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It is well known that a one-dimensional discrete-time model may yield endogenous fluctuations while this is impossible in a one-dimensional continuous-time model. Invernizzi and Medio (1991) recast this time-modeling issue into an aggregation issue. They have proposed a "random-lags approach" as a way of preserving fluctuations while relaxing the discrete-time assumption. The present paper applies this approach to the model of Aghion, Bacchetta and Banerjee (2000), and shows that their result that economies at an intermediate level of financial development may be prone to economic fluctuations continues to hold when the discrete-time assumption is relaxed.

Keywords: continuous time, discrete time, fluctuations, aggregation.

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1 Introduction

One explanation of economic fluctuations is based on financial frictions. Bernanke and Gertler (1989) have shown that borrowing constraints on firms can amplify and increase the persistence of temporary shocks. Kiyotaki and Moore (1997), Aghion, Banerjee, and Piketty (1999) and Azariadis and Smith (1998) have shown that these constraints can lead to oscillations in the context of a closed economy. Aghion, Bacchetta, Banerjee (2004), ABB from now on, study the case of a small open economy.

The goal of ABB's paper is to explain why economies at an intermediate level of development may be more unstable than either more or less developed economies. They propose a model in which fluctuations are more persistent for intermediate values of the borrowing constraint (which correspond to an intermediate level of financial development)[1]. In order to derive their result, ABB assume time to be discrete. The problem is that there is no reason (other than technical simplicity) to make this assumption. The present paper shows that their result still holds when the discrete-time assumption is

[1] They also show that in economies at an intermediate level of financial development full capital account liberalization may destabilize the economy (while foreign direct investment does not destabilize it). But I will focus here on their first result.

relaxed.

In order to prove ABB's result while relaxing the discrete-time assumption, I use the approach of Invernizzi and Medio (1991), IM from now on. They recast this time-modeling issue into an aggregation issue. IM's insight is that at the macro level the assumption that production takes place in discrete time implies in fact two assumptions: production at the firm level must occur at discrete intervals and production of all firms must be synchronized.[2] If firms are not synchronized, then at any given date some firms are finishing their production; in this case, aggregate production might best be seen as continuous although production is a discrete-time variable at the agent level. IM accept the lag assumption at the micro level, which is often realistic, but reject the synchronization assumption, which is usually unrealistic. In order to build a model that is not synchronized, they assume that lags are heterogeneous and random. Thus, the date of production of different firms cannot

[2] IM's approach is general and applies to any discrete-time model of the form $X_t = f(X_{t-1})$. In specific applications, the terminology "lags" may sometimes seem inappropriate. For example, in the production case, this lag is the exogenously-given time-interval between two production processes, which may include periods that one may not want to call "lags", such as the duration required to produce. But for simplicity I will stick to the lag terminology.

be synchronized, since their lags are different. IM show that their model converges toward the discrete-time model when the dispersion of lags tends toward zero. Then they show that if the dispersion of lags is small enough, the endogenous fluctuations of the discrete-time model are preserved.[3]

The present paper applies this approach to ABB's paper and shows not only that fluctuations are preserved, but also that the point of the ABB model (fluctuations are greater for economies at an intermediate level of financial development) holds while relaxing the discrete-time assumption. The plan of the paper is as follows: after presenting ABB's Model (§2), I apply IM's approach to it (§3) and present concluding remarks (§4).

2 A specific one-dimensional, discrete-time example: ABB's model

The goal of ABB's paper is to explain why economies at an intermediate level of financial development may be more unstable than either more or less

[3]In fact IM do not only show that fluctuations still yield: they are mainly interested in the chaotic properties of these fluctuations.

developed economies. I focus here on the simplest version of ABB's model. It features a small open economy with two types of agents: entrepreneurs and owners of a local input. Entrepreneurs produce a tradable good which is both a consumption and a capital good. The price of this tradable good is taken as given because of the small open economy assumption. The other input in the production of the tradable good is a local input that is not owned by entrepreneurs. Entrepreneurs can borrow at an interest rate $r - 1$, which is exogenous, given the assumption of a small open economy. Entrepreneurs, however, may not be able to borrow as much as they wish because they are subject to a borrowing constraint. This borrowing constraint takes the form of a constant credit-multiplier μ. Entrepreneurs can borrow up to μ times their wealth. The parameter μ captures the level of financial development. When $\mu = 0$ entrepreneurs cannot borrow, whereas when $\mu = \infty$ there is no limit to the amount entrepreneurs can borrow.

