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Endogenous Specialization of Heterogeneous Innovative Activities of Firms under the Technological Spillovers

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Abstract

This paper proposes a reduced form model of dynamic duopoly in the context of heterogeneous innovations framework. Two agents invest into product and process innovations simultaneously. Every newly introduced product has its own dimension of process-improving innovations and there is a continuum of possible new products. In the area of process innovations the costless imitation effect is modelled while in the area of product innovations agents are cooperating with each other. As a result the specialization of innovative activity is observed. This specialization arises from strategic interactions of agents in both fields of innovative activity and is endogenously defined from the dynamics of the model.

Keywords: Innovations, Dynamics, Multiproduct, Spillovers, Distributed Control, Differential Games
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1 Introduction

The main focus of the paper is on the modelling of strategic interactions of agents in the field of innovations. To this end the production activities of the agents is not explicitly modelled. Only the investment activities related to the introduction of new products and their further refinement are accounted for. The time-horizon of the model is set to be infinite for tractability reasons.

Despite of the simplicity of the framework suggested in the model, it allows to catch several major issues relevant for dynamic interactions in the field of innovations. First, it is shown that the costless imitation does not lead to incentive to decrease investments into process innovations for both agents simultaneously if one would account for their cooperation on the more fundamental level of product innovations. One of the agents appears to be the leader, which does not benefit from the imitation effect, while the other one positions herself as a follower, reducing her own investments to benefit from the technological spillover created from the activities of the first agent. Next, in the area of product innovations the united efforts of both agents are distributed unevenly. Instead, there is a natural specialization of investment activities of both agents. The agent, who is benefiting from the technology spillovers in process innovations (named 'the follower' in the sequel) puts more investments into the product introduction activities. However, in the simplest case of open-loop strategies the other agent invests non-zero amount into the product innovations also. In an effect the agent who is the most efficient in one or another type of innovative activities carries the major burden of investments in this direction while benefiting from the investments of the other agent in the other type of innovations.

One may consider the suggested model of strategic interactions as an extension and combination of the results from different directions. First, it contributes to the line of literature on strategic interactions between innovating agents in the spirit of (Reinganum 1982), (Judd 2003) and late (Lambertini and Mantovani 2010). These approaches are extended by considering the distributed parameter model and formulating the fully dynamical differential game with richer strategic sets for both players. Next, approaches to imitation , (Gallini 1992) and R&D joint ventures, (D'Aspremont and Jacquemin 1988) are combined in a single model and it is demonstrated that these two effects are complementary in nature and resulting strategies cannot be optimal in a dynamic context while taking into account only one of them. At last, everything is put together to obtain the model of strategic interactions of innovative agents in heterogeneous multi-product framework which might be considered more general in its nature then previous findings in the field.

The rest of the paper is organized as follows. In the next section a brief summary on relevant studies is presented. Next basic assumptions and the framework of the model are considered in brief. In the following section the model is extended to the two agents case. It is solved sequentially through employing the Hamilton-Jacobi-Bellman and Maximum Principle approaches. After obtaining analytical results the dynamics of the optimal investment strategies for both agents is considered and the nature and form of the resulting strategic interactions as well as some practical implications of the findings are discussed.

2 Related Research

To our knowledge one of the first attempts to model the strategic interactions of oligopolistic agents via the differential games approach is given in the work of Reinganum, (Reinganum 1982). In this paper the author combined static games approach with optimal control one to obtain the dynamic game of R&D competition in a n-firm industry. However this was not the first paper on the influence of the market structure on the outcome of R&D competition. One of the first works in the field is that of Loury, (Loury 1979). In this pioneering work the discrete single innovation is assumed and n firms compete for being the first to introduce the new product. The first firm which would introduce such a product obtains an exclusive right for its production and hence receives the perpetual stream of profits associated with this product. This model lacks the explicit formulation of strategic interactions and consists of identical optimization problems for all the firms. However the equilibrium outcome does depend on the number of firms in the industry. Another basic approach to modelling R&D competition consists mainly in static game formulation, as in the work of Dasgupta&Stigliz, (Dasgupta and Stiglitz 1980) where no explicit dynamical interactions appear. In their model they mainly tackle with the question how the market structure (e.g. monopoly versus oligopoly) would influence the equilibrium level and price of innovative products. They come to the conclusion that it is the elasticity of the market demand function which defines the optimal structure of the industry.

In the work of Reinganum these two basic approaches are combined in a single model and this is the basic paper one would compare the suggested framework with, since the same differential games approach is used here. On the other hand he pertains the general structure of Loury, namely there is a single innovative process and every player seeks to introduce this given product first to the market.

Concerning the particular form of interactions between players, the imitation effect at no cost in the dynamics of process innovations is assumed. There exists some literature on such kind of imitation. One of the examples is the work of Gallini, (Gallini 1992), although his approach is different from the one assumed here. Namely, he analyses the effect of imitation of the patented product which is costly, while in the suggested model it is assumed to be costless. In this respect the imitation effect is treated as the undirected technological spillover from the leading firm to the other one, while any firm may become a leader or the follower in the process innovations. It turns out that in the given framework the imitation effect may constitute the equilibrium only if one take into account the underlying process of variety expansion also. Another more recent work on dynamic interactions of R&D firms is that of Judd, (Judd 2003). In this paper the author analyses the multi-stage innovative race between multiple agents with multi-product situation and this is rather close to the suggested approach. Nevertheless he assumes the multi-stage structure of the game and hence a static situation with some transition between stages whereas here the dynamic game with continuous time is modelled. He finds out, that there is an ambiguity in the results of a game, namely a given player may increase his expenditure (investments in our case) when the other agent is ahead of him, while this is not profitable for him as an imitator. It is demonstrated that in the suggested model this is not the case and any ambiguity disappears if one would consider both aspects of innovation.

The most recent paper on product and process innovations in differential games framework is the work (Lambertini and Mantovani 2010). This paper assumes fully dynamical model of the duopoly competition of innovating firms. The suggested model differs from the work of Lambertini&Mantovani in two significant aspects. First, in their model authors assume uncertainty of innovations in the form of Poisson arrival rates, while the suggested model does not contain uncertainty in any form. However the same form of uncertainty may be easily introduced into the suggested model and will not change major results of the paper. Next, the discussed paper does not handle heterogeneity of innovations and hence is reduced to the differential game with two states, while the suggested model allows for distributed nature of innovations and all products differ from each other in their investment characteristics. This is more in line with the setup of Lin, (Lin 2004), but with fully dynamic context. The result on endogenous specialization of players is mainly due to dynamic context and heterogeneity of innovations being modeled simultaneously.

The last feature of the suggested model is the R&D cooperation on the level of products variety expansion (product innovations). It is argued that such a situation is more typical for R&D firms then the full-scale competition on both levels. First such a type of strategic interactions has been considered in D'Aspremaunt&Jacquemin, (D'Aspremont and Jacquemin 1988) where it is argued that in real economies the majority of R&D activities is performed in the form of joint R&D ventures if one would consider innovations of big enough size. As long as one assumes that the variety expansion represents the process of introduction of completely new products to the market it is rather natural to assume the precense of joint cooperative efforts on this level. This would mean that agents share common knowledge and efforts concerning this part of their innovative activity.

3 Assumptions and Basic Framework

In this section the formal model is introduced together with the underlying economic intuition.

3.1 Assumptions

Assume there are two firms in a given industry. The industry is mature and no growth of the demand is expected for existing products variety. There are equilibrium quantities for both firms which depend on the production costs. However, these costs are fixed for mature products and are not subject to change in the result of new process innovations. Both firms act as monopolists in their markets which are separated and cannot be entered by the other firm. In this respect the firms are regional monopolies. This implies that profit of every firm depends only on its own production costs but not of that of the other one.

