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A Behavioral Model of Bubbles and Crashes

Taisei Kaizoji[1]

Abstract

The aim of this paper is to propose a new model of bubbles and crashes to elucidate a mechanism of bubbles and subsequent crashes. We consider an asset market in which the risky assets into two classes, the risky asset, and the risk-free asset are traded. Investors are divided into two groups of investors who have the different rationality on decision-making respectively. One is arbitragers who maximize their expected utility of their wealth in the next period following their rational assessment of the fundamental values of risky assets. Another is noise traders who maximize their random utility of binary choice: buying the bubble asset and holding the risk-free asst. The noise trader's behavior is modeled in a framework of the theory of discrete choice with social interaction (Brock and Durlauf (1999, 2001)), which can be considered as a model of Keynse's beauty contest metaphor. We demonstrate that (i) if noise-traders' conformity effect (the extent that each noise-trader is influenced by the decisions of other noise-traders) is weak, then the market price converges to the fundamental price, so that the efficient market hypothesis holds, but that (ii) if noise-traders' conformity effect is strong, then noise-traders' herd behavior gives cause to a bubble, and their positive-feedback trading prolongs bubble, but a bubble is necessarily ended up with a crash. Furthermore, we describe that cycles of bubbles and crashes are repeated.

Key words: bubbles, crashes, arbitragers, noise traders, positive-feedback trading, efficient market hypothesis, and Keynesian beauty contest metaphor.

JEL:

C73 - Stochastic and Dynamic Games; Evolutionary Games; Repeated Games

G01 - Financial crises

G17 - Financial Forecasting

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1. Introduction

In the last few decades, the asset markets have been frequently visited by bubbles and the subsequent crashes. The increasingly frequent market crashes have attracted the attention of the general public. Although many academics, practitioners and policy makers have studied questions related to collapsing asset price bubbles, the questions, how asset bubbles come about, why it persists, and what causes a crash, have been the greatest myths. What is the origin of bubbles? Why are asset prices deviated away from fundamental value?

One recent growing body of empirical literature on stock price run-up is also devoted to the existence of the momentum trading (also referred to as positive-feedback trading). Many empirical studies documents that the momentum in stock prices is positive in the short term, but eventually reversed in the long term. Among many literatures, the coexistence of the short-run momentum and the long-run reversal in stock prices is documented in detail by Jegadeesh and Titman (1993), and DeBondt and Thaler (1985). Many researchers believe that the empirical evidences on the momentum trading (positive feedback trading) prove the existence of the noise-traders' herd behavior, and their herding have potential to explain speculative bubbles (see for example, DeLong et al. (1990)).

These empirical findings are also consistent with results of experiments in laboratory asset markets. Smith, Suchanek, and Williams (1988) find that (a) bubbles and crashes occur regularly in laboratory asset markets when market participants are inexperienced, but (b) price gradually approach fundamentals when the participants, who have experienced bubbles and crashes in prior trading sessions, interact repeatedly in similar markets. Haruvy, Lahav, and Noussair (2009) finds that the investors' expectations of prices are adaptive, and primarily based on past trends in the previous and current laboratory asset markets in which they have participated. Most traders do not anticipate market downturns the first time they participate in a laboratory market, and are more prone to the optimism that fuels the bubble. In the opposite direction, when experienced, they typically exercise caution about market bubbles and crashes. In summary, the studies mentioned above indicate that the bubble is caused by the non-rational investors who attempt to surf bubble. Greenwood and Nagel (2008) empirically study the portfolio decisions of experienced mutual fund managers, who have experienced market bubbles and the subsequent crashes, and inexperienced mutual fund managers, who have not yet directly experienced the consequences of a stock market downturn, during the internet bubble[2]. They found that increase their technology holdings during the run-up, and decrease them during the downturn. Furthermore, inexperienced managers, but not experienced managers, exhibit trend-chasing behavior in their technology stock investments[3]. Their results are in lines with Haruvy, Lahav, and Noussair (2009).

The recent theoretical literature on bubbles and crashes has evolved to increasingly recognize the evidence of bubbles which is defined as deviations from fundamental value. One important class of finance theories is devoted to the concept of noise-trader (also referred to as positive-feedback investors) which is introduced first by Kyle (1985) and Black (1986) to describe irrational investors, and is developed first by De Long, Shleifer, Summers and Waldmann (1990a, 1990b).) Their view of noise-traders has been motivated in part by George Soros' (1987) description of his own investment strategy. Soros has apparently been successful over the past two decades by betting not on fundamentals but, he claims, on future crowd behavior. Brunnermeier and Nagel (2004) extracted hedge fund holdings from Form 13F, including those of well-known managers such as Soros, Tiger, Tudor, and others in the period of the internet bubble. They found that, over the period of the DotCom bubble, many hedge fund managers tried to ride rather than attak bubbles, suggesting the existence of mechanisms that non-rational investors to surf bubbles rather than attempt to arbitrage. Abreu and Brunnermeier (2003) propose a different mechanism justifying why rational traders ride rather than arbitrage bubbles. These literatures agree in the point that the stock price is kept above its fundamental value by irrationally exuberant behavioral traders such as noise-traders.

Another extensive body of literature shows that there can be large movements in asset prices due to the combined effects of heterogeneous beliefs and short sales

[2] It is widely believed that the internet stocks were in the midst of stock price bubble in the period of the internet bubble from 1998 to 2000. The internet bubble is investigated by Ofek and Richardson (2003), and Battalio and Schultz (2006).
[3] Brennan (2004) insists that increased stock market participation by individuals with little investment experience may have been the driving factor of the internet bubbles.

constraints. The basic idea finds its root back to the Lintner (1969)'s CAPM model of asset prices with investors having heterogeneous beliefs. (See, for example, Miller (1977), Harrison and Kreps (1978), Hommes and Brock (1997, 1998), Chen, Hong and Stein (2000), Hong and Stein (1999, 2003) and Hong, Scheinkman and Xiong (2006)).