At time t, after consumption, entrepreneurs have wealth W_t at their disposal. Because of the borrowing constraint they can borrow up to μW_t. If they choose to borrow the maximum amount possible, they will have $(1 + \mu)W_t$ at their disposal. They buy the quantity z_t of local input at price p_t, and use the difference $K_t = (1 + \mu)W_t - p_t z_t$ as a tradable input. They choose z_t in such a way as to maximize their own production. Production is a function $y(K_t, z_t)$ of the tradable and local inputs. In their basic example, ABB assume that the production function is a Leontief: $y = min(\frac{K_t}{a}, z_t)$. Entrepreneurs receive an exogenous income e and at the end of the period repay the principal with interest $r\mu W_t$ to the lender. Then, entrepreneurs consume a fraction α of their wealth (this behavior can be derived from log utility).

The equilibrium price p_t adjusts to set z_t equal to the supply of local input assumed to be a constant z. If $z > \frac{K}{a}$ (this happens when W_t is so small that current investment cannot absorb the total supply of the non-tradable input), then there is excess supply of the non-tradable input and thus its price is null. If $z = \frac{K}{a}$ then it can be shown that $p_t = \frac{(1+\mu)W_t - az}{z}$. The case $z < \frac{K}{a}$ cannot exist because it cannot be optimal for the entrepreneurs to choose a quantity of the costly tradable input in excess of what is useful given the amount of local input.

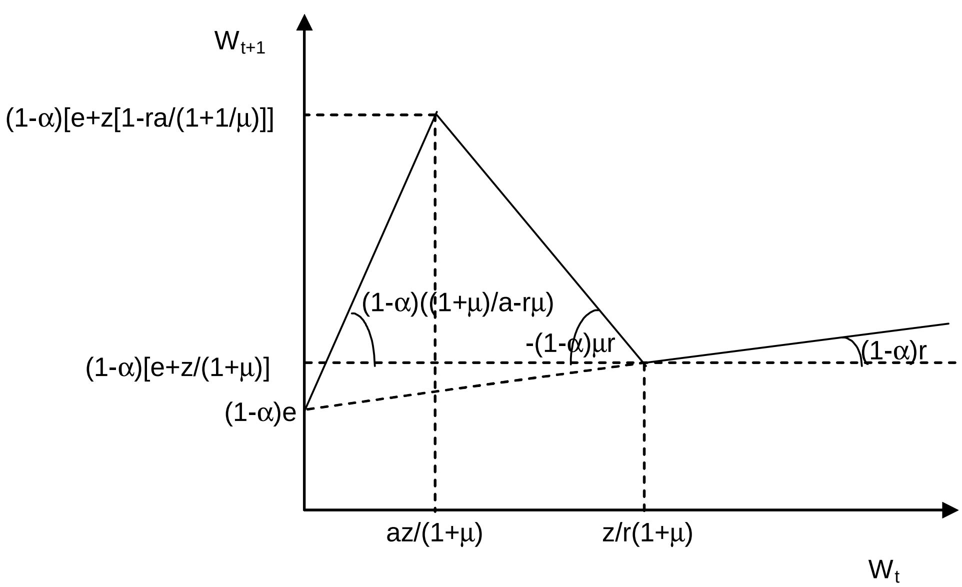
Entrepreneurs can also choose not to borrow the maximum amount possible (they are then not constrained). In this case the return on their investment is $r - 1$, and their wealth in the next period is $W_{t+1} = (1-\alpha)(e + rW_t)$.

