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## Abstract

This paper compares the theory of value and distribution of Arrow and Debreu [1954] with that of Sraffa [1960]. I consider such versions of the two models that capture their salient features, without aiming at the greatest possible generality, so as to isolate the precise nature of the differences between the two conceptions of the same economic reality, and *inter alia*, to quarantine both the *sources* and the *entailments* of the differences in the two theories that respectively purport to determine the values of commodities and distribution of income in society. Both theories are complete and consistent. Sraffa's model is based exclusively on factual information, so it achieves less in terms of determining endogenous variables. The Arrow-Debreu is based on counterfactual information regarding additional production scenarios that are unobserved, in addition to the factual information that Sraffa has, so it achieves more by way of determination of endogenous variables. In terms of entailments, in Sraffa's theory there is an insufficiency of determinants in the economic grounds of society, thereby requiring the political component of society to also play an influential role in the joint determination of values and distribution. In the Arrow-Debreu model this determination is made complete solely in the economic sphere of society, rendering this theory purely economic, rather than political-economic, as in Sraffa. Both the information content difference at source, and the purely-economic versus political-economic difference in the entailments of what it takes to determine values and distribution, render the two theories radically different. In addition, (1) the prices in the two theories are different both in terms of definitions and values, and (2) since Sraffa's model has only one set of numbers on the observed production of commodities by means of commodities and labor for a single year, it is impossible to define constant returns to scale, while in the Arrow-Debreu model, this property is admissible, and possible to define, because their model's information base is sufficiently larger than Sraffa's. Further, Sraffa's theory is invariant to (a) the interpretation of prices – market-clearing, long-period, or whatever, (b) multiplicity of profit rates across industries, instead of a uniform rate of profit, and (c) presence or absence of general aggregate demand functions for commodities, and is (d) more general than the Arrow-Debreu theory because it is based on weaker assumptions, in the sense of a strictly smaller information set, so that it is only to be expected that the Arrow-Debreu theory would be capable of determining more endogenous variables in the model of an economy. (416 words)

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# On Two Theories of Value and Distribution

## *1. INTRODUCTION*

If two economists, both Nobel Laureates, make very specific but contradictory claims regarding a well-defined issue, it calls for careful scrutiny. In this paper I report on the result of an investigation of one such pair of conflicting claims. And more. Sen [2003] writes (p. 1253),

> Sraffa [1960] … draws exclusively on *observed* information, rather than having to invoke any *counterfactual* presumptions. …. It also relates to other methodological features of Sraffa's analysis, including his strenuous – but entirely correct – insistence that his analysis does *not* need any assumption of constant returns to scale.

On the other hand, jointly with Erkko Etula, one of the greatest economists of our time, Samuelson [2006] claims to provide multiple proofs[1] (p. 183),

> to confirm that Leontief – Sraffa matrix equations for input/output must obey constant returns to scale[.]

There is no question that there is a conflict between the position taken by Sen [2003] and the one taken by Samuelson and Etula [2006] on the role of *constant returns to scale* in Sraffa [1960].[2] This is not new. Samuelson [1962] has held this position for almost half a century. Decades later, Hahn [1982] expresses a similar doubt regarding constant returns to scale in Sraffa.[3] To put this matter to rest, in this paper I conduct an information theoretic examination of two theories of value and distribution, one developed by Sraffa [1960] and the other developed by Arrow and Debreu [1954] as in Debreu [1959], based solely on these *three* specific contributions, and on *nothing* else.[4]

Fortunately, I am helped, enormously, by additional literature relating to the issues that arise here. However, my challenge is to communicate the comparison of the two theories in a language that communicates effectively to both (a) the Arrow-Debreu generation, and (b) the Sraffa generation. I have kept the jargon-differences between the two languages to a minimum; and where generality was not threatened I have made certain simplifying assumptions that would remove issues that are not crucial to a comparison of stripped-down models of both theories.[5] Whoever said demystifying was easy! Shorn off jargon, this is a simple story.

First, for clarification, in the next section I define the concept of constant returns to scale. I further argue that the information content is so sparse in Sraffa's model of a capitalist economy

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[1] These "proofs" are examined and debunked in Sinha (2007). Also see the Appendix to Naqvi (2007) for two possible interpretations on which the proofs fail.

[2] Every reference to Sraffa in this paper is exclusively to Sraffa [1960].

[3] "Sraffa prices can be found once the rate of profit is known without any appeal to the preferences of households between goods. This of course is also true of a special neoclassical model which … posits constant returns to scale which Sraffa claims not to posit. I have been at a loss to understand him here." Hahn [1982, p. 359]

[4] Every reference to Arrow and Debreu in this paper is exclusively to the rendition in Arrow and Debreu [1954], although Debreu [1959] is a complete and comprehensive classic.

[5] For example, considering the Arrow-Debreu model with durable goods, or considering the Sraffa model with long-lived commodities, does not vitiate any claim that I make here, which is why I leave such commodities out of consideration in the comparison conducted here.

that in his model it is *impossible* to define the concept of constant returns to scale. This implies that Sen's claim is true, and Samuelson and Etula's is, in fact, unfounded, as is the doubt expressed in Hahn [1982]. This claim is actually stronger than Sen's, in that even if Sraffa needed to invoke constant returns to scale, the impossibility of defining this property in his model, which, by assumption, is based solely on observationally factual information, would have prevented him from doing so.[6]

There are two distinct theories of value and distribution in economics that are fully developed, one by Arrow and Debreu and the other by Sraffa. In this paper, I develop an argument to demonstrate (i) the impossibility mentioned above, (ii) elucidate some of the implications of the informational differences between the Sraffa model and the Arrow-Debreu model, for two radically distinct economic theories of values of commodities and distribution of income, and (iii) on a salient distinguishing feature – *price-and-income-distribution indeterminateness* – identify, for relevance to comparing these distinct theories of value and distribution, both issues worthy of examination, and those that are not very useful to pursue.[7]

Further, it would be a grave error to treat the arguments that I present here as an issue only in the history of economic thought, important as that is. In fact, this contribution has significant implications for two existing, distinct, *bone fide* economic theories of value and distribution. It is not that the death knell has sounded for one and the other has won a resounding victory. Both theories of value and distribution are alive and kicking. This is precisely because they are based on entirely distinct information sets, so that the validity of one, *by itself*, does not invalidate the other.[8]

Second, *assertions* have continually been made that Sraffa's model is *not* based on the assumption of constant returns to scale, by Sraffa himself, Schefold [1985, 1989, 1996], and more recently, among many others, by Sen [2003], Sinha [2007] and Sinha and Dupertuis [2009]. However, in serious contributions from Samuelson [1962] to Hahn [1982] and Samuelson and Etula [2006], among others, claims to the contrary have also been made, that in Sraffa's model the constant returns to scale *restriction* is either *imposed* or *entailed*. To bring this matter to a final resolution, I do not merely make an assertion, but, in fact, construct an argument to provide an *explanation* as to why one assertion is true, and the other necessarily false. This is also a distinguishing feature of this contribution.

Very many academics in the economics profession are quite familiar with the Arrow-Debreu theory of value (and implicit distribution), as told by Debreu [1959]. By contrast, as students and teachers of economics, the understanding of Sraffa's theory of value and distribution is considerably less widespread. Is it that Sraffa's theory of value and distribution fails to qualify as a *bone fide* theory? No. On the contrary, one of the greatest economists of our times asserts that "[Sraffa's] pen writes as if a lawyer were at hand to ensure that no vulnerable sentence appears. I

[6] This is a consequence of what is sometimes called a "snapshot" of an economy.
[7] This is the subject matter of Section 6 below.
[8] There are other theories of value, such as those based on information regarding a continuum of commodities with or without a continuum of agents, but these theories share a common entailment of the determinateness of income distribution with the Arrow-Debreu model, which is violated by the Sraffa model. Thus, for the purpose of this investigation, the continuum-agent model is indistinguishable from the Arrow-Debreu model and hence not considered here.

honor him for that[.]"[9] With this assertion as the backdrop, therefore, I conduct an information-theoretic comparison of the Arrow-Debreu and the Sraffa models, and find them to be entirely distinct conceptions of a capitalist economy, with each constituting the basis of a distinct, coherent and *bone fide* theory of value and distribution.[10] That is how I attempt here to demystify Sraffa (for those who are mystified by him), in relation to the Arrow-Debreu contribution to the development of a distinct theory of value and distribution.

In Section 2, I construct an argument to demonstrate the impossibility of defining constant returns to scale in Sraffa's model, and draw out *six* distinct, though interrelated, implications of the informational difference between this model and the Arrow-Debreu model. Section 3 contains a more detailed description of a particularly simple version of the Sraffa model of a capitalist economy that is drawn exclusively from Part I of his book, which deals with single-product industries without any durable goods.[11] Section 4 outlines the standard Arrow-Debreu model, again in a particularly simple form. Section 5 deals with a comparison between the Sraffa and the Arrow-Debreu theories of value and distribution. Section 6 deals with some generalizations of Sraffa that highlight issues not worthy of pursuit, depending on the purpose. Finally, Section 7 contains some concluding remarks.

