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Abstract

The problem of optimizing a real valued function over an efficient set of the Multiple Objective Linear Fractional Programming problem (MOLFP) is an important field of research and has not received as much attention as did the problem of optimizing a linear function over an efficient set of the Multiple Objective Linear Programming problem (MOLP).In this work an algorithm is developed that optimizes an arbitrary linear function over an integer efficient set of problem (MOLFP) without explicitly having to enumerate all the efficient solutions.The proposed method is based on a simple selection technique that improves the linear objective value at each iteration.A numerical illustration is included to explain the proposed method.

Mathematics Subject Classification: 90C10; 90C26; 90C32; 90C29

Keywords: Integer programming, Optimization over the efficient set, Multiple objective linear fractional programming, Global optimization

1 Introduction

The Multiobjective Linear Fractional Programming problem (MOLFP) with continuous variables is an important class of problems arising in multicriteria decision making and has been studied extensively in the literature [8,11,12,18] and the references therein.

However, Integer Linear Fractional Programming problem with Multiple Objective (MOILFP) has not received as much attention as did the multiple objective linear fractional programming (MOLFP).To our knowledge there are very few algorithms [1,10,23] for (MOILFP) taking into account the integrity of the variables.Affine fractional functions as widely used as performance measures in some management situations, production planning and the analysis of financial enterpries.Thus the multicriteria programming problems with affine fractional criterion functions are important and have wide applications in various fields as financial planning [3], transportation problem [12], manpower planning models[19].

Mathematically, (MOILFP) is described as the problem of finding all efficient solutions of the problem

$$\begin{aligned} &maximize\{Z_1(x) = \frac{p^1x+\alpha^1}{q^1x+\beta^1}\} \\ &maximize\{Z_2(x) = \frac{p^2x+\alpha^2}{q^2x+\beta^2}\} \\ &\vdots \\ &maximize\{Z_r(x) = \frac{p^rx+\alpha^r}{q^rx+\beta^r}\} \end{aligned} \qquad (P)$$

subject to $x \in S$,
where α^i,β^i are scalars; $p^i, q^i \in R^n$ for each $i \in \{1, 2, ..., r\}$;$S = D \cap Z^n$;$r \geq 2$; $D = \{x \in R^n / Ax \leq b, x \geq 0\}$; $b \in R^m$; $A \in R^{m\times n}$; Z is the set of integers.
It is assumed that S is not empty and D is a bounded convex polyhedron in R^n and $q^ix + \beta^i > 0$ over D for all $i \in \{1, 2, ..., r\}$.
The set of all integer efficient solutions of (P) is denoted by $E(P)$.

As in multiple objective linear programming see [24,25], the solution to the problem (P) is to find all solutions that are efficient in the sense of the following definition.

Definition 1.1 *A point $x^0 \in S$ is said to be efficient of (P) if and only if there does not exist another point $x^1 \in S$ such that $Z_i(x^1) \geq Z_i(x^0)$ for all $i \in \{1, 2, ..., r\}$ and $Z_i(x^1) > Z_i(x^0)$ for at least one $i \in \{1, 2, ..., r\}$.*

Otherwise, x^0 is called a dominated solution and the vector $(Z_1(x^0), Z_2(x^0), ..., Z_r(x^0))$ is said a dominated $r-$tuple.

In practical application of multiple criteria decision making,the decision makers often have to choose some preferred point from the efficient set $E(P)$.

This involves the problem of finding efficient solutions and describing the structure of $E(P)$.Since in many cases the criteria are in conflict,the decision maker try to optimize a linear function representing his preferences over the efficient set $E(P)$.The problem of finding a most preferred (with respect to φ) efficient point can be stated as a mathematical programming problem

$$\begin{cases} maximize & \varphi = d.x \\ s.t. & x \in E(P) \end{cases} \qquad (P_E)$$

where dx is a linear function ($d \in R^n$).

