Adjoint expansions in local Lévy models

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Abstract

We propose a novel method for the analytical approximation in local volatility models with Lévy jumps. The main result is an expansion of the characteristic function in a local Lévy model, which is worked out in the Fourier space by considering the adjoint formulation of the pricing problem. Combined with standard Fourier methods, our result provides efficient and accurate pricing formulae. In the case of Gaussian jumps, we also derive an explicit approximation of the transition density of the underlying process by a heat kernel expansion: the approximation is obtained in two ways, using PIDE techniques and working in the Fourier space. Numerical tests confirm the effectiveness of the method.

Keywords: Lévy process, local volatility, analytical approximation, partial integro-differential equation, Fourier methods

JEL Classification G13

Mathematics Subject Classification (2000): 60J75, 35Kxx, 60Hxx

1 Introduction

We consider a one-dimensional *local Lévy model* where the log-price X solves the SDE

$$dX_t = \mu(t, X_{t-})dt + \sigma(t, X_{t-})dW_t + dJ_t. \tag{1.1}$$

In (1.1), W is a standard real Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ with the usual assumptions on the filtration and J is a

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pure-jump Lévy process, independent of W, with Lévy triplet $(\mu_1, 0, \nu)$. We denote by

$$T \mapsto X_T^{t,x}$$

the solution of (1.1) starting from x at time t and by

$$\varphi_{X_T^{t,x}}(\xi) = E\left[e^{i\xi X_T^{t,x}}\right], \qquad \xi \in \mathbb{R},$$

the characteristic function of $X_T^{t,x}$. Our main result, proved in Section 4, is a second order approximation formula of $\varphi_{X_T^{t,x}}$: in the case of time-homogeneous coefficients, it reads as follows

$$\begin{aligned}\varphi_{X_t^{0,x}}(\xi) \approx e^{i\xi x + t\psi(\xi)}\Bigg(1 + \frac{\psi'(\xi)}{2} i\alpha_1 t^2(\xi^2 + i\xi) + \psi''(\xi)\Bigg(&\frac{t^2}{2}\alpha_2\xi(i+\xi) \\ &- \frac{t^3}{6}\xi(i+\xi)\left(\alpha_1^2(i+2\xi) - 2\alpha_2\psi''(\xi) + \alpha_1^2\xi(i+\xi)\right) \\ &- \frac{t^4}{8}\alpha_1^2\xi^2(i+\xi)^2\psi''(\xi)\Bigg)\Bigg)\end{aligned} \tag{1.2}$$

where ψ, defined in (4.55), is the characteristic exponent of the Lévy process (4.56) which is the leading term of the expansion and the constants α_k are the coefficients of the Taylor expansion of the diffusion coefficient (see the general notations introduced in Section 2). In some particular cases, we also obtain an explicit approximation of the transition density of X.

Local Lévy models of the form (1.1) have attracted an increasing interest in the theory of volatility modeling (see, for instance, [1], [4] and [7]); however to date only in a few cases closed pricing formulae are available. The approximation formula (1.2) provides a way to compute efficiently and accurately option prices and sensitivities by using standard and well-known Fourier methods (see, for instance, Heston [17], Carr and Madan [5], Raible [27] and Lipton [22]).

We derive formula (1.2) by introducing an "adjoint" expansion method: this is worked out in the Fourier space by considering the adjoint formulation of the pricing problem. Generally speaking, our approach makes use of Fourier analysis and PDE techniques. In Section 2, we present the general procedure that allows to approximate analytically the transition density (or the characteristic function), in terms of the solutions of a sequence of nested Cauchy problems. Then we also prove explicit error bounds for the expansion that generalize in a new and nontrivial way some classical estimates. In the second part of the paper (Sections 3 and 4) the previous Cauchy problems are solved explicitly by using different approaches. In Section 3 we focus on the special class of local Lévy models with Gaussian jumps and we provide a heat kernel expansion of the transition density of the underlying process. The same results are derived in an alternative way in Subsection 3.1, by working in the Fourier space.

Section 4 contains the main contribution of the paper: we consider the general class of local Lévy models and provide high order approximations of the characteristic function. Since all the computations are carried out in the Fourier space, we are forced to introduce *a dual formulation* of the approximating problems, which involves the adjoint (forward) Kolmogorov operator. Even if at first sight the adjoint expansion method seems a bit odd, it turns out to be much more natural and simpler than the direct formulation. To the best of our knowledge, the interplay between perturbation methods and Fourier analysis has not been previously studied in finance. Actually our approach seems to be advantageous for several reasons:

- working in the Fourier space is natural and allows to get simple and clear results;
- we can treat the entire class of Lévy processes and not only jump-diffusions or processes which can be approximated by heat kernel expansions. Potentially, we can take as leading term of the expansion every process which admits an explicit characteristic function and not necessarily a Gaussian kernel;
- our method can be easily adapted to the case of stochastic volatility or multi-asset models;
- higher order approximations are rather easy to derive and the approximation results are generally very accurate. Potentially it is possible to derive approximation formulae for the characteristic function and plain vanilla options, at any prescribed order: for example, in Subsection 4.1 we provide also the 3^{rd} and 4^{th} order expansions of the characteristic function, used in the numerical tests of Section 5. The Mathematica notebook with the implemented formulae is freely available in the website of the authors.

For completeness, in the last part of Section 4, a standard pricing integral formula for European options is stated. Finally, in Section 5, we present some numerical tests under the Merton and Variance-Gamma models and show the effectiveness of the analytical approximations compared with Monte Carlo simulation.

Comparison with the literature. Analytical approximations and their applications to finance have been studied by several authors in the last decades because of their great importance in the calibration and risk management processes. The large body of the existing literature (see, for instance, [16], [18], [28], [15], [3], [8], [6]) is mainly devoted to purely diffusive (local and stochastic volatility) models or, as in [2] and [29], to local volatility

(LV) models with Poisson jumps, which can be approximated by Gaussian kernels.

The classical result by Hagan [16] is a particular case of our expansion, in the sense that for a standard LV model with time-homogeneous coefficients our formulae reduce to Hagan's ones (see Section 3.1). While Hagan's results are heuristic, here we also provide explicit error estimates for time-dependent coefficients as well.

