Arrow's Impossibility Theorem and the distinction between Voting and Deciding

Thomas Colignatus

Thomas Cool Consultancy & Econometrics

21 November 2011

Arrow's Impossibility Theorem and the distinction between Voting and Deciding

Thomas Colignatus
November 21, 2011
http://www.dataweb.nl/~cool

JEL A10, D71, P16

Abstract

Arrow's Impossibility Theorem in social choice finds different interpretations. Bordes-Tideman (1991) and Tideman (2006) suggest that collective rationality would be an illusion and that practical voting procedures do not tend to require completeness or transitivity. Colignatus (1990 and 2011) makes the distinction between voting and deciding. A voting field arises when pairwise comparisons are made without an overall winner, like in chess or basketball matches. Such (complete) comparisons can form cycles that need not be transitive. When transitivity is imposed then a decision is made who is the best. A cycle or deadlock may turn into indifference, that can be resolved by a tie-breaking rule. Since the objective behind a voting process is to determine a winner, then it is part of the very definition of collective rationality that there is completeness and transitivity, and then the voting field is extended with a decision.

I thank Nicolaus Tideman for clarification of his approach. All errors in this paper remain mine.

Introduction

The Impossibility Theorem in social choice by Arrow (1951, 1963) finds different interpretations. It can be illuminating to compare Bordes-Tideman (1991) and Tideman (2006, chapter 10) with *Voting Theory for Democracy* (VTFD) i.e. Colignatus (1990 and 2011a, chapter 9). Henceforth BT and VTFD.

Both BT and VTFD accept the mathematical structure of the impossibility. The theorem has been checked by many authors and it stands. The discussion is about its interpretation and meaning for social choice. The key difference between BT and VTFD concerns the properties of completeness & transitivity and the axiom of “Independence of Irrelevant Alternatives” (IIA). Both BT and VTFD accept the other axioms. There is no need to provide formulas here since the literature on Arrow’s Theorem abounds.

We will first look into a simple voting situation and then look deeper into the two different positions. We subsequently discuss various possible misunderstandings.

A voting situation

Consider three chess players (or basket ball or soccer teams) a, b and c. They are pairwise confronted in a tournament with the result $a > b > c > a$, meaning that a beats b etcetera. These results happen to be intransitive. The objective of the tournament may only have been to allow the players to play against each other. There need not be a notion to find the “best overall player”.

Even if the result had been $a > b > c$ and also $a > c$ so that the outcome happens to be transitive, then it need not be an issue that a would be the “best overall player”. The fact that a beats the two others need not be associated with a notion that this would be “best”. The question does not have to arise simply because it is not considered to be a relevant question, neither to the players nor the organisers of the tournament. (Indeed, a would be the best under the Condorcet rule but not necessarily under a Borda rule.)

This situation will be called a “voting field”.

There can be a drastic change in objectives. Namely, if the tournament wants to identify an “overall winner”. Then this becomes the issue of “direct single seat elections” – to distinguish the situation from the election of for example a multiple seat parliament or the indirect selection of the prime minister via such a parliament.

The notion of an overall winner amounts to using a “social decision function” (SDF), see Sen (1970). The SDF selects the winner from a list of candidates. It is the *definition* of the SDF that it does so.

There is the special effect that the SDF always implies a ranking. For example, if a = SDF[a, b, c] then the second might be b = SDF[b, c] and then the third would be c. The ranking arises by stepwise dropping the best of the remainder. The ranking means a transitive order of the candidates. Let us call this ranking “dynamic”, even though the theorem still applies to a static world (or universe of static worlds).

PM 1. This kind of ranking is rather natural. When you have a winner, the runner up would rather be found when the winner would not partake. Conceivably there could also be other ways to identify the runner up, but if such alternative method includes the winner, then this would not be entirely logical, since, from the viewpoint of winning, the runner up is only interesting if the overall winner would not partake. PM 2. Chess uses an Elo ranking discussed in VTFD.