Mathematically, the problem (P_E) can be classified as a global optimization problem.The main difficulty of this problem arises from the fact that its feasible domain, in general, is nonconvex and not given explicitly.

In the continuous case, the problem of optimizing a real valued function over an efficient set of the Multiple Objective Linear Programming problem (MOLP) have attracted much attention because of their important applications in decision making. This problem was first considered by Philip [22] in which an algorithm based on moving to adjacent efficient vertices is outlined.In Isermann and Steuer [16] the main idea of the algorithm is based on the use of a cutting plane procedure.Benson [4,5] has given two relaxation algorithms for solving this problem.The survey of Yamamoto [28] proposes a classification of the existing algorithms for optimization over the efficient set.Thi, Pham and Thoai [26] propose a branch and bound procedure based on some properties in Lagrange duality.Yamada, Tanino, Inuiguchi [27] propose a method for approximate minimization of a convex function over the weakly efficient set.Benson [6] suggested a more simple linear programming procedure for detecting and solving the problem in four special cases and many others references.

Although the discrete case has by no means seen a similar development.Linear functions optimization on an integer efficient set of (MOLP) is considered only by Nguyen [21] which gives an upper bound for the optimal objective value of the function φ.Abbas and Chaabane [2] where different types of cuts are imposed in such a way that the improvement of the objective value is guaranteed at each iteration.Chaabane and Pirlot [9]; Jesus M. Jorge [17] propose an algorithm which defines a sequence of progressively more constrained single-objective integer problems that successively eliminates undesirable points from further consideration.

Problem (P_E) of optimizing a linear function over a set of a vector affine fractional program with integer variables (MOILFP) has received no attention and we do not known any reference of numerical method specially designed for this kind of problem what justified our interest to study this problem.

The central problem of interest in this paper is the problem of optimizing a linear function φ over the efficient set $E(P)$ of problem (MOILFP).This problem is formulated as

$$\begin{cases} Maximize \quad w = d.x \\ s.t. \qquad\quad x \in E(P) \end{cases} \qquad (P_E)$$

The problem $(P_i(S))$, $i \in \{1, 2, ..., r\}$ is the following integer linear fractional programming problem

$$\begin{cases} maximize \quad Z_i(x) = \frac{p^i x + \alpha^i}{q^i x + \beta^i} \\ s.t\ . \quad x \in S = D \cap Z^n \end{cases} \qquad (P_i(S))$$

The outline of the paper is as follows:In this section we have presented the motivation for developing the approach.Section 2 compiles the notation and definitions used throughout the manuscript.In section 3, some preliminary results are given to justify the methodology.The detailed presentation of the algorithm is given in section 4.In section 5, a numerical illustration is included to explain the proposed method.

2 Notation and definitions

Along the present paper, the following notations will be used.

- $D_1 = \{x \in R^{n_1} : A_1 x \le b_1; A_1 \in R^{m_1 \times n_1}; b_1 \in R^{m_1}; x \ge 0\}$. D_1 is a current truncated region of D obtained by successive Gomory cuts introduced when optimizing problem $(P_1(S))$. Note that $S_1 = S = D_1 \cap Z^n$, because Gomory cuts do not eliminate integer solutions from D.

- $(Z_1^1, Z_2^1, ..., Z_r^1)$ is the first non-dominated $r-$tuple corresponding to the optimal integer solution x_1^1 obtained in D_1, where $Z_i^1 = \frac{p^i x + \alpha^i}{q^i x + \beta^i}$ for $i = 1, 2, ..., r$.

For $k \ge 2$,we have:

- $D_k = \{x \in R^{n_k} : A_k x \le b_k; A_k \in R^{m_k \times n_k}; b_k \in R^{m_k}; x \ge 0\}$. D_k is the current truncated region obtained after having applied the cut $\sum_{j \in N_{k-1} \setminus \{j_{k-1}\}} x_j \ge 1$ where $j_{k-1} \in \Gamma_{k-1}$ (see below) and successive Gomory cuts eventually.