The dynamics $W_{t+1} = f(W_t)$ of the entrepreneurs' wealth are therefore given by:

$$W_{t+1} = (1-\alpha)\left\{e + \max\left[\min\left[\left(\frac{1+\mu}{a} - r\mu\right)W_t, z - r\mu W_t\right], rW_t)\right]\right\} .$$

Assuming $e > 0$, $1 > ar$, and $(1-\alpha)r < 1$, these dynamics are represented graphically in Figure 1.

Figure 1: Dynamics of the entrepreneurs' wealth



The steady state is given by the intersection between this curve and the diagonal. There are fluctuations only if the curve has a negative slope at the steady state, i.e. if the intersection is on the second segment (and these fluctuations are permanent only if the slope is a negative number lower than -1). It can be shown that the steady state will be on the first segment if μ is small enough, and on the third segment if it is large enough. Thus, fluctuations occur (the steady state can be on the second segment) only for an intermediate level of financial development (i.e. for intermediate values of μ).

ABB explain the basic mechanism underlying their model as follows. It is a combination of two forces: on one side, greater investment leads to greater output and ceteris paribus, higher profits. Higher profits improve creditworthiness and fuel borrowing that leads to greater investment. Capital flows into the country to finance this boom. At the same time, the boom in investment increases the demand for the country-specific factor and raises its price relative to the output good. This rise in input prices leads to lower profits and therefore, reduced creditworthiness, less borrowing and less investment, and a fall in aggregate output. Of course, once investment falls all these forces get reversed and eventually initiate another boom. The reason why an intermediate level of financial development is important for this result is easy to comprehend: at very high levels of financial development, most firms' investment is not constrained by cash flow so shocks to cash flow are irrelevant. On the other hand, at very low levels of financial development, firms cannot borrow very much in any case and therefore their response to cash-flow shocks will be rather muted.

If there are fluctuations, one of the two forces described above should dominate sometimes and the other one should dominate at other times. But in between there should be a point at which the two forces cancel each other out. This point would be a steady state. In a single-variable, continuous-time model governed by a differential equation of degree 1,[4] the economy would be stuck at this steady state and would not fluctuate after all. But ABB assume that time is discrete. In this case the economy may overshoot the steady state, and then jump back over the steady state and be ready for a new cycle. It is this discrete-time assumption that I will try to relax.

[4]A single-variable, continuous-time model governed by a differential equation of degree n can be regarded as a n-variable, continuous-time model governed by n differential equations of degree 1.

3 Extension of ABB's model with random lags

I first discuss IM's random-lags approach (§3.1) on which my extension of the ABB model is based, then this extension is presented (§3.2).

3.1 IM's random-lags approach

Consider any variable X and assume that its dynamics in discrete time are given by:

$$X_t = f(X_{t-1}) \ . \qquad (1)$$

For example, it may be useful to think of X as representing aggregate production finished at time t.[5] The lag is the time required to produce (a new cycle of production starts right after the preceding is finished). The discrete-time dynamics equation (1) says that aggregate production finished at time t is a function of aggregate production finished at time $t-1$.

Instead of the single representative firm implied in equation (1), one may

[5] When applying this approach to ABB I will choose X=Wealth of the entrepreneurs.

consider an economy consisting of a large number of firms differing only by their production lags. Assume that this lag is random, and the density function ψ gives its distribution. Then equation (1) can be written as:

$$X_t = \int_0^\infty \psi(s) f(X_{t-s}) ds \ . \tag{2}$$

Equation (2) indicates that aggregate production finished at time t is the sum of production processes started in the past. Aggregate production carried out s periods ago, X_{t-s}, generates total production $f(X_{t-s})$. Only a fraction $\psi(s)$ of this production will, however, be finished at time t. Thus the production process beginning at time $t-s$ will contribute $\psi(s)f(X_{t-s})$ to aggregate production at time t. Notice that if $\psi(s) = 0$ for $s \neq 1$ then lags are not random anymore, and equation (2) can be simplified to $X_t = f(X_{t-1})$. The strength of the approach proposed by IM is to keep the discrete-time assumption at the micro level, a realistic assumption, but to dismiss the assumption of perfect synchronization, which is usually unrealistic.