Independently of the recent these studies, Keynes (1936 Chap. 12) proposes the completely different concept, that is, "beauty contest" to explain price fluctuations in stock markets. Keynes thought that similar behavior to his beauty contest metaphor was at work within the stock market. Investors evaluate shares not based on what they think their fundamental value is, but rather on what they think everyone else thinks their value is. More recently, the models of the stock market in terms of Keynes's beauty contest are proposed by Biais and Bossaerts, (1998), Allen, Morris, and Shin (2006), and Angeletos, Lorenzoni, and Pavan (2010).

In this paper we propose a new model to explore a mechanism of a bubble and its subsequent collapse. We consider a stock market that the two contrary types of investors coexist. One is arbitragers who invest based on their fundamental value they predict, whereas another is noise-traders whose investment is driven by expectations about what other investors predict, rather than expectations on their fundamentals. The arbitragers are corresponded to experienced managers who have a capability to predict accurately the fundamental price of the risky assets while noise traders are investors who have capability to understand the investors' crowd psychology which is considered in the Keynesian beauty contest metaphor. Our model shows that coexistence of the two contrary types of investors in the asset market is the key to understand a mechanism of stock market bubble and its subsequent crash. More concretely, we consider an asset market in which two assets: the risky asset, and the risk-free asset are traded. In accordance with a standard asst-pricing models proposed by Lintner (1969)), and developed by Brock and Hommes (1997, 1998)), the arbitragers chooses that the portfolio of two assets, the risky asset, and the risk-free asset which will maximize his expected utility of end-of-period wealth. On the other hand, noise traders maximize their random utility[4] of the discrete choice, that is, buying the risky stock or selling the risky asset. We

[4] The qualitative choice models based on maximization of the agent's random utility has been developed by McFadden (1974) .

assume that a noise trader's decision-making is influenced by (i) his expectation on the other noise-traders' decisions, and (ii) his expected price momentum on the risky asset. That is, the noise-traders adapt a positive feed-back strategy (or a momentum strategy) on the bubble asset. To model the interaction among noise-traders, the theory of discrete choice with social interactions proposed by Brock and Durlauf (1999, 2001) is applied. In our model, the market price of the risky asset is determined by the noise-traders' sentiment, which is defined as the difference of bullish noise-traders and the bearish noise-traders. Firstly, we demonstrate that as the conformity effect among noise traders (the extent that each noise-trader is influenced by the decisions of other noise-traders) is sufficiently weak, the arbitragers can stabilize the risky asset price, even if the risky asset price deviates temporally from the fundamental price by the noise-traders' trading. As the noise-traders' conformity effect is weak, the risky asset price converges to the fundamental price in a short run. Therefore, under the condition, the *efficient market hypothesis* holds. Secondly, however, as the conformity effect among noise traders is strengthened, noise traders begin to follow the herd, and the noise-traders' herding behavior destabilize the risky-asset price, and the deviation of the risky asset price from the fundamental price has been enlarging in a long run. Enhancing the noise-traders' bullish sentiment gives cause to a bubble, and their positive feedback trading prolongs bubble. In the second half of bubble, run up of risky asset price come to an end as the noise-traders' sentiment approaches to a limit of the bullish sentiment, that is, almost all the noise-trader 's demand for the risky asset are buyers of the risky asset. For the noise-traders' excess demand for the risky asset price is little or nothing. Thirdly, we demonstrate that decreasing the expected price momentum leads necessarily to market crash. Finally, we demonstrate that after a crash, the noise-trader's sentiment approaches to a limit of the bearish sentiment, the process of generating a bubble begins again.

The paper proceeds as follows. The model is described in Section 2. In Section 3, we give a theoretical explanation on a mechanism of bubble and crash. We give concluding remarks in Section 4.

2. Model

Consider a market on which a risky assets and a risk-free asset are traded. We divide into two groups of investors with different decision making. The first group of investors is a group of arbitragers who maximize their expected utility of wealth in the next period. The second group of investors is the group of the noise-traders who maximize the random utility of the binary choice: buying the risky asset or selling the risky asset.

2.1 Arbitragers

Let us consider the behavior of the arbitragers. We assume that there is a number M of arbitragers. There are two assets available, a risky and a riskless asset. The risk-free asset is in perfectly elastic supply and pays a constant return r. The risky asset pays am uncertain dividend Y_t in each period. The price of the risky asset in period t is denoted by p_t. The excess gain of the risky asset is defined as[5]

$$R_{t+1} = p_{t+1} + d_{t+1} - (1+r)p_t.$$

An arbitrager's wealth is written as

$$W_{t+1} = (1+r)W_t + (p_{t+1} + d_{t+1} - (1+r)p_t)x_t$$

Where X_t denotes the number of shares of the risky asset purchased at period *t*. Let E_t and V_t denote conditional expectation and conditional variance. The object of the arbitragers is to maximize the expected utility $EU(W_{t+1})$ of their wealth W_{t+1} in the next period, t+1. We assume that the arbitrager's preferences are characterized by the constant-absolute risk aversion (CARA) utility with the coefficient of risk aversion, γ. The maximization problem which the arbitragers solve is equivalent to the mean-variance model

$$\begin{aligned} &\underset{x_t, x_{f,t}}{Max}\, EU(W_{t+1}) = Max\left\{E(W_t) - \frac{\gamma}{2}V(W_t)\right\} \\ &s.t. \quad W_{t+1} = (1+r)W_t + (p_{t+1} + d_{t+1} - (1+r)p_t)x_t \end{aligned} \qquad (1)$$

[5] The CAPM, which is utilized in this paper, is proposed by Brock and Hommes (1997, 1998).

That is, in his choice among all the possible portfolios, the arbitrager is satisfied to be guided by its expected yields $E(W_t)$ and its variance $V(W_t)$. The demand x_t for risky asset by arbitrager is then

$$x_t = \frac{E_t\left[p_{t+1} + d_{t+1} - (1+r)p_t\right]}{V_t\left[p_{t+1} + d_{t+1} - (1+r)p_t\right]} = \frac{1}{\gamma\sigma^2}\left[E_t\left(p_{t+1} + d_{t+1}\right) - (1+r)p_t\right] \qquad (2)$$

where the conditional variance $V_t = \sigma^2$ is assumed to be constant. The term $E_t\left[p_{t+1} + d_{t+1}\right]$ denotes a next period payoff of the risky asset which the arbitrager expects, and the term, $(1+r)p_t$ denotes the cost of holding the risky asset for a period. rbitrager's investment decision is as follows. If the expected pay-off $E_t\left[p_{t+1} + d_{t+1}\right]$ is greater than the cost $(1+r)p_t$, arbitrager buys the risky asset. In the opposite direction, If the expected pay-off $E_t\left[p_{t+1} + d_{t+1}\right]$ is less than the cost $(1+r)p_t$, arbitrager sells the risky asset.