- $x_k^1 = (x_{k,j}^1)$ the k^{th} optimal integer solution of problem $P_1(S)$ obtained on D_k at step k. (Note that in place of $(P_1(S))$, one can similarly consider the problem $(P_i(S))$, $\ i \in \{2, ..., r\}$)

- B_k^1 is a basis associated with solution x_k^1.
- $a_{k,j}^1 \in R^{m_k \times 1}$ is the activity vector of $x_{k,j}^1$ with respect to D_k.
- $y_{k,j}^1 = (y_{k,ij}^1) = (B_k^1)^{-1} \times a_{k,j}^1$ where $y_{k,j}^1 \in R^{m_k \times 1}$
- $I_k = \{i / a_{k,i}^1 \in B_k^1\}$ (indices of basic variables)
- $N_k = \{j / a_{k,j}^1 \ \notin B_k^1\}$ (indices of non-basic variables)
- $p_j^1 =$ the j^{th} component of vector p^1
- $q_j^1 =$ the j^{th} component of vector q^1
- $p_{k,j}^1 = \sum_{i \in I_k} p_i^1 \cdot y_{k,ij}^1$
- $q_{k,j}^1 = \sum_{i \in I_k} q_i^1 \cdot y_{k,ij}^1$
- $Z_1(x_k^1) = \frac{Z_{k,1}^1}{Z_{k,2}^1} = \frac{p^1 x_k^1 + \alpha^1}{q^1 x_k^1 + \beta^1}$
- $\overline{\gamma}_{k,j}^1 = Z_{k,2}^1 (p_j^1 - p_{k,j}^1) - Z_{k,1}^1 (q_j^1 - q_{k,j}^1)$, the updated value of the j^{th} component of the reduce gradient vector $\overline{\gamma}_k^1$.
- $x_k^u = (x_{k,j}^u)$ are the $(t_k - 1)$ alternate integer solutions to x_k^1, if they exist, where t_k is an integer number and $u \in \{2, ..., t_k\}$.
- $\Gamma_k = \{j \in N_k \ / \ \overline{\gamma}_{k,j}^1 \leq 0 \quad \text{and} \quad \varphi_j^k - d_j^k \leq 0\}$, where $\varphi_j^k = d_{B_k^1} . y_{k,j}^1$ with $d_{B_k^1}$ the vector of cost-coefficients of basic variables associated with B_k^1 in vector d^k.

Theorem 2.1 [20] *The point x_k^1 of S is an optimal solution of problem $(P_1(S))$ if and only if the reduce gradient vector $\overline{\gamma}_k^1$ is such that $\overline{\gamma}_{k,j}^1 \leq 0$ for all $j \in N_k$.*

Remark 2.2 *Recall that a sufficient condition for the uniqueness of the optimal solution x_k^1 of $(P_1(S))$ is that the set $J_k = \{j \in N_k / \overline{\gamma}_{k,j}^1 = 0\}$ is empty.*

In this case, there does not exist any other integer feasible solution x in S such that $Z_1(x) = Z_1(x_k^1)$. We refer to x as an alternate optimal solution to x_k^1.

We recall a well known result[22].

Corollary 2.3 *A point x^0 that is unique solution of the integer linear fractional programming problem $(P_1(S))$ is efficient of (P).*

3 Theoretical results

Since the problem MOILFP is to determine the set of integer efficient solutions, we scan all integer points of the feasible region S by a cutting plane technique which is described in the present section.