Thus, this paper compares the theory of value and distribution of Arrow and Debreu (1954) with that of Sraffa [1960]. It finds both theories to be complete and consistent. However, it also finds that (1) in Sraffa's theory there is an insufficiency of determinants in the economic grounds of society, thereby requiring the political component of society to also play an influential role in the determination of values and distribution, whereas in the Arrow-Debreu theory this determination is made exclusively in the economic sphere of society. This distinction renders the two theories radically different. In addition, (1) the prices in the two theories are different both in terms of definitions and values, and (2) since Sraffa's model is based exclusively on factual information, he has only one set of numbers on the observed production of commodities by means of commodities and labor for a single year, which makes it is impossible to define constant returns to scale, while in the Arrow-Debreu model, this property is admissible, and possible to define, because the their model contains counterfactual information regarding additional production scenarios that are unobserved, in addition to the factual information that Sraffa has.

The purpose here is to consider such versions of the two models that capture their salient features, without aiming at the greatest possible generality, so as to isolate the precise nature of the differences between the two conceptions of the same economic reality, and *inter alia*, to quarantine both the *sources* of the differences and the *entailments* of the differences in the two theories of values of commodities and distribution of income that they respectively support.

## *2. A PRELIMINARY COMPARISON OF TWO THEORIES OF VALUE*

Constant Returns to Scale is a property of a production activity. An activity that transforms inputs of commodities and labor into outputs may or may not satisfy certain pre-specified requirements or axioms. An activity could also be thought of as occurring by distinct

[9] Samuelson [2000, p. 134, fn. 7.]
[10] Here "and distribution" only means interpersonal or inter-group income distribution in society.
[11] As an additional simplification, I assume that labor is employed directly in the production of every commodity. I also assume that every commodity is used in the production of all commodities.

*processes* insofar as different *quantities* of the *same* inputs and outputs are involved in production.

**Definition 1:** A production *activity* is said to satisfy the property of *constant returns to scale* if and only if, in any *pair-wise* comparison of distinct production processes of this activity, if *all* inputs of one process are *proportionate* positive multiples of the respective inputs of the other, then all outputs of the process will also be the *same* multiple of the other.

This definition is general enough to cover joint production in multi-product production correspondences, although such a feature is not entertained in this particular investigation.[12]

In what follows, I take a *production pattern* as a specific distribution of the quantities of all commodity *outputs* actually observed to have been produced in an economy. If this is taken together with the quantities of the various commodities and labor also actually observed as *inputs* in this 'pattern of production', then we have,

**Definition 2** (Sraffa): The set of all *actually observed* activities of production of all commodities by means of commodities and labor per period of time is called the set of *relations of production.*

It is noteworthy that, as defined here, the relations of production are based solely on *observed information*. This plays a crucial role in the argument that follows.

*2.1 Returns to Scale*

Consider the following (actually observed) relations of production of two commodities by means of two commodities and labor. This example is contained in Robinson and Naqvi [1967, p.585] and, as they state, these relations or production represent an "image in miniature of an actual system" [of observed inputs and outputs]. The image "represents a system in which one unit of current labor is employed and the surplus consists of a single commodity."

$$(1)\qquad \begin{array}{lllllll} \frac{14}{9}\,\text{t.wheat} & \& & \frac{20}{9}\,\text{t.iron} & \& & \frac{4}{5}\,\text{labor} & \rightarrow & \frac{10}{3}\,\text{t.wheat} \\ \frac{7}{9}\,\text{t.wheat} & \& & \frac{10}{9}\,\text{t.iron} & \& & \frac{1}{5}\,\text{labor} & \rightarrow & \frac{10}{3}\,\text{t.iron} \end{array}$$

In particular, the input of iron in both industries equals its gross output. However, the gross output of wheat is one ton more than its aggregate input usage in both sectors. Also, labor input in both sectors taken together is one unit. In (1) '$\rightarrow$' refers to 'is associated with the gross output of'. However, for Sraffa, $\forall\lambda > 0$,

[12] What is here called pair-wise comparison of distinct *processes* of a production *activity* is sometimes referred to in some works as proportionate positive 'change' in all inputs (say, all inputs get doubled). The assumption of constant returns to scale entails that, if this is the case, the output 'changes' by the same proportion (gets doubled too).

$$
(2)\qquad
\begin{array}{llllll}
\lambda\frac{14}{9}\ \text{t.wheat} & \& & \lambda\frac{20}{9}\ \text{t.iron} & \& \ \lambda\frac{4}{5}\ \text{labor} & \rightarrow & ?\ \text{t.wheat} \\
\lambda\frac{7}{9}\ \text{t.wheat} & \& & \lambda\frac{10}{9}\ \text{t.iron} & \& \ \lambda\frac{1}{5}\ \text{labor} & \rightarrow & ?\ \text{t.iron}
\end{array}
$$

The question mark (?) in (2) refers to 'Nobody knows for sure, so it is pure *counterfactual speculation*.' This is because Sraffa is willing to consider only observed information as in (1), and simply does not entertain any counterfactual information regarding what would happen to the output of a commodity if all its inputs *were to be*, counterfactually, higher (or lower) by the same proportion, as in (2). Never saw it, did not observe it, and Sraffa is unwilling to speculate as to what the outputs would have been in the case such as (2), in the *unobserved event* that all inputs were to have been higher (or lower) by the same proportion.[13]

Notice that the concept of constant returns to scale is constitutively counterfactual. And merely *to define* this property, information regarding at least *one additional* process of production of a commodity must be available, besides that contained in (1). Since Sraffa's model lacks the information pertaining to this additional process of production, it is impossible in his model to define constant returns to scale. Thus, it is not that the property of constant returns to scale is violated, nor that it is imposed, nor indeed that it is entailed in Sraffa's model. The fact is,

**Proposition 1**: *Constant returns to scale as a concept cannot be defined in Sraffa, because there is insufficient information in his model of a capitalist economy to define this property.*

In fact, in Sraffa's words,

> No changes in output and … no changes in the proportions in which different means of production are used by an industry are considered, so that no question arises as to the variation or constancy of returns. (p. v)

By contrast, Arrow and Debreu do, in fact, include counterfactual information in addition to observed information (1), so that in their model of a capitalist economy, an *entailment* of their assumptions regarding production is that *there exists* the possibility that, $\forall\lambda > 0$, the economy is described by (3)

$$
(3)\qquad
\begin{array}{llllll}
\lambda\frac{14}{9}\ \text{t.wheat} & \& & \lambda\frac{20}{9}\ \text{t.iron} & \& \ \lambda\frac{4}{5}\ \text{labor} & \rightarrow & \lambda\frac{10}{3}\ \text{t.wheat} \\
\lambda\frac{7}{9}\ \text{t.wheat} & \& & \lambda\frac{10}{9}\ \text{t.iron} & \& \ \lambda\frac{1}{5}\ \text{labor} & \rightarrow & \lambda\frac{10}{3}\ \text{t.iron}
\end{array}
$$

[13] In fact, Sraffa clearly states that "The investigation is concerned exclusively with such properties of an economic system as do not depend on changes in the scale of production or in the proportions of 'factors'." (p. v). Notice that in this paper I do not ask *why* Sraffa does what he does, and thus do not deal with any writings of his other than Sraffa [1960]. I therefore take his claims on face value. For a detailed examination that dwells extensively on Sraffa's unpublished writings, see Sinha [2007] and Sinha and Dupertuis [2009], among others.

Here (3) defines an additional set of infinite *processes* of production activities that are assumed to be possible, though unobserved and thus counterfactual, for all real, positive and finite values of $\lambda \neq 1$ *also*, when compared with the *less*-information-based economy described by (1) and (2).

Clearly, in this particular regard, the information on which the production part of the Arrow-Debreu model is based is *strictly greater* than the information on which Sraffa's model is based. About that, there should be no doubt. In particular, Sraffa's model assumes information contained in (3) *only* for $\lambda = 1$; by contrast the Arrow-Debreu model contains sufficient information to *admit the possibility* that (3) is true for (i) $\lambda = 1$ *and* for (ii) $\lambda \neq 1, \forall \lambda > 0$. This implies that the set of information on which Sraffa's model is based is a proper subset of the information set on which the Arrow-Debreu model is based. This should also settle the controversy regarding constant returns to scale in Sraffa's model.[14]

*2.2 Theories of Value*

Once adequate note is taken of this *informational difference* between the two models – the Arrow-Debreu model being based on strictly greater information than the Sraffa model – then any controversy regarding whether constant returns to scale is imposed or entailed in Sraffa can be dismissed altogether. That is why Sen [2003, p. 1253] writes, "Sraffa [1960] … draws exclusively on *observed* information, rather than having to invoke any *counterfactual* presumptions." This is a simple enough point, but it needs to be made, and emphasized, if only because there is much too much confusion surrounding this issue in the literature. Further, there are several significant implications of this fundamental informational difference between the models on which the theories of value are based in Sraffa versus Arrow-Debreu.