Assumption 1 *Both firms are perfectly separated with respect to their production.*

This assumption may be relaxed to allow for interactions of the firms on the product market also. However, this will not influence the main result of the paper since the production decision does not influence the innovative decisions, while the latter influence the production. We abstract from the explicit derivation of production policies here and note that this would be similar to the separation of production and knowledge accumulation for the firm as in,for example, (Dawid, Greiner, and Zou 2010).
These monopolists are maximizing their profits by developing new products, which are then introduced to the market.

Assumption 2 *The only source of new profit for both firms is the development of new products which leads to the increase in the existing range of products, common for both firms, over time,* $n(t) > 0$.

Of course, the profit is derived from existing and mature products also, but innovative activities do not influence this as well as the production decisions for these products.
Assume the process of development of products is continuous in time and yield new products (which are new versions of some basic for the industry product) with some rate. Let us call this rate the rate of variety expansion.

Assumption 3 *The product innovations are continuous in time and new products appear at a continuous basis,* $\dot{n}(t) \geq 0$.

Assume that the range of these new products is limited from above. The product innovations are limited to upgrades of some basic product which defines the industry (e.g. cell phones industry produces different versions of cell phones but not computers). We do not model fundamental inventions, which introduce totally new products to the economy by this model and hence it is natural to require that there is limited capacity of the industry for the variety of products which are somewhat similar to each other.

Assumption 4 *Product innovations are limited by the maximal possible range of products,* $n(t) \leq N$.

Assume these newly introduced products initially require very much resources for their production and hence each of the firms allocates part of its R&D capacity on process innovations related to these new products. Every new product is than intensively studied with respect to opportunities for its costs minimization. As there are numerous new products (continuum of) there are numerous streams of such cost-minimizing processes each of them being associated with every new product.

Assumption 5 *Every product has its own dimension of process innovations or 'quality' which depends on time and is different for both firms,* $\forall i \in n(t) \exists q_i^{[j],[l]}(t)$.

Since the introduction of new product is simultaneous for both firms, they start their investments into these new products simultaneously. It is natural to require that at the time of introduction, denoted by $t_i(0)$ for each product i the level of process technology is zero.

Assumption 6 *At the moment of introduction of the product* i *its 'quality' is zero for both firms,* $q_i^{[j],[l]}(t_i(0))|_{i=n(t)}$

Assume at each point in time, each of the firms has to choose optimally the level of investments being made into the development of new products (product innovations) and into the development of production technology of already existing products (process innovations). These investment streams cannot be negative.

Assumption 7 *Product innovations and process innovations require different types of investments, which vary over time, while process innovations for every product are also different* $u^{[j],[l]}(t) \geq 0; g_i^{[j],[l]}(t) \geq 0$

Assume also that firms are long-run players and do not restrict their planning to some certain length of time. Hence, the innovations of both types occur continuously up to infinite time.

Assumption 8 *There is no terminal time for both processes of innovations,* $0 \geq t < \infty$

The last point to mention is that we assume that all innovations are certain. This is rather strong limitation, but allows to concentrate on the key issues of this paper: endogenous specialization of innovative activities in the presence of heterogeneity of new products.

Assumption 9 *All innovations do not have any uncertainty associated with them.*

Under the given assumptions it must be clear that the only channels of interactions between the two firms should lie in the field of their innovative activity. That is, these firms share the knowledge on new products being created as a result of their investments $u^{[j],[l]}(t)$ while benefiting from spillovers of process technologies from the other firm, $q_i^{[j],[l]}(t)$. This last has yet to be specified. Up to the this point the assumptions and the suggested framework follows the model of a single monopolist as described in (Bondarev 2010b) with extension to two agents. Next we discuss the profit generation and objective functions of both firms in details.

3.2 Objective Function

In this section we introduce the objective functions of both players.
Consider first the situation in the absence of innovations for each of the firms. The natural objective of the firm is the maximization of its profits, $\pi^{j,l}(t) \to max$ for any given time period. This paper concentrates on just one part of activities of such a firm, namely on the process of its innovative activities. To put this in line with profit maximization behaviour we assumed

that markets for all existing products are mature, yield some constant profit with stable prices and output. Production policy of every firm is assumed to follow standard rules of monopolistic behaviour under profit maximization (since the perfect separation of markets): given (constant) demand, the monopolist is setting the price and production as to maximize its profit. In mature markets the process innovations reached their maximum and thus no further improvements to the production process may be made. Hence, the production costs are also constant in time. These considerations lead to the conclusion that in mature markets the monopolist's production and pricing (and hence profits) are constant.

Proposition 1 *For those products which are already in mature stage, the production, price and profit for each of the firms are constant.*

Because of this one may abstract from this part of firms' activities in the optimization problem.
The objective function of such a reduced optimization problem for both firms is then given by:

$$J^{[j],[l]} \stackrel{\text{def}}{=}$$

(1)

$$\stackrel{\text{def}}{=} \int_0^{\infty} e^{-rt}\Big[\int_0^{n(t)} \Big(q^{[j],[l]}(i,t) - \frac{1}{2}g^{[j],[l]}(i,t)^2\Big)di - \frac{1}{2}u^{[j],[l]}(t)^2\Big]dt \to \max_{u^{[j],[l]},g^{[j],[l]}} .$$

The basic intuition behind (1) is clear: every firm is maximizing the effect from its process innovations for every of the introduced products at every time t. The range of products, being introduced till time t is given by $n(t)$ and hence the difference in the effect of process innovations and the investments into them is evaluated over this range at any point in time. At the same time the investments into the creation of new products, $u^{[j],[l]}$, negatively influence the total generated by innovations value for the firm while the introduction of new products at some continuous rate enlarges the space of products, which production may be refined through process innovations. Observe that the introduction of the new product per se does not bring the increase in the value for the firm, since it is assumed that such a product has zero level of technology. The objective is to maximize the effect of innovations of both types at the firm performance at the infinite time horizon.

So far both firms are independent from each other in all their characteristics. Now we introduce the link between the firms in the field of their innovative activity. This comes from the dynamics of their state variables, $n(t), q_i^{[j],[l]}(t)$ and is described below.

3.3 Dynamics of Innovations

As seen from the objective, both firms are symmetric with respect to the value generated by the innovations of both types for them. Suppose the given firms join their efforts in the development of variety expansion over the same potential products' space, while having possibly different investment efficiencies. Then variety expansion dynamics is governed by the law:

$$\dot{n(t)} = \alpha_{[j]}u^{[j]}(t) + \alpha_{[l]}u^{[l]}(t). \tag{2}$$

where:

- $n(t)$ denoted the variety expansion level being reached at the time t;
- $u^{[j],[l]}(t)$ are investments of firms $[j],[l]$ into the process of variety expansion at time t;
- $\alpha_{[j],[l]}$ are investment efficiencies for both firms which are assumed to be constant in time.

This means both firms have the same underlying variety expansion process while freely choosing the level of efforts they would devote to the development of this variety. Note that this does not exclude the possibility for one or the other firm to have zero investments while benefiting from the investments being made by the other through using achieved variety level.