Assuming that $\psi(s)$ is a gamma density

$$\psi(s) = \frac{1}{(n-1)!} n^n s^{n-1} e^{-ns} \ , \tag{3}$$

with expectation 1 and variance $\frac{1}{n}$ (where n$\geq$1; the economic interpretation of this parameter is presented below), IM show that equation (2) is equivalent to the following differential equation:

$$\left(\frac{1}{n}D + 1\right)^n X = f(X) \ , \tag{4}$$

where $D = \frac{d}{dt}$ is the time-derivative operator.

Here the parameter n plays a crucial role. If n is infinite, then the variance of the distribution of lags is zero, and equation (4) describes a discrete-time model.[6] If $n = 1$, then equation (4) describes a single-variable, continuous-time model governed by a differential equation of degree 1. For intermediate values of n, equation (4) describes an intermediate case between discrete time and first-order continuous time.

n can be interpreted as the number of successive and independent elementary operations needed to complete production, the duration of each elementary operation being random and following an exponential distribution. For comparability, only production processes are considered for which the whole production process is expected to last one period. If there are n operations, then each operation is assumed to have an expected duration of

[6] It can be shown that the differential equation (4) tends toward (1) when n tends toward infinity.

$\frac{1}{n}$.[7] When n rises, the expected production lag stays the same (1 by construction), but the dispersion around this expected value decreases. The reason is that when there are many operations, it is very unlikely that operations are always short or always long. Thus by the law of large numbers the time gained on short operations tends to be canceled by the delay of some other, long operations. At the limit as $n \to \infty$ the distribution of lags is degenerate and one obtains the discrete-time model.

$n = 1$ corresponds to the continuous-time model: in this case equation (4) is a differential equation of degree 1. $n = 1$ is the opposite of $n = \infty$ (as the continuous-time model is the opposite of the discrete-time model) because the distribution of production duration for $n = 1$ is the opposite of the distribution of production duration in the discrete-time model in the following sense: the distribution for $n = 1$ has the property that production duration can take any positive value (instead of only one as in the discrete-time model) and that the probability of a firm finishing production in the next infinitesimal interval of time is completely independent of the time that

[7] Then it can be shown that the production duration will follow the gamma distribution given by equation (3).

has elapsed since production last occurred (instead of being completely determined by the time that has elapsed since production last occurred as in the discrete-time model).

Values of n between 1 and ∞ correspond to intermediate cases between continuous time and discrete time. IM show that permanent fluctuations that appear in discrete time still remain in intermediate cases close enough to discrete time. Intuitively, if n is large enough, then the standard deviation of production duration is small enough, and the tendency of production of various firms to get out of synchronization is weak enough, such that permanent fluctuations arising in the discrete-time model are not canceled out. Remember that fluctuations arise in the discrete-time model because all entrepreneurs can borrow large amounts when they start with large wealth, putting upward pressure on the price of the non-tradable input, leaving them with small profits and thus small wealth for the next period. This whole process collapses if production of various firms are sufficiently out of synchronization.

Formally IM show[8] that the condition for having a periodical solution is

[8]They don't explicitly write this equation, but it is a straightforward implication of their paper.

$$0 > f'(\overline{X}) = -\frac{1}{\cos^n(\frac{\pi}{n})} \ , \tag{5}$$

where $\overline{X}$ is the steady state of X defined by: $\overline{X} = f(\overline{X})$.