The results of Section 3 on the approximation of the transition density for jump-diffusions are essentially analogous to the results in [2]: however in [2] ad-hoc Malliavin techniques for LV models with Merton jumps are used and only a first order expansion is derived. Here we use different techniques (PDE and Fourier methods) which allows to handle the much more general class of local Lévy processes: this is a very significant difference from previous research. Moreover we derive higher order approximations, up to the 4^{th} order.

Our approach is also more general than the so-called "parametrix" methods recently proposed in [8] and [6] as an approximation method in finance. The parametrix method is based on repeated application of Duhamel's principle which leads to a recursive integral representation of the fundamental solution: the main problem with the parametrix approach is that, even in the simplest case of a LV model, it is hard to compute explicitly the parametrix approximations of order greater than one. As a matter of fact, [8] and [6] only contain first order formulae. The adjoint expansion method contains the parametrix approximation *as a particular case*, that is at order zero and in the purely diffusive case. However the general construction of the adjoint expansion is substantially different and allows us to find explicit higher-order formulae for the general class of local Lévy processes.

2 General framework

In a local Lévy model, we assume that the risk-neutral dynamics of the underlying asset process X is given by equation (1.1). In order to guarantee the martingale property for the discounted asset price $\widetilde{S}_t := S_0 e^{X_t - rt}$, we set

$$\mu(t,x) = \bar{r} - \mu_1 - \frac{\sigma^2(t,x)}{2}, \tag{2.3}$$

where

$$\bar{r} = r - \int_{\mathbb{R}} \left(e^y - 1 - y\mathbb{1}_{\{|y|<1\}}\right) \nu(dy). \tag{2.4}$$

We denote by

$$\Gamma(t,x;T,\cdot)$$

the law of $X_T^{t,x}$, which is the fundamental solution of the Kolmogorov operator

$$\begin{aligned} Lu(t,x) = &\frac{\sigma^2(t,x)}{2}\left(\partial_{xx} - \partial_x\right)u(t,x) + \bar{r}\partial_x u(t,x) + \partial_t u(t,x) \\ &+ \int_{\mathbb{R}} \left(u(t,x+y) - u(t,x) - \partial_x u(t,x) y \mathbb{1}_{\{|y|<1\}}\right)\nu(dy). \end{aligned} \tag{2.5}$$

Notice that the characteristic function of $X_T^{t,x}$ is equal to

$$\varphi_{X_T^{t,x}}(\xi) = \int_{\mathbb{R}} e^{i\xi y}\Gamma(t,x;T,y)dy.$$

Example 2.1. *Let J be a compound Poisson process with Gaussian jumps, that is*

$$J_t = \sum_{n=1}^{N_t} Z_n$$

where N_t is a Poisson process with intensity λ and Z_n are i.i.d. random variables independent of N_t with Normal distribution $\mathcal{N}_{m,\delta^2}$. In this case, $\nu = \lambda\mathcal{N}_{m,\delta^2}$ and

$$\mu_1 = \int_{|y|<1} y\nu(dy).$$

Therefore the drift condition (2.3) *reduces to*

$$\mu(t,x) = r_0 - \frac{\sigma^2(t,x)}{2}, \tag{2.6}$$

where

$$r_0 = r - \int_{\mathbb{R}} \left(e^y - 1\right)\nu(dy) = r - \lambda\left(e^{m+\frac{\delta^2}{2}} - 1\right). \tag{2.7}$$

Moreover, the characteristic operator can be written in the equivalent form

$$\begin{aligned} Lu(t,x) = &\frac{\sigma^2(t,x)}{2}\left(\partial_{xx} - \partial_x\right)u(t,x) + r_0\partial_x u(t,x) + \partial_t u(t,x) \\ &+ \int_{\mathbb{R}} \left(u(t,x+y) - u(t,x)\right)\nu(dy). \end{aligned} \tag{2.8}$$

Example 2.2. *Let J be a Variance-Gamma process (cf. [23]) obtained by subordinating a Brownian motion with drift θ and standard deviation ϱ, by a Gamma process with variance κ and unitary mean. In this case the Lévy measure is given by*

$$\nu(dx) = \frac{e^{-\lambda_1 x}}{\kappa x}\mathbb{1}_{\{x>0\}}(x)dx + \frac{e^{\lambda_2 x}}{\kappa|x|}\mathbb{1}_{\{x<0\}}(x)dx \tag{2.9}$$

where

$$\lambda_1 = \left(\sqrt{\frac{\theta^2\kappa^2}{4} + \frac{\varrho^2\kappa}{2}} + \frac{\theta\kappa}{2}\right)^{-1}, \qquad \lambda_2 = \left(\sqrt{\frac{\theta^2\kappa^2}{4} + \frac{\varrho^2\kappa}{2}} - \frac{\theta\kappa}{2}\right)^{-1}.$$

The risk-neutral drift in (1.1) *is equal to*

$$\mu(t,x) = r_0 - \frac{\sigma^2(t,x)}{2}$$

where

$$r_0 = r + \frac{1}{\kappa}\log\left(1-\lambda_1^{-1}\right)\left(1+\lambda_2^{-1}\right) = r + \frac{1}{\kappa}\log\left(1-\kappa\left(\theta+\frac{\varrho^2}{2}\right)\right), \quad (2.10)$$

and the expression of the characteristic operator L is the same as in (2.8) *with ν and r_0 as in* (2.9) *and* (2.10) *respectively.*

Our goal is to give an accurate analytic approximation of the characteristic function and, when possible, of the transition density of X. The general idea is to consider an approximation of the volatility coefficient σ. More precisely, to shorten notations we set

$$a(t,x) = \sigma^2(t,x) \quad (2.11)$$

and we assume that a is regular enough: more precisely, for a fixed $N \in \mathbb{N}$, we make the following

Assumption $\mathbf{A}_N$. *The function $a = a(x,t)$ is continuously differentiable with respect to x up to order N. Moreover, the function a and its derivatives in x are bounded and Lipschitz continuous in x, uniformly with respect to t.*