First, based on very different information sets, the two theories of value are profoundly different: there are *Sraffa prices* in the Sraffa model and there are *Walrasian prices* in the Arrow-Debreu model, and these two sets are completely distinct – both by their respective definitions and in their values.[15]

Sraffa writes (p. 3), "each commodity, which initially was distributed between the industries according to their needs, is found at the end of the year to be entirely concentrated in the hands of its producer." And he seeks "a unique set of exchange-values which if adopted by the market

[14] Notice that in the Arrow-Debreu model, the returns to scale are required to be non-increasing, which permits the possibility that they could be constant.

[15] Without using this terminology, Sen [2003] has attempted to bring this matter out by making a distinction between a *mathematical* determination of prices in Sraffa versus a *causal* determination of prices, (as, for instance, in Arrow-Debreu). In the context of "Prices and Two Senses of Determination" [2003, p.1247], Sen [p.1253] writes, "The sense of "determination" invoked by Sraffa concerns the mathematical determination of one set of facts from another set. To illustrate the point … a sundial may allow us to "determine" what time it is by looking at the shadow of the indicator (gnomon), but it is not the case that the shadow of the indicator "causally determines" what time it is. The value of the clock does not lie in its ability to "fix" – rather than "tell" – the time of day."

restores the original distribution of products and makes it possible for the process to be repeated; such values spring directly from the relations of production."

*Property **R*** (*Property of Production Replication*): is the capability of Sraffa prices *to redistribute commodities back to the original order in the different industries so that the entire production cycle can be replicated* in the next period, as a perfect duplicate of the previous period, and of the one before that, and so on.

Instead, Arrow and Debreu ask, as we shall see in Section 4, if there exists a unique commodity price vector (and a uniquely entailed interpersonal income distribution), unique up to a price normalization, at which excess of demand over output net of endowment for every commodity is zero. They provide a set of conditions sufficient for the existence of this vector.

*Property **E***: is a set of prices at which *excess demand for every commodity net of the economy's endowment is zero*.

Notice that the aggregate endowments of the economy are recouped at the end of a period, so in principle there is no reason that replication of the production and consumption processes of the previous period would be infeasible in the next period. Out of the aggregate, each commodity has to find its way back to a person in the amount of the initial personal endowment. However, since they are based on distinct sets of information, one should expect distinct sets of *accounting values* consistent with the respective axiomatic structures of each theory of value and distribution. More significantly, the Sraffa accounting values and the Arrow-Debreu accounting values represent constitutively different *contents*.

Second, neither theory has so far been *rigorously* shown to be false, in terms of making a claim *A*, and reaching the conclusion *not A*. Third, to ascertain the truth or falsity of either theory of values of commodities and distribution of income, the investigation must be conducted within the respective model, not from outside it. It is, in this sense, *not* a legitimate exercise to use the Arrow-Debreu model to criticize Sraffa's theory of value and distribution, nor is it legitimate to use Sraffa's model to criticize the Arrow-Debreu theory of value. Fourth, due to this third implication, the entire Cambridge-Cambridge debate on Capital theory, with hindsight, ends up being utterly futile, because it led to no definitive conclusion. The issues raised in that debate remain unresolved to date. Indeed, as Sen [2005, p.6] writes in a different context, "We need to take note not only of the opinions that won – or allegedly won – in debates, but also of other points of view that were presented and are recorded or remembered. A defeated argument that refuses to be obliterated can remain very alive."

Fifth, additional issues that simply do not arise in Sraffa's model, as, for example pertaining to consumers' demand functions for commodities, simply must not be raised in the context of his theory of value and distribution. Sraffa has *nothing* to say about such matters, *one way or the other*, so that asserting anything regarding them is tantamount to putting words in his mouth, to

which he has gone on record to object quite strenuously.[16] Similarly, it is not legitimate to raise issues pertaining to matters for which the Arrow-Debreu model was not designed, as, for instance, of economy-wide replication of production activities, because they also have nothing to say on the matter, one way or the other. I do, however, ask about generalizations, which is the subject matter of Section 6.

Sixth, 'capital', in the sense of the value of intermediate goods, does not arise as a concept in the Arrow-Debreu model, because Arrow and Debreu do not engage in the aggregation of the intermediate goods (which are used as inputs rather than net final output) in the production of a commodity, so that the issue of distribution of income between capital owners and workers is completely absent in their theory of value.[17]

Regarding the two theories of value, there are additional implications that can be inferred, over and above the six mentioned so far, but their discussion requires a more complete description of the full-blown models of both Sraffa and Arrow and Debreu. To these, I turn next.

### *3. SRAFFA'S THEORY OF VALUE AND DISTRIBUTION*

As noted, the theory of value and distribution I describe here is taken exclusively from Part I of Sraffa's book, which deals only with the simplest case in which each industry produces only one commodity and there is only one process of production of each commodity, i.e., there is neither any joint production nor any issue of choice of technique, and all commodities have a life of one period.

The observed relations of production from which Sraffa starts are a matrix of inputs, $A$, and a matrix of outputs, $C$, both assumed to be non-singular. The entry in row $i$, column $j$ in matrix $A = [a_{ij}]$, $\forall i, j = 1, \ldots m$, represents the amount of commodity $j$ actually observed to have been used as input in the production of $c_i$ quantity of commodity $i$. Since each industry produces only one product, $C$ is a diagonal matrix with the amounts of outputs produced $[c_1, \ldots c_m]$ along the main diagonal.

[16] Witness the Arun Bose [1965] case. In a letter to Bose in 1964, Sraffa wrote:

"I am sorry to have kept your MS so long—and with so little result.

The fact is that your opening sentence is for me an obstacle which I am unable to get over. You write: 'It is a basic proposition of the Sraffa theory that prices are determined exclusively by the physical requirements of production and the social wage-profit division with consumers demand playing a purely passive role.'

Never have I said this: certainly not in the two places to which you refer in your note 2. Nothing, in my view, could be more suicidal than to make such a statement. You are asking me to put my head on the block so that the first fool who comes along can cut it off neatly.

Whatever you do, *please* do not represent me as saying such a thing."

[17] The concept of capital does appear in other renditions of economic theory, especially the "neoclassical" rendition identified by Hahn [1982], which is not the concern of this paper. Instead I deal only with Arrow-Debreu and Sraffa.

Suppose also that at least one industry produces a surplus over and above the total input requirement of that commodity in all industries. The value of the surplus is $\boldsymbol{p}^s.(\boldsymbol{u}.C - \boldsymbol{u}.A)$ in that event, where $C$ = diag $[c_1 \ldots c_m]$ such that $\left(\forall j: c_j \geq \sum_{i=1}^{m} a_{ij}\right)\&(\exists j: c_j > \sum_{i=1}^{m} a_{ij})$, $\boldsymbol{u}$ is a $m$-vector of ones, and $\boldsymbol{p}^s$ is an $m$-vector of the Sraffa prices of the $m$ commodities. For such an information-set-based economy, Sraffa asserts the following $m$ independent relationships in $m + 2$ unknowns

$$(1 + r)A.\boldsymbol{p}^s + w\boldsymbol{l} = C.\boldsymbol{p}^s. \tag{4}$$

This is on the assumption that wages are paid *post factum*. In (4), $\boldsymbol{l}$ is a strictly positive $m$-vector of (current or direct) labor actually employed in each of the $m$ industries.[18]

Thus the only information – factual information – that Sraffa considers as available for ascertaining the "unique set of exchange values" embedded in the relations of production is that which is contained in $A$, $C$, and $\boldsymbol{l}$, and nothing else whatsoever.

Since every element of $\boldsymbol{l} = (l_1, l_2, \ldots l_m)$ is a given number in the relations of production, so is its sum, which is never considered to change in any examination of the model. It is harmless, therefore, to set this sum equal to unity, which is what Sraffa does, $\sum_{j=1}^{m} l_i = 1$, with the understanding that $\forall i: l_i > 0$.

In (4), among the $m + 2$ unknowns,

(i) $w \geq 0$, though finite, is the wage rate that is assumed to be the same in all industries,
(ii) $r \geq 0$, though finite, is the rate of profit on the value of capital that is assumed to be the same in every industry, and
(iii) the remaining $m$ unknowns are the Sraffa prices $\boldsymbol{p}^s = (p_1^s, \ldots p_m^s)$ $\forall i: p_i^s > 0$.

**Definition 3**: A *focal point of the Sraffa model* is defined as a set of values of ($\boldsymbol{p}^s$, $w$, $r$) consistent with (4), the price normalization rule that the value of net output of the economy equals one,[19] and (i) – (iii) above.

To ensure that all Sraffa prices are strictly positive, some conditions have to be imposed on $A$ and $C$. These are

[18] Sraffa does *not* require that labor be employed *directly* in the production of every commodity, but that is another simplification I adopt, in addition to assuming that every commodity is used directly in the production of all commodities, to do away with the distinction between what Sraffa calls basic and non-basic commodities, simply because this distinction, while clearly important for Sraffa's theory, is not of significance for the purpose of a comparison with the Arrow-Debreu model.
[19] This is formalized below as (5).