It is easy to derive equation (5) by taking the following linear approximation of equation (4) around the steady state (using $f(X) \approx f(\overline{X}) + \left(X - \overline{X}\right) f'(\overline{X})$):

$$\left[\left(\frac{1}{n}D + 1\right)^n - f'(\overline{X})\right]\left(X - \overline{X}\right) = 0 \ . \tag{6}$$

The eigenvalues λ are given by the solutions of $\left(\frac{1}{n}\lambda + 1\right)^n = f'(\overline{X})$. Notice that for $f'(\overline{X}) < 0$ the eigenvalues with the higher real component are a complex number (with the imaginary component different from zero) and its complex conjugate. Their real component is $n\left\{\left[\left|f'(\overline{X})\right|\right]^{\frac{1}{n}}\cos(\frac{\pi}{n}) - 1\right\}$.

For $n = 1$, this maximal real component is equal to $-\left|f'(\overline{X})\right| - 1$, which is negative. Thus all real components are negative and the system is stable.

For $n = 2$, this maximal real component is equal to -1 (except if $\left|f'(\overline{X})\right| = \infty$), which is negative. Thus all real components are negative and the system is stable.

For $n > 2$, the real component of all eigenvalues is negative and the system is stable if $f'(\overline{X}) > -\frac{1}{\cos^n(\frac{\pi}{n})}$, whereas there is at least one dimension in which the system is unstable if $f'(\overline{X}) < -\frac{1}{\cos^n(\frac{\pi}{n})}$. If $f'(\overline{X}) = -\frac{1}{\cos^n(\frac{\pi}{n})}$, it can be shown that there is a periodical solution.

If $n \to \infty$, then $-\frac{1}{\cos^n(\frac{\pi}{n})} \to -1$ and, as usual in discrete-time models, there are permanent fluctuations if the slope of f at the steady state is smaller than -1. Notice that $\frac{1}{\cos^n(\frac{\pi}{n})}$ is already close to 1 for fairly small n.

3.2 Robustness of ABB's results

I now show that qualitatively ABB's result is still valid for intermediate cases close enough to a discrete-time model.

Using $X \equiv W$ in equation (4), the dynamics are given by

$$\left(\frac{1}{n}D + 1\right)^n W = f(W) \ ,$$

where

$$f(W) = (1-\alpha)\left\{e + \max\left[\min\left[\left(\frac{1+\mu}{a} - r\mu\right)W, z - r\mu W\right], rW)\right]\right\} \ .$$

How do the properties of the steady state depend on μ? First the steady state $\overline{W}$ must be computed. The steady state satisfies the following equation $\left(\frac{1}{n}D+1\right)^n \overline{W} = f(\overline{W})$, which, since $\overline{W}$ is constant, reduces to $\overline{W} = f(\overline{W})$. Thus, the steady state is the same as in ABB's discrete-time case. Assuming a is big enough, the steady state will be either on the second or the third segment. Let's discuss the stability of the steady state. Linearizing around the steady state yields:

$$\left[\left(\frac{1}{n}D+1\right)^n - f'(\overline{W})\right]\left(W-\overline{W}\right) = 0 \ .$$

The eigenvalues λ are given by the solution of $\left(\frac{1}{n}\lambda+1\right)^n = f'(\overline{W})$. If the steady state is on the third segment, then $0 < f'(\overline{W}) < 1$ and all eigenvalues have negative real components. Thus the steady state is stable and there will be no permanent fluctuations. If the steady state is on the second segment, then $f'(\overline{W})$ is negative, and there will be permanent fluctuations if $f'(\overline{W})$ is sufficiently negative. The difference with respect to the discrete case is that "sufficiently negative" no longer means that $f'(\overline{W}) < -1$, but that $f'(\overline{W}) < -\frac{1}{\cos^n(\frac{\pi}{n})}$. Thus as long as n>2 the difference from the discrete-time model is quantitative (how negative $f'(\overline{W})$ needs to be in order to get permanent fluctuations) rather than qualitative.