Definition 3.1 *Assume that $j_k \in N_k$. An edge E_{j_k} incident to a solution x_k^1 is defined as the set*

$$E_{j_k} = \left\{ x = (x_i) \in D_k : \begin{cases} x_i = x_{k,i}^1 - \theta_{j_k}.y_{k,ij_k}^1 & \text{for } i \in I_k \\ x_{j_k} = \theta_{j_k} & \\ x_v = 0 & \text{for all } v \in N_k \backslash \{j_k\} \end{cases} \right. \tag{1}$$

where $0 < \theta_{j_k} \leq \min_{i \in I_k}\{\frac{x_{k,i}^1}{y_{k,ij_k}^1} / y_{k,ij_k}^1 > 0\}$, θ_{j_k} *is a positive integer and* $\theta_{j_k}.y_{k,ij_k}^1$ *are integers for all* $i \in I_k$ *if such integer values exist.*

Remark 3.2 *Note that equation (1) enables us to compute the integer feasible alternate solutions when the optimal solution obtained by solving $(P_1(S))$ is not unique ($J_k \neq \emptyset$).*

The following theorem addresses the case in which the optimal solution of $(P_1(S))$ is not unique.

Theorem 3.3 [1] *All integer feasible solutions x_k^u, $u \in \{2, ..., t_k\}$ of problem $(P_1(S))$ alternate to x_k^1 on an edge E_{j_k} of region D (or truncated region D_k) emanating from it, in the direction of a vector a_{k,j_k}^1, $j_k \in J_k$, exist in the open half space.*

$$\sum_{j \in N_k \backslash \{j_k\}} x_j < 1 \tag{2}$$

The following theorem suggests a cut that can be viewed as a generalization of Dantzig's cut.

Theorem 3.4 [1] *An integer feasible solution of problem $(P_1(S))$ that is distinct from x_k^1 and not on an edge E_{j_k}, $j_k \in J_k$ of the truncated region D_k (or D) through an integer feasible point x_k^1 of problem $(P_1(S))$ exists in the closed half space*

$$\sum_{j \in N_k \backslash \{j_k\}} x_j \geq 1 \tag{3}$$

1- We calculate the value φ'_k of the linear function φ at any solution $x^u_k = (x^u_1, x^u_2, ..., x^u_n)$ lying on the edge E_{j_k}.

$$\varphi'_k = \sum_{j=1}^{n} d^k_j.x^u_j = \sum_{i\in I_k} d^k_i(x_{k,i} - \theta_{j_k}.y_{k,ij_k}) + d^k_{j_k}.\theta_{j_k}$$

$$\varphi'_k = (d^k_{j_k} - \sum_{i\in I_k} d^k_i.y_{k,ij_k}).\theta_{j_k} + \sum_{i\in I_k} d^k_i.x_{k,i}$$

where θ_{j_k} is an integer verifying $0 < \theta_{j_k} \leq \theta^0_{j_k}$ and $\theta^0_{j_k}$ is the integer part of $\min_{i\in I_k}\{\frac{x_{k,i}}{y_{k,ij_k}}/y_{k,ij_k} > 0\}$.

We put :

$$\upsilon_k = (d^k_{j_k} - \sum_{i\in I_k} d^k_i.y_{k,ij_k}) \tag{4}$$

Then along an edge $E_{j_k},\ j_k \in \Gamma_k$, we have $\upsilon_k \geq 0$. Therefore, the values of φ'_k are increasing and φ'_k reaches its maximum for $\theta_{j_k} = \theta^0_{j_k}$.

Definition 3.5 *Let* $f : S \subset R^n \longrightarrow R$ *and* $\overline{x} \in S$. *Then*
$L_{\geq}f(\overline{x}) = \{x \in S : f(x) \geq f(\overline{x})\}$ *is called the level set of* $\overline{x}$ *for* f.
$L_{=}f(\overline{x}) = \{x \in S : f(x) = f(\overline{x})\}$ *is called the level curve of* $\overline{x}$ *for* f.

2- The following theorem is used in various steps of our algorithm to test the efficiency of a given integer feasible solution of Multiobjective Integer Linear Fractional Programming problem (P).

Theorem 3.6 [14] *Characterization of Pareto optimal solutions*
$\overline{x} \in S$ *is Pareto optimal of (P) if and only if* $\bigcap_{i=1}^{i=r} L_{\geq}Z_i(\overline{x}) = \bigcap_{i=1}^{i=r} L_{=}Z_i(\overline{x})$.