Hence, we distinguish between the *voting field* and *deciding*. In everyday parlance we tend to associate voting with deciding. Voting thus tends to mean: using both a voting field and a decision. Hence there is a distinction. Sometimes “voting” can be used in the sense of a “voting field” where the “field” is dropped. “Voting” thus is a somewhat ambiguous term, with some ambiguity about what it is ambiguous about. If one keeps track of the context the meaning however will be clear.

An idiosyncracy in Arrow (1951, 1963)

Arrow (1951, 1963) puts completeness in Axiom I and transitivity in Axiom II. Possibly the axiom format is the most eclectic approach since one might reject an axiom and then consider only voting fields.

Axiom I: For all x and y, either $x \, R \, y$ or $y \, R \, x$.

Axiom II: For all x, y, and z, $x \, R \, y$ and $y \, R \, z$ imply $x \, R \, z$.

However, the objective of his study is social choice. Hence subsequent authors like Sen (1970) have put completeness and transitivity into the very *definition* of the collective preference ordering. Thus what Arrow calls "condition" now is "axiom" in VTFD.

Like transitivity holds for rational individuals it is a defining property of rational collective choice. The issue becomes not whether you accept transitivity or not, but it becomes one of existence versus non-existence of collective choice. A person arguing that "there is no rational collective choice" implicitly accepts this definition in order to accurately express what is rejected.

Domain versus budget

An important distinction in this discussion area is between the domain X of all possible candidates and the budget of B of the actually available candidates.

The axiom "Independence of Irrelevant Alternatives" (IIA) is not the same as the distinction between X and B. Even if B is given we can still look at subsets of that budget.

IIA imposes that choices made over one particular budget would be "the same" as choices over a different budget. IIA imposes that outcomes are independent from the budget. For example, the ranking between *a, b* and *c* should not be affected if a *d* is included. (The domain may be much larger and contain the whole alphabet.)

Imposing transitivity creates a dependence upon the budget. The dependence upon the budget appears to create the logical inconsistency, that causes the Impossibilty that Arrow's Theorem is about. Voting fields, with intransitive "better than" for pairs only, are not the same as Decisions that generate a ranking, with "better than" for the whole. The message of the Impossibility Theorem is: the winner and the implied ranking depend upon the budget.

The outcome also depends upon the voting procedure. But even if one selects "the best possible procedure" then there will still be dependence upon the budget. For example, in a tournament of $\{a, b, c\}$ then *a* could win but for $\{a, b, c, d\}$ then *c* could win.

The VTFD approach

In VTFD, Arrow's theorem concerns the inconsistency of various general properties. Next to this, VTFD tackles the problem of the interpretation of these properties.

BT suggest that there is a distinction between practical and impractical properties. My reasoning differs. All Arrowian properties combined result into contradiction: so that none can be practical in combination with the others. Hence, I see Arrow's theorem as a theoretical and philosophical exercise. Hence, I would stop here. There is little use in delving into Arrow's original thesis – also since the formulation in Sen (1970) is more streamlined – and we should rather proceed with research into practical democracy.

There is the question whether it would be practical in Arrow's approach to assume an (infinite) domain *X* while in reality there is only a limited budget *B*. However it is still practical to allow for arbitrary budgets. Thus a discussion about what is "practical" is at risk of confusing the general mathematical setting and the interpretation.

The main problem is indeed the interpretation. For this, I found Arrow (1988) in the Palgrave encyclopedia to be most enlightening about what many researchers apparently have started to think. Arrow gave the axioms the interpretation that they would be "reasonable" and "morally desirable". Hence the impossibility has caused problems in reasoning and morality. VTFD casts this interpretation into formulas in deontic logic (the logic of morals), and then shows that Arrow's interpretation cannot hold. The axioms seem attractive only individually but jointly they don't work and thus cannot be "reasonable" and "morally desirable".

VTFD §9.5 rebaptises IIA into the "axiom of pairwise decision making" (APDM). IIA allows for subsets of flexible sizes but then it also allows for pairs. Using *both* pairs *and* changing voting into deciding, is sufficient to create the impossibility theorem. Arrow has a point that budgets consists of larger sets. It is an idiosyncracy of Arrow's original result that he mixes the issue of practical world budget dependence with the mathematical reason why the impossibility arises. An original result however is not always the most expressive form of it.

