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Abstract

We extend the war of attrition and all-pay auction analysis of Krishna and Morgan (1997) to a stochastic competition setting. We determine the existence of equilibrium bidding strategies and discuss the potential shape of these strategies. Results for the war of attrition contrast with the characterization of the bidding equilibrium strategies in the first-price all-pay auction as well as the winner-pay auctions. Furthermore we investigate the expected revenue comparisons among the war of attrition, the all-pay auction and the winner-pay auctions and discuss the Linkage Principle as well. Our findings are applicable to future works on contests and charity auctions.

KEYWORDS: All-pay auction, war of attrition, number of bidders

JEL CLASSIFICATION: D44, D82

1 Introduction

The wide and growing literature on all-pay auctions assumes that the number of bidders is common knowledge. Yet, in many situations where all-pay auctions illustrate economic, social and political issues, participants do not know the number of their opponents. Indeed, in lobbying contests, R&D races or battles to control some markets, agents do not know the exact number of their rivals. In a lobbying contest, some groups of interest give a bribe to the decision maker in order to obtain a market or a political favor. In R&D races, firms compete each other to be the first one to obtain a patent. The money spent in this race is

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not refundable. More generally, the effect of an unknown number of bidders is an important question in auction theory (see the recent papers of Harstad, Pekec, and Tsetlin (2008) and Pekec and Tsetlin (2008)). However, to our knowledge there is no analysis of all-pay auctions with an uncertain number of bidders.

Krishna and Morgan (1997) analyzed these auction designs with affiliated signals where the number of bidders is fixed and common knowledge. In this paper, we extend their analysis to a stochastic competition framework. In the following we call "all-pay auction" the first-price all-pay auction and "war of attrition" the second-price all-pay auction. We focus on equilibrium bidding strategies analysis and expected revenue comparisons as most of previous papers on winner-pay auctions with uncertain number of bidders.

McAfee and McMillan (1987) and Matthews (1987) studied first-price auctions with a stochastic number of bidders. They determined whether it is better to conceal or to reveal the information about the number of bidders for first and second-price winner-pay auctions in different frameworks.[1] However, they did not characterize the equilibrium strategies. Using a model *à la* Milgrom and Weber (1982) with independent private signals instead of affiliated ones, Harstad, Kagel, and Levin (1990) established that equilibrium bids with stochastic competition are weighted averages of the equilibrium bids in auctions where the number of bidders is common knowledge. Krishna (2002) investigated this result in another way with an independent private value model. In a recent paper Harstad, Pekec, and Tsetlin (2008) found the same result in multi-unit winner-pay auctions with common value.[2] Pekec and Tsetlin (2008) also investigate multi-unit auctions with unknown number of bidders. Indeed they determine the ranking of the expected revenues for uniform and discriminatory auctions. In addition they compare the expected revenues for each auction design when the number of bidders is known and unknown.

In this paper we determine the equilibrium strategies for the all-pay auction and the war of attrition under a monotonicity assumption when the number of bidders is unknown. Indeed we assume the Bayesian assessment of the bidder's value times a hazard rate given a stochastic number of bidders is an increasing function in the bidder's signal. It is a generalization of an assumption of Krishna and Morgan (1997) when the number of bidders is fixed and common knowledge. The consistency of this assumption is discussed through an example. The equilibrium strategies of the all-pay auction, as well as winner-pay auctions (Harstad, Kagel, and Levin, 1990), is a weighted average of equilibrium strategies that would be chosen for each number of bidders. However, it is not obvious for the war of attrition. Indeed, contrary to the – first and second-price – winner-pay auctions, it does not directly follow from the first order condition that the equilibrium strategy should be equal to a weighted

[1] Matthews (1987) considered bidders with an increasing, a decreasing or a constant absolute risk-aversion and McAfee and McMillan (1987) focused only on the risk-averse bidders and determined the optimal auction.

[2] In their framework, the number of identical prizes is proportional to the number of bidders. They showed that an unknown number of bidders could change the results on information aggregation. Common knowledge of the proportional ratio allows to find the results on information aggregation when the number of bidders is sufficiently high.

average. Using an example, this result is discussed. Moreover an answer for the independent-private-values model is provided.

Expected revenues are not only compared for the war of attrition and the all-pay auction but also among all-pay and winner-pay mechanisms. Then, we show that the stochastic competition does not affect the ranking of the expected revenues and the Linkage Principle as well. It is not an intuitive result. Indeed, we prove that the unknown number of bidders affects bidding strategies differently for the war of attrition, the all-pay auction and the winner-pay auctions. Moreover bidding strategy comparisons are provided among the all-pay and winner-pay mechanisms.

The paper is organized as follows. The model and preliminaries are described in Section 2. The analysis of the war of attrition and the all-pay auctions are given in Sections 3 and 4. Section 5 compares expected revenues and bidding strategies. Some computational details are provided in Appendix.

2 Model with Stochastic Competition

The model follows and generalizes the preliminaries of Krishna and Morgan (1997) (henceforth K-M) in a stochastic competition setting (as McAfee and McMillan (1987) and Harstad, Kagel, and Levin (1990) used in the study of winner-pay auctions). There is an indivisible object that can be allocated to $N = \{1, 2, ..., n\}$ potential bidders, with $n < \infty$. Every potential bidder is risk neutral. Firstly, we consider a set of bidders $A \subset N$. Denote $|A| = a$ the cardinality of set A.

Prior to the auction, each bidder i observes a real-valued signal $X_i \in [0, \bar{x}]$. The value of the object to bidder i, which depends on his signal and those of the other bidders, is denoted by

$$V_{a,i} = V_{a,i}(\boldsymbol{X}) = V_a(X_i, \boldsymbol{X}_{-i})$$

where V_a, which is the same function for all bidders, is symmetric in the opponent bidders' signals $\boldsymbol{X}_{-i} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_a)$. It is assumed that V_a is non-negative, continuous, and non-decreasing in each argument. Moreover, the bidders' valuation for the object is supposed bounded for all a: $\mathbb{E}V_{a,i} < \infty$.
Let f be the joint density of $X_1, X_2, ..., X_a$, a symmetric function in the bidders' signals. Besides, for any a-tuple $y, z \in [0, \bar{x}]^a$ with $\bar{m} = \{\max(y_i, z_i)\}_{i=1}^a$ and $\underline{m} = \{\min(y_i, z_i)\}_{i=1}^a$, f satisfies the affiliation inequality

$$f(\bar{m})f(\underline{m}) \geq f(y)f(z).$$

Affiliation is a strong form of positive correlation as discussed by Milgrom and Weber (1982). It means that if a bidder's signal is high, then other bidders' signals are likely high too. As a consequence, the competition is likely to be strong. Let $F_{Y_a^1}(.|x)$ be the conditional distribution of Y_a^1, where $Y_a^1 = \max\{X_j\}_{j=2}^a$, given $X_1 = x$ and $f_{Y_a^1}(.|x)$ the corresponding density function.