Let these firms have different and separate process innovations processes for each product within the products range. Firms do not benefit from investments of each other into these process innovations, but they may copy the progress of the other in the case their own innovations level for the same product is lower. In the effect one has two dynamic processes linked with each other and two different independent streams of investments:

$$\begin{aligned} \dot{q_i^{[j]}(t)} &= \gamma_{[j]}\sqrt{N-i}g_i^{[j]}(t) - \beta_{[j]}q_i^{[j]}(t) + \theta \times \max\{0, (q_i^{[l]}(t) - q_i^{[j]}(t))\}; \\ \dot{q_i^{[l]}(t)} &= \gamma_{[l]}\sqrt{N-i}g_i^{[l]}(t) - \beta_{[j]}q_i^{[l]}(t) + \theta \times \max\{0, (q_i^{[j]}(t) - q_i^{[l]}(t))\}. \end{aligned} \tag{3}$$

where:

- $q_i^{[j],[l]}(t)$ are technology levels achieved by process innovations into the product i by the agent $[j],[l]$ at the time t;
- $g_i^{[j],[l]}(t)$ are investments of both agents into the development of the production process for product i at time t;

- $\gamma_{[j],[l]}$ are efficiencies of investments into the process innovations for agents $[j],[l]$, constant across products and time;
- $\beta_{[j],[l]}$ are technology decay rates in the absence of investments for both agents;
- θ is the speed of costless imitation, equal for both agents.

Observe that the term $\sqrt{N-i}$ in both processes defines the decreasing efficiency of investments into the quality growth for both players across products' range. It is assumed that the higher is the index i of the product, the harder it is to improve its production technology due to the increased complexity of the product. The specific form of this decreasing function is chosen to linearise the resulting dynamical systems. This is done following the same arguments as in (Bondarev 2010a).

As a result the firm's level of process innovations may grow in two different ways. As long as the given firm is the leader in process innovations for a given product, e.g. $t : q_i^{[j]}(t) > q_i^{[l]}(t)$, its 'quality' grows only due to its own investments in process innovations for this product, $g_i^{[j]}(t)$. At the same time, the other firm's production technology is inferior to the one being developed by the first firm and it benefits not only from its own investments but also from the 'imitation effect': it benefits from the difference between the leader's technology level and its own one. Clearly this effect will boost the second firm's process innovations but will wear down eventually while the follower's technology approaches that of the leader.
The dynamics of product and process innovations is subject to a number of static constraints which are formalizations of assumptions given above:

$$\begin{aligned} u^{[j],[l]}(t) &\geq 0; \\ g^{[j],[l]}(i,t) &\geq 0; \\ 0 \leq n(t) &\leq N; \\ q^{[j],[l]}(i,t) \mid_{i=n(t)} &= 0; \\ q^{[j],[l]}(i,0) &= 0, \forall i \in \mathbf{I}; \\ n(0) &= n_0 \geq 0. \end{aligned} \tag{4}$$

These are essentially non-negativity and boundedness requirements for state variables for both firms as well as irreversibility of investments into both types of innovations. **I** stands for the notion of the products space.

Observe that it follows from (2),(3), that the only form of strategic interaction in process innovations is the costless imitation effect which influences the dynamic of state variables but does not influence directly the objective function. At the same time even such simple introduction of interdependencies between firms' strategies allows for the endogenous specialization of investments between them.

4 Solution

In this section the solution techniques applied to the problem formulated above are discussed.

4.1 Decomposition of the Problem

Given the basic formulation of dynamical problems of both agents above, it is clear that the optimal solution has to be found in the form of the equilibrium pair of strategies in the differential game framework. From the general point of view the model considered here is the infinite-dimensional one as long as one have the continuum of process innovations associated with each product for every player. This may provide some difficulties in formal construction of the game. However due to the special structure of the dynamic framework being used it is possible to decompose the problem into 'quality' growth problem and variety expansion problem. This can be done due to the fact that process innovations do not depend on the variety expansion except for the time of emergence of new products. Then every such a problem should be the finite-dimensional one and as long as it is of the linear-quadratic form, one may be assured that equilibrium exists for each such a game of process innovations under the same conditions as in standard linear-quadratic differential game, (Dockner, Jorgensen, Long, and Sorger 2000). Then the results obtained for this game may be used for solution of variety expansion problem which is also the differential game but with only one state, $n(t)$. For this one may rewrite the objective functional of both players (1) in terms of values generated by the process innovations and by the product innovations games.

To decompose the value function of the overall model, first we make use of the observation above. Starting from the time of emergence (denoted by $t(0)_i$) value of each product's process innovations is independent of variety expansion process:

$$V_i^{[j],[l]}(q) = \max_{g^{[j],[l]}} \int_{t(0)_i}^{\infty} e^{-r(t-t(0)_i)} (q_i^{[j],[l]}(t) - \frac{1}{2} g_i^{[j],[l]}(t)^2) dt. \quad (5)$$

where $t(0)_i$ is the time of emergence of the product i, which is defined from the dynamics of the variety expansion process and is similar for both agents due to the form of dynamic constraints (2),(3). With infinite time horizon the problem of process innovations management is the time-autonomous one and hence the time of the products emergence does not influence the value generation process and can be normalized for all products to zero, (Bondarev 2010b).

Proposition 2 *Value functions of process innovations management game for both firms,* $V_i^{[j],[l]}(q)$ *are invariant to the ratio of investments of both firms into new products development,* $u^{[j],[l]}(t)$ *and hence to the emergence time of this product,* $t(0)_i$*.*

Second part of the overall value generation consists of the intensity of introduction of new products at every time given the expected value of the stream of profit derived from the reduced costs of production (which comes from process innovations) of the newly introduced products. This part may be represented by the integral over all potential stream of process innovations for each product over it's life-cycle. At the same time this information is already contained in the value function of the 'quality' problem above, so it suffices to integrate over all potential products at initial time. Last observation to be made is that at the moment of the emergence of the new product it's production technology is zero, as it is required by (4). These yield the value function for variety expansion problem in the following form:

$$V^{[j],[l]}(n) =$$

$$= \max_{u^{[j],[l]}} \int_0^\infty e^{-rt}\Big((\alpha^{[j]}u^{[j]}(t) + \alpha^{[l]}u^{[l]}(t)) \times V_i^{[j],[l]}(0) \mid_{i=n(t)} -\frac{1}{2}u^{[j],[l]}(t)^2\Big)dt. \tag{6}$$

In the last equation value generated by the process innovations management game for each firm is estimated at the zero technology level for the product next to be invented. Hence one may sequentially solve the process innovations game for an arbitrary i, then calculate the associated value function at the zero technology level and $i = n(t)$ and use this last as an input for the variety expansion game.

Observe that a decomposition method is valid here since there is no competition on the level of variety expansion. Joint variety expansion process yields the coincidence of emergence times of all new products for both firms. There is no dependence of value creation at the production technology level from the relative speed of variety expansion. Moreover, every firm is able to estimate the potential accumulated value from the production (process innovations) of each product in the potential products' space, because it may estimate it at zero technology levels not only for itself but for the other firm also, since the time of emergence is the same. Then the value function for the variety expansion does not depend on technology levels or process innovations themselves, but only from the potential value generated by the refinement of the production technology for each product as a whole. In the effect this value function although different for both firms (as their process innovations' value functions are different) is invariant to the future process innovations associated with every new product i.

Proposition 3 *Value function of the product innovations game for both players, $V^{[j],[l]}(n)$ is invariant to the investments being made into the process innovations of all the products except the boundary one, $g^{[j],[l]}_{i=n(t)}$. Moreover, it depends only on the total value generated by this product at the time of its emergence, $V^{[j],[l]}_i(0)\mid_{i=n(t)}$.*

In an effect one may observe that there is an influence of process innovations on the intensity of product innovations but the inverse effect is almost absent. This form of the interdependence of product and process innovations is in line with the empirical literature on the subject, (Faria and Lima 2009) (Kraft 1990).