Next, we fix a basepoint $\bar{x} \in \mathbb{R}$ and consider the N^{th}-order Taylor polynomial of $a(t,x)$ about $\bar{x}$:

$$\alpha_0(t) + 2\sum_{n=1}^{N} \alpha_n(t)(x-\bar{x})^n,$$

where $\alpha_0(t) = a(t,\bar{x})$ and

$$\alpha_n(t) = \frac{1}{2}\frac{\partial_x^n a(t,\bar{x})}{n!}, \qquad n \leq N. \quad (2.12)$$

Then we introduce the n^{th}-order approximation of L:

$$L_n := L_0 + \sum_{k=1}^{n} \alpha_k(t)(x-\bar{x})^k \left(\partial_{xx} - \partial_x\right), \qquad n \leq N, \quad (2.13)$$

where

$$
\begin{aligned}
L_0u(t,x) = {} & \frac{\alpha_0(t)}{2}\left(\partial_{xx}u(t,x) - \partial_x u(t,x)\right) + \bar{r}\partial_x u(t,x) + \partial_t u(t,x) \\
& + \int_{\mathbb{R}} \left(u(t,x+y) - u(t,x) - \partial_x u(t,x) y \mathbb{1}_{\{|y|<1\}}\right) \nu(dy).
\end{aligned} \tag{2.14}
$$

Following the perturbation method proposed in [25], and also recently used in [12] for the approximation of Asian options, the n^{th}-order approximation of the fundamental solution Γ of L is defined by

$$
\Gamma^n(t,x;T,y) := \sum_{k=0}^{n} G^k(t,x;T,y), \qquad t < T, \ x, y \in \mathbb{R}. \tag{2.15}
$$

The leading term G^0 of the expansion in (2.15) is the fundamental solution of L_0 and, for any $(T,y) \in \mathbb{R}_+ \times \mathbb{R}$ and $k \le N$, the functions $G^k(\cdot,\cdot;T,y)$ are defined recursively in terms of the solutions of the following sequence of Cauchy problems on the strip $]0,T[\times\mathbb{R}$:

$$
\begin{cases}
L_0 G^k(t,x;T,y) = -\sum\limits_{h=1}^{k} \left(L_h - L_{h-1}\right) G^{k-h}(t,x;T,y) \\
\qquad\qquad\qquad\quad = -\sum\limits_{h=1}^{k} \alpha_h(t)(x-\bar{x})^h \left(\partial_{xx} - \partial_x\right) G^{k-h}(t,x;T,y), \\
G^k(T,x;T,y) = 0.
\end{cases} \tag{2.16}
$$

In the sequel, when we want to specify explicitly the dependence of the approximation Γ^n on the basepoint $\bar{x}$, we shall use the notation

$$
\Gamma^{\bar{x},n}(t,x;T,y) \equiv \Gamma^n(t,x;T,y). \tag{2.17}
$$

In Section 3 we show that, in the case of a LV model with Gaussian jumps, it is possible to find *the explicit solutions* to the problems (2.16) by an iterative argument. When general Lévy jumps are considered, it is still possible to compute the explicit solution of problems (2.16) *in the Fourier space*. Indeed, in Section 4, we get an expansion of the characteristic function $\varphi_{X_T^{t,x}}$ having as the leading term the characteristic function of the process whose Kolmogorov operator is L_0 in (2.14).

We explicitly notice that, if the function σ only depends on time, then *the approximation in* (2.15) *is exact at order zero.*

We now provide global error estimates for the approximation in the purely diffusive case. The proof is postponed to the Appendix.

Theorem 2.3. *Assume the parabolicity condition*

$$
m \le \frac{a(t,x)}{2} \le M, \qquad (t,x) \in [0,T] \times \mathbb{R}, \tag{2.18}
$$

where m, M are positive constants and let $\bar{x} = x$ or $\bar{x} = y$ in (2.17)*. Under Assumption A_{N+1}, for any $\varepsilon > 0$ we have*

$$\left|\Gamma(t,x;T,y) - \Gamma^{\bar{x},N}(t,x;T,y)\right| \leq g_N(T-t)\bar{\Gamma}^{M+\varepsilon}(t,x;T,y), \tag{2.19}$$

for $x, y \in \mathbb{R}$ and $t \in [0,T[$, where $\bar{\Gamma}^M$ is the Gaussian fundamental solution of the heat operator

$$M\partial_{xx} + \partial_t,$$

and $g_N(s) = O\left(s^{\frac{N+1}{2}}\right)$ as $s \to 0^+$.

Theorem 2.3 improves some known results in the literature. In particular in [3] asymptotic estimates for option prices in terms of $(T-t)^{\frac{N+1}{2}}$ are proved under a stronger assumption on the regularity of the coefficients, equivalent to Assumption A_{3N+2}. Here we provide error estimates for the transition density: error bounds for option prices can be easily derived from (2.19). Moreover, for small N it is not difficult to find the explicit expression of g_N.

Estimate (2.19) also justifies a time-splitting procedure which nicely adapts to our approximation operators, as shown in detail in Remark 2.7 in [25].

3 LV models with Gaussian jumps

In this section we consider the SDE (1.1) with J as in Example 2.1, namely J is a compound Poisson process with Gaussian jumps. Clearly, in the particular case of a constant diffusion coefficient $\sigma(t,x) \equiv \sigma$, we have the classical Merton jump-diffusion model [24]:

$$X_t^{\text{Merton}} = \left(r_0 - \frac{\sigma^2}{2}\right)t + \sigma W_t + J_t,$$

with r_0 as in (2.7). We recall that the analytical approximation of this kind of models has been recently studied by Benhamou, Gobet and Miri in [2] by Malliavin calculus techniques.

The expression of the pricing operator L was given in (2.8) and in this case the leading term of the approximation (cf. (2.14)) is equal to

$$\begin{aligned} L_0 v(t,x) = {} & \frac{\alpha_0(t)}{2}\left(\partial_{xx}v(t,x) - \partial_x v(t,x)\right) + r_0\partial_x v(t,x) \\ & + \partial_t v(t,x) + \int_{\mathbb{R}} \left(v(t,x+y) - v(t,x)\right)\nu(dy). \end{aligned} \tag{3.20}$$

The fundamental solution of L_0 is the transition density of a Merton process, that is

$$G^0(t,x;T,y) = e^{-\lambda(T-t)}\sum_{n=0}^{+\infty}\frac{(\lambda(T-t))^n}{n!}\Gamma_n(t,x;T,y), \tag{3.21}$$

where

$$\Gamma_n(t,x;T,y) = \frac{1}{\sqrt{2\pi\left(A(t,T)+n\delta^2\right)}}\, e^{-\frac{\left(x-y+(T-t)r_0-\frac{1}{2}A(t,T)+nm\right)^2}{2\left(A(t,T)+n\delta^2\right)}},$$
$$A(t,T) = \int_t^T \alpha_0(s)ds. \tag{3.22}$$

In order to determine the explicit solution to problems (2.16) for $k \geq 1$, we use some elementary properties of the functions $(\Gamma_n)_{n\geq 0}$. The following lemma can be proved as Lemma 2.2 in [25].