When the number of potential bidders a is common knowledge, we can define

$$v_a(x, y) = \mathbb{E}(V_{a,1}|X_1 = x, Y_a^1 = y), \tag{1}$$

the Bayesian assessment of bidder 1 when his private signal is x and the maximal signal of his opponents is y. As in K-M, we assume that $v_a(x, y)$ is increasing.[3]

We consider the situation in which bidders do not know the number of their rivals when they choose their strategy. For any subset A of N, we denote π_A the probability that A is the set of active bidders. Moreover, the probabilities π_A are independent of the bidders' identities and auction rules. Sets with equal cardinality have equal probabilities. Therefore, the *ex ante* probability to have a participants in the auction is the sum of probabilities with the same cardinal a:

$$s_a := \sum_{|A|=a, A\subset N} \pi_A$$

Let p_a^i bidder i's updated probability that there are a bidders conditional upon the event that he is an active bidder. We suppose that these probabilities are common knowledge and symmetric such as $p_a^i = p_a$. Therefore[4]

$$p_a^i := \frac{\sum_{|A|=a, i\in A\subset N} \pi_A}{\sum_{i\in B\subset N} \pi_B} \text{ and } p_a = p_a^i = \frac{a s_a}{\sum_{i=1}^{n} i s_i}$$

3 Analysis of the War of Attrition

In this section we determine the equilibrium strategies for the war of attrition with affiliated signals. It is not clear from the first order condition that the equilibrium strategies are weighted average of the equilibrium strategies that would be chosen for each number of bidders. Then we consider an independent-private-values model to investigate further this question.

3.1 General Case with Affiliated Signals

Assume that the number of bidders is common knowledge and each bidder i bids an amount b_i. Thus, the payoff of the bidder i if $\boldsymbol{b}$ is the vector of bids is

$$U_{a,i}(\boldsymbol{b}, \boldsymbol{X}) = \begin{cases} V_{a,i}(\boldsymbol{X}) - \max_{j\neq i} b_j & \text{if } b_i > \max_{j\neq i} b_j \\ \dfrac{1}{\#Q(\boldsymbol{b})} V_{a,i}(\boldsymbol{X}) - b_i & \text{if } b_i = \max_{i\neq j} b_j \\ -b_i & \text{if } b_i < \max_{j\neq i} b_j \end{cases}$$

[3]As Milgrom and Weber (1982) and K-M remark, since X_1 and Y_a^1 are affiliated, $v_a(x, y)$ is a non-decreasing function of its arguments. But they adopted the same assumption.

[4]For detail, see McAfee and McMillan (1987).

where $i \neq j$ and $Q(\boldsymbol{b}) := \{\text{argmax}_i\, b_i\}$ is the collection of the highest bids. Strategies at the symmetric equilibrium are noted β_a when the number of bidders a is known. K-M show that the bidding equilibrium strategy when the bidders are informed about the number of bidders a is

$$\beta_a(x) = \int_0^x v_a(y,y)\lambda(y|y,a)dt \tag{2}$$

where $\lambda(y|x,a) = \dfrac{f_{Y_a^1}(y|x)}{1 - F_{Y_a^1}(y|x)}$ and with the following boundary conditions:

$$\beta_a(0) = 0 \text{ and } \lim_{x \to \bar{x}} \beta_a(x) = \infty.$$

Let us assume the same mechanism for a stochastic number of bidders and denoted $\beta^i : [0, \bar{x}] \to \mathbb{R}_+$ a bidder's i pure strategy, mapping signals into bids. As we consider only the symmetric equilibria, we focus on the symmetric and increasing pure strategies $\beta \equiv \beta^1 = \beta^2 = ... = \beta^a$. As the number of bidders is stochastic, the definition of the equilibrium strategy concerns bidders' beliefs about the number of active bidders. Strategy β is called a equilibrium strategy if for all bidders i

$$\beta(x) \in \text{argmax}_{b_i}\, \mathbb{E}_a \mathbb{E}[U_{a,i}(b_i, \boldsymbol{\beta(X_{-i})}, \boldsymbol{X})|X_i = x] \;\; \forall x \in [0, \bar{x}] \tag{3}$$

where $\boldsymbol{\beta(X_{-i})} = (\beta(X_1), ...\beta(X_{i-1}), \beta(X_{i+1}), ..., \beta(X_a))$ and $\mathbb{E}_a$ is the expectation operator with respect to the distribution of the bidders' beliefs.

The uncertain number of bidders enters the expected utility through the value of the object for the bidder and the size of the vector of bids $\boldsymbol{b}$.[5]. Assume that all bidders except bidder 1 follow a symmetric – and differentiable – equilibrium strategy. Bidder 1 receives a signal x and bids an amount b. The expected utility of bidder 1 is

$$\begin{aligned}
\Pi^W(b,x) &= \mathbb{E}_a \mathbb{E}[U_{a,1}(b, \boldsymbol{\beta(X_{-1})}, \boldsymbol{X})|X_1 = x] \\
&= \mathbb{E}_a \mathbb{E}\{[V_{a,1} - \beta(Y_a^1)]\mathbb{1}_{\beta(Y_a^1) \leq b} - b\mathbb{1}_{\beta(Y_a^1) > b}|X_1 = x\} \\
&= \mathbb{E}_a \mathbb{E}\{\mathbb{E}\{[V_{a,1} - \beta(Y_a^1)]\mathbb{1}_{\beta(Y_a^1) \leq b} - b|X_1, Y_a^1\}|X_1 = x\} \\
&= \sum_a p_a \int_0^{\beta^{-1}(b)} [v_a(x,y) - \beta(y))] f_{Y_a^1}(y|x)dy - b\left[1 - \sum_a p_a F_{Y_a^1}(\beta^{-1}(b)|x)\right]
\end{aligned} \tag{4}$$

with $\beta^{-1}(.)$ the inverse function of $\beta(.)$. The maximization of (4) with respect to b leads to:

$$\sum_a p_a v_a(x, \beta^{-1}(b)) f_{Y_a^1}(\beta^{-1}(b)|x) \frac{1}{\beta'(\beta^{-1}(b))} - \left[1 - \sum_a p_a F_{Y_a^1}(\beta^{-1}(b)|x)\right] = 0 \tag{5}$$

At the symmetric equilibrium $b = \beta(x)$, thus (5) yields

$$\begin{aligned}
\beta'(x) &= \sum_a \frac{p_a v_a(x,x) f_{Y_a^1}(x|x)}{1 - \sum_i p_i F_{Y_i^1}(x|x)} \\
&= \sum_a w_a(x)\beta_a'(x)
\end{aligned} \tag{6}$$

[5]It also enters through the collection of the highest bids $Q(\boldsymbol{b})$. Yet, when $\#Q(\boldsymbol{b}) > 1$ the value of the integral is zero: at least one of the support is an atom. Thus, we do not need to consider it.

with the weights

$$w_a(x) = \frac{p_a(1 - F_{Y_a^1}(x|x))}{1 - \sum_i p_i F_{Y_i^1}(x|x)} \tag{7}$$

By (2) and (6) we know that $\beta(.)$ is increasing. It follows that an equilibrium strategy must be given by

$$\beta(x) = \sum_a w_a(x)\beta_a(x) - \sum_a \int_0^x w_a'(t)\beta_a(t)dt \tag{8}$$

Thus, we have a necessary condition about the shape of β. We prove that it is indeed an equilibrium strategy under an additional assumption, as stated in the next theorem. This assumption provides a sufficient condition for the existence of the symmetric monotonic equilibrium bidding strategies.