**Conditions 1**: Define $B = C^{-1}A$. Note that $B$ is a square matrix, and is non-singular because, by assumption, $A$ and $C$ are square, of the same order, and non-singular. In addition, if (1) $B$ is indecomposable, (2) all elements of $B$ are non-negative and at least one element is strictly positive, and (3) $\boldsymbol{p}^s$ is a non-negative characteristic vector associated with the maximal real-valued characteristic root of $B$, then it follows from a theorem of Perron and Frobenius that the price vector $\boldsymbol{p}^s$ is (a) strictly positive and (b) unique up to multiplication by a positive real number.[20]

As already noted, the object of Sraffa's exercise is to ascertain the values of $w$, $r$ and the $m$ Sraffa prices $\boldsymbol{p}^s$. Of course, it is not possible to obtain unique values of these $m + 2$ unknowns from the $m$ independent relationships in (4). Arrow and Debreu face a similar indeterminacy problem. To get around this, as we shall see in Section 4, they assume that the $m$-vector of prices in *their* model belongs to the unit simplex, which solves the problem of indeterminacy of the $m$ prices. It is also common in many fields such as international trade theory to take one of the commodities as the *numéraire*, so that its price is set at unity, and all other prices and the wage rate are expressed in terms of, say, $p_i$ units of that *numéraire* commodity per unit of commodity $i$.[21] On the other hand, the rate of profit on the value of capital is, of course, a unit-free number such as 0.25 that refers to a 25% rate.

Sraffa uses a different, though equally legitimate normalization rule. He takes the national income of the economy as equal to one. This is the value of net output of the economy. All Sraffa prices and the wage rate are then expressed in units of net national product. Formally, the normalization rule that national income equals one is

$$\boldsymbol{p}^s.(\boldsymbol{u}.C - \boldsymbol{u}.A) = 1. \tag{5}$$

Equations (4) and (5) consist of $m + 1$ independent equations in $m + 2$ unknowns, thereby rendering any focal point ($\boldsymbol{p}^s$, $w$, $r$) of Sraffa's model consistent with (4) and (5) to be *under*determined.

This is not so at all at

**Definition 4**: A *focal point of the Arrow-Debreu model*, with its normalization rule that the Walrasian price vector belongs to the unit simplex, exists as a strictly positive price vector at which there is zero excess demand, net of the economy's endowment vector, for every commodity, and an interpersonal income distribution is uniquely determined at this focal point, if Conditions 2a and 2b of Section 4 hold.

[20] For details, see Kurz and Salvadori [1995, p. 517]. Henceforth, by Conditions 1, I mean that $C^{-1}A$ is square, non-singular and satisfies properties (1), (2) and (3) mentioned in Conditions 1.

[21] If Commodity $m$ is taken to be the *numéraire*, then $(p_1, \ldots p_{m-1})$ would be the relative prices of the $(m - 1)$ commodities that are expressed in units of Commodity $m$, and the wage rate $w$ would also be expressed in the quantity of Commodity $m$ per year of labor.

The distinction highlighted by Definitions 3 and 4, between the characteristics of the focal points of Sraffa and Arrow-Debreu, is crucial for the development of the argument. It will be referred to as

***Property I*:** *Axiom of Price and Income Distribution Indeterminateness* states that at a focal point of a model of an economy, a set of strictly positive commodity prices *and* an associated income distribution in society are not uniquely determined.

The Arrow-Debreu model violates *Property **I*** but the Sraffa model satisfies it. I make much of this distinction later, based on

**Proposition 2**: *In general, consistent exclusively with the factual information contained in Sraffa's model, every conceivable distribution of national income between workers and capital owners can arise as the actual distribution at a focal point of the model.*

This is can be seen as one of the central messages of Sraffa, from the perspective of an economic theory of value and distribution. There is nothing counterfactual in Proposition 2.

To gain additional insight, it is helpful to return to the example of a Sraffa economy utilized in Section 2. Consider the counterpart of (4) in the example referred to in Robinson and Naqvi [1967, p. 585-86]. This takes the form

$$\begin{aligned} (\tfrac{14}{9} + \tfrac{20}{9}\,p)(1+r) \quad + \tfrac{4}{5}\mathrm{w} \quad &= \quad \tfrac{10}{3} \\ (\tfrac{7}{9} \quad + \tfrac{10}{9}\,p)(1+r) \quad + \tfrac{1}{5}\mathrm{w} \quad &= \quad \tfrac{10}{3}p \end{aligned} \qquad \text{(4a)}$$

Noting that the net output of this economy is one ton of wheat in the Robinson-Naqvi case, using Sraffa's normalization rule of setting the value of net output of the economy equal to one, with the price of wheat multiplied by 1 ton of wheat equal to one, it follows that the price of wheat is one unit of national income per ton of wheat. Further, using (4a), $p$, the price of iron in terms of national income (and also in terms of wheat, in this example) can be eliminated to solve for the wage rate as a function of the profit rate. This yields the equation of the *w*-*r* curve. This relatioship for the specific Robinson-Naqvi example is contained in Figure 1. Figure 1 shows the inverse, though non-linear, relationship between the rate of profit on the value of capital on the horizontal axis, and the *total wages* accruing to all the workers in the economy on the vertical axis, and this relationship is embedded entirely in the observed relations of production.

With total labor employment equal to one, in general the wage rate, *w*, equals the total income of workers, which, in turn, equals the fraction of national income accruing to workers; the remainder of the value of net output of the economy naturally accrues to capital owners. Capital owners, also known as capitalists, are defined as those persons who have property rights over the

commodities that constitute the means of production. For the Robinson-Naqvi example, Figure 1 displays such a tradeoff involved in the distribution of national income between workers and capitalists.

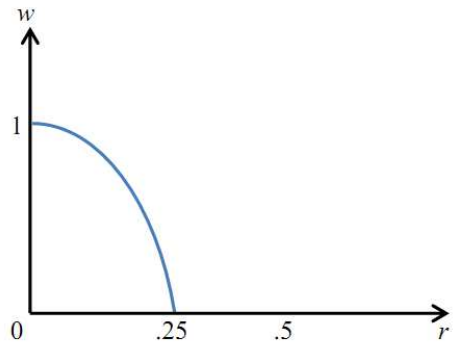


Figure 1

At the one extreme, $w = 0$, so that workers get nothing out of national income and capitalists get the entire national income, which corresponds in the Robinson-Naqvi example to the maximum rate of profit on the value of capital of 25% in each industry, and thus in the economy, and also reveals as embedded in these relations of production the Sraffa price of iron equal to one-half of the national income per ton of iron, which happens to equal to ½ ton of wheat per ton of iron.[22] At the other extreme, however, the entire national income accrues to workers, with $w = 1$, so that $r = 0$, and the Sraffa price is 0.44 tons of wheat per ton of iron. Thus the price of iron varies from 0.44 to 0.5 tons of wheat per ton of iron as (i) $r$ varies from 0 to 0.25, or equivalently, as (ii) $w$ varies from 1 to 0.

Further, define $r_{max}$ as a finite value of $r$ at $w = 0$. Then, $r_{max}$ refers to the maximum rate of profit embedded in the relations of production (matrices $A$ and $C$). Since the Sraffa model violates *Property I* due to one degree of freedom (with $n + 2$ unknowns in $n + 1$ independent relationships), at a focal point of the Sraffa model, as $w$ varies between 0 and 1, $r$ varies uniquely between $r_{max}$ and 0, and conversely. Moreover, with $w$ representing the share of national income that accrues to workers, it follows that the remainder accrues to capitalists. Thus

**Proposition 3**: *There is, in fact, a one-to-one, real-valued, inverse relationship between the share of national income that accrues to workers, w, and the rate of profit, r, on the value of capital (to which corresponds a unique share of national income that accrues to capital owners), and to each such value of w (or of r) corresponds a unique Sraffa price vector and unique value of r (or of w) at a focal point of the Sraffa model.*

From Proposition 3, it follows that

$$\{\forall w \in [0,1] \rightarrow \exists[\boldsymbol{p}^s(w) > 0 \,\&\, r(w) \geq 0]\} \,\&\, \{\forall r \in [0, r_{max}] \rightarrow \exists[\boldsymbol{p}^s(r) > 0 \,\&\, w(r) \geq 0]\},$$

[22] The example, one might recall, was designed with a one ton surplus output of wheat and no surplus output of iron.

where $\boldsymbol{p}^s(w), r(w)$, $\boldsymbol{p}^s(r)$ and $w(r)$ are unique if Conditions 1 hold.