Lemma 3.1. *For any $x, y, \bar{x} \in \mathbb{R}$, $t < s < T$ and $n, k \in \mathbb{N}_0$, we have*

$$\Gamma_{n+k}(t,x;T,y) = \int_{\mathbb{R}} \Gamma_n(t,x;s,\eta)\Gamma_k(s,\eta;T,y)d\eta, \tag{3.23}$$
$$\partial_y^k \Gamma_n(t,x;T,y) = (-1)^k \partial_x^k \Gamma_n(t,x;T,y), \tag{3.24}$$
$$(y-\bar{x})^k \Gamma_n(t,x;T,y) = V_{t,T,x,n}^k \Gamma_n(t,x;T,y), \tag{3.25}$$

where $V_{t,T,x,n}$ is the operator defined by

$$\begin{aligned} V_{t,T,x,n} f(x) = &\left(x - \bar{x} + (T-t)r_0 - \frac{1}{2}A(t,T) + nm\right) f(x) \\ &+ \left(A(t,T) + n\delta^2\right) \partial_x f(x). \end{aligned} \tag{3.26}$$

Our first results are the following first and second order expansions of the transition density Γ.

Theorem 3.2 (1st order expansion)**.** *The solution G^1 of the Cauchy problem* (2.16) *with $k = 1$ is given by*

$$G^1(t,x;T,y) = \sum_{n,k=0}^{+\infty} J_{n,k}^1(t,T,x)\Gamma_{n+k}(t,x;T,y). \tag{3.27}$$

where $J_{n,k}^1(t,T,x)$ is the differential operator defined by

$$J_{n,k}^1(t,T,x) = e^{-\lambda(T-t)} \frac{\lambda^{n+k}}{n!k!} \int_t^T \alpha_1(s)(s-t)^n (T-s)^k V_{t,s,x,n} ds\, (\partial_{xx} - \partial_x). \tag{3.28}$$

Proof. By the standard representation formula for solutions to the non-homogeneous parabolic Cauchy problem (2.16) with null final condition, we have

$$\begin{aligned} G^1(t,x;T,y) = &\int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta)\alpha_1(s)(\eta-\bar{x})\cdot \\ &\cdot(\partial_{\eta\eta} - \partial_\eta)G^0(s,\eta;T,y)d\eta ds = \end{aligned}$$

(by (3.25))

$$
\begin{aligned}
&= \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} \int_t^T \alpha_1(s) e^{-\lambda(s-t)} (s-t)^n \cdot \\
&\quad \cdot V_{t,s,x,n} \int_{\mathbb{R}} \Gamma_n(t,x;s,\eta)(\partial_{\eta\eta} - \partial_\eta) G^0(s,\eta;T,y) d\eta ds =
\end{aligned}
$$

(by parts)

$$
\begin{aligned}
&= e^{-\lambda(T-t)} \sum_{n,k=0}^{+\infty} \frac{\lambda^{n+k}}{n!k!} \int_t^T \alpha_1(s)(T-s)^k (s-t)^n \cdot \\
&\quad \cdot V_{t,s,x,n} \int_{\mathbb{R}} (\partial_{\eta\eta} + \partial_\eta) \Gamma_n(t,x;s,\eta) \Gamma_k(s,\eta;T,y) d\eta ds =
\end{aligned}
$$

(by (3.24) and (3.23))

$$
\begin{aligned}
&= e^{-\lambda(T-t)} \sum_{n,k=0}^{\infty} \frac{\lambda^{n+k}}{n!k!} \int_t^T \alpha_1(s)(T-s)^k (s-t)^n V_{t,s,x,n} ds \cdot \\
&\quad \cdot (\partial_{xx} - \partial_x) \Gamma_{n+k}(t,x;T,y)
\end{aligned}
$$

and this proves (3.27)-(3.28). □

Remark 3.3. *A straightforward but tedious computation shows that the operator $J^1_{n,k}(t,T,x)$ can be rewritten in the more convenient form*

$$
J^1_{n,k}(t,T,x) = \sum_{i=1}^{3} \sum_{j=0}^{1} f^1_{n,k,i,j}(t,T)(x-\bar{x})^j \partial_x^i, \tag{3.29}
$$

for some deterministic functions $f^1_{n,k,i,j}$.

Theorem 3.4 (2nd order expansion)**.** *The solution G^2 of the Cauchy problem* (2.16) *with $k=2$ is given by*

$$
\begin{aligned}
G^2(t,x;T,y) = &\sum_{n,h,k=0}^{+\infty} J^{2,1}_{n,h,k}(t,T,x) \Gamma_{n+h+k}(t,x;T,y) \\
&+ \sum_{n,k=0}^{\infty} J^{2,2}_{n,k}(t,T,x) \Gamma_{n+k}(t,x;T,y),
\end{aligned} \tag{3.30}
$$

where

$$
\begin{aligned}
J^{2,1}_{n,h,k}(t,T,x) &= \frac{\lambda^n}{n!} \int_t^T \alpha_1(s) e^{-\lambda(s-t)} (s-t)^n V_{t,s,x,n} (\partial_{xx} - \partial_x) \widetilde{J}^1_{h,k}(t,s,T,x) ds \\
J^{2,2}_{n,k}(t,T,x) &= e^{-\lambda(T-t)} \frac{\lambda^{n+k}}{n!k!} \int_t^T \alpha_2(s)(s-t)^n (T-s)^k V^2_{t,s,x,n} ds \, (\partial_{xx} - \partial_x)
\end{aligned}
$$