Definition 1. *Let $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $\phi(x, y|a) = v_a(x, y)\tilde{\lambda}(y|x, a)$ where $\tilde{\lambda}(y|x, a) = \dfrac{f_{Y_a^1}(y|x)}{1 - \sum_i p_i F_{Y_i^1}(y|x)}$.*

$\phi(., y|a)$ is the product of $v_a(., y)$, an increasing function, and $\tilde{\lambda}(y|x, a)$, a non-increasing function.[6] Besides, ϕ is equivalent to $v_a(x, y)\lambda(y|x, a)$ defined by K-M when the number of agents a is common knowledge.

Assumption 1. *$\phi(x, y|a)$ is increasing in x for all y.*

Theorem 1. *Under assumption 1, a symmetric equilibrium in a war of attrition is represented by*

$$\beta(x) = \sum_a w_a(x)\beta_a(x) - \sum_a \int_0^x w_a'(t)\beta_a(t)dt$$

with $\beta_a(t)$ and $w_a(t)$ given by (2) and (7).

Proof. First, $\beta(.)$ is a continuous and differentiable function. Indeed, by K-M we know that $\beta_a(.)$ is a continuous and differentiable function. We have to verify the optimality of $\beta(z)$ when bidder 1's signal is x. Using equation (5), we find that

$$\begin{aligned}
\frac{\partial \Pi^W}{\partial \beta(z)}(\beta(z), x) &= \sum_a p_a v_a(x, z) f_{Y_a^1}(z|x) \frac{1}{\beta'(z)} - 1 + \sum_a p_a F_{Y_a^1}(z|x) \\
&= \frac{1}{\beta'(z)} \left[\sum_a p_a v_a(x, z) f_{Y_a^1}(z|x) - \sum_a p_a v_a(z, z)\tilde{\lambda}(z|z, a)(1 - \sum_i p_i F_{Y_i^1}(z|x)) \right] \\
&= \frac{1}{\beta'(z)}(1 - \sum_i p_i F_{Y_i^1}(z|x)) \sum_a p_a[\phi(x, z|a) - \phi(z, z|a)]
\end{aligned}$$

When $x > z$, as $\phi(x|y, a)$ is increasing in x, it follows that $\dfrac{\partial \Pi^W}{\partial \beta(z)}(\beta(z), x) > 0$. In a similar manner, when $x < z$, $\dfrac{\partial \Pi^W}{\partial \beta(z)}(\beta(z), x) < 0$. Thus, $\dfrac{\partial \Pi^W}{\partial \beta(z)}(\beta(x), x) = 0$. As a result, the maximum of $\Pi^W(\beta(z), x)$ is achieved for $z = x$. ∎

[6] This fact can be proved in a similar way that the hazard rate $\lambda(y|x, a)$ of the distribution $F_{Y_a^1}(y|x)$ is non-increasing in x.

K-M discussed assumption 1 when the number of bidders is common knowledge. This assumption means that $v_a(., y)$ increases faster than $\tilde{\lambda}(y|x, a)$ decreases. However, as in the war of attrition with a fixed number of bidders, this is not a problem. Indeed, this assumption holds if the affiliation between X and Y_a^1 is not so strong. We give an example below to illustrate this discussion with a stochastic number of bidders.[7]

Example 1. *Let $f(x) = \frac{2^a}{2^a+1}(1 + \prod_{i=1}^a x_i)$ on $[0, 1]^a$ with X_i bidder i's signals and let us denote $f_{\mathbf{Y_a}}(x, y_1, y_2, ..., y_{a-1})$ the joint density of $(X_1, Y_a^1, Y_a^2, ..., Y_a^{a-1})$ with Y_a^k the k^{th}-highest order statistic of $(X_2, ..., X_a)$ such as $Y_a^1 \geq Y_a^2 \geq ... \geq Y_a^{a-1}$. Let us consider $a \in \{2, 3\}$. Therefore,*

$$\begin{array}{ll} f_{\mathbf{Y_2}}(x, y) = \frac{4}{5}(1 + xy) & \text{on } [0,1]^2 \\ f_{\mathbf{Y_3}}(x, y_1, y_2) = \frac{16}{9}(1 + xy_1y_2)\mathbb{1}_{y_1 \geq y_2} & \text{on } [0,1]^3 \end{array}$$

First of all, we can easily verify that the affiliation inequality given holds. We also assume that $v_a(x, y) = a(x + y)$. Then computations lead to

$$\begin{array}{lcl} f_{Y_2^1}(y|x) = 2\dfrac{1 + xy}{2 + x} & \text{and} & F_{Y_2^1}(y|x) = y\dfrac{2 + xy}{2 + x} \\ f_{Y_3^1}(y|x) = 4y\dfrac{2 + xy^2}{4 + x} & \text{and} & F_{Y_3^1}(y|x) = y^2\dfrac{4 + xy^2}{4 + x} \end{array}$$

We can also verify that $F_{Y_a^1}(y|x)$ is non-increasing in x. We obtain

$$\phi(x, y|2) = 2(x + y)\frac{2(1 + xy)(x + 4)}{(x + 4)(x + 2) - p_2y(2 + xy)(4 + x) - p_3y^2(4 + xy^2)(2 + x)}$$
$$\phi(x, y|3) = 3(x + y)\frac{4y(2 + xy^2)(2 + x)}{(x + 4)(x + 2) - p_2y(2 + xy)(4 + x) - p_3y^2(4 + xy^2)(2 + x)}$$

Thus, assumption 1 holds (some details are given in appendix).

Using the results where the number of bidders is common knowledge, the boundary condition $\beta(0) = 0$ follows. Thus, if the expected value is bounded whatever the number of potential bidders, then the bidding strategy will be bounded too. Following the same logic than K-M, we could determine that $\lim_{x \to \bar{x}} \beta(x) = \infty$. Indeed, in this situation,

$$\beta(x) \geq \sum_a p_a \int_0^x v_a(y, y)\tilde{\lambda}(y|y, a)dy + \min_a v_a(z, z) \ln\left(\frac{1 - \sum_a p_a F_{Y_a^1}(z|z)}{1 - \sum_a p_a F_{Y_a^1}(x|z)}\right)$$

Harstad, Kagel, and Levin (1990) and Harstad, Pekec, and Tsetlin (2008) show that the form of the equilibrium strategies for winner-pay auctions is such that $\beta(x) = \sum_a w_a(x)\beta_a(x)$. However, this result is not obvious for the war of attrition. Indeed, contrary to winner-pay auctions and the all-pay auction (cf infra.), in the case of the war attrition, it is not a direct result of the first order condition that the equilibrium strategy should be equal to a weighted average. Yet, the following example illustrates in a simple case that the bidding strategy in the war of attrition with stochastic competition could be written as a weighted average of the bidding strategies that would have been chosen for each number of competitors.