Moreover, at the Sraffa prices, measured in terms of the quantity of national income per unit of a commodity,

(6a) $$K_i(A, C, r) \equiv \sum_{j=1}^{m} p_j(A, C, r) a_{ij}\,, \forall r \in [0, r_{max}\,]$$

Or

(6b) $$K_i(A, C, w) \equiv \sum_{j=1}^{m} p_j(A, C, w) a_{ij}\,, \forall w \in [0, 1]$$

are, by definition, the values of capital in industry *i*, and $K = \sum_{i=1}^{m} K_i(A, C, x)$ is the value of aggregate capital in the economy, for *x* = *w*, *r*. Naturally, since Sraffa commodity prices vary depending on the share of national income that accrues to workers contained in the value of the parameter *r* (or *w*), the value of capital is not unique, nor is the concept of ranking industries, or anything else, in terms of "capital intensity" construed as the ratio of capital to labor. While this argument follows from Sraffa's analysis, and is thoroughly devastating for explaining the rate of profit as determined by the marginal productivity of capital, it is still not the fundamental issue that Sraffa is after, despite considerable interest in the matter in the subsequent literature. I return to this issue in Section 5.

In the Robinson-Naqvi example, the relations of production imply that there is a 25% maximum rate of profit on the value of capital. Equation (4a) and Figure 1 reveal how the 1 t. wheat surplus is distributed across the two industries in proportion to their respective values of capital, and that *the value of capital itself depends on the value of the distribution parameter* $r \in [0, 0.25\,]$ in (6a), which is *external* to the factual relations of production in Sraffa's model of a capitalist economy. Of course, *the value of capital itself can also be seen as parametrically dependent on the parameter* $w \in [0, 1]$ in (6b), which is a distributional issue that is also external to Sraffa's model. In particular, in the two polar cases,

Table 1

| Units = t. wheat | Value of Capital at *r* = 0.25 | Value of Capital at *r* = 0 |
|---|---|---|
| Wheat industry | 120/45 | 114/45 |
| Iron industry | 60/45 | 57/45 |
| Total | 4.0 | 3.8 |

Some economists have tended to make more of the *specificity* of the inverse relationship between the share of national income going to workers and the rate of profit on the value of capital embodied in Proposition 3, and seen this as displaying a class conflict between workers and capitalists. They have made less of the *indeterminacy* of income distribution and prices, that is also embodied in Proposition 3, but which is quite visible in Proposition 2 and highlighted in

*Property* ***I***. I return to this issue in Sections 5 and 6, after briefly describing the Arrow-Debreu model in the next section.[23]

*4. ARROW-DEBREU THEORY OF VALUE*

Just as I have taken the model from Part I of Sraffa's book, I shall also take up the Arrow-Debreu model in a particularly simple form, while retaining its essential features. There is a society of finite *n* persons. Each person *i* is characterized by:

(i) a finite *m*-vector of vector of commodities, called the personal endowment, $\boldsymbol{\omega}_i = (\omega_{i1} \dots \omega_{im}), \omega_{ij} \geq 0, \forall j = 1, \dots, m, \forall i = 1, \dots, n$, with at least one commodity in a strictly positive quantity that a person is endowed with. This commodity could be labor – say 40 hours per week over 52 weeks – that the person is endowed with. Each commodity has a life of one period.

(ii) by a scalar $\theta_{is} \geq 0$, a non-negative fraction that represents the *share* of a producing-unit $s$ called a *firm* $\forall s = 1, \dots, l$, and there exist a finite $l$ number of firms, each with a one-period production cycle. Also, $\forall s, \exists i = 1, \dots, n, \theta_{is} > 0$ & $\sum_{i=1}^{n} \theta_{is} = 1$, indicating that any given firm is entirely owned privately by some persons.

(iii) $R_i$ , which is a *binary* relation of weak preference that stands for "is at least as good as" defined on a subset $S_i$ of the *m*-dimensional real commodity space, with the convention that if a person ends up being a net buyer (seller) of a commodity then its quantity is denoted by a strictly positive (negative) real number, zero otherwise.

Each firm $s$ produces exactly one commodity in the period under consideration.[24] A firm $s$ buys commodities, including labor, from persons and transforms them into a single commodity, so that it is characterized by an *m*-vector called its *net* output vector $\boldsymbol{y}_s = (y_{s1}, \dots, y_{sm})$, with exactly one positive element, at least one negative element and the rest non-positive, with $\boldsymbol{y}_s \in Y_s \subset \Re^m$, which is the *m*-dimensional real space, where $Y_s$ is called the production set of firm $s, \forall s = 1, \dots l$. Further, there are $s_i$ number of firms that produce commodity *i*, where $s_i$ is a large positive, though finite, number. Thus the economy ends up with $l = \sum_{i=1}^{m} s_i$ firms.

A firm is a legal entity owned entirely by some or all of the *n* persons. A commodity that is produced is identified with an industry that produces that commodity, so that there are *m* industries, each populated by a large number of firms that produce that commodity. Each firm buys commodities, including labor, from the persons who sell these commodities to the firms from their respective endowments. A firm *chooses* the quantities of commodities it buys as inputs and the quantity of its net output that it produces and sells at a *parametrically given*,

[23] See Hahn [1982] among others.

[24] In the original Arrow and Debreu [1954] contribution, joint production and multi-product correspondences are admissible. The case of joint production in Sraffa's formulation of the Standard Commodity is an extension by Schefold [1989]. The Arrow-Debreu model has also been extended in a number of important ways, including by Chipman [1970] to include parametric (or external to a firm but internal to an industry) increasing returns to scale, and to the case of asymmetric information between buyers and sellers by Akerlof [1970], among other significant extensions. However, we do not deal with these issues here.

strictly positive, $m$-vector of prices $\boldsymbol{p}^w = (\, p_1 \ldots\, p_m)$ of the $m$ commodities.[25] The sole motive of a firm in choosing these quantities of its net output vector $\boldsymbol{y}_s$ is the maximization of $\pi_s = \boldsymbol{p}^w.\boldsymbol{y}_s : \boldsymbol{y}_s \in Y_s$.

Each person chooses the quantities of each of the $m$ commodities $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{im})$, to buy from firms, at the *parametrically given* strictly positive price vector $\boldsymbol{p}^w$. Thus a person can play a double role, one as a buyer and seller of commodities out of the endowment $\boldsymbol{\omega}_i$, including selling labor (as a negative quantity purchased), and the second role as a possible fractional owner of a firm or firms. Thus the income of person $i$ is $v_i = \boldsymbol{p}^w.\boldsymbol{\omega}_i + \sum_{s=1}^{l} \theta_{is}\pi_{is} > 0$. The sole motivation of every person $i$ is to *choose* such quantities $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{im})$ as to maximize personal preference $R_i$ subject to the budget constraint $\boldsymbol{p}^w.\boldsymbol{x}_i \leq v_i$.

Further, by assumption, $\sum_{i=1}^{n} \boldsymbol{\omega}_i = \boldsymbol{\omega}$, the economy's endowment vector of the $m$ commodities, all elements of which are strictly positive, and $\boldsymbol{v} = (v_1 \ldots\, v_n)$ is the interpersonal income distribution vector that is also strictly positive. Moreover, $\boldsymbol{x} = \sum_{i=1}^{n} \boldsymbol{x}_i$ is the aggregate $m$-vector of commodities chosen to be consumed by all the $n$ persons in the economy. Let, $\boldsymbol{y} = \sum_{s=1}^{l} \boldsymbol{y}_s$, where $\boldsymbol{y}$ is the net output vector of the economy. Then $\boldsymbol{y} \in Y$, where $Y$ is the production set of the economy, provided $\boldsymbol{y} + \boldsymbol{\omega} \geq \boldsymbol{0}$. An entailment of the closure of the economy is $\boldsymbol{p}^w.\boldsymbol{x} = \boldsymbol{p}^w.(\boldsymbol{y} + \boldsymbol{\omega})$.[26]

To obtain the existence of a strictly positive Arrow-Debreu price vector $\boldsymbol{p}^w$, some restrictions are placed on each firm's production set $Y_s$, the production set of the economy $Y$, and on the personal preference relation $R_i$ defined on $S_i$, to be discussed presently. Labor is a commodity like any other, except that a person may have an endowment of labor, but no firm produces labor. That is, labor is a primary factor of production, and it may be considered the only non-produced commodity.[27] Sufficient restrictions are

**Conditions 2a**: In the capacity of a buyer of commodities, a person's preference relation $R_i$ $\forall 1 = 1, \ldots, n$ defined on $S_i$ is: (i) Binary, (ii) Reflexive, Transitive and Complete, (iii) Strongly monotonic, (iv) Convex, and (v) Continuous. Also, $S_i$ is (a) closed, and (b) bounded below.

[25] In the general model, contained in Arrow and Debreu [1954], some prices can be zero if they end up with negative excess demands in the aggregate, but that is not a generalization to our purpose, because our goal is to make a comparison of the Arrow-Debreu model with the Sraffa model. The Sraffa model, Part III onwards, is also more general than the one presented here.