Proof: $\overline{x}$ is Pareto optimal of (P)
$\Longleftrightarrow$ *There does not exist* $x \in S$ *such that* $[Z_i(x) \geq Z_i(\overline{x})\ \forall\ i = 1...r$ *and* $Z_j(x) > Z_j(\overline{x})$ *for some* $j]$
$\Longleftrightarrow$ *There does not exist* $x \in S$ *such that* $[x \in \bigcap_{i=1}^{i=r} L_{\geq}Z_i(\overline{x})$ *and* $\exists\ j : x \in L_{>}Z_j(\overline{x})]$
$\Longleftrightarrow \bigcap_{i=1}^{i=r} L_{\geq}Z_i(\overline{x}) = \bigcap_{i=1}^{i=r} L_{=}Z_i(\overline{x})$

3- Before starting the description of the algorithm we introduce the following inequality $(dx \geq \varphi_{opt})$ which eliminate only solutions that are strictly worse than the current optimal solution.

4 Development of the algorithm

The algorithm that we propose here is proved to provide an optimal solution of (P_E) without having to compute the set of all efficient solutions of the underlying problem (P).

Step 0: Initialization let $\varphi_{opt} = -\infty$

Solve the relaxed problem $(P_R) : \max\{dx \ / \ x \in S\}$

- If (P_R) is unfeasible $\Rightarrow$ STOP, (P_E) is unfeasible.
- Otherwise, let x^0 be an optimal solution of (P_R).

Step 1: This solution is tested for efficiency by applying the Theorem 3.6

- If $x^0 \in E(P) \Rightarrow$ STOP, x^0 is an optimal solution of (P_E).
- Otherwise, go to step 2.

Step 2: Solve the problem $(P_1(S))$ [one may similarly consider any of the problems $(P_i(S)) \quad i = 2, 3, ..., r$ instead of $(P_1(S))$].

2.1 If $J_1 = \{j \in N_1/\overline{\gamma}^1_{1,j} = 0\} = \emptyset$ then the optimal solution found x^1_1 is unique and it is efficient (corollary 2.3). Set $(x_{opt} = x^1_1, \varphi_{opt} = dx^1_1)$ and go to step 3.

2.2 If $J_1 \neq \emptyset$ then x^1_1 may not be unique, test the efficiency of x^1_1.

- If it is not efficient go to step 3.
- Otherwise, set $(x_{opt} = x^1_1, \varphi_{opt} = dx^1_1)$ and go to step 3.

Step 3: Let $k = 1$ and perform the following sub-steps:

3.1 Construct the set $\Gamma_k = \{j \in N_k/\overline{\gamma}^k_{k,j} \leq 0 \text{ and } \varphi^k_j - d^k_j \leq 0\}$.

– If $\Gamma_k = \emptyset$, then go to step 3.3 and apply the Dantzig cut $\sum_{j \in N_k} x_j \geq 1$.

– Otherwise, let $\gamma = \Gamma_k$. go to (a).

a - If $\gamma = \emptyset$, then let $j_k \in \Gamma_k$ and go to 3.3.

- Else, select $j_k \in \gamma$ and calculate $\theta^0_{j_k}$ the integer part of $\min_{i \in I_k}\{\frac{x^1_{k,i}}{y^1_{k,ij_k}}/y^1_{k,ij_k} > 0\}$
- If $\theta^0_{j_k} = 0$ then there is no integer feasible solution on the edge E_{j_k}, put $\gamma = \gamma \backslash \{j_k\}$ and go to (a).
- Else, if $\theta^0_{j_k} \geq 1$, then go to (b).

b - If x_k^1 is efficient and $dx_k^1 \geq \varphi_{opt}$, then calculate the value of the parameter υ_k defined in equation (4).

- If $\upsilon_k \neq 0$, then go to (c).
- If $\upsilon_k = 0$, put $\gamma = \gamma \backslash \{j_k\}$ and go to (a).