When we assume that the arbitragers are assumed to be identical, the aggregated demands for the risky assets by arbitragers are obtained by multiplying the number M of arbitragers:

$$M\,x_t = \frac{E_t\left[p_{t+1} + d_{t+1} - (1+r)p_t\right]}{V_t\left[p_{t+1} + d_{t+1} - (1+r)p_t\right]} = \frac{M}{\gamma\sigma^2}\left\{E_t\left[p_{t+1} + d_{t+1}\right] - (1+r)p_t\right\}\ . \qquad (3)$$

We assume that the arbitrager calculates the fundamental price f_t using the dividend discount model.

$$\begin{aligned} f_t &\equiv \frac{E[p_{t+1} + d_{t+1}]}{1+\tilde{r}} = \frac{E[D_{t+1}]}{1+\tilde{r}} + \frac{E[D_{t+2}]}{(1+\tilde{r})^2} + \frac{E[D_{t+3}]}{(1+\tilde{r})^3} + \frac{E[D_{t+4}]}{(1+\tilde{r})^4} + \cdots \qquad (4) \\ &= \sum_{i=1}^{\infty} \frac{E[D_{t+i}]}{(1+\tilde{r})^i} \end{aligned}$$

where $\tilde{r}$ denotes the discount rate. Using the formula of the fundamental price, the term The aggregated demands for the risky assets by arbitragers are obtained by multiplying the number M of arbitragers:

$$M\,x_t = \frac{E_t\left[p_{t+1} + d_{t+1} - (1+r)p_t\right]}{V_t\left[p_{t+1} + d_{t+1} - (1+r)p_t\right]} = \frac{M}{\gamma\sigma^2}\left\{E_t\left[p_{t+1} + d_{t+1}\right] - (1+r)p_t\right\}\ . \qquad (5)$$

2.2. Noise traders

Let us consider the problem of a noise trader's decision on investment. a noise trader's choice is influenced from the other noise-traders' choice, especially his choice has a tendency to be in favor of the majority decision as Keynes's beauty contest metaphor. To formalize the behavior of a noise-trader, we utilize that the theory of discrete choice with social interactions proposed by Brock and Durlauf (1999, 2001). We assume that there is a number N of noise traders. Individual noise-trader is indexed by *i*. We assume the noise-traders' decision is to choose buying the risky asset or buying the risky asset for each period. The noise-trader i's choice at time t is $s_{i,t}$ with associated support $\{-1,1\}$. If noise-trader i buys the risky asset, then $s_{i,t}=1$, and if noise-trader i sells the risky asset, then $s_{i,t}=-1$.

The individual noise-trader maximizes the random utility a noise trader receives form holding of an asset[6].

$$\max_{s_{i,t}\in\{-1,+1\}} U(s_{i,t}) \tag{6.}$$

We assume that the noise-trader's random utility function U is decomposed into three components[7]

$$U(s_{i,t}) = s_{i,t}\left[(\lambda s^e_{i,t} + H_{i,t+1}) + \varepsilon(s_{i,t})\right]. \tag{7}$$

where $s^e_{i,t}=\sum_{j\neq i} s^e_{i\,j,t}/(N-1)$ .denotes the subjective expectation of $s^e_{i\,j,t}$ denotes the subjective expected value from the noise-trader i's anticipation of noise-trader j 's choice at time t, so that $\overline{s}^e_{i,t}$ denotes the mean value of all noise traders' behavior perceived by noise-trader *i*. other than noise-trader *i*'s behavior. The parameter λ measures the degree of dependence across noise-traders. It indicates the so-called *conformity effect*. Given λ is a positive, an increase in $\overline{s}^e_{i,t}$ raises his utility of holding the risky asset, and lowers his

[6] The qualitative choice models based on maximization of the agent's random utility function were developed by McFadden (1974).

[7] The utility function (9) is in line with the general model of discrete choice in the presence of social interactions proposed by Brock and Durlauf (1999, 2000). See also Lux (1995) and Kaizoji (2000).

utility of holding the risk-free asset. The second term H_t [8]denotes *the momentum* of the risky asset price which is anticipated by noise-traders. The noise-traders' expectation on the expected price momentum is assumed to be *adaptive,*

$$H_{t+1} = (1-\theta)H_t + \theta(p_t - p_{t-1}), \quad H_{t=0} = H_0 \tag{8}$$

where $0 < \theta < 1$. As the expected price momentum H_t of the risky asset is higher, his utility of holding the risky asset is higher, and the utility of holding the risk-free asset is lower. The equation (8) means that the noise-traders adapt a momentum strategy which is a strategy that buys risky assets with high capital gains and sells risky assets with poor capital gains over the previous periods[9].

Finally, the term $\varepsilon(s_{i,t})$ denotes the random term that there are unobserved characteristics of the individual noise-trader's trading strategies and unobserved attributes of the assets perceived by individual noise-trader independently. The random term may let noise-trader *i* to make different choice with that of the noise-traders who has the same deterministic utility term as him/her. According to the standard assumption (see MacFadden (1974)), the term $\varepsilon(s_i)$ is assumed to be independently and identically distributed across noise-traders with the *Gumbell distribution*[10],

$$\Pr[\varepsilon(s_i) \le \varepsilon] = \exp[-\exp[-\varepsilon]]. \tag{9}$$

We introduce new variables, U^+ and U^- where U^+ denotes the utility U when noise-trader *i* buys the risky asset ($s_{i,t} = +1$), and U^- the utility when noise-trader *i* sells the risky asset ($s_{i,t} = -1$). The deterministic utility of noise-trader *i* is defined as

$$\begin{cases} U^+ = \lambda \bar{s}^e_{i,t} + H_{t+1} \\ U^- = -(\lambda \bar{s}^e_{i,t} + H_{t+1}) \end{cases} \tag{10}$$

(12)

[8] $H_{i,t}$ corresponds to the private utility in the words of Brock and Durlauf (2001).