[7] This example generalizes an example of K-M with two – fixed – bidders.

Example 2. *Let* $f(x) = 2^a \prod_{i=1}^a x_i$ *on* $[0,1]^a$ *with* X_i *bidder* i*'s signals and let* $a \in \{2,3\}$*. As in Example 1 we assume that* $v_a(x,y) = a(x+y)$*. Therefore,*

$$\begin{aligned} f_{\mathbf{Y_2}}(x,y) &= 4xy && \text{on } [0,1]^2 \\ f_{\mathbf{Y_3}}(x,y_1,y_2) &= 16xy_1y_2\mathbb{1}_{y_1\geq y_2} && \text{on } [0,1]^3 \end{aligned}$$

We can easily verify that the affiliation inequality and the assumption 1 hold. Then the equilibrium strategies for a fixed number of bidders are given by

$$\begin{aligned} \beta_2(x) &= 8\int_0^x \frac{y^2}{1-y^2}dy \\ &= -8x + 4\ln\frac{1+x}{1-x} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \beta_3(x) &= 24\int_0^x \frac{y^4}{1-y^4}dy \\ &= 24\left(-x + \frac{1}{4}\ln\frac{1+x}{1-x} + \arctan x\right) \end{aligned}$$

When the number of bidders is stochastic and $p_2 = p_3 = 0.5$

$$\begin{aligned} \beta(x) &= 8\int_0^x y^2 \frac{1+3x^2}{2-x^2-x^4}dy \\ &= -\frac{8}{3}\int_0^x 2\frac{y}{y+1} + 2\frac{y}{y-1} + 5\frac{y^2}{y^2+2}dy \\ &= -12x + \frac{16}{3}\ln\frac{1+x}{1-x} + \frac{16\sqrt{2}}{3}\arctan\frac{x}{\sqrt{2}} \end{aligned}$$

All these bidding strategies are depicted in Figure 1. The bidding strategy with a stochastic number of bidders β *(solid line) is always higher than the bidding strategy with 2 bidders (long dashed line) and lower than the bidding strategy with 3 bidders (short dashed line) for all value of* x*. Then we can find a vector of weights such as the bidding strategy with stochastic competition would be written as a weighted average of the bidding strategies with a fixed number of bidders.*

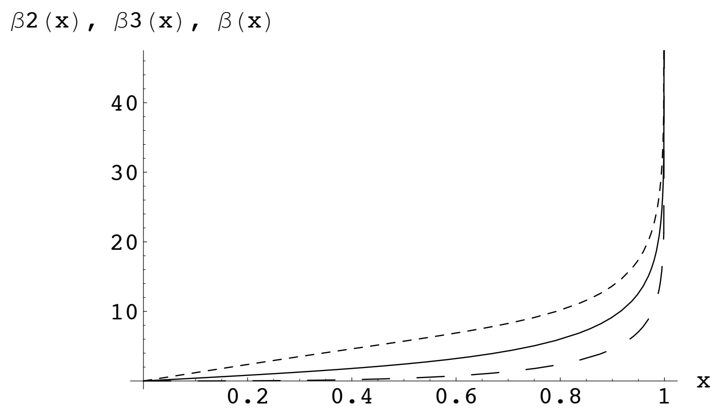

Figure 1: Bidding strategies β_2, β_3 and β.

3.2 An Example: Independent-Private-Values Model

As we have seen previously, and despite Example 2, it is not obvious that the equilibrium strategy in the war of attrition is equal to a weighted average such that $\beta(x) = \sum_a w_a(x)\beta_a(x)$.

In this section, we provide an answer for the IPV model.

Let us consider that each bidder i assigns value X_i to the object, independently distributed on $[0, \bar{x}]$ from the identically distribution F. Therefore, the bidding strategy where the number of bidders a is common knowledge is

$$\beta_a(x) = (a-1)\int_0^x \frac{yf(y)F^{a-2}(y)}{1-F^{a-1}(y)}dy$$

and the bidding strategy with stochastic competition is given by

$$\beta(x) = \sum_a p_a(a-1)\int_0^x \frac{yf(y)F^{a-2}(y)}{1-\sum_i p_i F^{i-1}(y)}dy$$

.

Lemma 1. *The equilibrium strategy in a war of attrition is decreasing in a for all $a \geq 2$.*

Proof.

$$\frac{\partial \beta_a}{\partial a}(x) = \int_0^x \frac{yf(y)F^{a-2}(y)}{(1-F^{a-1}(y))^2}[1-F^{a-1}(y)+(a-1)\ln F(y)]dy$$

As $1-F^{a-1}(y)+(a-1)\ln F(y)$ is negative for all a, y, the result follows. ∎

If $\beta(x) \in [\beta_{\underline{a}}(x), \beta_{\bar{a}}(x)]$ for all x with $\beta_{\underline{a}}(x) = \min_a\{\beta_a(x) \forall a \in N | s_a > 0\}$ and $\beta_{\bar{a}}(x) = \max_a\{\beta_a(x) \forall a \in N | s_a > 0\}$ then we can find a vector of weights $(z_a(.))_a$ with $\sum_a z_a(.) = 1, z_a(.) \geq 0$ for all x such that $\beta(x) = \sum_a z_a(x)\beta_a(x)$. Thus, we state:

Proposition 1. *In an IPV model, the equilibrium strategy in the war of attrition with stochastic competition is a weighted average of equilibrium strategies where the number of bidders is common knowledge.*

Proof. We have to distinguish two cases. Indeed from Lemma 1 either $p_1 = 0$ and then $\beta_{\bar{a}}(x) = \beta_2(x)$ or $p_1 > 0$ and $\beta_{\underline{a}}(x) = \beta_n(x)$.

$$\beta(x) - \beta_2(x) = \int_0^x \frac{yf(y)}{[1-\sum_i p_i F^{i-1}(y)][1-F(y)]}\left[\sum_a p_a(a-1)F^{a-2}(y) - \sum_a p_a(a-2)F^{a-1}(y) - 1\right]dy$$

As $\sum_a p_a(a-1)F^{a-2}(y) - \sum_a p_a(a-2)F^{a-1}(y) - 1$ is negative, $\beta(x) \leq \beta_2(x)$.

If $p_1 > 0$ $\beta_{\underline{a}}(x) = \beta_1(x) = 0$ then the result follows. However if $p_1 = 0$:

$$\beta(x) - \beta_n(x) = \int_0^x \frac{yf(y)}{[1-\sum_{i>1} p_i F^{i-1}(y)][1-F^{n-1}(y)]}\sum_{a>1} p_a k(y,a)dy$$

where $k(y,a) = (a-1)F^{a-2}(y) + (n-a)F^{n+a-3}(y) - (n-1)F^{n-2}(y)$ is positive for all $a \geq 2$ and y.