and $\widetilde{J}^1_{h,k}$ *is the "adjoint" operator of* $J^1_{h,k}$*, defined by*

$$\widetilde{J}^1_{h,k}(t,s,T,x) = \sum_{i=1}^{3}\sum_{j=0}^{1} f^1_{h,k,i,j}(s,T) V^j_{t,s,x,h+k}\partial^i_x \tag{3.31}$$

with $f^1_{h,k,i,j}$ *as in* (3.29)*. Also in this case we have the alternative representation*

$$J^{2,1}_{n,h,k}(t,T,x) = \sum_{i=1}^{6}\sum_{j=0}^{2} f^{2,1}_{n,h,k,i,j}(t,T)(x-\bar{x})^j\partial^i_x \tag{3.32}$$

$$J^{2,2}_{n,k}(t,T,x) = \sum_{i=1}^{6}\sum_{j=0}^{2} f^{2,2}_{n,k,i,j}(t,T)(x-\bar{x})^j\partial^i_x, \tag{3.33}$$

with $f^{2,1}_{n,h,k,i,j}$ *and* $f^{2,2}_{n,k,i,j}$ *deterministic functions.*

Proof. We show a preliminary result: from formulae (3.29) and (3.31) for J^1 and $\widetilde{J}^1$ respectively, it follows that

$$\int_{\mathbb{R}} \Gamma_n(t,x;s,\eta) J^1_{h,k}(s,T,\eta)\Gamma_{h+k}(s,\eta;T,y)d\eta =$$

(by (3.24) and (3.25))

$$\begin{aligned}
&= \int_{\mathbb{R}} \widetilde{J}^1_{h,k}(s,T,x)\Gamma_n(t,x;s,\eta)\Gamma_{h+k}(s,\eta;T,y)d\eta \\
&= \widetilde{J}^1_{h,k}(s,T,x)\int_{\mathbb{R}} \Gamma_n(t,x;s,\eta)\Gamma_{h+k}(s,\eta;T,y)d\eta =
\end{aligned}$$

(by (3.23))

$$= \widetilde{J}^1_{h,k}(s,T,x)\Gamma_{n+h+k}(x,t;T,y). \tag{3.34}$$

Now we have

$$G^2(t,x;T,y) = I_1 + I_2,$$

where, proceeding as before,

$$\begin{aligned}
I_1 &= \int_t^T\int_{\mathbb{R}} G^0(t,x;s,\eta)\alpha_1(s)(\eta-\bar{x})(\partial_{\eta\eta}-\partial_\eta)G^1(s,\eta;T,y)d\eta ds \\
&= \sum_{n,h,k=0}^{+\infty}\frac{\lambda^n}{n!}\int_t^T \alpha_1(s)e^{-\lambda(s-t)}(s-t)^n \cdot \\
&\quad \cdot V_{t,s,x,n}\int_{\mathbb{R}}\Gamma_n(t,x;s,\eta)(\partial_{\eta\eta}-\partial_\eta)J^1_{h,k}(s,T,\eta)\Gamma_{h+k}(s,\eta;T,y)d\eta ds
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n,h,k=0}^{+\infty} \frac{\lambda^n}{n!} \int_t^T \alpha_1(s) e^{-\lambda(s-t)} (s-t)^n \cdot \\
&\quad \cdot V_{t,s,x,n}(\partial_{xx} - \partial_x) \int_{\mathbb{R}} \Gamma_n(t,x;s,\eta) J^1_{h,k}(s,T,\eta) \Gamma_{h+k}(s,\eta;T,y) d\eta ds =
\end{aligned}
$$

(by (3.34))

$$
\begin{aligned}
&= \sum_{n,h,k=0}^{+\infty} \frac{\lambda^n}{n!} \int_t^T \alpha_1(s) e^{-\lambda(s-t)} (s-t)^n V_{t,s,x,n}(\partial_{xx} - \partial_x) \widetilde{J}^1_{h,k}(s,T,x) ds \cdot \\
&\quad \cdot \Gamma_{n+h+k}(x,t;T,y) \\
&= \sum_{n,h,k=0}^{+\infty} J^{2,1}_{n,h,k}(t,T,x) \Gamma_{n+h+k}(t,x;T,y)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) \alpha_2(s) (\eta - \bar{x})^2 (\partial_{\eta\eta} - \partial_\eta) G^0(s,\eta;T,y) d\eta ds \\
&= e^{-\lambda(T-t)} \sum_{n,k=0}^{+\infty} \frac{\lambda^{n+k}}{n!k!} \int_t^T \alpha_2(s) (T-s)^k (s-t)^n \cdot \\
&\quad \cdot V^2_{t,s,x,n} \int_{\mathbb{R}} \Gamma_n(t,x;s,\eta) (\partial_{\eta\eta} - \partial_\eta) \Gamma_k(s,\eta;T,y) d\eta ds \\
&= e^{-\lambda(T-t)} \sum_{n,k=0}^{+\infty} \frac{\lambda^{n+k}}{n!k!} \int_t^T \alpha_2(s) (T-s)^k (s-t)^n \cdot \\
&\quad \cdot V^2_{t,s,x,n} (\partial_{xx} - \partial_x) \int_{\mathbb{R}} \Gamma_n(t,x;s,\eta) \Gamma_k(s,\eta;T,y) d\eta ds \\
&= e^{-\lambda(T-t)} \sum_{n,k=0}^{+\infty} \frac{\lambda^{n+k}}{n!k!} \int_t^T \alpha_2(s) (T-s)^k (s-t)^n \cdot \\
&\quad \cdot V^2_{t,s,x,n} ds \, (\partial_{xx} - \partial_x) \Gamma_{n+k}(t,x;T,y) \\
&= \sum_{n,k=0}^{+\infty} J^{2,2}_{n,k}(t,T,x) \Gamma_{n+k}(t,x;T,y).
\end{aligned}
$$