The Bordes-Tideman approach

Bordes-Tideman (1991) reject completeness and transitivity and accept IIA. Thus they reduce voting to voting fields only.

However, they somehow still think that a winner can be selected. The selection of the winner in their eyes needs not be transitive and apparently can be found by all kinds of "manipulations" depending upon the situation e.g. on the agenda of voting rounds. Tideman (2006:140) holds that there somehow is a selection by "voting". In his words the outcome need not be called "indisputably best" but only "better (…) than if they do not agree to make such decisions by voting".

I don't think that this is consistent. Even if the winner would be selected by random choice, it would still be a SDF with a "winner". Then the "dynamic ranking" above will generate a transitive ranking. There is only a voting field if there is no selection or even the very notion of a winner.

An example of BT

Tideman (2006:126-7):

> "In interpreting Axiom I, it is important to distinguish between its meaning as a statement about individual preferences and its meaning as a condition to be applied to collective rankings of options. As a statement about individual preferences, Axiom I means that individuals are able to compare options. As a condition to be applied to collective rankings, Axiom I embodies an absolutism and infallibility that are in fact unattainable. It says that either the collectivity will regard *x* as at least as good as *y* or it will regard *y* as at least as good as *x*. If a collectivity were to pronounce upon the relative merit of *x* and *y* only once, on the basis of a fixed amount of information about the preferences of the members of the collectivity, then Axiom I could be satisfied. But as Arrow employs Axiom I, the question of whether *x R y* or *y R x* is true, or both, is not allowed to vary when the collectivity gains information about the views of its members concerning any other option, *z*, which might be involved in a

> majority-rule cycle with x and y. It is for this reason that Axiom I is not satisfied by actual voting procedures."

In my analysis this however confuses with IIA. The notion "as Arrow employs Axiom I" is undefined as it is merely a definition of what social choice is about.

Tideman might mean to say that the ranking determined on some budget B will not be the ranking based upon X (for eternity), so that we will not attain the 'true' ranking *TR*. This forgets that you need a ranking to be able to express this. For this 'true' ranking you need completeness and transitivity, and assume that it exists, to say that it cannot be attained. Subsequently, Arrow's Theorem is not quite about that we don't live long enough. Even assuming X already creates the impossibility.

Tideman's position is that the ideal ranking cannot be attained and thus need not be thought of as existing. Arrow's Theorem is for him about the ideal so that actual rankings created by voting procedures do not feature in it. But Arrow's Theorem is not just about the ideal. It is a mathematical result. Voting procedures are subject to that math and actual choices still cause paradoxes due to changing budgets. We would still like to have "the best possible procedure".

Tideman's view might create an epicycle upon Arrow's Theorem, where we distinguish an ideal or true ranking *TR* on the whole domain X from practical procedures *PR* on budgets B. It is a fair point that we cannot determine *TR* so that practical rankings with *PR* can be misleading with respect to that 'true' order. This still leaves my point that we would reject IIA to create a practical winner in the budget and to find the practical ranking by dynamic eliminiation.

Is and ought

If people think that they ought to lose weight but still eat too much and exercise too little, then the current Arrowian model under discussion assumes that there is no real conflict and that people still prefer what they actually do. The model allows no conflict between Is and Ought.

A conflict between Is and Ought perhaps might be modelled as the distinction between X and B. Something that ought to happen may be unattainable. This is somewhat difficult however. Compare $\{a, b, c\}$ and unattainable d. A vote on "shall do" gives a. A vote on "should do" may cause people to consider d as an option even though it is unattainable. But this gives c. Well, this only shows that the distinction between Is and Ought is not easy to bring into this model.

VTFD §9.2 extends the model with Is and Ought by applying morals to the choice of voting rules themselves.

Let us then consider Tideman (2006:140): "it is only necessary to remember that voting establishes not what a collectivity should do, but only what the collectivity shall do."