[9] The fact that momentum strategies yield significant profits have been well investigated. Jegadeesh and Titman (1993) examines a variety of momentum strategies and documents that momentum strategies earn profits.

[10] For simplicity of analysis, we set the scale factor for a Gumbel probability distribution of the random utility error to unity and reduce the distribution to simplest form (11).

The optimal probability that a noise trader chooses buying the risky asset or selling the risky asset is given respectively as[11]:

$$\begin{cases} P_t^+ = \dfrac{\exp[\bar{U}_t^+]}{\exp[\bar{U}_t^+] + \exp[\bar{U}_t^-]} \\ P_t^- = \dfrac{\exp[\bar{U}_t^-]}{\exp[\bar{U}_t^+] + \exp[\bar{U}_t^-]} \end{cases} \tag{11}$$

where $P_+ + P_- = 1$. The effects of λ and H_t on the probabilities can be described as follows:

i) As the positive parameter λ pulls up, the noise-traders' conformity effect is enforced.

ii) As the variable H_t which denotes the noise-trader's expectation of the momentum of the risky asset, increases (decreases), the probability that a noise trader holds the risky asset rises (falls), and the probability of holding the risk-free asset falls (rises), and vice versa for H_t.

2.2.1. The self-consistent equilibrium

The expectation of noise-trader i's choice, conditional on his belief concerning the behavior of all noise-traders other than him, can be written as

$$\begin{aligned} E(s_{i,t}) &= +1 \cdot \frac{\exp[\bar{U}_t^+]}{\exp\left[\bar{U}_t^+\right] + \exp\left[\bar{U}_t^-\right]} - 1 \cdot \frac{\exp[\bar{U}_t^-]}{\exp\left[\bar{U}_t^+\right] + \exp\left[\bar{U}_t^-\right]} \\ &= \tanh\left[H_t + \sum\nolimits_{j \neq i} s_{i\,j,t}^e / N\right] \end{aligned} \tag{12}$$

Following Brock and Durlauf (1999, 2001), we assume rational expectations, that is,

$$s_{i\,j,t}^e = E(s_{j,t}) \tag{13}$$

for all i and j. Then, one can prove that there exists a self-consistent equilibrium s^* such that

$$s = \tanh(H_t + \lambda s) \equiv F(s \mid H_t, \lambda)\,. \tag{14}$$

[11] For mathematical derivations of (13), see, for example, MacFadden (1974) and Ben-Akiva, and Lerman (1985).

At the self-consistent equilibrium, it is held that $E(s_{i,t}) = E(s_{j,t}) \quad \forall i, j$. That is, at a self-consistent equilibrium, the common noise-trader's expected value equals the expected value of the average choice for the other noise-traders. The self-consistent equilibrium can be interpreted as a solution of Keynes's beauty-contest metaphor (Keynes (1936)). An implication of his metaphor is that an understanding of financial markets requires an understanding not just of market participants' beliefs, but also an understanding of market participants' beliefs about other market participants' beliefs.

2.2.3. Properties of the self-consistent equilibria

We summarize the properties of the self-consistent equiribria as solutions to equation (17). The self-consistent equilibrium depends on λ and H_{t+1}. For simplicity of analysis, let us assume that the expected price momentum H_{t+1} is a parameter H. The properties of the self-consistent equilibrium, $s^*(H,\lambda)$ with respect to λ and H as follows:

i) The case of $0 < \lambda < 1$ and arbitrary H:

   There is only one possible self-consistent equilibrium $s^{**}(H,\lambda)$. For $H = 0$ and $0 < \lambda < 1$, the only one possible solution is zero. In the case which the noise-traders' conformity effect is weak, herding among noise-traders dose not function. In Figure 1 the graphical solution to (14) is plotted for $\lambda < 1$ and the different values of H. An increase in $H_A(>0)$ shifts the curve which draw the transcendental equation (14) upward. Therefore, the self-consistent equilibrium $s^{**}(H,\lambda)$ moves from the origin to point s_A^{**}. In contrast, a decrease in $H_B(<0)$ shifts the curve which draw the transcendental equation (14) downward. The self-consistent equilibrium $s^{**}(H,\lambda)$ moves from the origin to from the origin to s_B^{**}.

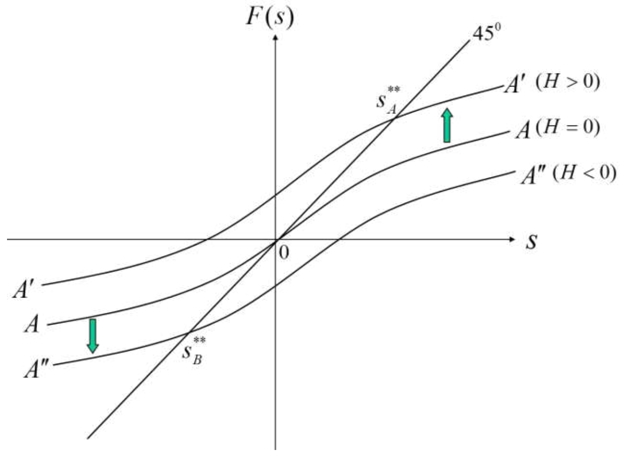

Figure 1: The solutions of the equation (14) for $\lambda < 1$ and the three values of H. There exists a unique self-consistent equilibrium. The straight line is 45 degree line.

ii) The case of $\lambda > 1$ and $|H| < \bar{H}$:

$\bar{H}$ is determined by the equation $\cosh^2[\bar{H} - \sqrt{\lambda(\lambda-1)}] = \lambda$. Under the conditions, there are three self-consistent equilibria $s^* < s^{**} < s^{***}$. The equilibrium s^* and s^{***} are called the bear-market equilibrium and the bull-market equilibrium respectively. As the parameter λ increases and exceeds unity, the equilibrium s^{**} is unstable, and appears the bear-market equilibrium $s^*(<0)$ and bull-market equilibrium $s^{***}(>0)$ anew. This bifurcation is called as the second-order phase transition that we consider as the origin of a bubble. In Figure 2 the graphical solution to (14) is plotted for $\lambda > 1$ and $H = 0$.