This concludes the proof. □

Remark 3.5. *Since the derivatives of a Gaussian density can be expressed in terms of Hermite polynomials, the computation of the terms of the expansion* (2.15) *is very fast. Indeed, we have*

$$
\frac{\partial_x^i \Gamma_n(t,x;T,y)}{\Gamma_n(t,x;T,y)} = \frac{(-1)^i h_{i,n}(t,T,x-y)}{\left(2\left(A(t,T) + n\delta^2\right)\right)^{\frac{i}{2}}}
$$

where

$$h_{i,n}(t,T,z)=\mathbf{H}_i\left(\frac{z+(T-t)\mu_0-\frac{1}{2}A(t,T)+nm}{\sqrt{2\left(A(t,T)+n\delta^2\right)}}\right)$$

and $\mathbf{H}_i=\mathbf{H}_i(x)$ *denotes the Hermite polynomial of degree* i. *Thus we can rewrite the terms* $\left(G^k\right)_{k=1,2}$ *in* (3.27) *and* (3.30) *as follows:*

$$\begin{aligned}
G^1(t,x;T,y) &= \sum_{n,k=0}^{\infty}\mathbf{G}^1_{n,k}(t,x;T,y)\Gamma_{n+k}(t,x;T,y)\\
G^2(t,x;T,y) &= \sum_{n,h,k=0}^{\infty}\mathbf{G}^{2,1}_{n,h,k}(t,x;T,y)\Gamma_{n+h+k}(t,x;T,y)\\
&\quad+\sum_{n,k=0}^{\infty}\mathbf{G}^{2,2}_{n,k}(t,x;T,y)\Gamma_{n+k}(t,x;T,y),
\end{aligned}\tag{3.35}$$

where

$$\begin{aligned}
\mathbf{G}^1_{n,k}(t,x;T,y) &= \sum_{i=1}^{3}(-1)^i\sum_{j=0}^{1}f^1_{n,k,i,j}(t,T)(x-\bar{x})^j\frac{h_{i,n+k}(t,T,x-y)}{\left(2\left(A(t,T)+(n+k)\delta^2\right)\right)^{\frac{i}{2}}}\\
\mathbf{G}^{2,1}_{n,h,k}(t,x;T,y) &= \sum_{i=1}^{6}(-1)^i\sum_{j=0}^{1}f^{2,1}_{n,h,k,i,j}(t,T)(x-\bar{x})^j\frac{h_{i,n+h+k}(t,T,x-y)}{\left(2\left(A(t,T)+(n+h+k)\delta^2\right)\right)^{\frac{i}{2}}}\\
\mathbf{G}^{2,2}_{n,k}(t,x;T,y) &= \sum_{i=1}^{6}(-1)^i\sum_{j=0}^{1}f^{2,2}_{n,k,i,j}(t,T)(x-\bar{x})^j\frac{h_{i,n+k}(t,T,x-y)}{\left(2\left(A(t,T)+(n+k)\delta^2\right)\right)^{\frac{i}{2}}}.
\end{aligned}$$

In the practical implementation, we truncate the series in (3.21) *and* (3.35) *to a finite number of terms, say* $M\in\mathbb{N}$. *Therefore we put*

$$\begin{aligned}
G^0_M(t,x;T,y) &= e^{-\lambda(T-t)}\sum_{n=0}^{M}\frac{(\lambda(T-t))^n}{n!}\Gamma_n(t,x;T,y),\\
G^1_M(t,x;T,y) &= \sum_{n,k=0}^{M}\mathbf{G}^1_{n,k}(t,x;T,y)\Gamma_{n+k}(t,x;T,y),\\
G^2_M(t,x;T,y) &= \sum_{n,h,k=0}^{M}\mathbf{G}^{2,1}_{n,h,k}(t,x;T,y)\Gamma_{n+h+k}(t,x;T,y)\\
&\quad+\sum_{n,k=0}^{M}\mathbf{G}^{2,2}_{n,k}(t,x;T,y)\Gamma_{n+k}(t,x;T,y),
\end{aligned}$$

and we approximate the density Γ *by*

$$\Gamma^2_M(t,x;T,y):=G^0_M(t,x;T,y)+G^1_M(t,x;T,y)+G^2_M(t,x;T,y).\tag{3.36}$$

Next we denote by $C(t, S_t)$ the price at time $t < T$ of a European option with payoff function φ and maturity T; for instance, $\varphi(y) = (y-K)^+$ in the case of a Call option with strike K. From the expansion of the density in (3.36), we get the following second order approximation formula.

Corollary 3.6. *We have*

$$C(t, S_t) \approx e^{-r(T-t)} u_M(t, \log S_t)$$

where

$$\begin{aligned} u_M(t,x) &= \int_{\mathbb{R}^+} \frac{1}{S} \Gamma_M^2(t,x;T,\log S)\varphi(S)dS \\ &= e^{-\lambda(T-t)} \sum_{n=0}^{M} \frac{(\lambda(T-t))^n}{n!} \mathrm{CBS}_n(t,x) \\ &\quad + \sum_{n,k=0}^{M} \left(J^1_{n,k}(t,T,x) + J^{2,2}_{n,k}(t,T,x) \right) \mathrm{CBS}_{n+k}(t,x) \\ &\quad + \sum_{n,h,k=0}^{M} J^{2,1}_{n,h,k}(t,T,x) \mathrm{CBS}_{n+h+k}(t,x) \end{aligned} \tag{3.37}$$

and $\mathrm{CBS}_n(t,x)$ *is the BS price*[1] *under the Gaussian law* $\Gamma_n(t,x;T,\cdot)$ *in* (3.22)*, namely*

$$\mathrm{CBS}_n(t,x) = \int_{\mathbb{R}^+} \frac{1}{S} \Gamma_n(t,x;T,\log S)\varphi(S)dS.$$

3.1 Simplified Fourier approach for LV models

Equation (1.1) with $J = 0$ reduces to the standard SDE of a LV model. In this case we can simplify the proof of Theorems 3.2-3.4 by using Fourier analysis methods. Let us first notice that L_0 in (3.20) becomes

$$L_0 = \frac{\alpha_0(t)}{2} (\partial_{xx} - \partial_x) + r\partial_x + \partial_t, \tag{3.38}$$

and its fundamental solution is the Gaussian density

$$G^0(t,x;T,y) = \frac{1}{\sqrt{2\pi A(t,T)}} e^{-\frac{\left(x-y+(T-t)r-\frac{1}{2}A(t,T)\right)^2}{2A(t,T)}},$$

with A as in (3.22).