This statement causes various questions.

(a) In itself it is true that the Arrowian model does not distinguish between Is and Ought. But is it correct to present a paradoxical voting result as a sign of that property ? Perhaps it is an Ought: there ought be no paradoxes. Instead, though, the paradoxes in the realm of Is cause real paradoxes (only seeming inconsistencies) that can find an explanation there too.

(b) Tideman presents the acceptance of IIA and its effects as either the distinction between X and B (interpreted in morality) or as something else (for which there is no place in the model).

I however don't think that IIA can be described as merely the distinction between *X* and *B*. I don't think that IIA can be described in these terms.

(c) In practical votes some people vote for what we shall do and others vote for what we should do, and each try to convince the others of their approach. This observation is OK, but little has been done with it in terms of modelling.

Tideman (2006:140): "When there are just two options and preferences are not unanimous, an [choice function] will select one (presumably by majority rule). But that does not imply that the [choice function] has identified the better option."

IIA implies that it *is* the better result. IIA is to assume that it does *not* depend upon the budget. Tideman accepts IIA. But to make this statement he has to reject it.

(Thus, the vote result is the better option, given the budget and the voting rule. Of course, if you change the budget or the voting rule than another outcome can be better. Perhaps people have ideas that another budget or voting rule ought to have been used, so that this gives a sense that there ought to have been another choice. But it is confusing to accept IIA and interprete outcomes in terms of such sentiments that are not in the model.)

Choice consistency

Tideman (2006:141): "When the outcomes of [choice functions] are viewed as estimates rather than pronouncements of truth, choice consistency loses its appeal as a condition that [choice functions] must satisfy."

Here we must distinguish two kinds of "choice consistency": (a) dependence upon the budget, (b) transitivity given the budget. Assuming completeness and transitivity is merely a definition required to select a winner, and should not be confused with "truth".

The "choice consistency" that Tideman refers to actually is IIA and not transitivity that is needed to find the winner or estimate.

Discussion of BT

The BT paper is rather complex. It assigns some back-guessing of Arrow's reasoning, and gives interpretations to some misunderstandings in the literature. It is not clear whether this is really relevant, also given their fundamental confusion of voting and deciding. It is better to take Sen (1970) as the place to start from, and then observe that Sen neither makes a proper distinction between voting and deciding so runs into similar problems of interpretation.

A definition of a "Social Choice Function" (SCF) depends upon the domain, and then BT ask the proper question whether it still is the same function if the domain changes ? It seems to me that such issues have been clarified in Sen (1970). I myself prefer the term "social welfare function – generating mechanism" (SWF-GM) such that the voting mechanism (like e.g. plurality or Condorcet or Borda) can generate a social welfare function plus its choice whatever the input. It is a bit an epicycle in math. Originally we had freedom in interpreting Arrow's properties, say for arbitrary domains, but now we record that we have freedom in interpretation, say for overlapping domains.

BT call this SWF-GM the "Arrow function". They hold that Arrow's original IIA uses plain functions and not the generating mechanism. Their answer is a regularity condition that generates IIA for such SCF:

> "Regularity means that, given a voting rule, a set of actual candidates, a set of voters and a preference profile, if the set of potential-but-not-actual candidates shrinks but the voters' preferences over the remaining potential candidates (including the actual ones) do not change, then the choice from the set of actual candidates does not change." (p168)

It remains an epicycle upon IIA but this does not seem crucial. In my impression it suffices that IIA can be interpreted for SWF-GM's such that we do not need a new discussion about "regularity". Instead, it is better to rebaptise IIA into APDM.

BT claim:

> "So to ask a voting rule to be a [sic] regular or an SCF to satisfy IIA means exactly that we want it to represent a possible real-world voting rule and not some fairy-world voting rule." (p169)

This does not seem accurate. Borda ranking does not satisfy IIA and is not fairy-world.

In general, in real world situations, voters only vote on the budget and not on the unavailable candidates. To test whether IIA is satisfied you would need the additional information on the unavailable candidates. This is generally not provided by voting procedures.