4.2 Process Innovations

Consider first the problem of process innovations management for each product i for both firms $[j], [l]$.

$$\begin{aligned} V^{[j]}(q_i^{[j]}(t)) &= \int_0^\infty e^{-rs}\Big(q_i^{[j]}(s) - \frac{1}{2}g_i^{[j]}(s)\Big)ds \to \max_{g^{[j]}}; \\ V^{[l]}(q_i^{[l]}(t)) &= \int_0^\infty e^{-rs}\Big(q_i^{[l]}(s) - \frac{1}{2}g_i^{[l]}(s)\Big)ds \to \max_{g^{[l]}}. \end{aligned} \tag{7}$$

with the respect to dynamic constraints (3) for every product i.

These two problems constitute the differential game with two states, $\{q_i^{[j]}(t), q_i^{[l]}(t)\}$ and two controls which are strategies of the firms, $\{g_i^{[j]}(t), g_i^{[l]}(t)\}$ for every i. Note that this formulation is of the same form across all products' production technologies and they are independent of each other. Hence solution of this game is valid for any i. The associated pair of HJB equations is dependent on both states for each firm as well as on investments of both firms.

Depending on the realization of the $\max\{0, \big(q_i^{[j,l]}(t) - q_i^{[l,j]}(t)\big)\}$ functions in (3) one has 3 different formulations of HJB equations which correspond to the leadership of one or the other firm and symmetric dynamics with no leadership.

Due to the specifics of dynamic constraints on process innovations, (3), value functions of both firms are not differentiable along the line $q_i^{[j]}(t) = q_i^{[l]}(t)$. This creates difficulties in the formulation of optimal strategies for the symmetric case. Hence in this paper we consider only the situation with constant leadership $q_i^{[j]}(t) > q_i^{[l]}(t)$ for simplicity. In this case technology evolution path of the leading firm is always higher than that of the following one. It is also the case with only one steady state for the dynamical system (3). The exact conditions on parameters for such an outcome of the process innovations game are derived in (Bondarev 2011). The symmetric case strategies are to be considered as the immediate extension of the results of the current paper. In the current paper one may assume higher investment efficiency of the leading firm and equal decay rates. This is not the only alternative when the constant leadership and single steady state of the game takes place, but it is the simplest one. So, from now on assume

$$\begin{aligned} &\gamma_{[j]} > \gamma_{[l]}; \\ &\beta_{[j]} = \beta_{[l]}. \end{aligned} \tag{8}$$

Such conditions guarantees the existence of only one steady state with firm j being the leader in process innovations.

As long as one of the firms has the leadership in the process innovations, that is, $q_i^{[j]}(t) > q_i^{[l]}(t)$, its dynamics does not depend on the imitation effect, while the other's does. Then subsequent pair of HJB equations may be written as:

$$
\begin{aligned}
&rV_i^{[j]} = \\
&\max_{g^{[j]}(\bullet)}\Bigg\{ q_i^{[j]}(t) - \frac{1}{2} g_i^{[j]}(t)^2 + \frac{\partial V_i^{[j]}}{\partial q_i^{[j]}(t)} \left(\gamma_{[j]}\sqrt{(N-i)}g_i^{[j]}(t) - \beta_{[j]} q_i^{[j]}(t) \right) \\
&+ \frac{\partial V_i^{[j]}}{\partial q_i^{[l]}(t)} \left(\gamma_{[l]}\sqrt{(N-i)}g_i^{[l]}(t) - \beta_{[l]} q_i^{[l]}(t) - \theta \times \left(q_i^{[j]}(t) - q_i^{[l]}(t) \right) \right) \Bigg\};
\end{aligned}
$$

$$
\begin{aligned}
&rV_i^{[l]} = \\
&\max_{g^{[l]}(\bullet)}\Bigg\{ q_i^{[l]}(t) - \frac{1}{2} g_i^{[l]}(t)^2 + \frac{\partial V_i^{[l]}}{\partial q_i^{[j]}(t)} \left(\gamma_{[j]}\sqrt{(N-i)}g_i^{[j]}(t) - \beta_{[j]} q_i^{[j]}(t) \right) \\
&+ \frac{\partial V_i^{[l]}}{\partial q_i^{[l]}(t)} \left(\gamma_{[l]}\sqrt{(N-i)}g_i^{[l]}(t) - \beta_{[l]} q_i^{[l]}(t) + \theta \times \left(q_i^{[j]}(t) - q_i^{[l]}(t) \right) \right) \Bigg\}.
\end{aligned}
\tag{9}
$$

Observe that in this formulation only the second firm l, which is called the 'follower' is benefiting from the technological spillover resulting from superior production technology of the other firm j. This can be seen from the form of the dynamic constraint on the dynamics of technologies which is now different between firms and includes the spillover effect only for the follower.

The first-order conditions for optimal investments depend only on the own firm's value function but not on that of the other's:

$$
\begin{aligned}
g_i^{[j]}(t)^* &= \gamma_{[j]}\sqrt{(N-i)} \times \frac{\partial V_i^{[j]}}{\partial q_i^{[j]}(t)}; \\
g_i^{[l]}(t)^* &= \gamma_{[l]}\sqrt{(N-i)} \times \frac{\partial V_i^{[l]}}{\partial q_i^{[l]}(t)}.
\end{aligned}
\tag{10}
$$

Hence the form of the optimal control is defined by the form of the underlying value function for both firms separately.

In this paper we limit ourselves to open-loop type equilibria which corresponds to the linear value functions of both firms. It may be shown that no other value functions of the polynomial form may fit the problem. Hence the set of strategies derived further on is the only one optimal in the class of at most linear feedback controllers with constant leadership of one of the firms. Formally, assume the following form of value functions for both firms:

$$V_i^{[j],[l]} = A^{[j],[l]} \times q_i^{[j]} + B^{[j],[l]} \times q_i^{[l]} + C^{[j],[l]}. \tag{11}$$

The following result follows from first order conditions immediately:

Proposition 4 *The optimal investments rule for each of the firms does not depend on the level of technology of the other firm nor on its value generation process.*

Consider first the HJB equation for the leading firm. This firm does not benefit from the imitation effect but its problem is influenced by the imitation effect present for the firm l. The position of the leader is characterized by the condition:

$$\forall t : q_i^{[j]}(t) > q_i^{[l]}(t).$$

Of course, as long as one of the firms is the leader in process innovations, the other is the follower.