[1] Here the BS price is expressed as a function of the time t and of the log-asset x.

Corollary 3.7 (1st order expansion)**.** *In case of $\lambda = 0$, the solution G^1 in* (3.27) *is given by*

$$G^1(t, x; T, y) = J^1(t, T, x) G^0(t, x; T, y) \tag{3.39}$$

where $J^1(t, T, x)$ is the differential operator

$$J^1(t, T, x) = \int_t^T \alpha_1(s) V_{t,s,x} ds \, (\partial_{xx} - \partial_x), \tag{3.40}$$

with $V_{t,s,x} \equiv V_{t,s,x,0}$ as in (3.26)*, that is*

$$V_{t,T,x} f(x) = \left(x - \bar{x} + (T - t) r - \frac{1}{2} A(t, T) \right) f(x) + A(t, T) \partial_x f(x).$$

Proof. Although the thesis follows directly from Theorem 3.2, here we propose an alternative proof of formula (3.40). The idea is to determine the solution of the Cauchy problem (2.16) in the Fourier space, where all the computation can be carried out more easily; then, using the fact that the leading term G^0 of the expansion is a Gaussian kernel, we are able to compute explicitly the inverse Fourier transform to get back to the analytic approximation of the transition density.

Since we aim at showing the main ideas of an alternative approach, for simplicity we only consider the case of time-independent coefficients, precisely we set $\alpha_0 = 2$ and $r = 0$. In this case we have

$$L_0 = \partial_{xx} - \partial_x + \partial_t$$

and the related Gaussian fundamental solution is equal to

$$G^0(t, x; T, y) = \frac{1}{\sqrt{4\pi(T - t)}} e^{-\frac{(x - y - (T - t))^2}{4(T - t)}}.$$

Now we apply the Fourier transform (in the variable x) to the Cauchy problem (2.16) with $k = 1$ and we get

$$\begin{cases} \partial_t \hat{G}^1(t, \xi; T, y) = \left(\xi^2 - i\xi\right) \hat{G}^1(t, \xi; T, y) \\ \qquad\qquad\qquad\qquad + \alpha_1 (i\partial_\xi + \bar{x}) \left(-\xi^2 + i\xi\right) \hat{G}^0(t, \xi; T, y), \\ \hat{G}^1(T, \xi; T, y) = 0, \qquad \xi \in \mathbb{R}. \end{cases} \tag{3.41}$$

Notice that

$$\hat{G}^0(t, \xi; T, y) = e^{-\xi^2 (T - t) + i\xi(y + (T - t))}. \tag{3.42}$$

Therefore the solution to the ordinary differential equation (3.41) is

$$\hat{G}^1(t, \xi; T, y) = -\alpha_1 \int_t^T e^{(s - t)(-\xi^2 + i\xi)} (i\partial_\xi + \bar{x}) \left((-\xi^2 + i\xi) \hat{G}^0(s, \xi; T, y) \right) ds =$$

(using the identity $f(\xi)(i\partial_\xi + \bar{x})(g(\xi)) = (i\partial_\xi + \bar{x})(f(\xi)g(\xi)) - ig(\xi)\partial_\xi f(\xi)$)

$$
\begin{aligned}
&= -\alpha_1 \int_t^T (i\partial_\xi + \bar{x}) \left((-\xi^2 + i\xi) e^{(s-t)(-\xi^2+i\xi)} \hat{G}^0(s,\xi;T,y) \right) ds \\
&\quad + i\alpha_1 \int_t^T (-\xi^2 + i\xi) \hat{G}^0(s,\xi;T,y) \partial_\xi e^{(s-t)(-\xi^2+i\xi)} ds =
\end{aligned}
$$

(by (3.42))

$$
\begin{aligned}
&= -\alpha_1 \int_t^T (i\partial_\xi + \bar{x}) \left((-\xi^2 + i\xi) e^{i\xi(y+(T-t))-\xi^2(T-t)} \right) ds \\
&\quad + i\alpha_1 \int_t^T (-\xi^2 + i\xi)(s-t)(-2\xi + i) e^{i\xi(y+(T-t))-\xi^2(T-t)} ds =
\end{aligned}
$$

(again by (3.42))

$$
\begin{aligned}
&= -\alpha_1 (T-t)(i\partial_\xi + \bar{x}) \left((-\xi^2 + i\xi) \hat{G}^0(t,\xi;T,y) \right) \\
&\quad + i\alpha_1 \frac{(T-t)^2}{2} (-\xi^2 + i\xi)(-2\xi + i) \hat{G}^0(t,\xi;T,y).
\end{aligned}
$$

Thus, by inverting the Fourier transform, we get

$$
\begin{aligned}
G^1(t,x;T,y) &= \alpha_1 (T-t)(x-\bar{x})(\partial_x^2 - \partial_x) G^0(t,x;T,y) + \\
&\quad - \alpha_1 \frac{(T-t)^2}{2} (-2\partial_x^3 + 3\partial_x^2 - \partial_x) G^0(t,x;T,y) \\
&= \alpha_1 \left((T-t)^2 \partial_x^3 + \left((x-\bar{x})(T-t) - \frac{3}{2}(T-t)^2 \right) \partial_x^2 + \right. \\
&\quad \left. + \left(-(x-\bar{x})(T-t) + \frac{(T-t)^2}{2} \right) \partial_x \right) G^0(t,x;T,y),
\end{aligned}
$$

where the operator acting on $G^0(t,x;T,y)$ is exactly the same as in (3.40). $\square$

Remark 3.8. *As in Remark 3.3, operator $J^1(t,T,x)$ can also be rewritten in the form*

$$
J^1(t,T,x) = \sum_{i=1}^{3} \sum_{j=0}^{1} f^1_{i,j}(t,T)(x-\bar{x})^j \partial_x^i, \tag{3.43}
$$

where $f^1_{i,j}$ are deterministic functions whose explicit expression can be easily derived.

The previous argument can be used to prove the following second order expansion.