[26] It is noteworthy that this condition holds as a weak inequality. However, it turns out that with the requirement of (i) strong monotonicity of personal preferences, and (ii) finite markets, this relationship holds as a strict equality, and thus entails a strictly positive price vector under the full set of conditions 2a and 2b. It may also be noted that if there are infinite markets, as in the typical *Overlapping Generations* model, then this condition holds as a strict inequality, which renders the stationary outcome Pareto suboptimal, so that the introduction of financial intermediation, among other possibilities, raises it to a Pareto optimal level.

[27] Of course, the general model does not require that there be only one primary factor of production, but that, again, is not a generalization to our purpose.

**Conditions 2b**: The production set of a firm, (i) $Y_s$ is closed, $\forall s = 1, \ldots, l$. The economy's production set, $Y$ is (ii) Convex, (iii) admits of the possibility of Inaction ($0 \in Y$), (iv) satisfies Irreversibility of production, in so far as $[\forall s: \boldsymbol{y}_s \in Y \;\&\; -\boldsymbol{y}_s \in Y \;\rightarrow \boldsymbol{y}_s = 0]$, and the property that (v) nothing can be produced out of thin air, in that $[\forall s: \boldsymbol{y}_s \geq 0 \;\&\; \boldsymbol{y}_s \in Y \;\rightarrow \boldsymbol{y}_s = 0]$.

The following existence result can be proved, and has been proven by Arrow and Debreu [1954]:[28]

**Proposition 4**: *There exists a price vector* $\boldsymbol{p}^w$*, with the normalization that* $\boldsymbol{p}^w$ *belongs to the unit simplex, such that quantities demanded and supplied are equal for every commodity, rendering all excess demands* $\boldsymbol{z}(\boldsymbol{p}^w)$ *zero at strictly positive prices,* $\boldsymbol{z}(\boldsymbol{p}^w) = \boldsymbol{x}(\boldsymbol{p}^w) - \boldsymbol{y}(\boldsymbol{p}^w) - \boldsymbol{\omega} = \boldsymbol{0}$*, if Conditions 2a and 2b are satisfied.*[29]

Arrow and Debreu seek a set of (strictly positive) commodity prices $\boldsymbol{p}^w$ such that quantity demanded equals net quantity supplied in the market for every commodity, *Property* $\boldsymbol{E}$, and the price vector belongs to the unit simplex. And, they accomplish this goal with the greatest of rigor possible. They ask a different question, come up with a different answer, based on their distinct – though entirely complete and entirely correct – characterization of a capitalist economy, based on an informationally distinct model.

*5. COMPARISON OF THE TWO THEORIES OF VALUE*

It is noteworthy that throughout his book Sraffa never refers to the concept of equilibrium. Arrow and Debreu, by contrast, prove the existence of equilibrium by specifying a sufficient set of conditions. In the light of Propositions 2, in Sraffa's conception of a capitalist economy, unique values of Sraffa prices and the distribution of income between workers and owners of commodities that constitute inputs are *undetermined* based exclusively on observed facts pertaining to an economy. By contrast, based on facts and counterfactual presumptions about a capitalist economy, it is clear from Proposition 4 that a set of Walrasian commodity prices and a unique corresponding interpersonal income distribution are *completely determined* in equilibrium.

Recall that a *focal point* of the Sraffa model is ($\boldsymbol{p}^s$, $w$, $r$). Recall also that a focal point of the Arrow Debreu model is $\boldsymbol{p}^w$, which implies a unique income distribution $\boldsymbol{v} = (v_1 \ldots v_n)$, where $v_i$ is person $i$'s income. As already noted, Sraffa seeks prices $\boldsymbol{p}^s$ that will admit replication of the economy-wide production process, whereas Arrow and Debreu want prices $\boldsymbol{p}^w$ at which net excess demand for every commodity is zero. Based on different information sets: (a) factual for Sraffa, and factual-and-counterfactual for Arrow-Debreu, and looking for different characteristics of a focal point: (b) *Property* $\boldsymbol{E}$ for Arrow-Debreu and *Property* $\boldsymbol{R}$ for Sraffa, distinct sets of *accounting values* representing constitutively different *contents*, will necessarily

[28] Formalization of conditions 2a and 2b is straightforward. For details, see Mas-Colell, Whinston and Green (1995), especially Sections B and C.

[29] For a proof, see Arrow and Debreu [1954], where the function $\boldsymbol{z}(\boldsymbol{p})$ is allowed to be multi valued, so that it can be called an excess demand correspondence.

typify the Sraffa accounting values and the Arrow-Debreu accounting values, regardless of their interpretation.

While Sraffa sets out to find a unique set of exchange values that would redistribute the commodities concentrated at the end of the production cycle in the hand of their respective producers back to the industries so as to permit a replication of the production activities, he finds that such exchange-values and the associated income distribution *cannot* be determined if the sole basis of ascertaining them is the factual information about inputs of commodities and labor and commodity outputs that is contained in *A*, *C*, and $\boldsymbol{l}$, and nothing else whatsoever. Thus, the Sraffa theory of value and distribution violates *Property* ***D***, whereas this property is an entailment of the Arrow-Debreu theory of value.

Proposition 2 can therefore be reformulated to reflect this indeterminate character of values and distribution based on economic information in Sraffa

**Proposition 5**: *There is nothing factual about an economy that endogenously determines what the actual distribution of income between workers and capital owners will be. The explication of this distribution is to be found within society but outside the economic system.*

This claim is completely at odds with the conclusion reached by Arrow and Debreu, based in turn on both factual and counterfactual information. For, given Conditions 2a and 2b, once the prices of all commodities at zero excess demands are causally determined, so is the income distribution $\boldsymbol{v} = (v_1 \ldots \ v_n)$, and Proposition 4 can be reformulated to reflect this additional feature:

**Proposition 6**: *There exists an endogenously determined interpersonal income distribution at strictly positive Walrasian prices with zero net excess demand for every commodity.*

Proposition 6 asserts that in the Arrow-Debreu economy, that is characterized by both factual and counterfactual information, the values of commodities and the distribution of income are completely determined endogenously, whereas Proposition 5 says that this is simply not true in the Sraffa economy that is based solely on facts.

The question is not whether Propositions 5 is true or Proposition 6 is true. Indeed both are true, in their respective models of a capitalist economy. The question, then, turns on which characterization of a capitalist economy is a more accurate description of reality – Sraffa's based only on facts or the Arrow-Debreu explanation based on facts and counterfactual information.

This issue can be examined at different levels. The first is purely information theoretic. If the information set on which Sraffa's model is based is *S*, and the information set on which the Arrow-Debreu model is based is *D*, then *S* is a proper subset of *D*. Moreover, while *S* contains only factual information, *D* contains both the factual information in *S and* additional counterfactual information not in *S*. Since Sraffa's theory of value is based on weaker assumptions – in the sense of depending on less information – than the assumptions on which the Arrow-Debreu theory of value is based, it necessarily follows that Sraffa's is a more general theory. There are no ifs, ands or buts about it, regardless of what the conclusions of the two might be. This is a simple point of logic. Since the information set on which Sraffa's model

is based is a proper subset of the information set on which the Arrow-Debreu model is based, it is also true that challenging the accuracy of Sraffa's model as the basis of the description of capitalism would prove fatal for the Arrow-Debreu model as well.

Second, at a philosophical level, there is a well-known, long-standing debate in epistemology regarding the use of counterfactual information. It has been argued that there is an element of unreliability in propositions that are predicated on counterfactuals that is absent, or is considerably less pronounced, in purely observational propositions that are based exclusively on facts.[30] In this regard, Sraffa's approach of eschewing counterfactual information bypasses such difficulties in his purely descriptive theory of value. The Arrow-Debreu descriptive theory of value, however, remains open to this epistemological critique. On the other hand, it is also evident that the sole concern of economics is not with *description*. The concern with *prescription* is also inescapable and significant. Therefore, it is not altogether clear why one would want to eschew counterfactuals in descriptive economics, only to take them on board in the normative exercise of social evaluation aimed at policy prescription. However, just because counterfactual presumptions are necessary for normative social evaluation, it does not follow that there is any need to swallow counterfactuals hook, line and sinker in descriptive economics as well. In addition, it could be argued that it is useful to know how far one can go in reaching conclusions solely on the basis of factual information, which is the approach adopted by Sraffa.

There is a third significant ground on which Sraffa's theory of value and distribution differs from that of Arrow and Debreu, which is implied by Propositions 5 and 6 *if Sraffa's work is seen as a continuation of an investigation initiated by classical political economists*. This is the political dimension, which the concluding section addresses.

### *6. GENERALIZATION: INTERPRETATION AND MATERIALITY*[31]

Considerable attention has been paid in the literature as to why Sraffa assumed a uniform rate of profit on the value of capital in every industry, and what it does or not does not imply regarding its empirical content. This issue can be addressed at two levels at least. One is interpretational, and the other deals with the robustness of some results of Sraffa.

If the assumption of a single rate of profit is dropped, and replaced by multiple rates of profit in different industries, does it overturn some crucial property of the Sraffa model – say *Property **I***?