Inserting (11) into the pair of HJB equations (9) one obtains the following set of coefficients for the leaders' value function:

$$\begin{cases} A^{[j]} = \frac{1}{\beta_{[j]}+r}; \\ B^{[j]} = 0; \\ C^{[j]} = \frac{1}{2}\frac{\gamma_{[j]}^2}{(\beta_{[j]}+r)^2 r}(N-i). \end{cases} \tag{12}$$

Hence coefficients for the leader's value function do not depend on the optimal investments of the follower. This set of coefficients corresponds to the linear value function of the leader with the absence of cross-effects and hence the optimal strategy is constant as long as $\forall t : q_i^{[j]}(t) > q_i^{[l]}(t)$. Together with first-order conditions on controls the derived value function of the leader constitutes optimal (constant) control for the leader:

$$g_i^{[j]} = \frac{\gamma_{[j]}\sqrt{(N-i)}}{r+\beta_{[j]}} = const. \tag{13}$$

Proposition 5 *The optimal investments rule for the leader is constant for each product i and does not depend on the imitation speed θ nor on the achieved technology level $q_i^{[j]}(t)$. However, process innovations are different across products and this difference is proportional to the position of the porduct in the product space, $(N-i)$.*

Now consider the problem of the follower. Inserting (11) into the second equation in (9) results in a system of equations for coefficients of the followers' value function:

$$\begin{cases} A^{[l]} = \frac{1}{\theta+\beta_{[l]}+r}; \\ B^{[l]} = \frac{\theta}{(\beta_{[j]}+r)(\beta_{[l]}+\theta+r)}; \\ C^{[l]} = \frac{1}{2}\frac{(r^2\gamma_{[l]}^2+\beta_{[j]}^2\gamma_{[l]}^2+2\gamma_{[j]}^2\theta^2+2\theta\beta_{[l]}\gamma_{[j]}^2+(2\theta\gamma_{[j]}^2+2\beta_{[j]}\gamma_{[l]}^2)r)}{r(\beta_{[j]}+r)^2(r+\beta_{[l]}+\theta)^2}(N-i). \end{cases} \tag{14}$$

Here value function of the follower depends on the level of technology achieved by process innovations of the leader. Since this last is known already, one has the explicit formulation of the value function for the follower. One may derive the optimal investments of the follower according to first order conditions.

$$\begin{aligned} g_i^{[l]} &= \frac{\gamma_{[l]}\sqrt{N-i}}{r+\theta+\beta_{[l]}} = const; \\ &\forall t : q_i^{[l]}(t) < q_i^{[j]}(t). \end{aligned} \tag{15}$$

The constant strategy does not depend on the follower's or the leader's technology levels except for the fact that this strategy is effective only for follower's technology being inferior to that of the leader. This investment rule defines constant rate of investments but lesser then that for the leader (also constant). It differs from the latter by the term θ in the denominator. Provided θ is the imitation speed and is defined from zero to one, this decreases the overall investment rate for the follower. These observations are summarized below.

Proposition 6 *The optimal investments rule for the follower is also constant for each product i and does not depend on the achieved technology level* $q_i^{[l]}(t)$*. However it depends on the imitation speed* θ *and is decreasing in it. It also differs for every product and decreases with the position of the product in the products space* $\boldsymbol{I}$*. Moreover,* $0 \leq g_i^{[l]} \leq g_i^{[j]} \forall i \in \boldsymbol{I}$ *with strict inequalities on both sides for all* $i < N$

Provided formulation of optimal controls, the dynamic system for process innovations in the constant leader-follower regime being considered here is:

$$\begin{cases} \dot{q_i^{[j]}}(t) = \frac{\gamma_{[j]}^2}{\beta_{[j]}+r}(N-i) - \beta_{[j]}q_i^{[j]}(t); \\ \dot{q_i^{[l]}}(t) = \frac{\gamma_{[l]}^2}{\beta_{[l]}+\theta+r}(N-i) + \theta q_i^{[j]}(t) - (\beta_{[l]}+\theta)q_i^{[l]}(t). \end{cases} \tag{16}$$

From this system one may observe that the improvements of production technology of the follower is faster for higher technology level of the leader while it's investments' rate is lesser then for the leader, provided (8). At the same time the already reached level of own technology decreases the process innovations rate for the follower to a higher extent, since θ is positive. Hence one may conclude:

Proposition 7 *The level of production technology being reached by the process innovations of the follower is always smaller then that of the leader for any product i for all times,* $\forall t > 0, i < N : q_i^{[j]}(t) > q_i^{[l]}(t)$.

This dynamical system has the following solution:

$$q_i^{[j]}(t) =$$

$$\frac{\gamma_{[j]}^2 (N-i)}{\beta_{[j]}(r+\beta_{[j]})} \times (1 - e^{-\beta_{[j]}t});$$

$$q_i^{[l]}(t) =$$

$$\left(1 + (N-i) \times (\frac{E_1(e^{(\beta_{[l]}+\theta)t}+1)}{\beta_{[l]}+\theta} - \frac{E_2(e^{(\beta_{[l]}-\beta_{[j]}+\theta)t}+1)}{\beta_{[l]}-\beta_{[j]}+\theta}\right) \times e^{-(\beta_{[l]}+\theta)t}. \tag{17}$$

where $E_1, E_2 = f(\gamma_{[j],[l]}, \beta_{[j],[l]}, \theta)$ are some functions of parameters only. The subsequent values generated by the process innovations with constant leadership:

$$V_i^{[j]} = \frac{q_i^{[j]}}{r+\beta_{[j]}} + \frac{1}{2} \frac{\gamma_{[j]}^2}{r(r+\beta_{[j]})^2}(N-i);$$

$$V_i^{[l]} = 2\left(\frac{q_i^{[l]}}{\theta+r+\beta_{[l]}} + \frac{\theta}{r+\beta_{[j]}} \times \frac{q_i^{[j]}}{\theta+r+\beta_{[l]}}\right) +$$

$$+ \left(\frac{\theta\gamma_{[j]}^2}{r(r+\beta_{[j]})(r+\theta+\beta_{[l]})} \frac{1}{r+\beta_{[j]}} + \frac{1}{2} \frac{\gamma_{[l]}^2}{r(r+\theta+\beta_{[l]})}\right)(N-i). \tag{18}$$

From this it might be seen that it is not profitable for the firm which is the leader in process innovations to choose the investments rate lower than optimal. If one would do so, it would like to benefit from the spillover effect as the follower does. To this end the leader should choose the investments rule in the same way as the following firm. Then its value function will be of the same type as of the follower. To demonstrate, that this cannot be optimal for the leading firm, it is sufficient to compare its value functions for the case of being the leader and for the case of imitating the other firm. Since we consider open-loop strategies case here, the investment path is chosen at the time of emergence of product i when technologies for both players are at zero level. Hence the value functions for the firm j at this point are given by:

$$V_i^{[L]} = \frac{1}{2}\frac{\gamma_{[j]}^2}{r(r+\beta_{[j]})^2}(N-i);$$

$$V_i^{[F]} = \left(\frac{\theta\gamma_{[l]}^2}{r(r+\beta_{[l]})(r+\theta+\beta_{[j]})}\frac{1}{r+\beta_{[l]}} + \frac{1}{2}\frac{\gamma_{[j]}^2}{r(r+\theta+\beta_{[j]})}\right)(N-i). \tag{19}$$

for being the leader and being the follower respectively. Direct comparison of these values while (8) holds shows that the first value is always higher than the second one.

Proposition 8 *The value of process innovations game is always higher for the firm which leads in investment efficiencies when it invests as a leader in this game, $V_i^{[L]} > V_i^{[F]}$ while the opposite holds for the firm which has lower efficiency γ.*

For illustration of the difference in investment policies caused by leader-follower patterns we take the following set of parameters which correspond to leadership of firm j in process innovations. Efficiency of investments into variety expansion is assumed to be equal for both firms:

$$SETJL := [\gamma_{[j]} = 0.7, \gamma_{[l]} = 0.4, \beta_{[j]} = 0.2, \beta_{[l]} = 0.2]; \tag{20}$$

with

$$[n_0 = 1, \alpha_{[j]} = \alpha_{[l]} = 0.5, r = 0.01, \theta = 0.15, N = 1000]$$

for both variants.
First consider the form of process innovations paths for both firms on Figure

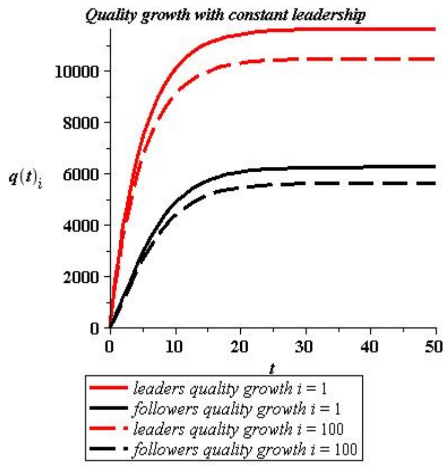

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Figure 1: Difference in technologies for different products and between firms

1.