Thus in both cases, $p_1 = 0$ and $p_1 > 0$, $\beta(x) \in [\beta_{\underline{a}}(x), \beta_{\bar{a}}(x)]$ for all x and the equilibrium strategy with stochastic competition can be written as a weighted average of equilibrium strategies with a fixed number of bidders. ∎

The next example considers uniform distributions and at most three bidders. Then an explicit shape of the vector of weights is determined. Even in this simple case, this vector cannot be written as easily as for the winner-pay auctions.

Example 3. *Let us consider the value X_i is given by a uniform distribution on $[0,1]$ and the number of bidders a could be 2 or 3. Then the equilibrium strategies for a fixed number of bidders are given by*

$$\begin{aligned}\beta_2(x) &= \int_0^x \frac{y}{1-y}dy \\ &= -x - \ln(1-x)\end{aligned} \qquad \textit{and} \qquad \begin{aligned}\beta_3(x) &= 2\int_0^x \frac{y^2}{1-y^2}dy \\ &= -2x + \ln\frac{1+x}{1-x}\end{aligned}$$

When the number of bidders is stochastic

$$\begin{aligned}\beta(x) &= \int_0^x \frac{p_2 y + 2p_3 y^2}{1 - p_2 y - p_3 y^2}dy \\ &= -2x - \int_0^x \frac{2 - p_2 y}{p_3(y-1)(y-y_o)}dy \\ &= -2x - \frac{1}{p_3}\frac{2-p_2}{1-y_o}\ln(1-x) + \frac{1}{p_3}\frac{2-p_2 y_o}{1-y_o}\ln[-y_o(x-y_o)]\end{aligned}$$

where $y_o = \dfrac{-p_2 - \sqrt{p_2^2 + 4p_3}}{2p_3}$ *and belongs to* $(-2,-1]$.

Using Proposition 1 there exists a vector of weights $(z_2(.), z_3(.))$ such that $z_2(x)\beta_2(x) + z_3(x)\beta_3(x) = \beta(x)$ for all $x \in (0,1]$. It follows that

$$z_3(x) = \frac{-x + \ln(1-x) - \displaystyle\int_0^x \frac{2-p_2 y}{p_3(y-1)(y-y_o)}dy}{-x + \ln(1+x)} \quad \textit{and } z_2(x) = 1 - z_3(x) \textit{ for all } x \in (0,1].$$

Remark that if $p_2 = 0$ then $z_3(x) = 1$ for all x.[8] *Moreover it is routine to verify that $z_3(x) \in [0,1]$.*

4 Analysis of the All-Pay Auction

As before assume the number of bidders is common knowledge and each bidder i bids an amount b_i. Thus, the payoff of the bidder i is

$$U_{a,i}(\boldsymbol{b}, \boldsymbol{X}) = \begin{cases} V_{a,i}(\boldsymbol{X}) - b_i & \text{if } b_i > \max\limits_{j\neq i} b_j \\ \dfrac{1}{\#Q(\boldsymbol{b})}V_{a,i}(\boldsymbol{X}) - b_i & \text{if } b_i = \max\limits_{i\neq j} b_j \\ -b_i & \text{if } b_i < \max\limits_{j\neq i} b_j \end{cases}$$

where $i \neq j$ and $Q(\boldsymbol{b}) := \{\text{argmax}_i\, b_i\}$ is the collection of the highest bids. Strategies at the symmetric equilibrium are noted α_a when the number of bidders a is known. K-M show that

[8] Indeed $-\displaystyle\int_0^x \frac{2-p_2 y}{p_3(y-1)(y-y_o)}dy = 2\int_0^x \frac{dy}{1-y^2}$.

the bidding equilibrium strategy when the bidders are informed about the number of bidders a is

$$\alpha_a(x) = \int_0^x v_a(t,t) f_{Y_a^1}(t|t)dt \tag{9}$$

with the following boundary conditions:

$$\alpha_a(0) = 0 \text{ and } \lim_{x\to\bar{x}} \alpha_a(x) = \lim_{x\to\bar{x}} v_a(x,x). \tag{10}$$

As for the war of attrition, we focus only on the symmetric pure strategies $\alpha : [0,\bar{x}] \to \mathbb{R}_+$, called an equilibrium strategy if for all bidders i (such that $i \leq a$)

$$\alpha(x) \in \text{argmax}_{b_i} \mathbb{E}_a \mathbb{E}[U_{a,i}(b_i, \boldsymbol{\alpha(X_{-i})}, \boldsymbol{X})|X_i = x] \ \ \forall x \in [0,\bar{x}]$$

where $\boldsymbol{\alpha(X_{-i})} = (\alpha(X_1), ...\alpha(X_{i-1}), \alpha(X_{i+1}), ..., \alpha(X_a))$.

Assume that all bidders except bidder 1 follow a symmetric – and differentiable – equilibrium strategy. Bidder 1 receives a signal x and bids an amount b. The expected utility of bidder 1 is

$$\begin{aligned}
\Pi^A(b,x) &= \mathbb{E}_a \mathbb{E}[U_{a,1}(b, \boldsymbol{\alpha(X_{-1})}, \boldsymbol{X})|X_1 = x] \\
&= \mathbb{E}_a \mathbb{E}[V_{a,1} \mathbb{1}_{\alpha(Y_a^1)\leq b} - b|X_1 = x] \\
&= \mathbb{E}_a \mathbb{E}[\mathbb{E}[V_{a,1} \mathbb{1}_{\alpha(Y_a^1)\leq b} - b|X_1, Y_a^1]|X_1 = x] \\
&= \sum_a p_a \int_0^{\alpha^{-1}(b)} [v_a(x,y) - \alpha(y))] f_{Y_a^1}(y|x)dy - b
\end{aligned} \tag{11}$$

with $\alpha^{-1}(.)$ the inverse function of $\alpha(.)$. The maximisation of (11) with respect to b leads, at the symmetric equilibrium $b = \alpha(x)$, to

$$\alpha'(x) = \sum_a p_a \alpha'_a(x) \tag{12}$$

By (9) and (12) the bidding strategy $\alpha(.)$ is an increasing function. It follows from the boundary condition (10) that an equilibrium strategy must be given by

$$\alpha(x) = \sum_a p_a \alpha_a(x) \tag{13}$$

Once again, we have only a necessary condition about the shape of the equilibrium strategy. Under assumption[9] 1 we prove that $\alpha(.)$ is indeed an equilibrium strategy, as stated in the next theorem.

[9] Indeed, this assumption implies that $v_a(.,y) f_{Y_a^1}(y|.)$ is increasing for all y. The proof is similar to the proof of Proposition 3 of K-M.