To investigate this matter, let there be two possible rates of profit in the Robinson-Naqvi economy, and let us denote these rates by $r_1$ and $r_2$ . As noted above, one set of values consistent with $w = 0$ is $r_1$ and $r_2$ both equal to 0.25. However, with two rates, it is evident that in fact $r_1 = f(w)$ and $r_2 = g(w)$ , and $f(w) \neq g(w), \exists w \in [0, 1]$. Consider,

[30] There is, for instance, a distinction made between *Prima Facie* and *Ultima Facie* justification in epistemology. On this, see Senor [1996], among others.

[31] I owe this entire section to Ajit Sinha whose comments on a previous version of this paper provoked the claims that I make here, whether or not he intended this to be the effect.

$$(\frac{14}{9} + \frac{20}{9}\,p)(1 + r_1) \quad + \frac{4}{5}\mathrm{w} \;=\; \frac{10}{3}$$

(7)
$$\left(\frac{7}{9} + \frac{10}{9}\,p\right)(1 + r_2) \quad + \frac{1}{5}\mathrm{w} \;=\; \frac{10}{3}p$$

Suppose $r_2 = 0$. Then $r_1 = \left(\frac{20}{9} + \frac{w}{5}\right)(1-w)^{-1} = f(w)$. *Property **I*** is still satisfied by Sraffa's model, despite non-uniform rates of profit on the value of capital in different industries. In fact, if $r_2 = 0$, then for $w = 0$, $r_1 = \frac{3}{7}$ and $p = 0.35$; however, as seen before, for $w = 1$, $r_1 = 0$ & $p = 0.44$ tons of wheat per ton of iron. Although multiple profit rates open up a new question of the distribution of income among capital owners who have their capital employed in one industry rather than in another, *it still leaves open to indeterminacy the distribution of income distribution between workers and capital owners at a focal point of the Sraffa model*.

The "price-and-income-distribution indeterminateness" feature of Sraffa's model, I would argue, is a crucial message of Sraffa that is worth preserving, simply because (a) it stands in sharp contrast to the message of the only other fully-developed theory of value, viz., of Arrow and Debreu, and (b) it is a recurring theme in the work of classical political economics, particularly Ricardo and Marx. Thus, it is quite unnecessary to defend the assumption of a uniform rate of profit. For the case of multiple rates of profit on the value of capital, Sraffa could have considered $r = \sqrt[K]{\prod_{i=1}^{m} r_i^{K_i(A,C,w)}}$, as the economy-wide rate of profit, endogenously determined, though not uniquely, only if $w$ is exogenously specified, with the understanding that K= $\sum_{i=1}^{m} K_i(A,C,w)$. Of course, multiple wage rates that differ across industries is also a generalization that permits the indeterminateness property of income distribution between workers and capitalists to be preserved, and only raises the new issue of the internal distribution of income within the working class. Here, $w = \sum_{i=1}^{m} l_i w_i(A,C,r)$ could be taken as the weighted-average economy-wide wage rate, again endogenously determined but not uniquely, only if $r$ is exogenously specified.

Therefore, Sraffa did not have to posit a unique rate of profit in all industries, nor a unique wage rate paid in every industry, for his theory of value and distribution to stand in contrast with that of Arrow and Debreu, in terms of indeterminacy of prices and income distribution. Sraffa chose to provide a minimal model, with only one *r*, and only one *w*, because his result in this regard is invariant to generalization to multiple rates of profit or multiple wage rates.

This argument also casts serious doubt on Hahn's [1982, p. 359] claim regarding the uniform rate of profit being "the only falsifiable entailment of the Sraffa equations". In addition, any particular reading of how a uniformity of profit rates comes about – any interpretation – is rendered completely irrelevant to the basic requirement of indeterminacy in *Property **I*** in Sraffa's theory, which, in turn, is precipitated by *Axiom **F***: Factual information in *A*, *C*, and ***l***, and *Property **R**,* to find such set of prices that would redistribute commodities back to the original order in the different industries so that the entire production cycle can be replicated in the next period.

The lesson to be drawn from the foregoing discussion is that if Sraffa's central message is construed as the focal point of a model satisfying *Property* ***I***, then it is invariant to possible non-uniformity of the rates of profit on the value of capital in different industries. This is, therefore, a front on which no war needs to be waged. Although Sraffa provides a minimal model, with only one rate of profit in all industries, this is not a necessary requirement. The essential requirement for ascertaining the Sraffa prices and income distribution embedded in a set of relations of production is the capability of Sraffa prices *to redistribute commodities back to the original order in the different industries so that the entire production cycle can be replicated* in the next period. But, uniformity of the rate of profit on the value of capital is distinct from, and not material to, the preservation of *Property* ***R***. The bite of the restriction cuts by requiring the replication of production, not by imposing a single rate of profit requirement, contrary to what some may have thought regarding the equilibrating role of a uniform rate of profit.

Another line not worth pursuing is whether Sraffa prices are market-clearing, long-period, or center of gravity, or anything else. The accounting values of prices and income distribution consistent with Sraffa's model are constitutively distinct from the corresponding values at a focal point of the Arrow-Debreu model. However, the two theories remain complete and distinct regardless of the interpretation adopted for either set of values. Of course, this is not to deny the usefulness of such interpretations for different purposes, such as gaining insights into price movements.

The question is sometimes raised as to what role demand functions for final goods play in Sraffa's model. Sraffa is mocked for not noticing that at higher prices quantities demanded of the more expensive commodities will be lower. The fact is, the indeterminacy in his model – for the determination of prices and income distribution – is unaffected by the presence or absence of consumer demand functions for each of the commodities.

Suppose $\forall i = 1, \dots, m$: $\forall x = r \in [0, r_{max}]$ or $\forall x = w \in [0, 1]$. Let the demand function be $d_i = h^i(\boldsymbol{p}^s,\ x)$ for commodity *i*. We already know that the supply of commodity *i*, net of the quantities used up in the production process, is $b_i(\boldsymbol{p}^s,\ x) = c_i - \sum_{j=1}^{m} a_{ij}$. The reason that the net supply function is a function of prices and distribution is that there is no additional relationship that can determine which particular value of $\boldsymbol{p}^s$ will show up as consistent with what specific distribution state *x* that actually obtains in society, *keeping in mind that x is determined from outside of the economic relations of production in the Sraffa model*.

We are one equation short on the supply side: we need to know *m + 1* values in $(\boldsymbol{p}^s, x)$ due to normalization, but have only *m* independent relationships to determine them. The demand side also suffers from the same problem: *m + 1* unknowns in $d_i = h^i(\boldsymbol{p}^s,\ x)$, but only *m*-independent relationships to determine them. So, how can one indeterminacy help resolve another? Set $\tilde{z}_i(\boldsymbol{p}^s,\ x) = h^i(\boldsymbol{p}^s,\ x) - b_i(\boldsymbol{p}^s,\ x) = 0, \forall i$. Based on economic relations alone, we *still* do not know which state of distribution *x* will obtain, and thus which set of Sraffa prices $\boldsymbol{p}^s$ will bring about production replication.

So, why bother about demand-side issues unless they can help remove the indeterminacy already present on the supply side of the economy. Sraffa thus has nothing to say about consumer

demand. This is because demand issues are important in their own right, but even they cannot scoop up both prices and distribution solely from the economic pond. To understand both values and income distribution in a society, one has to go beyond economic relations, into the political domain, which sits right outside both the Sraffa and the Arrow-Debreu models.

*7. CONCLUDING REMARKS*

This paper covers much ground in dealing with two very comprehensive theories of value. Both theories of value provide answers to the "determination" of commodity prices and income distribution in a society, in their own distinctive ways. Both are based on perspicaciously articulated characterizations of a capitalist economy. Yet, they offer conclusions that are quite different due to the fact that the characterizations of economies upon which these conclusions are based are themselves distinct, philosophically, informationally, and as a consequence, politically. Sraffa's theory of value falls squarely in the domain of classical political economy – a theory that sees society as consisting constitutively of classes, with individuals acting in the interest of the class to which they belong. Arrow and Debreu's theory of value, by contrast, is the culmination of the tradition of neoclassical economics that sees society as constitutively made up of individuals who act in their personal self-interest.