It might be seen that leader's 'quality' is always higher then that of the following firm while both firms' technology levels are lower for every next invented product then for the preceding one. This last comes from the assumption of decreasing returns on investments into every next product technology, which is reflected in the $\sqrt{N-i}$ term entering the evolution equations. It can also be seen that process innovations for each product eventually reach the steady-state level and do not increase further on. This steady state levels are different for the leader and the follower and also differ across products.

Proposition 9 *For each product i and each firm there is a unique steady-state level of production technology, $q_i^{[\bar{j},l]}$. For each product i this level is higher for the leading firm, $q_i^{[\bar{j}]} > q_i^{[l]}$. It is lower for every next product $i+\epsilon$ for both firms than for all the preceding ones,$q_{i+\epsilon}^{[\bar{j},l]} > q_i^{[\bar{j},l]}$.*

Inserting solutions (17) into value functions for both firms one obtains values generated by the process innovations management game as functions of exogenous parameters only. We need values generated by this game at the zero level of technology and at initial time to proceed to the variety expansion part. These are:

(21)

$$V_i^{[j]}(0,0)\mid_{(i=n(t))}=\frac{\gamma_{[j]}}{(r+\beta_{[j]})^2 r}(N-n(t));$$

$$V_i^{[l]}(0,0)\mid_{(i=n(t))}=$$

$$=\frac{1}{2}\frac{r^2\gamma_{[l]}^2+r(2\gamma_{[l]}^2\beta_{[j]}+2\gamma_{[j]}^2\theta)+\gamma_{[l]}^2\beta_{[j]}^2+2\theta\gamma_{[j]}^2\beta_{[l]}+2\theta^2\gamma_{[j]}^2}{r(r+\beta_{[j]})^2(r+\theta+\beta_{[l]})^2}(N-n(t)).$$

These values are used for the solution of the variety expansion problem in the way discussed previously.

4.3 Variety Expansion Problem

Variety expansion problem is the differential game with one state and two controls. Both firms invest simultaneously in the variety expansion and benefit from the resulting variety on common base thus sharing all the information on this level of innovations. The dynamic problem for both firms is to maximize the potential output of innovations given the costs of investments. Note that the potential profit in this part of the model consists only from the future accumulated profit from development of production technologies (process innovations) of newly invented products. Since we limit ourselves to the case of open-loop strategies in the process innovations game the variety expansion problem is also solved in this class of strategies. The other strategies, of piecewise-constant and of closed-loop type may also be considered as an immediate extension.

For the characterization of the open-loop solution for the variety expansion game the maximum principle method is convenient. One may rewrite the problem of variety expansion as following:

$$J^{[j]} = \int_0^{\infty} e^{-rt}\Big((\alpha_{[j]}u^{[j]}(t) + \alpha_{[l]}u^{[l]}(t))V_{[j]}(0,0)\mid_{(i=n(t))} -\frac{1}{2}u^{[j]}(t)^2\Big)dt \to \max_{u_{[j]}(\bullet)};$$

$$J^{[l]} = \int_0^{\infty} e^{-rt}\Big((\alpha_{[j]}u^{[j]}(t) + \alpha_{[l]}u^{[l]}(t))V_{[l]}(0,0)\mid_{(i=n(t))} -\frac{1}{2}u^{[l]}(t)^2\Big)dt \to \max_{u^{[l]}(\bullet)};$$

$s.t.$

$$\dot{n(t)} = \alpha_{[j]}u^{[j]}(t) + \alpha_{[l]}u^{[l]}(t);$$

(22)
$$u^{[j]}(t), u^{[l]}(t) \geq 0, \forall t \geq 0.$$

where $V_{[j]}(0,0)\ \mid_{(i=n(t))}, V_{[l]}(0,0)\ \mid_{(i=n(t))}$ are given by (22) and depend on $n(t)$ linearly. Denote the value functions from the process innovations game associated with the technology of the boundary product $i = n(t)$ by

$$\begin{aligned} V_{[j]}(0,0)\mid_{(i=n(t))} &= C_v^{[j]} \times (N - n(t)); \\ V_{[l]}(0,0)\mid_{(i=n(t))} &= C_v^{[l]} \times (N - n(t)). \end{aligned} \tag{23}$$

where N is the maximal range of products variety.

The constant part may vary depending on the leadership in the process innovations game, but the variety expansion is analysed parametrically and then the dynamics corresponding to different regimes of the process innovations game are compared. This may be done since these constant parts of value functions above do not depend on the state variable and controls nor time. This constitutes the one-state differential game with common state constraint which may be solved using standard techniques. First we construct Hamiltonians of the given problem and derive first-order conditions on controls. Substituting these into Hamiltonian functions and writing down co-state equations yield the canonical system for the variety expansion game:

$$\dot{\lambda_{[j]}} = r\lambda_{[j]} - \frac{\partial \mathcal{H}^{[j]}}{\partial n(t)} =$$

$$= (r + \alpha_{[j]}^2 C_v^{[j]})\lambda_{[j]}(t) + \alpha_{[l]}^2 C_v^{[j]}\lambda_{[l]}(t) + (\alpha_{[j]}^2 (C_v^{[j]})^2 + \alpha_{[l]}^2 C_v^{[j]} C_v^{[l]})(N - n(t));$$

$$\dot{\lambda_{[l]}} = r\lambda_{[l]} - \frac{\partial \mathcal{H}^{[l]}}{\partial n(t)} =$$

$$= (r + \alpha_{[l]}^2 C_v^{[l]})\lambda_{[l]}(t) + \alpha_{[j]}^2 C_v^{[l]}\lambda_{[j]}(t) + (\alpha_{[l]}^2 (C_v^{[l]})^2 + \alpha_{[j]}^2 C_v^{[j]} C_v^{[l]})(N - n(t));$$

$$\dot{n(t)} = \alpha_{[j]}^2 \lambda_{[j]}(t) + \alpha_{[l]}^2 \lambda_{[l]}(t) + (\alpha_{[j]}^2 (C_v^{[j]})^2 + \alpha_{[l]}^2 (C_v^{[l]})^2)(N - n(t));$$

$$n(0) = n_0;$$

$$lim_{t\to\infty} e^{-rt}\lambda_{[j]}(t) = 0;$$

(24)

$$lim_{t\to\infty} e^{-rt}\lambda_{[l]}(t) = 0.$$

The first-order conditions on the investments into the variety expansion for both firms define investments as functions of co-state variables:

$$\frac{\partial \mathcal{H}^{[j]}}{\partial u^{[j]}} = 0 : u^{[j]}(t)^* = \alpha_{[j]}\lambda_{[j]}(t) + C_v^{[j]}(N - n(t));$$

$$\frac{\partial \mathcal{H}^{[l]}}{\partial u^{[l]}} = 0 : u^{[l]}(t)^* = \alpha_{[l]}\lambda_{[l]}(t) + C_v^{[l]}(N - n(t)). \tag{25}$$