If you accept IIA then you impose some condition. But this is not the same as restricting your attention to only the budget. Because IIA also applies to subsets of the budget. BT erroneously think that IIA means only that you restrict your attention to the budget.

Thus it appears to be erroneous that they conclude (and Dummett would be right):

> "In fact, in the Arrow Theorem we can do without the conditions of Universal Domain, Choice Consistency, and Unanimity and still have a recognizable voting rule, but we cannot do without IIA. For it is IIA (along with Nondictatorship) that ensures that the Arrow Theorem is a theorem about real-world voting rules. Without IIA it would be a theorem about a fairy-world, and hence quite uninteresting. Dummett (1970, p. 54) is therefore wrong when he writes that IIA "... lacks complete intuitive justification"."

Defence of democracy

Tideman (2006:141):

> "People would like a basis for saying, "Now that we have voted we know the right thing to do; even if you disagree, you have an obligation to do what we all now know to be the right thing. We will tolerate no uncooperativeness." Some people may have thought that the virtues of appropriate [choice functions] provided a basis for such an assertion. Thus the Arrow theorem may leave these people in shocked disbelief. A person who believes that voting can identify the indisputably correct course of action and is concerned about the defensibility of his or her beliefs is required by the Arrow theorem to do some serious re-thinking of political philosophy."

Tideman interpretes Arrow's theorem plus Arrow's interpretation as contributing to a sense of democracy ! Instead, I regard Arrow's interpretation (not the math) that IIA would be "reasonable" and "morally desirable" as hugely destructive to democracy. People are goaded into thinking that democracy is impossible. Since Arrow's thesis in 1951 sixty years have passed and many countries in the world – like the USA, UK and France – still use very undemocratic rules to select their leaderships and parliaments, see for example Colignatus (2011b) and the interview by Stavrou (2011).

The suggestion that an "appropriate" voting procedure would be a basis to say "you must co-operate" is wrong. There is no such basis, even allowing for the vagueness of "appropriate". This can be discussed without reference to Arrow's theorem. Even majority decisions may cause people to vote with their feet, to flee the country or bring their money to Swiss banks, even when that is not considered "appropriate". In VTFD there is proper attention to the condition of the Status Quo that gives minorities veto powers against infringement of their rights. Arrow's defence on his treatment of the Status Quo is rather weak.

A "review" of VTFD

This present paper and its comparison between BT and VTFD has been written in the context of a "review" of VTFD by Schulze (2011) in the October 2011 issue of *Voting Matters*, of which professor Tideman is the editor. I submitted Colignatus (2011c) on November 9 to *Voting Matters* for publication as a response to that "review".

My proposal was that the "review" was retracted and removed from the journal's website and that Schulze or someone else did a real review. Tideman as editor thought it better to first exhange views between him and me about the proper analysis of Arrow's Theorem. I don't think that such a delay is necessary for removing a "review" that is not a review. I cannot do much about this procedure however. The editor has full power here. An appeal to the McDougall Trust on inappropriate treatment might be possible but is also an ultimum remedium, and it is very likely that they don't understand voting matters as well (even though they exist for it).

As explained in my response (2011b) I regard Tideman's invitation however also as a suggestion in the tradition of Leibniz: *let us sit down and look at the formulas*. I ran into bad luck again. As explained in this present paper that you are reading now, Tideman fundamentally misunderstands Arrow's Theorem and commits a logical error in thinking that intransitive voting fields still generate a "winner". As a general principle, and I have experienced it again, it is awfully complex to have a discussion with someone who does not understand the basics of the subject matter, but thinks that he does and you don't.