Theorem 2. *Under assumption 1, a symmetric equilibrium in an all-pay auction, denoted $\alpha(.)$, is a weighted average of equilibrium strategies, denoted $\alpha_a(.)$, that would be chosen for each number of bidders such that $\alpha(x) = \sum_a p_a \alpha_a(x)$.*

Proof. To prove that α is optimal, we follow the same way that for the war of attrition. $\alpha(.)$ is a continuous and differentiable function. Indeed, by K-M we know that $\alpha_a(.)$ is a continuous and differentiable function. We verify the optimality of $\alpha(z)$ when bidder 1's signal is x. Using equation (12), we find that

$$\begin{aligned}\frac{\partial \Pi^A}{\partial \alpha(z)}(\alpha(z), x) &= \sum_a p_a v_a(x, z) f_{Y_a^1}(z|x) \frac{1}{\alpha'(z)} - 1 \\ &= \frac{1}{\alpha'(z)} \sum_a p_a [v_a(x, z) f_{Y_a^1}(z|x) - v_a(z, z) f_{Y_a^1}(z|z)]\end{aligned}$$

As we said before, assumption 1 implies that $v_a(x, y) f_{Y_a^1}(y|x)$ is increasing in x for all y. When $x > z$, it follows that $\frac{\partial \Pi^A}{\partial \alpha(z)}(\alpha(z), x) > 0$. In a similar manner, when $x < z$, $\frac{\partial \Pi^A}{\partial \alpha(z)}(\alpha(z), x) < 0$. Thus, $\frac{\partial \Pi^A}{\partial \alpha(z)}(\alpha(x), x) = 0$. As a result, the maximum of $\Pi^A(\alpha(z), x)$ is achieved for $z = x$. ■

Using the results where the number of bidders is common knowledge, the boundary condition $\alpha(0) = 0$ follows. Thus, if the expected value is bounded whatever the number of potential bidders, then the bidding strategy will be bounded too. Following the same logic than K-M, we could determine that $\lim_{x \to \bar{x}} \alpha(x) = \lim_{x \to \bar{x}} \max_a v_a(x, x)$.

Thus, the bidders' beliefs about the number of competitors is crucial to determine the equilibrium strategies. Indeed, the stochastic number of bidders does not affect the bidders' strategies at the equilibrium of the all-pay auction and the war of attrition in the same way.

5 Bidding Strategy and Revenue Comparisons

In this section we investigate the expected revenue comparisons for the war of attrition and the all-pay auction. We also compare the expected revenues and the equilibrium strategies obtained from the all-pay and winner-pay mechanisms. Finally the Linkage Principle is discussed.[10] The probability that a potential bidder i is taking part of the auction is given by $\sum_{i \in A} \pi_A$. Let us denote $e^d(.)$ the expected payment of the current bidder i in an auction design d. Then the expected revenue is $\sum_{i=1}^n [\sum_{i \in A} \pi_A] \mathbb{E} e^d(X)$.

5.1 War of Attrition versus All-Pay Auction

K-M show that the expected revenue from the war of attrition is greater than the expected revenue from the all-pay auction when the number of bidders is known and signals affiliated.

[10] Note that the proofs of the expected revenue comparisons use the same logic than the proofs of K-M.

In our stochastic setting, it is not obvious that this result still holds. Indeed, the uncertainty about the number of bidders has various consequences on the bidders' strategies at the equilibrium. As opposed to the all-pay auction, the equilibrium bidding strategy in the war of attrition is not average with weight p_a of the bidding strategies for each fixed number of bidders. Intuitively it is difficult to determine from the equilibrium bidding strategies how the stochastic competition modifies the ranking of the expected revenues. However, as we state in the next proposition, the stochastic competition does not affect the ranking of the expected revenues.

Proposition 2. *Under assumption 1, the expected revenue from the war of attrition is greater than or equal to the expected revenue from the all-pay auction.*

Proof. Denote $e^A(.)$, the bidders' expected payment in the all-pay auction at the symmetric equilibrium and $e^W(.)$ in the war of attrition. Then, under assumption 1,

$$
\begin{aligned}
e^W(x) &= \int_0^x \beta(y) \sum_a p_a f_{Y_a^1}(x|x)dy + \beta(x)(1 - \sum_a p_a F_{Y_a^1}(y|x)) \\
&= \beta(x) - \int_0^x \beta'(y) \sum_a p_a F_{Y_a^1}(y|x)dy \\
&= \sum_a \int_0^x w_a(y)\beta_a'(y)dy - \sum_a \int_0^x w_a(y)\beta_a'(y) \sum_i p_i F_{Y_i^1}(y|x)dy \\
&= \sum_a \int_0^x w_a(y)\beta_a'(y)(1 - \sum_i p_i F_{Y_i^1}(y|x))dy \\
&= \sum_a p_a \int_0^x v_a(y,y) f_{Y_a^1}(y|y) \frac{1 - \sum_i p_i F_{Y_i^1}(y|x)}{1 - \sum_i p_i F_{Y_i^1}(y|y)} dy \\
&\geq \alpha(x)
\end{aligned}
$$

As $e^A(x) = \alpha(x)$ and $F_{Y_i^1}(y|.)$ is a non-increasing function for all y, the war of attrition outperforms the all-pay auction. ■

5.2 War of Attrition versus Second-Price Auction

Our second result describes, under Assumption 1, the ranking of the equilibrium strategies from the war of attrition and the second-price auction.

Proposition 3. *Under assumption 1, the equilibrium strategies from the war of attrition and the second-price auction intersect at least once.*

Proof. Denote $\omega^{II}(.)$, the bidding strategy at the symmetric equilibrium in the second-price winner-pay auction. Following Harstad, Kagel, and Levin (1990) the equilibrium strategy is given by $\omega^{II}(x) = \sum_a \frac{p_a f_{Y_a^1}(x|x)}{\sum_i p_i f_{Y_i^1}(x|x)} v_a(x,x)$.

Then,

$$\mathbb{E}[\omega^{II}(Y)|X_1 = x, Y_a^1 < x] = \sum_a p_a \int_0^x v_a(y,y) f_{Y_a^1}(y|y) \frac{\sum_i p_i f_{Y_i^1}(y|x)}{\sum_i p_i f_{Y_i^1}(y|y) \sum_i p_i F_{Y_i^1}(x|x)} dy$$

In addition,

$$\begin{aligned}
\mathbb{E}[\beta(Y)|X_1 = x, Y_a^1 < x] &= \frac{\int_0^x \beta(y) \sum_a p_a f_{Y_a^1}(y|x) dy}{\sum_i p_i F_{Y_i^1}(x|x)} \\
&= \beta(x) - \int_0^x \beta'(y) \frac{\sum_a p_a F_{Y_a^1}(y|x)}{\sum_i p_i F_{Y_i^1}(x|x)} dy \\
&= \sum_a \int_0^x w_a(y) \beta_a'(y) dy - \sum_a \int_0^x w_a(y) \beta_a'(y) \frac{\sum_i p_i F_{Y_i^1}(y|x)}{\sum_i p_i F_{Y_i^1}(x|x)} dy \\
&= \sum_a \int_0^x w_a(y) \beta_a'(y) \frac{\sum_i p_i F_{Y_i^1}(x|x) - \sum_i p_i F_{Y_i^1}(y|x)}{\sum_i p_i F_{Y_i^1}(x|x)} dy \\
&= \sum_a p_a \int_0^x v_a(y,y) f_{Y_a^1}(y|y) \frac{\sum_i p_i F_{Y_i^1}(x|x) - \sum_i p_i F_{Y_i^1}(y|x)}{(1 - \sum_i p_i F_{Y_i^1}(y|y)) \sum_i p_i F_{Y_i^1}(x|x)} dy
\end{aligned}$$

From the affiliation inequality it follows for all $y \leq x$ that $\dfrac{\int_y^x \sum_i p_i f_{Y_i^1}(t|x) dt}{\sum_i p_i f_{Y_i^1}(y|x)} < \dfrac{\int_y^{\bar{x}} \sum_i p_i f_{Y_i^1}(t|y) dt}{\sum_i p_i f_{Y_i^1}(y|y)}$ if x is sufficiently low and $\dfrac{\int_y^x \sum_i p_i f_{Y_i^1}(t|x) dt}{\sum_i p_i f_{Y_i^1}(y|x)} > \dfrac{\int_y^{\bar{x}} \sum_i p_i f_{Y_i^1}(t|y) dt}{\sum_i p_i f_{Y_i^1}(y|y)}$ if x sufficiently high.

