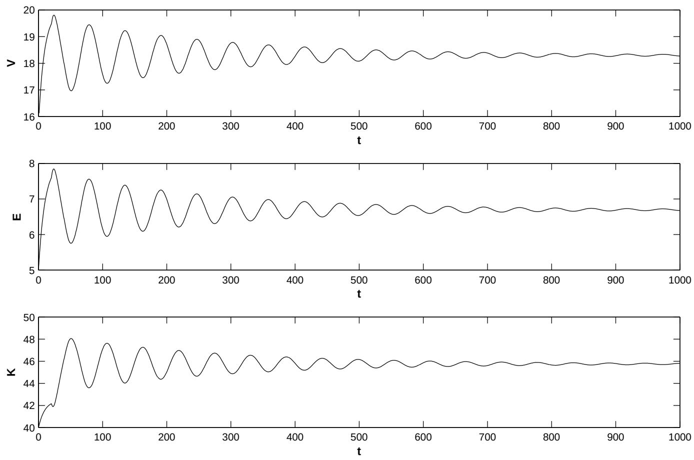
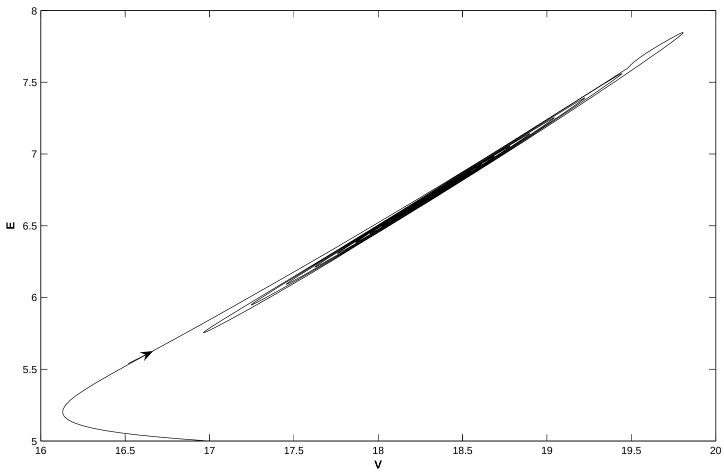
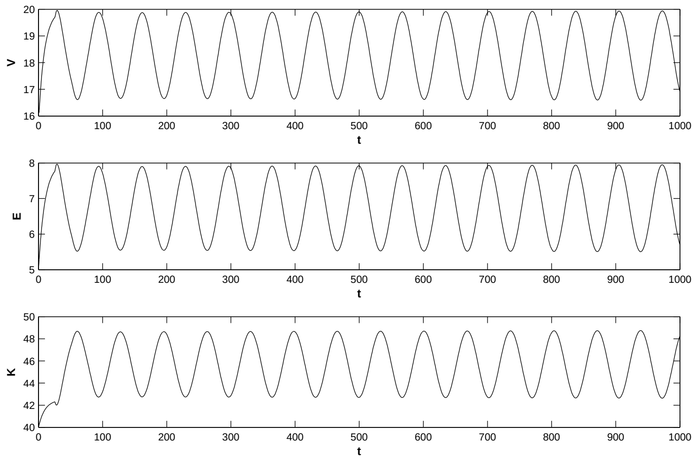


(a)

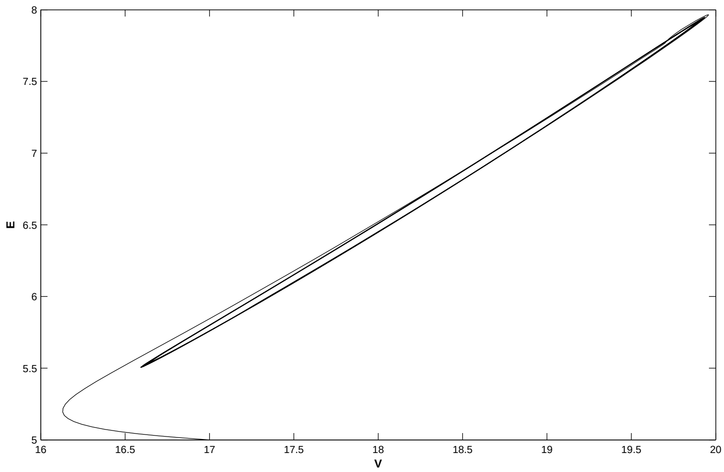


(b)

Figure 1: **Example 1**, with $\tau = 20 < \tau_0$, the fixed point is attractive: $a)$ plane $t - V$, $t - E$, $t - K$, $b)$ plane $V - E$.