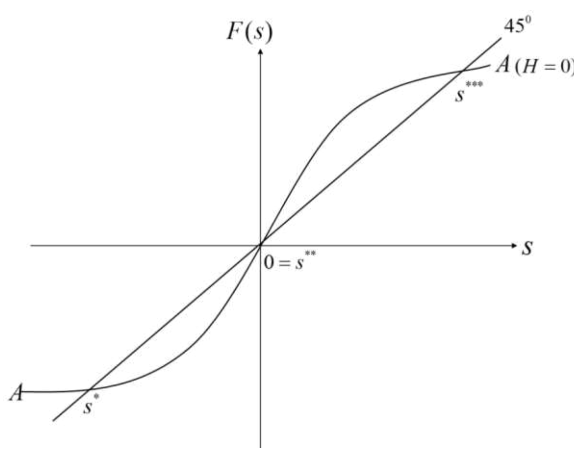

Figure 2: The solutions of the equation (14) for $\lambda > 1$ and $H = 0$. The straight line is 45 degree line. There are the three self-consistent equilibria, s^*, s^{**}, and s^{***}.

iii) The case of $\lambda > 1$ and $|H| = \bar{H}$:

Two of the tree solutions $s^* < s^{**} < s^{***}$ coincide at $s_c = \pm\sqrt{(\lambda - 1)/\lambda}$.

Given that λ is constant, an increase (a decrease) in H causes the curve which indicates the transcendental equation (14) to shift up (down), so that the solutions rise (fall). Figure 3 shows the states that two of the tree solutions coincide.

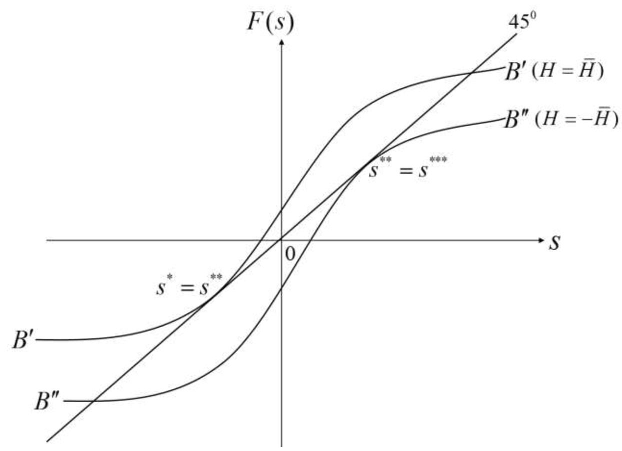

Figure 3: The solutions of the equation (14) for $\lambda > 1$ and $H = \pm\bar{H}$. The straight line is 45 degree line.

iv) There case of $\lambda > 1$ and $|H| > \bar{H}$:

There is one self-consistent equilibrium. When $\lambda > 1$, and H is negative and decreasing continuously, the equilibrium jumps down from s^{***} to s^* at the moment that H falls below $-\bar{H}$. Inversely, the self-consistent equilibrium jumps up from s^* to s^{***} at the moment that H exceeds $\bar{H}$. This bifurcation is called as the first-order phase transition that is related to market crash in Section 3.

2.2.2. Dynamics of the noise-trader's sentiment

We consider the mean s_t of the noise-traders' choice is adjusted by the error correction model,

$$\Delta s_{t+1} \equiv s_{t+1} - s_t = \nu[\tanh(\lambda s_t + H_t) - s_t] \tag{15}$$

where the right hand side of (15) is the error adjustment term and the parameter λ describes the adjustment speed and is between 0 and 1.

We can rewrite the average of noise-traders' choices s_t using the arithmetic average,

$$s_t = \frac{\sum_{i=1}^{N} s_{i,t}}{N} = \frac{(n_t^+ - n_t^-)}{N} \tag{16}$$

where n_t^+ denotes the number of noise traders who buy the risky asset at the time t, and n_t^- denotes the number of noise traders who sell the risk free asset at time t. The variable s_t is the proportion of the number of the bullish noise-traders to the number of the bearish noise-traders. Thus, the variable s_t can be interpreted as a measurement of the noise-traders' sentiment. Hereafter we simply call the variable s_t *the noise-traders' sentiment*.

Using n_t in period t, the aggregate demand $D(s_t)$ for the risky asset over the noise traders is defined as

$$D(s_t) = QNs_t \tag{17}$$

where the parameter Q denotes the number of shares of the risky asset which is exchanged in any transaction by a noise trader, and is assumed to be constant. The equation (17) is utilized when the market prices of the risky assets are calculated under the market clearing conditions in section 2.3.

2.3. Market-clearing prices

The market clearing condition requires that the aggregated demand (supply) for each asset by rational investors is equal to the aggregated supply (demand) by noise traders from the period t. That is, if one noise-trader changes from a holder of the risk-free asset to a holder of the bubble stocks, then the prices are adjusted such that rational investors supply the corresponding number of the risky assets. The temporal market-clearing condition is described as

$$Mx_t + QNs_t = \frac{(1+r)M}{\gamma\sigma^2}\left[\frac{E_t(p_{t+1}+d_{t+1})}{(1+r)} - p_t\right] + QNs_t = 0 \tag{18}$$

Solving the equations (18), we can obtain the price of the risky assets which satisfy the market-clearing conditions,

$$p_t = \frac{E_t(p_{t+1}+d_{t+1})}{1+r} + \frac{\gamma\sigma^2 QN}{(1+r)M} s_t \tag{19}$$

and the noise-traders' sentiment s_t is described as**,**

$$s_t = \frac{1}{\kappa}\left[p_t - \frac{E_t(p_{t+1}+d_{t+1})}{(1+r)}\right] \tag{20}$$

where $\kappa = \dfrac{\gamma\sigma^2 QN}{(1+r)M}$**.** Substituting (20) into the dynamic equation of the noise-traders' sentiment s_t**,** the equilibrium dynamics of stock markets can be described as:

$$\begin{cases} \tilde{p}_{t+1} - \tilde{p}_t = \nu\kappa[\tanh(\frac{\lambda}{\kappa}\tilde{p}_t + H_{t+1}) - \tilde{p}_t] \equiv F\left[\tilde{p}_t, H_{t+1}\right] & (21) \\ H_{t+1} - H_t = \theta((p_t - p_{t-1}) - H_t) & (22) \end{cases}$$