- If x_k^1 is not efficient or $dx_k^1 < \varphi_{opt}$, then go to (c).

c- Explore the edge E_{j_k}, searching for a feasible integer solutions of $(P_1(S))$ corresponding to θ_{j_k} (θ_{j_k} is an integer verifying $0 < \theta_{j_k} \leq \theta_{j_k}^0$) and test for efficiency starting from $\theta_{j_k} = \theta_{j_k}^0$ until $\theta_{j_k} = 1$.
Once a first integer efficient solution is found, say x_k^u such that $dx_k^u > \varphi_{opt}$, set ($x_{opt} = x_k^u$, $\varphi_{opt} = dx_k^u$), and go to sub-step 3.2.
If there is no integer efficient solution on this edge, then put $\gamma = \gamma \backslash \{j_k\}$ and go to (a).

3.2 Let $k = k + 1$. Define the new truncated region D_k as the subset of D_{k-1} obtained by applying the cut ($dx \geq dx_{k-1}^u$) and using the dual simplex method and gomory cuts, if necessary, to find a new optimal solution x_k^1.Set ($x_{opt} = x_k^1$, $\varphi_{opt} = dx_k^1$) and go to (3.1).

3.3 Let $k = k + 1$. The new truncated region D_k is obtained as a subset of D_{k-1} by applying the specified cut (Dantzig cut $\sum_{j \in N_k} x_j \geq 1$ or cut $\sum_{j \in N_k \backslash \{j_k\}} x_j \geq 1$) and using the dual simplex method and Gomory cuts, if necessary, to find a new optimal solution x_k^1.

- If the solution x_k^1 is efficient and $dx_k^1 > \varphi_{opt}$, set ($x_{opt} = x_k^1$, $\varphi_{opt} = dx_k^1$) and go to (3.1)
- Else, go to (3.1).

Terminal step: The process terminates when infeasibility of pivot operations appears indicating that the current region contains no integer feasible point x_k^1 such that $dx_k^1 > \varphi_{opt}$. The optimal solution of problem (P_E) is then x_{opt} and its value on criterion φ is φ_{opt}.

Proposition 4.1 *Under the hypothesis that S is not empty and D bounded, the algorithm ends up with an efficient solution of problem (P).*

Proof: Since D is bounded, S is non-empty and finite. Each cut of Dantzig $\sum_{j \in N_k} x_j \geq 1$ or a cut of type $\sum_{j \in N_k \backslash \{J_k\}} x_j \geq 1$ reduces strictly the domain. Hence the procedure terminales with an efficient solution of (P) because at least one such solution exists in S.

Theorem 4.2 *If S is non-empty and D is bounded, then*

1. *The algorithm terminates in a finite number of iterations.*

2. *The solution x_{opt} is an optimal solution of problem (P_E).*

Proof: Proposition (4.1) guarantees that we can obtain an initial efficient solution of (P), at iteration p, $p \geq 1$. We see also from the description of the algorithm that, during iteration k, either a cut of Dantzig $\sum_{j \in N_k} x_j \geq 1$ or a cut of type $\sum_{j \in N_k \setminus \{J_k\}} x_j \geq 1$ is applied which strictly reduces the domain or a new efficient solution is found that improves φ_{opt}. Obviously, since the domain S is finite, it may not be strictly reduced an infinite number of times. For the same reason, only a finite number of improvements of $\varphi = dx$ may be observed when x moves in the finite set S. This proves that the algorithm stops after a finite number of iterations.

Provided S is non-empty and D is bounded, the algorithm stops at iteration $k > p$ if and only if the problem $(P_1(S_k))$ is unfeasible, this is seen from the fact that, the dual simplex algorithm, at some stage, possibly after the adjunction of Gomory cuts, can not perform any pivot operation. The current value of φ_{opt} at that iteration is optimal and x_{opt} is an optimal solution of problem (P_E).