Inserting this into the dynamic constraint for variety expansion together with (24) constitutes the system of linear ODEs with one initial condition and two boundary conditions (transversal ones) which is then solved. The solution is:

$$n(t)^* = N - (N - n_0)e^{\frac{1}{2}(r - \sqrt{r(r+4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})})t};$$

$$\lambda_{[j]}(t)^* = -\frac{C_v^{[j]}(\alpha_{[j]}^2 C_v^{[j]} + \alpha_{[l]}^2 C_v^{[j]})}{2\alpha_{[j]}^2 C_v^{[j]} + r + \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})}} \times$$
$$\times\, 2(N - n_0)e^{\frac{1}{2}(r - \sqrt{r(r+4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})})t};$$

$$\lambda_{[l]}(t)^* = -\frac{C_v^{[l]}(\alpha_{[j]}^2 C_v^{[j]} + \alpha_{[l]}^2 C_v^{[j]})}{2\alpha_{[j]}^2 C_v^{[j]} + r + \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})}} \times$$
$$\times\, 2(N - n_0)e^{\frac{1}{2}(r - \sqrt{r(r+4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})})t}. \tag{26}$$

The product innovations increase the available for both firms variety of products $n(t)$. Speed of this increase is slowing down to zero while the process approaches the maximal available variety, N.

Proposition 10 *With common product innovations process the available for both firms products variety $n(t)$ is increasing over time and does not reach the maximal available level in finite time.*

Due to the nature of the problem analysed here the co-states' dynamics is negative. This happens because of the form the variety expansion problem is reformulated in this section. Every firm cares only about future investments into variety expansion. From this point of view shadow price of investments is negative since every marginal addition to investments reduces future possibilities to invest. This happens because one has bounded space of products in the model and inventions reduce the dimensionality of this space. Firms take into account the profit generated only by the next potential product but neglect all the products which are already invented before. Hence the shadow price of investing into the expansion of products variety is negative. Still, investments are positive for both firms as well as the growth of variety.

Explicit formulation of investments into variety expansion for both firms is:

$$u^{[j]}(t)^* = \frac{\alpha_{[j]} C_v^{[j]} (r + \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})})}{2\alpha_{[j]}^2 C_v^{[j]} + 2\alpha_{[l]}^2 C_v^{[l]} + r + \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})}} \times$$
$$\times (N - n_0) e^{\frac{1}{2}(r - \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})})t};$$

$$u^{[l]}(t)^* = \frac{\alpha_{[l]} C_v^{[l]} (r + \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})})}{2\alpha_{[j]}^2 C_v^{[j]} + 2\alpha_{[l]}^2 C_v^{[l]} + r + \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})}} \times$$
$$\times (N - n_0) e^{\frac{1}{2}(r - \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})})t}. \tag{27}$$

These are completely symmetric except for the term $\alpha_{[j,l]} C_v^{[j,l]}$ which depends on investment efficiencies and value generated by the process innovations into the production of the next product to be invented for both firms. Hence relative scale of investments into the variety expansion depends on the outcome of the process innovations game.

Proposition 11 *Investments of both firms into the product innovations are positive and depend on the value generated by subsequent process innovations into the next product*

One also may compute the value function of the variety expansion game as the optimized Hamiltonian function. The value of the variety expansion game for each of the firms is the respective Hamiltonian function at time $t = 0$ and optimal co-state and variety values:

$$V^{[j]}(n) = \frac{1}{r} \mathcal{H}^{[j]}(n(0)^*, \lambda_{[j]}(0)^*) =$$

$$= C_v^{[j]} \times \frac{(N - n_0)^2 (2\alpha_{[l]}^2 C_v^{[l]} + \alpha_{[j]}^2 C_v^{[j]})}{2\alpha_{[l]}^2 C_v^{[l]} + 2\alpha_{[j]}^2 C_v^{[j]} + r + \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})}};$$

$$V^{[l]}(n) = \frac{1}{r} \mathcal{H}^{[l]}(n(0)^*, \lambda_{[l]}(0)^*) =$$

$$= C_v^{[l]} \times \frac{(N - n_0)^2 (2\alpha_{[j]}^2 C_v^{[j]} + \alpha_{[l]}^2 C_v^{[l]})}{2\alpha_{[l]}^2 C_v^{[l]} + 2\alpha_{[j]}^2 C_v^{[l]} + r + \sqrt{r(r + 4\alpha_{[j]}^2 C_v^{[j]} + 4\alpha_{[l]}^2 C_v^{[l]})}}. \tag{28}$$

It can be seen that value function of the variety expansion game is just the combination of efficiencies of investments into the variety expansion, $\alpha_{[j],[l]}$ and value functions of the process innovations game of both firms. Observe that these are independent of the index of the product, i, as they include estimation of value generation of all products to be invented at the point $i = n_0$. Hence these two functions give the total value of the combined game of process and product innovations. However, they may take different values depending on the leadership regime in process innovations game. Due to the special and symmetric form of investment efficiencies $\gamma(\bullet)$ the same regime is preserved for all product indices i and values of $C_v^{[j]}, C_v^{[l]}$ are independent of i. As long as shadow costs of investments in process innovations game are independent of i, optimal investments for both firms depend on i in the same way. Then value functions and process innovations dynamics will depend on i also in the same way for all regimes and hence conditions for realisation of one or another regime are independent on i also.

This of course is not necessarily the case with more general (e.g. defined differently for different i) specification of efficiency functions $\gamma_{[j],[l]}(i)$. No claims concerning general properties of these functions are made here. One would stop on the conclusion that with the adopted specification of $\gamma(\bullet)$ functions the regime of leadership in process innovations game is constant across products and hence the value function for the variety expansion part may be defined independently on i or $n(t)$.

Consider now the shape of the product innovations dynamics. At the

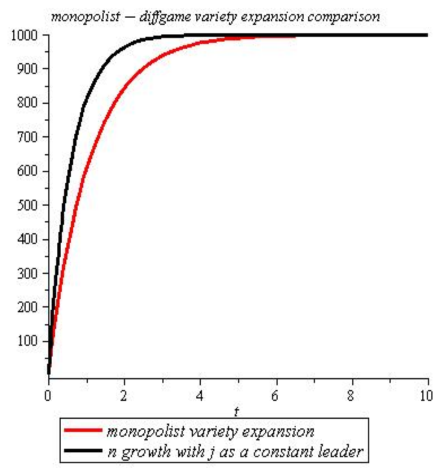

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Figure 2: Product innovations for cooperative investments and a single firm

Figure 2 one may find the comparison of product innovations dynamics of the differential game with constant leadership in process innovations of one of the firms with that of the single firm in the market. The model with one firm is previously considered in (Bondarev 2010b). Here it suffices to note that the problem of one firm is the same one as that of the leading firm in process innovations part (with the same investment rule) and variety expansion part is obtained by assuming the single firm investing into the product innovations instead of two. In this figure the same set $SETJL$ of parameters is adopted for illustration purposes, while the single monopolist's efficiency of process innovations investments is set in between of the two firms at the level $\gamma_M = \frac{\gamma_{[j]}+\gamma_{[l]}}{2}$ and all other parameters kept similar.

One may see that under the cooperative investments the product innovations speed is higher than for the single monopolistic firm. This happens due to non-zero investments of both firms into this kind of innovations in cooperative case. It may be shown that the cooperative investments are possible in this model due to the specialization of innovative activities of both firms with natural selection of these activities. This main feature of the suggested model is discussed in the last section of the paper. One thing which is important to note at this stage is summarized below.