where $\tilde{p}_t = p_t - \dfrac{E(p_{t+1}+d_{t+1})}{1+r}$. Let us consider that arbitragers calculate the fundamental price using the *dividend discount model*, the market price, p_t is equal to the fundamental price,

$$\begin{aligned} f_t &\equiv \frac{E[p_{t+1}+d_{t+1}]}{1+r} = \frac{E[D_{t+1}]}{1+r} + \frac{E[D_{t+2}]}{(1+r)^2} + \frac{E[D_{t+3}]}{(1+r)^3} + \frac{E[D_{t+4}]}{(1+r)^4} + \cdots \\ &= \sum_{i=1}^{\infty} \frac{E[D_{t+i}]}{(1+r)^i} \end{aligned} \tag{23}$$

where r denotes the discount rate which is equal to the risk-free rate. Using the fundamental price, we can rewrite $\tilde{p}_t$ as the derivation of the price of the risky asset from the fundamental price, that is, $\tilde{p}_t = (p_t - f_t)$. Hereafter, we simply call the variable $\tilde{p}_t$ *the deviation*. The dynamics of $\tilde{p}_t$ (21) is essentially equivalent to the dynamics (15) on the imbalance s_t of noise-traders' trading. Thus, we can investigate the properties of $\tilde{p}_t$ by using the same method as the method in which we investigated the properties of the self-consistent equilibrium of s_t in section 2.2.1.

It is clear that a solution of the above dynamics is given at $(\tilde{p}_t, H_{t+1}) = (0,0)$. In the equilibrium, $(\tilde{p}_t, H_{t+1}) = (0,0)$, the market price, p_t is equal to the fundamental price f_t, that is, the equality, $p_t = \dfrac{E(p_{t+1} + d_{t+1})}{1+r}$ is satisfied. For simplicity of analysis, we consider the case that the dividend of the risky asset is constant at $\bar{d}$. In the simple case, the fundamental price, which arbitrager expects, is constant at the value, $\bar{f} = \dfrac{E(p_{t+1} + d_{t+1})}{1+r} = \dfrac{\bar{d}}{r}$. Then, we can obtain the stability condition of the equilibrium, $(\tilde{p}_t, H_{t+1}) = (0,0)$. (See Figure 4).

Proposition 1: Stability of the fundamental price

If $\lambda < \dfrac{\kappa}{F_p}$ where $\kappa = \dfrac{\gamma\sigma^2 QN}{(1+r)M}$ and $F_p = \left.\dfrac{\partial F}{\partial p}\right|_{\tilde{p}=0}$, there is a unique equilibrium, $(\tilde{p}_t, H_t) = (0,0)$, and the unique equilibrium is stable.

A proof of the stability is provided in Appendix 1.

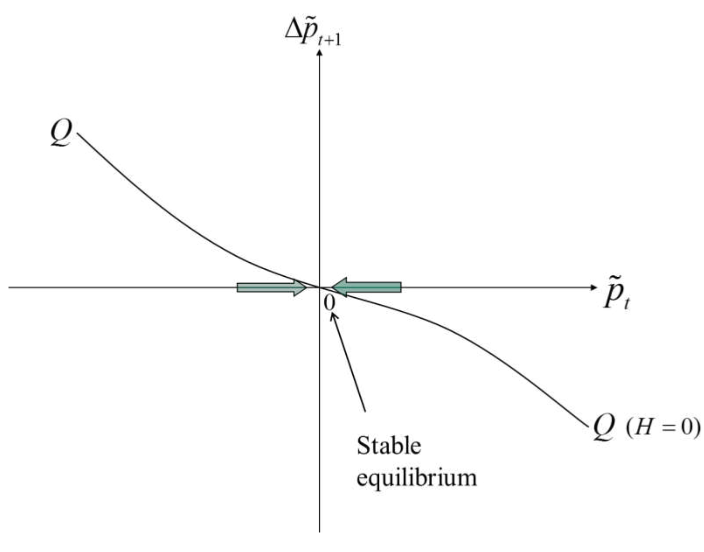

Figure 4: Stability of the equilibrium $(\tilde{p}_t, H_{t+1}) = (0,0)$. $\lambda < \kappa$ and $\bar{f} = \frac{E(p_{t+1} + d_{t+1})}{1+r} = \frac{\bar{d}}{r}$.

The proposition 1 demonstrates that when the degree of the noise-trader's conformity effect, λ is sufficiently weak, the risky asset prices p_t converge the fundamental price f_t. As Friedman (1953) thought, even if noise-traders attempt to destabilize the risky asset price by buying when price is overvalued and selling when price is undervalued, arbitragers can stabilize the risky-asset price when they counter the deviations of the risky asset prices from fundamentals. In brief, when the stability condition demonstrated in Proposition 1 is satisfied, the *efficient market hypothesis* is justified.

3. Bubbles and crashes

3.1. How does a bubble come about?

When the noise-trader's conformity effect is strong ($\lambda > \kappa / F_p$), a bifurcation of the equilibrium $\tilde{p}^*(\lambda, H_{t+1}) = 0$ in the dynamic system (21) is caused by strengthening of noise-traders' conformity effect λ, and the equilibrium, $\tilde{p}^*(\lambda, H_{t+1}) = 0$ becomes unstable. As discussed in subsection 2.3., when the parameter λ is large, the two temporal equilibria, the bull-market equilibrium $\tilde{p}^{**} > 0$ and the bear-market equilibrium $\tilde{p}^{***} < 0$, are generated anew when $H_t = 0$. It is apparent that in the bull-market equilibrium $\tilde{p}^{**} > 0$, the risky-asset price p_t is overvalued against the fundamental price f_t. In contrast, in the bear-market equilibrium $\tilde{p}^{***} < 0$, the risky-asset price p_t is undervalued against the fundamental price f_t.