5 Numerical illustration

To illustrate the use of this algorithm, we consider the following integer linear fractional program with three objectives [18].

$$Maximize\left\{Z_1(x) = \frac{-x_1+4}{x_2+1}; Z_2(x) = \frac{x_1-4}{-x_2+3}; Z_3(x) = -x_1+x_2\right\} \qquad (P)$$

subject to $x \in S$ where

$$S = \left\{x \in R^2 : -x_1 + 4x_2 \leq 0; x_1 - \frac{1}{2}x_2 \leq 4; x_1, x_2 \geq 0 \;\; and \;\; integers\right\}$$

Let the main problem be

$$\begin{cases} \max \quad \varphi = 2x_1 - 3x_2 \\ s.t. \;\; x_1, x_2 \in E(P) \end{cases} \qquad (P_E)$$

Step 0: Initialization Let $\varphi_{opt} = -\infty$

We solve the relaxed problem $(P_R)\begin{cases}\max(2x_1 - 3x_2)\\ x \in S\end{cases}$

The optimal solution is $x^0 = (4,0)'$ and $\varphi^0 = 8$.

Step 1: This solution x^0 is tested for efficiency (theorem 3.6) and we obtain:

$$\bigcap_{i=1}^{3} L_{\geq} f_i(4,0) = \{(4,0)';\ (4,1)'\} \neq \bigcap_{i=1}^{3} L_{=} f_i(4,0) = \{(4,0)'\}$$

Thus x^0 is not efficient go to step 2.

Step 2: We solve the problem $(P_1(S))\begin{cases}\max(\frac{-x_1+4}{x_2+1})\\ x \in S\end{cases}$

The results of solving problem $(P_1(S))$ using the procedure developed in [7] or [15] are summarized in table I.

B	x_B	x_1	x_2
x_3	0	-1	4
x_4	4	1	$-\frac{1}{2}$
$-p^1$	-4	-1	0
$-q^1$	-1	0	1
$\overline{\gamma}^1_{1,j}$		-1	-4
$\varphi^1_j - d^1_j$	0	-2	3

table I

The optimal solution $x^1_1 = (0,0)'$ is unique, thus it is efficient. Let it be a first efficient solution that corresponds to $\varphi^1 = 0$. We have $dx^1_1 = 0 > -\infty$ then $\varphi_{opt} = 0$ and $x_{opt} = (0,0)'$.

Step 3:

3.1 $k = 1$

$I_1 = \{3,4\}$; $N_1 = \{1,2\}$, $\Gamma_1 = \{j \in N_1/\overline{\gamma}^1_{1,j} \leq 0 \ \ and \ \ \varphi^1_j - d^1_j \leq 0\} = \{1\} \neq \emptyset$. We put $\gamma = \Gamma_1 = \{1\}$.

Let $j_1 = 1 \in \gamma$. Since x^1_1 is efficient and $dx^1_1 = 0 > \varphi_{opt} = -\infty$ then we calculate the value of υ_1.

$\upsilon_1 = d^1_1 - d^1_3.y_{1,31} - d^1_3.y_{1,41} = 2 - 0.(-1) - 0(1) = 2 > 0$, we start exploring the edge E_1; we calculate $\theta^0_1 = \min\{\frac{4}{1}\} = 4$; for $\theta_1 = 4$, $x^2_1(4) = (4,0)'$ which is not efficient.

For $\theta_1 = 3$ (the best value of θ_1 yielding a great increase in φ), the

corresponding solution an the edge E_1 is $x_1^3(3) = (3,0)'$.
The solution $x_1^3(3)$ is being tested for efficiency and we obtain:

$$\bigcap_{i=1}^{3} L_{\geq} f_i(3,0) = \bigcap_{i=1}^{3} L_{=} f_i(3,0) = \{(3,0)'\}$$

Thus $x_1^3(3)$ is efficient. We calculate $\varphi_1^1 = d.x_1^3(3) = 6$.
As $\varphi_1^1 > \varphi_{opt} = 0$, then $\varphi_{opt} = 6$ and $x_{opt} = (3,0)'$.