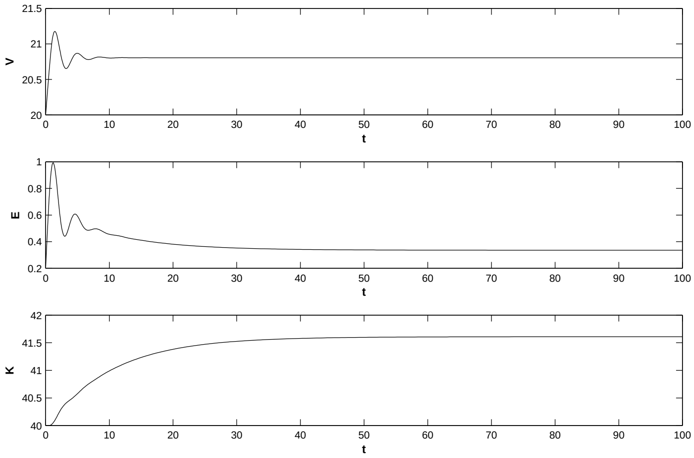


(a)

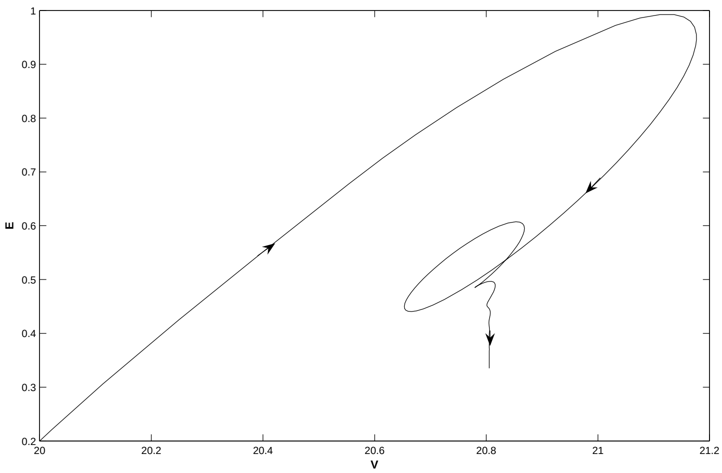


(b)

Figure 2: **Example 1**, with $\tau = 25.6 > \tau_0$, and sufficiently near to τ_0, the bifurcation periodic solutions from positive equilibrium S occur and are stable: $a)$ plane $t - V$, $t - E$, $t - K$, $b)$ plane $V - E$.

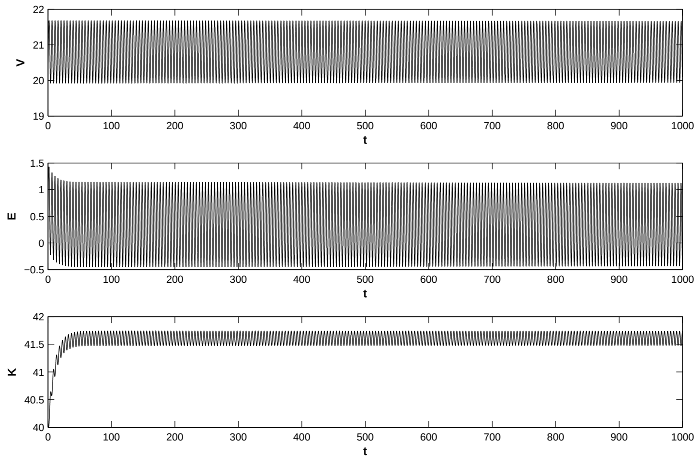


(a)

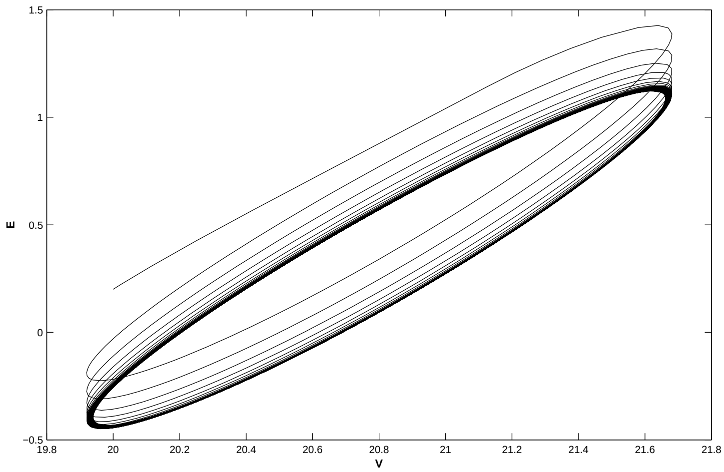


(b)

Figure 3: **Example 2**, with $\tau = 0.5 < \tau_0$, the fixed point is attractive: a) plane $t - V$, $t - E$, $t - K$, b) plane $V - E$.



(a)



(b)

Figure 4: **Example 2**, with $\tau = 1.1 > \tau_0$, and sufficiently near to τ_0, the bifurcation periodic solutions from positive equilibrium S occur and are stable: a) plane $t - V$, $t - E$, $t - K$, b) plane $V - E$.