It follows that $\mathbb{E}[\beta(Y)|X_1 = x, Y_a^1 < x] < \mathbb{E}[\omega^{II}(Y)|X_1 = x, Y_a^1 < x]$ if x is sufficiently low and $\mathbb{E}[\beta(Y)|X_1 = x, Y_a^1 < x] > \mathbb{E}[\omega^{II}(Y)|X_1 = x, Y_a^1 < x]$ if x is sufficiently high.

■

K-M also show that the expected revenue from the war of attrition is greater than the expected revenue from the second-price winner-pay auction when the number of bidders is known and signals affiliated. For similar reasons than above, it is not obvious that this result still holds here. Yet, as we state in the next proposition, the stochastic competition still does not affect the ranking of the expected revenues.

Proposition 4. *Under assumption 1, the expected revenue from the war of attrition is greater than or equal to the expected revenue from the second-price auction.*

Proof. Denote $e^{II}(.)$ the expected payment at the symmetric equilibrium in the second-price winner-pay auction such as

$$e^{II}(x) = \sum_i p_i F_{Y_i^1}(x|x) \mathbb{E}[\omega^{II}(Y)|X_1 = x, Y_a^1 < x]$$

with $\omega^{II}(x) = \sum_a \frac{p_a f_{Y_a^1}(x|x)}{\sum_i p_i f_{Y_i^1}(x|x)} v_a(x,x)$.

$$\begin{aligned} e^W(x) &= \sum_a p_a \int_0^x v_a(y,y) f_{Y_a^1}(y|y) \frac{1-\sum_i p_i F_{Y_i^1}(y|x)}{1-\sum_i p_i F_{Y_i^1}(y|y)} dy \\ &\geq \sum_a p_a \int_0^x v_a(y,y) f_{Y_a^1}(y|y) \frac{\sum_i p_i f_{Y_i^1}(y|x)}{\sum_i p_i f_{Y_i^1}(y|y)} dy \\ &= e^{II}(x) \end{aligned}$$

To get this result remark that

$$\frac{\sum_i p_i f_{Y_i^1}(y|y)}{1-\sum_i p_i F_{Y_i^1}(y|y)} \geq \frac{\sum_i p_i f_{Y_i^1}(y|x)}{1-\sum_i p_i F_{Y_i^1}(y|x)}$$

holds for all $y \leq x$.[11] ■

5.3 All-Pay Auction versus First-Price Auction

The next Proposition describes, under assumption 1, the ranking of the equilibrium strategies from the all-pay auction and the first-price auction. We show in an example that these two bidding strategies are not strictly ordered for a fixed number of bidders for all range of x.

Proposition 5. *Under assumption 1, the equilibrium strategies from the all-pay auction and the first-price auction intersect at least once.*

Proof. Denote $\omega^I(.)$, the bidding equilibrium strategy in the first-price winner-pay auction such as (see Harstad, Kagel, and Levin (1990)) $\omega^I(x) = \sum_a \frac{p_a F_{Y_a^1}(x|x)}{\sum_i p_i F_{Y_i^1}(x|x)} \omega_a^I(x)$ with $\omega_a^I(x) = \int_0^x v_a(y,y) \frac{f_{Y_a^1}(y|y)}{F_{Y_a^1}(y|y)} exp\left\{-\int_y^x \frac{f_{Y_a^1}(t|t)}{F_{Y_a^1}(t|t)} dt\right\} dy.$

Let us consider the Example 1 for $v_a(x,y) = ax$. If bidding strategies cannot be strictly ordered for $p_2 = 1$ they cannot be strictly ordered neither for $p_2 < 1$. Computations lead to

$$\begin{aligned} \alpha_2(x) &= \int_0^x y \frac{1+y^2}{2+y} dy \\ &= \frac{4}{3}x^3 - x^2 + 20x - 40 \ln \frac{x+2}{2} \end{aligned}$$

and

[11] This fact can be proved in a similar way that the hazard rate $\lambda(y|x,a)$ of the distribution $F_{Y_a^1}(y|x)$ is non-increasing in x.

$$
\begin{aligned}
\omega_2^I(x) &= \int_0^x y\frac{1+y^2}{2+y}dy \\
&= 4\int_0^x \frac{1+y^2}{2+y^2}\exp\left\{-\int_y^x \frac{1+t^2}{2t+t^3}dt\right\}dy \\
&= 4\int_0^x \left(1-\frac{1}{2+y^2}\right)\exp\left\{-\int_y^x \left(\frac{1}{t}+\frac{t}{2+t^2}\right)dt\right\}dy \\
&= 4\int_0^x \frac{y}{x}\left(\frac{2+y^2}{2+x^2}\right)^{1/2} - \frac{y}{x}\frac{(2+y^2)^{-1/2}}{(2+x^2)^{1/2}}dy \\
&= \frac{4}{3x}(x^2-1)+\frac{4\sqrt{2}}{3x(2+x^2)^{1/2}}
\end{aligned}
$$

As $\omega_2^I(0.15)=0.15>\alpha_2(0.15)=0.09$ and $\omega_2^I(0.75)=0.79<\alpha_2(0.75)=2.26$ the result follows. ■

Our next result compares the expected revenues obtained from the all-pay auction and the first-price auction. Equilibrium bidding strategies in the first-price winner-pay auction and the all-pay auction with stochastic competition can be written as weighted average of equilibrium strategies that would be chosen for each number of bidders. However the weight of the average are different and cannot be strictly ranked. Then once again, it is not obvious that results with exogenous number of bidders still holds. Yet, as we state in the next proposition, the stochastic competition does not affect the ranking of the expected revenues.