Let us consider the mechanism by which the risky asset prices are derived from the fundamental values, and a bubble is caused. Let us start from a small and positive value s_0 at the initial time near the unstable equilibrium $s^{**}(=0)$. According to the dynamic equation of the noise-traders' sentiment (15), the number of the bullish noise-traders increases and the number of the bearish noise-trader decreases, and so the noise-traders' excess demand (QNs_t) for the risky asset increases. Since the deviation $\tilde{p}_t$ reflects the noise-traders' sentiment s_t, the deviation $\tilde{p}_t$ raises proportionally with respect to increases in s_t. The deviation $\tilde{p}_t$ raises toward the bull market equilibrium $\tilde{p}^{***}(>0)$.

Run-up in the deviation $\tilde{p}_t$ increases the expected price momentum H_{t+1} perceived by noise traders. Since an increase in H_{t+1} shifts the curve of hyperbolic tangent function (21) upward, the bull-market equilibrium $\tilde{p}^{***}(>0)$ moves to point A to point B in Figure 5. The noise-traders' sentiment s_t, and the noise-traders' demand for the risky asset (QNs_t) is increased further by an increase in the expected price momentum H_{t+1}. The increase in the noise-traders' sentiment s_t generates an increase in the deviation $\tilde{p}_t$, and an increase in the expected price momentum H_{t+1}. This inflationary spiral gives cause to a bubble of the risky-asset price. As demonstrated in section 2.3, the bear-market equilibrium $\tilde{p}^{*}(<0)$ disappears for $H_{t+1} > \bar{H}$, and the bull-market equilibrium $\tilde{p}^{***}$ only remains. (See point C in Figure 5.) The bubble persists as long as the noise traders' bullish sentiment is enhanced, and the bull-market equilibrium $\tilde{p}^{***}(>0)$ moves upward by continuous rises in the expected price momentum H_{t+1}.

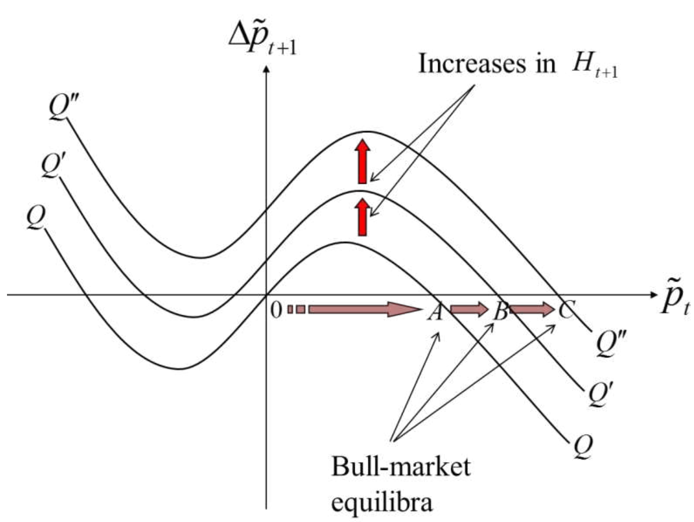

Figure 5: A mechanism of bubble of the risky asset price from the asset price dynamics of (21) for $\lambda > \kappa$ and $\bar{f} = \dfrac{E(p_{t+1} + d_{t+1})}{1+r} = \dfrac{\bar{d}}{r}$. $\Delta\tilde{p}_t = \tilde{p}_t - \tilde{p}_{t-1}$

3.2. Why does a bubble burst?

In the first half of bubbles, the noise-traders' excess demand for the risky asset is sharply increasing, so that the price of the risky asset is also sharply increasing, but in the second half of bubbles, as the noise-traders' sentiment s_t is necessarily approaching approaches the upper limit of the unity. Then, almost all noise traders are buyer. Therefore, the noise-traders' excess demand for the risky asset is little or nothing. The risky-asset price almost never rises. The end of price run-up lowers the expected price momentum H_{t+1}. In contrast to the process of the bubble, decreasing the expected price momentum H_{t+1} move the bull-market equilibrium $\tilde{p}^{***}$ downward bit by bit, so that the noise traders' bullish sentiment starts declining. A decrease in the noise-traders' sentiment gives cause to a decrease in the risky asset. A decrease in the risky asset price then decreases the expected price momentum H_{t+1} again. In this way, the turning from the bullish sentiment to the bearish sentiment necessarily begins to decrease as a reaction of the bubble. This deflationary spiral continues until the expected price momentum H_{t+1} declines by the critical value, $-\bar{H}$ at which the bull-market equilibrium $\tilde{p}^{***}$ disappears. In the next instant when H_{t+1} falls below $-\bar{H}$, the bear-market equilibrium s^* become a unique equilibrium and a market crash can be suddenly caused[12]. In our model, the noise-traders' panic selling of the bubble asset is caused by that the utility of the noise trader's selling the risky asset is progressively higher than that of the noise trader's buying the risky asset. (See Figure 6.) After a crash, the arbitragers buy the risky asset, which they sell and/or go short in the period of a bubble, back at a price which is lower than the fundamental price f_t.

[12] The market crash in our model is considered as the first-order phase transition which is a kind of transformation of a thermodynamic system from one phase to another.

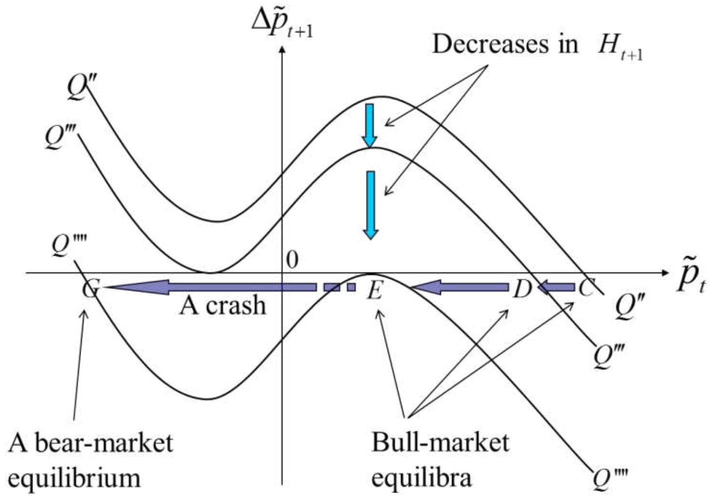

Figure 6: A mechanism of burst of bubble of the risky asset price from the asset price dynamics of (21) for $\lambda > \kappa$ and $\bar{f} = \dfrac{E(p_{t+1} + d_{t+1})}{1+r} = \dfrac{\bar{d}}{r}$. $\Delta\tilde{p}_t = \tilde{p}_t - \tilde{p}_{t-1}$.