The set of values of e for which ABB's result still holds becomes, however, more restrictive. For example, given our parameters' values, fluctuations cannot be permanent for $n = 3$. Simple algebra shows that for $n > 2$, if $\frac{e}{z} < \frac{1-(1-\alpha)r}{\frac{1}{\cos^n(\frac{\pi}{n})}+(1-\alpha)r}$ there will always be permanent fluctuations for some value of μ: there will always be a μ such that the steady state is on the second segment and the negative slope is steep enough for fluctuations to be permanent. With our parametrization, however, there will be no such μ for $n = 3$ since this inequality is not satisfied ($\frac{e}{z} = \frac{1}{100}$ and $\frac{1-(1-\alpha)r}{\frac{1}{\cos^3(\frac{\pi}{3})}+(1-\alpha)r} = 9.1 \times 10^{-3}$). Changing the value of e would change the results. If e were small enough there would be permanent fluctuations for intermediate values of μ also for $n = 3$. On the other hand, for any n we could choose a value e high enough such that there are no permanent fluctuations. Compared to the similar condition prevailing in the discrete-time model $\frac{e}{z} < \frac{1-(1-\alpha)r}{1+(1-\alpha)r}$, the condition $\frac{e}{z} < \frac{1-(1-\alpha)r}{\frac{1}{\cos^n(\frac{\pi}{n})}+(1-\alpha)r}$ becomes more restrictive when n gets smaller (that is, when we move away from the discrete-time case). ABB's result that permanent fluctuations occur for intermediate values of μ is true only for a particular set of values for the parameters (for example, e must be small enough). As n decreases this set shrinks. But as long as $n > 2$, this set is never empty. In this sense the result ABB obtain in discrete time is still qualitatively valid for any $n > 2$, but quantitatively the set shrinks.

The following intuition explains why the lower bound of e values for which endogenous fluctuations cannot occur (whatever the value of μ) is an increasing function of n. Remember that endogenous fluctuations occur because of cash-flow shocks to firms' capacity to borrow. For endogenous fluctuations to occur, two conditions must be satisfied. First, μ has to be large enough for borrowing to be substantial. Second, μ has to be small enough for firms to be financially constrained. When n gets larger, the tendency of firms to get out of synchronization diminishes, and a smaller μ will suffice to generate enough borrowing for endogenous fluctuations to occur. With smaller μ, the second condition will also be easier to satisfy: firms will still be financially constrained even if their exogenous endowment e is a bit larger. Thus, it is easier to get endogenous fluctuations when n is larger: more pairs (μ,e) are compatible with endogenous fluctuations.

4 Conclusion

Applying Invernizzi and Medio's approach, the present paper has shown that Aghion, Bacchetta and Banerjee's explanation of why economies at an intermediate level of financial development may be more unstable than either more or less financially developed economies is fairly robust to the continuous–time versus discrete-time choice. When the discrete-time assumption is dropped in favor of a random-lags assumption that is an intermediate case between discrete and continuous time, the argument stays qualitatively the same except in extreme cases when the variance of the lags is large (larger than half the variance corresponding to the first-order, continuous-time model).

Possible directions for further research would be to apply the random-lags approach to models other than ABB's model, or to examine whether it can be applied to issues purely related to aggregation rather than to time-modeling.

References

Aghion Ph., Ph. Bacchetta, and A. Banerjee (2004), "Financial Development and the Instability of Open Economies," *Journal of Monetary Economics* 51(6), 1077-1106.

Aghion Ph., A. Banerjee, and T. Piketty (1999), "Dualism and Macroeconomic Volatility," *Quaterly Journal of Economics* 114(4), 1359-1397.

Azariadis C., and B. Smith (1998), "Financial Intermediation and Regime Switching in Business Cycles," *American Economic Review* 88, 516-536.

Bernanke B., and M. Gertler (1989), "Agency Costs, Net Worth, and Business Fluctuations," *American Economic Review* 79, 14-31.

Invernizzi S., and A. Medio (1991), "On Lags and Chaos in Dynamic Economic Models," *Journal of Mathematical Economics* 20, 521-550.

Kiyotaki N., and J. Moore (1997), "Credit Cycles," *Journal of Political Economy* 105, 211-248.