Schulze's "review" was published in October, I was not notified and saw it only on November 6. After some email exchanges that clarified that the "review" was not retracted, I logged my response on November 9 and submitted it for publication. Since then some 8 substantial emails have been exchanged between me and professor Tideman. If he had read VTFD he could have understood my argumentation on the distinction between voting and deciding, and the deontic logic in §9.1 and 9.2. Now the procedure turned into a personal educational course where I was forced to guide an unwilling student through his personal misunderstandings. Professor Tideman insisted on using Arrow's original book while it has idiosyncraties and while Sen is more streamlined and while we both agree that there is no discussion about the Impossibility but only on the interpretation. Tideman kept treating completeness (Axiom I) and transitivity (Axiom II) as properties that can be rejected and not as a definition of what social choice is about. While I was trying to teach him on my view, he was trying to teach me on his view and gallantly taking time to do so. For me all this is Hell. It is all in VTFD that has been submitted for a review, so read it. Only in the 8th email on substance on November 15 professor Tideman acknowledged: "I'm going to read your 9.1 and 9.2".

Since then I have sent some more detailed comments plus a draft of this paper on the published view in BT, but have not received a reply yet. I presume that professor Tideman is thinking. Possibly he needs that time. When you are convinced for 20 years that you have the right view about a mathematical theorem and that others don't understand it, and when you even have written a book about it explaining your view to your students, then it is tough to recognize that you yourself have fundamentally misunderstood it. Instead of waiting for the outcome, I have waited long enough now for this stage in the editorial process at *Voting Matters*. This present paper can be submitted to that journal too.

I once met a mathematician who had the urgent need to test whether I had a decent understanding of mathematics so that he wanted me to give the derivative of x^x. It is curious how when you say something that runs counter to people's view causes them to question your abilities instead of their own views. Schulze (2011) takes this to the next level and there is no guarantee where the present editorial cognitive dissonance may take us. The proper response of professor Tideman should be "I am happy that I now finally understand it" but given the many misunderstandings that I encountered I have no idea where it all will lead.

This paper thus discusses the distinction between voting and deciding, using the publicly available sources of Bordes-Tideman (1991) and Tideman (2006). I thank professor Tideman for his comments that allowed me to quicker identify the crucial points there.

Conclusion

There is a distinction between voting and deciding, Colignatus (1990). Voting consists of a voting field and a decision. If this distinction is not properly made then serious misunderstandings can arise about what Arrow's Impossibility Theorem is about. When the subject matter is social choice then we are considering decisions. For that subject matter it is another issue whether Arrow's conditions are "reasonable" or "morally desirable".

References

Arrow, K. (1951, 1963), "Social choice and individual values", J. Wiley

Arrow, K. (1988), entry on his theorem in The New Palgrave, Macmillan

Bordes, G. and N. Tideman (1991), "Independence of Irrelevant Alternatives in the Theory of Voting" Theory and Decision 30 (1991), 163-86.

Colignatus, Th. (1990), "Why a social welfare (meta) function does exit: The Arrow Impossibility Theorem for Social Choice resolved, A better analysis suggested," internal note Central Planning Bureau 90-III-37, The Hague

Colignatus, Th. (2011a), "Voting Theory for Democracy", 3rd edition, T. Cool (Consultancy & Econometrics), http://www.dataweb.nl/~cool/Papers/VTFD/Index.html

Colignatus, Th. (2011b), "Yes we can, in Cannes", http://www.dataweb.nl/~cool/Papers/Drgtpe/Crisis-2007plus/2011-10-28-YesWeCan-in-Cannes.html

Colignatus, Th. (2011c), "Response to a review of Voting Theory for Democracy, in the light of the economic crisis and the role of mathematicians", http://mpra.ub.uni-muenchen.de/34615

Schulze, M. (2011), "Review "Voting Theory for Democracy"", Voting Matters, Issue 29, October 2011, http://www.votingmatters.org.uk/ISSUE29/INDEX.HTM and http://www.votingmatters.org.uk/ISSUE29/I29P5.pdf

Sen, A. (1970), “Collective choice and social welfare”, North Holland

Stavrou, P. (2011), “Chaos in Europe, the G20 in Cannes and the need for constitutional changes – Interview with Thomas Colignatus”, November 3, http://protes-stavrou.blogspot.com/2011/11/chaos-in-europe-g20-in-cannes-and-need.html

Tideman, N. (2006), “Collective Decisions and Voting: The Potential for Public Choice”, Ashgate Publishing