3.4. Cycles of bubble and crash

Once a market crash occurs, the market downturn continues until the noise-traders' sentiment s_t approaches to the bear-market equilibrium $\tilde{p}^*$. When the noise-traders' sentiment s_t approaches to the lower-limitation, minus one, the noise-traders' excess supply for the risky asset is approaching gradually zero. A fall in the risky-asset price comes to an end. This enhances the value of the expected price momentum H_{t+1} though the value of H_{t+1} is negative. An increase in the expected price momentum H_{t+1} moves the bear-market equilibrium $\tilde{p}^*$ upward, and the noise traders' sentiment improves so that the deviation $\tilde{p}_t$ rises. Next, a rise in the deviation $\tilde{p}_t$ increases the expected price momentum H_{t+1}. The risky-asset price necessarily begins to rise as a reaction of the bubble, again after the noise-traders' sentiment starts improving. When the expected price momentum H_{t+1} rises by the critical value, $\bar{H}$, the bear-market equilibrium $\tilde{p}^{***}$ disappears. When H_{t+1} exceeds $\bar{H}$, the bull-market equilibrium $\tilde{p}^*$ become a unique equilibrium, and the risky-asset prices enter a bubble phase. In this way, the above process of the bubble and the burst phase of the bubble are repeated. Figure 7 shows the cycle on bubble and crash. The dynamic process which is described above is summarized in the following.

The dynamics of the risky asset prices (21) is globally unstable given that the parameter λ *, which describes the noise-traders' conformity effect, is greater than unity, the dynamics of the risky asset prices are globally unstable, and the bubble and crash cycles of risky assets, which are described above, are continually repeated.*

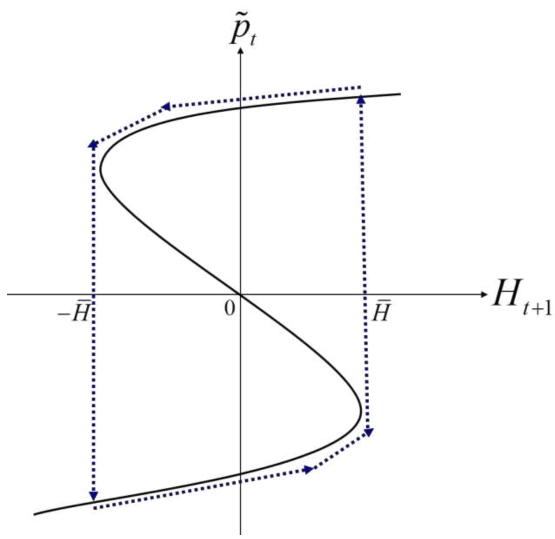

Figure 7: A cycle of bubble and crash appears from the asset price dynamics of (21) for $\lambda > \kappa$ and $\bar{f} = \dfrac{E(p_{t+1} + d_{t+1})}{1+r} = \dfrac{\bar{d}}{r}$.

6. Concluding Remarks

This paper provides a new model that gives one potential theoretical explanation for asset bubble followed by crash. We consider the two groups of investors, which have the different sort of rationality regarding their decision making respectively. One is a group of arbitragers who employ the CAPM which their demands for shares depend on their assessment of fundamental value, and maximize their expected utility of wealth. Another is a group of noise-traders whose demand for bubble asset depend on their expectations on the average value of other noise-traders' investment, and the price momentum they expected. The noise trader's behavior is modeled in a framework of the self-consistent equilibrium which can be considered as a modeling of Keynes's beauty contest metaphor. We elucidate a mechanism that (i) noise-traders' herd behavior gives cause to a bubble, and that (ii) their positive feedback trading prolongs bubble, and that (iii) a bubble is necessarily ended up with a crash, and that (iv) the cycles of bubble and crash are repeated.

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Appendix 1

To demonstrate Proposition 1, we liberalize the nonlinear equation (23) using Taylor expansion around the equilibrium $(\tilde{p}_t, H_{t+1}) = (0,0)$.

The linearized dynamic equation is the following homogenous equation,

$$\tilde{p}_{t+1} + a_1\tilde{p}_t + a_2\tilde{p}_{t-1} = 0 ,$$

where $A = (1-\nu\kappa) + \nu\lambda$, $a_1 = -\left[1-\theta + A + F_H\theta\right]$, $a_2 = \left[(1-\theta)A + F_H\theta\right]$.

The set of necessary and sufficient conditions for the root of the characteristic equation to be less than unity in absolute value are the following inequalities (See Chapter 5 in Gandolfo (1980)):

$$1 + a_1 + a_2 = 1 - \big((1-\theta) + A + F_H\theta\big) + (1-\theta)A + F_H\theta = \theta(1-A) > 0, \qquad \text{(A-1)}$$

$$1 - a_2 = 1 - (1-\theta)A - F_H\theta > 0, \qquad \text{(A-2)}$$

$$1 - a_1 + a_2 = (2-\theta)(1+A) + 2F_H\theta > 0, \qquad \text{(A-3)}$$

When the above inequalities are satisfied, the oscillation is damped to the equilibrium $\tilde{p}_t = 0$. It is clear that the inequalities (A-1), (A-2) , and (A-3) holds when $A < 1$ and $0 < \theta < 1$. Rewriting the inequality $A < 1$, we obtain $\lambda < \kappa$.

The function F is defined as $F[\tilde{p}_t, H_{t+1}] = \tanh(\frac{\lambda}{\kappa}\tilde{p}_t + H_{t+1})$. The derivative, $F_H = \left.\frac{\partial F}{\partial H}\right|_{H=0}$ are greater than zero and less than unity. Note that at the equilibrium $\tilde{p}_t = 0$, the derivative $F_p = \left.\frac{\partial F}{\partial p}\right|_{\tilde{p}=0}$ is equal to unity, Therefore, if the parameter λ is sufficiently small, the inequality $\lambda < \kappa$ holds given that the function, κ is defined as

$$\kappa = \frac{\gamma\sigma^2 QN}{(1+r)M} > 0.$$

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