Proposition 6. *Under assumption 1, the expected revenue from the all-pay auction is greater than or equal to the expected revenue from the first-price auction.*

Proof. Denote $e^I(.)$ the expected payment at the symmetric equilibrium in the first-price winner-pay auction such as

$$e^I(x)=\sum_i p_i F_{Y_i^1}(x|x)\omega^I(x)$$

Then,

$$
\begin{aligned}
e^I(x) &= \sum_a p_a F_{Y_a^1}(x|x)\sum_i \frac{p_i F_{Y_a^1}(x|x)}{\sum_a p_a F_{Y_a^1}(x|x)}\omega_i^I(x) \\
&= \sum_a p_a \int_0^x v_a(y,y) f_{Y_a^1}(y|y)\frac{F_{Y_a^1}(x|x)}{F_{Y_a^1}(y|y)} exp\left\{-\int_y^x \frac{f_{Y_a^1}(t|t)}{F_{Y_a^1}(t|t)}dt\right\}dy \\
&\leq \sum_a p_a \int_0^x v_a(y,y) f_{Y_a^1}(y|y)dy \\
&= e^A(x)
\end{aligned}
$$

To get this result remark that[12] $exp\left\{-\int_y^x \frac{f_{Y_a^1}(t|t)}{F_{Y_a^1}(t|t)}dt\right\} \leq \frac{F_{Y_a^1}(x|x)}{F_{Y_a^1}(y|y)}$ for all $y \leq x$. ■

[12] This fact is proved by K-M page 353.

5.4 Linkage Principle

When the number of bidders is common knowledge, Milgrom and Weber (1982) and K-M determine a ranking relationship in the expected revenue among first and second-price in winner-pay and all-pay auctions. That derives from the comparison of the statistical linkages between the bidder's expected payment and his signal. This result, called *linkage principle*, is based on the affiliation.

Let us consider bidder 1. Let $e^M(z, x)$ be his expected payment with a bid z and a signal x in the auction mechanism M and $e_2^M(x, x)$ be the derivative with respect to the second argument at $z = x$.

Theorem 3 (K-M's Linkage Principle, 1997)**.** *Suppose M and L are two auction mechanisms with symmetric and increasing equilibria such that $e^M(0, 0) = e^L(0, 0) = 0$. If for all x, $e_2^M(x, x) \geq e_2^L(x, x)$ then for all x $e^M(x, x) \geq e^L(x, x)$.*

The linkage principle is still satisfied with the stochastic competition. To see this formally, consider the auction mechanism M and let $\Pi^M(z, x)$ be the expected payoff of a bidder with a bid z and a signal x. Then,

$$\begin{aligned}\Pi^M(z, x) &= R(z, x) - e^M(z, x)\\ &= \sum_a p_a \int_0^z v_a(x, y) f_{Y_a^1}(y|x)dy - e^M(z, x)\end{aligned}$$

The expected gain of winning is the same in all mechanisms with stochastic competition (as in the case of a fixed number of bidders). Moreover the stochastic number of bidders is integrated in the expected payment and then does not affect the linkage principle properties. We could apply the linkage principle to compare the expected payment between winner-pay and all-pay mechanisms and then get the same results than above.

6 Conclusion

In this paper we determine the equilibrium strategies in the war of attrition and the all-pay auction with affiliated values and stochastic competition. We establish a sufficient condition for the existence of the monotonic equilibrium bidding strategies. We have shown that in the war of attrition, in opposite to the all-pay auction and the winner-pay auctions, it does not directly follow from the first order condition that the equilibrium strategy is equal to a weighted average. Even if stochastic competition affects the all-pay auction and the war of attrition in different ways, we prove that it does not modify the ranking of the expected revenues and the K-M's linkage principle.

Our results can be useful for many applications of all-pay designs such as in contest theory and charity auctions. Indeed, recent papers compare all-pay and winner-pay auctions to raise money for charity and suggest to use an all-pay design. In particular, Goeree, Maasland, Onderstal, and Turner (2005) show that the second-price all-pay auction is better to raise money for charity than the first-price all-pay auction and the winner-pay auctions. Charity auctions may be implemented for special events or on the Internet. A large number of charity auctions take place while potential bidders do not know the number of competitors.[13] As we do not introduce externalities in the bidders' payoff, our results could not be applied to charity auctions. However, as they change some insights in the second-price all-pay auction this work lets us open questions for future research on charity auctions.

7 Appendix

Boundary Condition of the Equilibrium Strategy for the war of attrition.

$$\begin{aligned}
\beta(x) &= \sum_a p_a \int_0^z v_a(y,y)\tilde{\lambda}(y|y,a)dy + \sum_a p_a \int_z^x v_a(y,y)\tilde{\lambda}(y|y,a)dy \\
&\geq \sum_a p_a \int_0^z v_a(y,y)\tilde{\lambda}(y|y,a)dy + \sum_a p_a \int_z^x v_a(z,z)\tilde{\lambda}(y|z,a)dy \\
&\geq \sum_a p_a \int_0^z v_a(y,y)\tilde{\lambda}(y|y,a)dy + \min_a v_a(z,z) \int_z^x \sum_a p_a \tilde{\lambda}(y|z,a)dy \\
&= \sum_a p_a \int_0^z v_a(y,y)\tilde{\lambda}(y|y,a)dy + \min_a v_a(z,z) \ln\left(\frac{1-\sum_a p_a F_{Y_a^1}(z|z)}{1-\sum_a p_a F_{Y_a^1}(x|z)}\right)
\end{aligned}$$

Boundary Condition of the Equilibrium Strategy for the all-pay auction.

$$\begin{aligned}
\alpha(x) &= \sum_a p_a \int_0^x v_a(y,y) f_{Y_a^1}(y|y)dy \\
&\leq \sum_a p_a \int_0^x v_a(x,y) f_{Y_a^1}(y|x)dy \qquad (14)\\
&\leq \max_a v_a(x,x) \int_0^x \sum_a p_a f_{Y_a^1}(y|x)dy \\
&\leq \max_a v_a(x,x)
\end{aligned}$$

(14) is a consequence of assumption 1.

Derivation of Example 1.

$$\begin{aligned}
\frac{\partial}{\partial x}\phi^1(x,y|2) =& \frac{4}{(x+2)(1-p_2F_{Y_2^1}(y|x)-p_3F_{Y_3^1}(y|x))}\Bigg[y^2+2xy^2+1-\frac{(x+y)(xy+1)}{x+2} \\
&-(x+y)(xy+1)\frac{-\frac{p_3y^4}{x+4}+\frac{p_3(xy^2+4)y^2}{(x+4)^2}-\frac{p_2y^2}{x+2}+\frac{p_2(xy^2+2y)}{(x+2)^2}}{1-p_2F_{Y_2^1}(y|x)-p_3F_{Y_3^1}(y|x)}\Bigg]
\end{aligned}$$

[13] They can know the number of their potential opponents but not the number of their active rivals.

$$\frac{\partial}{\partial x}\phi^1(x,y|3) = \frac{12y}{(x+4)(1-p_2F_{Y_2^1}(y|x)-p_3F_{Y_3^1}(y|x))}\left[y^3+2xy^2+2-\frac{(x+y)(xy^2+2)}{x+4}\right.$$
$$\left. -(x+y)(xy^2+2)\frac{-\frac{p_3y^4}{x+4}+\frac{p_3y^2(xy^2+4)}{(x+4)^2}-\frac{p_2y^2}{x+2}+\frac{p_2y(xy+2)}{(x+2)^2}}{1-p_2F_{Y_2^1}(y|x)-p_3F_{Y_3^1}(y|x)}\right]$$

Computations lead to non-negative derivatives.

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