There is a significant political issue at stake here. In the Arrow-Debreu conception, the interpersonal distribution of income is endogenously determined. It is what it is, and it comes out in the wash. Not so, in Sraffa's conception of the economy, however. There simply isn't enough cutting power in a replication requirement imposed on the economic system of factual relationships that is capable of pinning down the distribution of income and commodity prices in a society. This matter of nailing down commodity prices and income distribution rests, in part, outside the purely economic sphere of society, and thus falls in the political domain *also*. This is a position taken by classical political economists for centuries – which finally receives formalization in Sraffa's work, so that a claim can be made about *distribution-indeterminateness*,

***Proposition 7***: *The distribution of income between workers and capitalists and the associated vector of exchange-values* $\boldsymbol{p^s}$ *cannot* be *determined solely in the economic sphere of society. A political engagement between the two classes, based on the relative bargaining power of workers and capitalists, is an additional consideration that has to be brought to bear on the determination of both commodity prices and income distribution in a society. This is true if based exclusively on Axiom* ***F***: *Factual information in A*, *C*, and $\boldsymbol{l}$, and *Property* ***R****, to find such a set of prices that would redistribute commodities back to the original order in the different industries so that the entire production cycle can be replicated in the next period.*[32]

Sen [2003, p.1247] writes, "I must confess that I find it altogether difficult to be convinced that one's skepticism of unrestrained capitalism must turn on such matters as the usefulness of aggregate capital as a factor or production and the productivity attributed to it, rather than on the mean streets and strained lives that capitalism can generate[.]" Actually, Sraffa achieves considerably more than that. The fact is, Sraffa provides an *explanation* of the makings of "the

[32] As it happens, the German constitution actually embodies this feature of collective bargaining by trade unions with corporate management. It is somewhat odd that the country that once expelled Karl Marx, the most forceful proponent of this position on the determination of income distribution, has, in a significant way, embraced his ideas on this matter.

mean streets and strained lives that capitalism can generate" by identifying the underlying cause of such "strained lives" in the inequality of income distribution that falls in the domain of political negotiation and balance of power between workers and capitalists. That is the true power of Sraffa's contribution – to provide an answer to a question as old as economics itself – that the factual relations of production, by themselves, fail to determine the actual income distribution.

In some sense, Sraffa was the last of the greats among classical political economists. He saw the disassociation between economics and politics, inherent in the trend of neoclassical economics, as a diminution of the study of society. It is in this precise sense that, in his words, "It is, however, a peculiar feature of the set of propositions now published that, although they do not enter into any discussion of the marginal theory of value and distribution, they have nevertheless been designed to serve as the basis for the critique of that theory."[33] This politico-economic approach of Sraffa – as articulated in Proposition 7 – may be contrasted with the politics-free neoclassical approach to rehabilitating the *status quo* income distribution in the Arrow-Debreu formulation (as in Proposition 6), which, along with the entire allocation, can be, and has been, shown to be optimal in a specific sense, viz., that of Pareto.[34]

Notice that one could be tempted to make more than is warranted of the economic indeterminacy of commodity values and income distribution revealed by the work of Sraffa. Merely because of fact (a) that there is an inverse relationship between the share of national income that accrues to workers versus capitalists, and fact (b) that the economic relations fail to resolve the income distribution matter, thereby leaving it to the political process to determine, it does not follow that a violent (or non-violent) revolution is called for. As in several countries, including Germany, both facts can be embraced with tranquility, and a state-aided political resolution to the income distribution issue can be reached under constitutional rule of law, without any call to arms.

There is another perspective of significance to political action that emerges from a comparison of Propositions 6 and 7. Proposition 6, arising from the Arrow-Debreu theory of value, lulls one into thinking that the reality of "mean streets and strained lives that capitalism can generate" is what it is, to the extent that the distribution of income is completely determined in the economic sphere, and thus gives room for a conservative orthodoxy to argue that "all is for the best," in the sense of Pareto, "in the best of all possible worlds," even if misery is pervasive. This claim puts a conscientious objector on the defensive in having to justify state intervention for ameliorating widespread misery. However, Proposition 7, based on Sraffa's theory of value and distribution, constitutes a basis for a critique of the Arrow-Debreu theory of value by establishing a completely divergent position (that is based solely on factual information), viz., that the economic system does not uniquely dictate any specific income distribution or prices, and that if the income distribution that emerges with prices is fraught with deprivation for the many, it is due, at least in part, to a political failure in empowering *les misérables*. Public action in the political domain is the answer to such discontent, not a resigned and passive acceptance of the

[33] p. vi. Notice that "marginal" is *not* the operative word here. It is the existing, alternative "theory of value and distribution" to which Sraffa makes reference as aiming to critique. And that is the theory of Arrow and Debreu.
[34] The reference to the "critique" in Sraffa is to a "theory of value and distribution" that is bereft of any political consideration, rather than to one that relies on differential calculus.

*status quo* that arises from reliance on counterfactuals for intellectual justification of the ideology of conservatism.

Oddly enough, Sraffa writes only about commodities, and there are no people visible in his model of a capitalist economy or in his theory of value, even though his overriding concern is – as that of classical political economists – with the uncovering of the lack of economic determinism of the distribution of income. By contrasts, the Arrow-Debreu model is visibly populated by persons, and based on a larger quantity of information it ends up with the conclusion of economic determinism of interpersonal income distribution. It must be recognized, though, that some doubt has been cast on the concept of persons in Arrow and Debreu. In the words of Sen and Williams [1982, p.4], "Persons do not count as individuals in this any more than petrol tanks do in the analysis of the national consumption of petroleum." Actually, Arrow and Debreu refer to them as "consumption units."

Thus, one finds an absence of people in Sraffa, and persons with highly circumcised, inorganic individuality in Arrow and Debreu. This leaves room for a theory of value that gives greater play to persons as individuals in society, with the full set of cultural identities, political affiliations, familial associations, and personal predicaments and preferences. A way out appears to be Sen's [1985] theory of *Commodities and Capabilities* of persons that aims to capture the freedoms of individuals to achieve and be what they have reason to value constitutively and instrumentally, and of which Kuklys [2005] has done a phenomenal job of operationalizing.

**REFERENCES**


Akerlof, George A. [1970]. "The Market for 'Lemons': Quality Uncertainty and the Market Mechanism". *Quarterly Journal of Economics* **84** (3), 488–500.

Arrow, Kenneth, J. and Debreu, Gerard. [1954], "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, **22** (3), 265-290.

Bose, Arun. [1965], "Consumers' Demand, Distributive Shares and Prices," *Economic Journal*, **LXXV**, 771-786.

Chipman, John, S. [1970], "External Economies of Scale and Competitive Equilibrium," *Quarterly Journal of Economics*, **84**, 347-385.

Debreu, Gerard. [1959], *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*, New Haven: Yale University Press.

Hahn, Frank. [1982], "The Neo-Ricardians," *Cambridge Journal of Economics*, **6**, 353-374.

Kuklys, Wiebke. [2005], *Amartya Sen's Capability Approach: Theoretical Insights and Empirical Applications*, Berlin: Springer-Verlag.

Kurz, Heinz D. and Salvadori, Neri. [1995], *Theory of Production: A Long-Period Analysis*, Cambridge: Cambridge University Press.

Mas-Colell, Andreu. Whinston, Michael D. and Green. Gerry R. [1995], *Microeconomic Theory*, New York: Oxford University Press.

Naqvi, Nadeem. [2007], "Constant Returns to Scale and Economic Theories of Value," *MPRA Paper 5306*, available at http://mpra.ub.uni-muenchen.de/5306/.

Robinson, Joan and Naqvi, K.A. [1967], "The Badly Behaved Production Function," *Quarterly Journal of Economics*, **81** (4), 579-591.

Samuelson, Paul A. [1962], "Parable and Realism in Capital Theory: The Surrogate Production Function," *Review of Economic Studies*, **29** (3), pp. 193-206.

**_____.** [2000], "Sraffa's Hits and Misses," Chapter 3 in Kurz, Heinz D. (ed.), *Critical Essays on Piero Sraffa's Legacy in Economics*, Cambridge: Cambridge University press, 111-152.

**_____.** and Etula, Erkko M. [2006], "Testing to Confirm that Leontief-Sraffa Matrix Equations for Input/Output must obey Constancy of Returns to Scale," *Economics Letters*, **90**, 183-188.

Schefold, Bertram. [1985], "Cambridge Price Theory: Special Model or General Theory of Value," *American Economic Review*, **75**, 2, 140-145.

**_____.** [1989], *Mr. Sraffa on Joint Production and other Essays*, London: Unwin Hyman.

**_____.** [1996], "Piero Sraffa 1898-1983," *The Economic Journal*, **106**, 438, 1314-25.

Sen, Amartya. [1985], *Commodities and Capabilities*, Amsterdam: North Holland.

**_____.** [2003], "Sraffa, Wittgenstein, and Gramsci," *Journal of Economic Literature*, **41** (4), pp. 1240-1255.

**_____.** [2005] *The Argumentative Indian: Writings on Indian History, Culture and Identity*, London: Penguin Books.

_____ and Willams, Bernard. [1982], "Introduction," in Sen, Amartya and Williams, Bernard (eds.) *Utilitarianism and Beyond*, Cambridge: Cambridge University Press, 1-21.

Senor, Thomas D. [1996], "The Prima/Ultima Facie Justification Distinction in Epistemology," *Philosophy and Phenomenological Research*, **56**, 3, 551-566.

Sinha, Ajit. [2007], "Sraffa and the Assumption of Constant Returns to Scale: A Critique of Samuelson and Etula," *Contributions to Political Economy*, **26**, 61-70.

_____. and Dupertuis, Michel-Stéphane [2009], Cahiers d'économie politique, **56**, L'Harmattan, 91-100.

Sraffa, Piero, [1960], *Production of Commodities by means of Commodities*, Cambridge University Press: Cambridge.