3.2 $k = k+1 = 2$

We cut by $2x_1 - 3x_2 \geq 6$
After adjusting the table I for the reduced feasible region and applying the dual simplex method.The optimal feasible solution is $x_2^1 = (3,0)'$ which is efficient. It corresponds to $\varphi^2 = 6$; $\varphi_{opt} = 6$ and $x_{opt} = (3,0)'$ (see table II)

B	x_B	x_2	x_5
x_3	3	$\frac{5}{2}$	$-\frac{1}{2}$
x_4	1	1	$\frac{1}{2}$
x_1	3	$-\frac{3}{2}$	$-\frac{1}{2}$
$-p^2$	-1	$-\frac{3}{2}$	$-\frac{1}{2}$
$-q^2$	-1	1	0
$\overline{\gamma}_{2,j}^1$		$-\frac{5}{2}$	$-\frac{1}{2}$
$\varphi_j^2 - d_j^2$	6	0	-1

table II

$I_2 = \{1,3,4\}$, $N_2 = \{2,5\}$, $\Gamma_2 = \{2,5\} \neq \emptyset$
Let $\gamma = \Gamma_2$. Let $j_2 = 2$. Since x_2^1 is efficient and $dx_2^1 = \varphi_{opt} = 6$, then we calculate the value of υ_2; $\upsilon_2 = -3 - (2(-\frac{3}{2}) + 0 + 0) = 0$.We do not explore the edge E_2.
Let $\gamma = \gamma\backslash\{2\}$ and consider the second index $j_2 = 5 \in \gamma$,$\theta_5^0 = \min\{\frac{1}{\frac{1}{2}}\}$ $= 2$ .Since x_2^1 is efficient and $dx_2 = \varphi_{opt} = 6$, then we calculate the value of υ_2; $\upsilon_2 = 0 - (2 \times (-\frac{1}{2})) = 1 > 0$ we explore the Edge $E_{j_2} = E_5$.
The corresponding solution on the edge E_5 is $x_2^2(2) = (4,0)$ which is not efficient and $x_2^3(1) = (\frac{7}{2},0)$ which is not integer.
We have $\gamma = \gamma\backslash\{5\} = \emptyset$.

3.3 Let $k = k+1 = 3$

and we cut the current feasible region by $\sum_{j \in N_2\backslash\{5\}} x_j \geq 1 \Leftrightarrow x_2 \geq 1$.
We add this constraint at the bottom of table II and apply the dual simplex method and Gomory method to obtain table III

B	x_B	x_6	x_7
x_3	1	3	-1
x_4	$-\frac{1}{2}$	$\frac{1}{2}$	1
x_1	5	-1	-1
x_2	1	-1	0
x_5	1	1	-2
$-p^3$	1	-1	-1
$-q^3$	-2	1	0
$\overline{\gamma}^1_{3,j}$		-1	-2
$\varphi^3_j - d^3_j$	7	1	-2

table III

The dual is not feasible then the algorithm terminates.
The optimal solution of problem (P_E) is then $x_{opt} = (3,0)'$ and $\varphi_{opt} = 6$.

This example was first presented in [18] to find the set of integer efficient solutions : $E(P) = \{(4,1)'; (3,0)'; (2,0)'; (1,0)'; (0,0)'\}$.
However, our algorithm optimizes the linear function $\varphi = 2x_1 - 3x_2$ without having to determine all these solutions but only $\{(0,0)', (3,0)'\}$.

6 Conclusion

In this work we have presented a new algorithm for optimizing a linear function over an efficient set of the Multiple Objective Integer Linear Fractional Programming problem (MOILFP).
The proposed algorithm solves problem (P_E) by using classical linear programming procedures without having to enumerate all the efficient solutions. The algorithm may generate several dominated solutions but it provides a shorter way to the optimal one.

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