Proposition 12 *In the case of the constant leader in process innovations cooperative product innovations of two firms are higher than those of the single monopolistic firm,* $u^{[j]}(t) + u^{[l]}(t) > u^{[M]}(t)$*. This effect is observed as long as* $\gamma_{[j]} > \gamma_{[M]} > \gamma_{[l]}$ *or vice versa.*

5 Specialization of Innovative Activities

Now consider the overall strategic profile and value generated for both firms. It turns out, that one of the firms invests more into the development of process innovations while the other one invests more into the creation of new products. Thus the specialization of innovative activities is observed. It may be shown, that this effect is robust to parameters value changes as well as the leadership regime.

5.1 Process Innovations Investments

To observe this specialization effect, consider first the set of optimal strategies for the process innovations game for both firms in the case of constant leader.

$$g_{L,CON}^{[j]} = \frac{\gamma_{[j]}\sqrt{(N-i)}}{r+\beta_{[j]}};$$

$$g_{F,CON}^{[l]} = \frac{\gamma_{[l]}\sqrt{N-i}}{r+\theta+\beta_{[l]}}. \tag{29}$$

It is straightforward that the leaders' investments are higher than those of the follower for each i, if (8) hold. At the same time leader's value of the process innovations game is always lower than that of the follower since its investments do not depend on the achieved technology level and are higher than those of the follower. At zero technology level leader's value is lower than that of the follower also. This constitutes the specialization of innovative activities of both firms in the area of process innovations.

Figure 3 shows differences in investments of the leading and the following firm in the area of process innovations for two different products from the products' space. The same parameter set, $SETJL$, is used for this illustration.

This figure illustrates some additional effect also. Namely, the investments

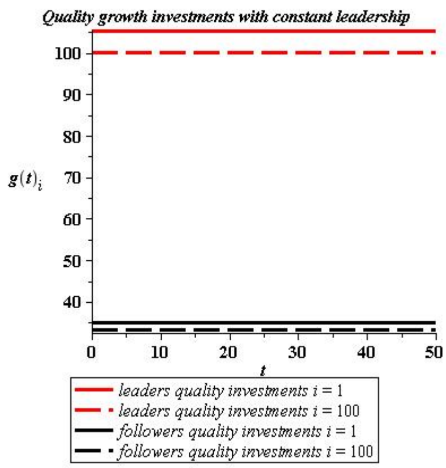

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Figure 3: Specialization: the most efficient investor invests more into process innovations despite of the spillover effect.

of both firms decrease with the increase of the position of the product, i, but they are decreasing more rapidly for the leading firm than for the following one. This happens due to the presence of the imitation speed term in the formulation of the optimal investments of the follower which slows down the decrease of investments over products. It is more profitable for the follower to develop the production technology of every next product than for the leader since this first one benefits from spillover effect and thus has less incentives to reduce investments. At the same time it has lower efficiency of investments by assumption (8) and hence its investments for each product are lower. These observations are summarized below.

Proposition 13 *The investments of the leader in process innovations are higher than those of the follower,* $g^{[j]}_{L,CON} > g^{[l]}_{F,CON}$ *for every product* $i \in N$*. At the same time, across products investments of the follower decrease at a slower pace than those of the leader,* $\frac{\partial g^{[j]}_{L,CON}}{\partial i} < \frac{\partial g^{[l]}_{L,CON}}{\partial i} < 0$.

5.2 Product Innovations Investments

Now consider differences in product innovations. As it has been noted, the value of the process innovations game estimated at zero technology level is higher for the follower, as it may be seen from (22). Hence,the investments into the product innovations of the follower are higher than those of the leader, as (27) are completely symmetric except for the value functions of the process innovations game. It follows, that the higher is the difference in values generated for both firms by the process innovations game, the higher is the difference in variety expansion investments.

The difference in product innovations strategies of the firms is illustrated on the Figure 4. The same set of parameters as before is used.

With constant leadership in process innovations variety expansion investments are rather large for both firms. The rate of investments is not constant and depends negatively on the already achieved level of variety $n(t)$. It decreases rapidly until zero. The firm which is the leader in process innovations invests less then the follower all the time. The decrease in intensity of product innovations over time is explained through the increasing complexity of the development of process innovations for every next product within the products' range. Since the only source of new value is the value generated by the development of the production technologies for new products through process innovations, it becomes less attractive to introduce new products in comparison to the development of already existing ones as the process of variety expansion approaches its limit N.

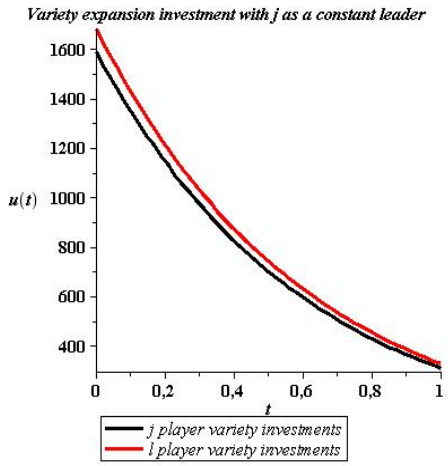

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Figure 4: Specialization: the following firm invests more into product innovations because of the spillover effect.

Proposition 14 *Product innovations investments are higher for the following firm all the time, $u^{[j]}(t) < u^{[l]}(t)$, and they slow down for both firms while the variety expansion process approaches its limit, $\frac{\partial u^{[j]}_{L,CON}}{\partial n(t)} < 0, \frac{\partial u^{[l]}_{L,CON}}{\partial n(t)} < 0$. In particular, they slow down over time, $\frac{\partial u^{[j]}_{L,CON}}{\partial t} < 0, \frac{\partial u^{[l]}_{L,CON}}{\partial t} < 0$.*

There is substantial difference between firms' investments into production technologies of new products and the spillover effect is strong enough to boost variety expansion investments of the firm which is the follower in process innovations. At the same time it has to be noted that due to the open-loop nature of the investment strategies analysed here both firms invest non-zero amounts into variety expansion irrespective of their positions in the process innovations game. It may be shown that this is not the case in closed-loop situation, where total variety expansion investments are made by the follower only while the leader is investing strictly zero amount.

The endogenous specialization of innovative activities between firms in the model follows the natural selection criteria: the firm which is more efficient in investing into one or the other type of innovations is specializing in this kind of innovations. Yet this specialization is not the full one, as both firms invest non-zero amounts in both directions of their activities. The overall process of generation of innovations may be described by the 3-dimensional reconstruction at Figure 5.

Here one may observe the underlying process of generation of products variety, $n(t)$, together with associated processes of technology improvements

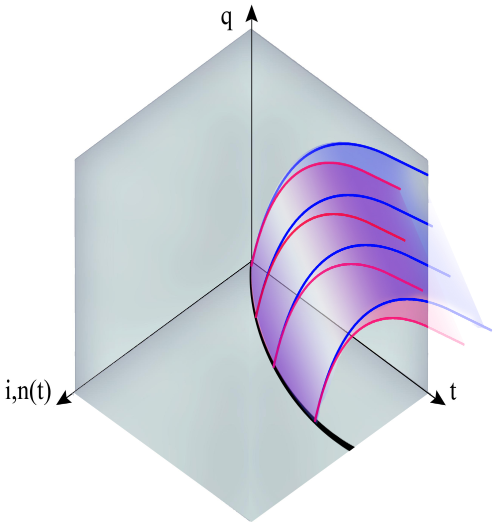

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Figure 5: Common product innovations is the generator of process innovations.

for both firms and for different products. The domination in technology levels is preserved along all the range of products to be invented. The same is true for the specialization of activities: the follower remains the follower in all products process innovations and continues to invest more into the variety expansion all the time.

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