Corollary 3.9 (2nd order expansion)**.** *In case of $\lambda = 0$, the solution G^2 in* (3.30) *is given by*

$$G^2(t, x; T, y) = J^2(t, T, x)G^0(t, x; T, y)$$

where

$$\begin{aligned} J^2(t, T, x) = &\int_t^T \alpha_1(s)V_{t,s,x}(\partial_{xx} - \partial_x)\widetilde{J}^1(t, s, T, x)ds \\ &+ \int_t^T \alpha_2(s)V_{t,s,x}^2 ds\,(\partial_{xx} - \partial_x) \end{aligned} \tag{3.44}$$

and $\widetilde{J}^1$ is the "adjoint" operator of J^1, defined by

$$\widetilde{J}^1(t, s, T, x) = \sum_{i=1}^{3}\sum_{j=0}^{1} f_{i,j}^1(s, T)V_{t,s,x}^j \partial_x^i$$

with $f_{i,j}^1$ as in (3.43)*.*

Remark 3.10. *In a standard LV model, the leading operator of the approximation, i.e. L_0 in* (3.38)*, has a Gaussian density G^0 and this allowed us to use the inverse Fourier transform in order to get the approximated density. This approach does not work in the general case of models with jumps because typically the explicit expression of the fundamental solution of an integro-differential equation is not available. On the other hand, for several Lévy processes used in finance, the characteristic function is known explicitly even if the density is not. This suggests that the argument used in this section may be adapted to obtain an approximation of the characteristic function of the process instead of its density. This is what we are going to investigate in Section 4.*

4 Local Lévy models

In this section, we provide an expansion of the characteristic function for the local Lévy model (1.1). We denote by

$$\hat{\Gamma}(t, x; T, \xi) = \mathcal{F}\left(\Gamma(t, x; T, \cdot)\right)(\xi)$$

the Fourier transform, with respect to the second spatial variable, of the transition density $\Gamma(t, x; T, \cdot)$; clearly, $\hat{\Gamma}(t, x; T, \xi)$ is the characteristic function of $X_T^{t,x}$. Then, by applying the Fourier transform to the expansion (2.15), we find

$$\varphi_{X_T^{t,x}}(\xi) \approx \sum_{k=0}^{n} \hat{G}^k(t, x; T, \xi). \tag{4.45}$$

Now we recall that $G^k(t,x;T,y)$ is defined, as a function of the variables (t,x), in terms of the sequence of Cauchy problems (2.16). Since the Fourier transform in (4.45) is performed with respect to the variable y, in order to take advantage of such a transformation it seems natural to characterize $G^k(t,x;T,y)$ as a solution of the *adjoint operator* in the dual variables (T,y).

To be more specific, we recall the definition of adjoint operator. Let L be the operator in (2.5); then its adjoint operator $\widetilde{L}$ satisfies (actually, it is defined by) the identity

$$\int_{\mathbb{R}^2} u(t,x)Lv(t,x)dxdt = \int_{\mathbb{R}^2} v(t,x)\widetilde{L}u(t,x)dxdt$$

for all $u,v \in C_0^\infty$. More explicitly, by recalling notation (2.11), we have

$$\begin{aligned}\widetilde{L}^{(T,y)}u(T,y) = {} & \frac{a(T,y)}{2}\partial_{yy}u(T,y) + b(T,y)\partial_y u(T,y) \\ & - \partial_T u(T,y) + c(T,y)u(T,y) \\ & + \int_{\mathbb{R}} \left(u(T,y+z) - u(T,y) - z\partial_y u(T,y)\mathbb{1}_{\{|z|<1\}}\right)\bar{\nu}(dz),\end{aligned}$$

where

$$b(T,y) = \partial_y a(T,y) - \left(\bar{r} - \frac{a(T,y)}{2}\right), \qquad c(T,y) = \frac{1}{2}(\partial_{yy} + \partial_y)a(T,y),$$

and $\bar{\nu}$ is the Lévy measure with reverted jumps, i.e. $\bar{\nu}(dx) = \nu(-dx)$. Here the superscript in $\widetilde{L}^{(T,y)}$ is indicative of the fact that the operator $\widetilde{L}$ is acting in the variables (T,y).

By a classical result (cf., for instance, [14]) the fundamental solution $\Gamma(t,x;T,y)$ of L is also a solution of $\widetilde{L}$ in the dual variables, that is

$$\widetilde{L}^{(T,y)}\Gamma(t,x;T,y) = 0, \qquad t < T,\ x,y \in \mathbb{R}. \tag{4.46}$$

Going back to approximation (4.45), the idea is to consider the series of the dual Cauchy problems of (2.16) in order to solve them by Fourier-transforming in the variable y and finally get an approximation of $\varphi_{X_T^{t,x}}$.

For sake of simplicity, from now on we only consider the case of time-independent coefficients: the general case can be treated in a completely analogous way. First of all, we consider the integro-differential operator L_0 in (2.14), which in this case becomes

$$\begin{aligned}L_0^{(t,x)}u(t,x) = {} & \frac{\alpha_0}{2}(\partial_{xx} - \partial_x)u(t,x) + \bar{r}\partial_x u(t,x) + \partial_t u(t,x) \\ & + \int_{\mathbb{R}} \left(u(t,x+y) - u(t,x) - y\partial_x u(t,x)\mathbb{1}_{\{|y|<1\}}\right)\nu(dy),\end{aligned} \tag{4.47}$$

and its adjoint operator

$$\begin{aligned}\widetilde{L}_0^{(T,y)}u(T,y) &= \frac{\alpha_0}{2}(\partial_{yy}+\partial_y)u(T,y) - \bar{r}\partial_y u(T,y) - \partial_T u(T,y)\\ &\quad + \int_{\mathbb{R}} \left(u(T,y+z) - u(T,y) - z\partial_y u(T,y)\mathbb{1}_{\{|z|<1\}}\right)\bar{\nu}(dz).\end{aligned} \tag{4.48}$$

By (4.46), for any $(t,x)\in\mathbb{R}^2$, the fundamental solution $G^0(t,x;T,y)$ of L_0 solves the dual Cauchy problem

$$\begin{cases}\widetilde{L}_0^{(T,y)}G^0(t,x;T,y) = 0, & T>t,\ y\in\mathbb{R},\\ G^0(t,x;t,\cdot) = \delta_x.\end{cases} \tag{4.49}$$

It is remarkable that a similar result holds for the higher order terms of the approximation (4.45). Indeed, let us denote by L_n the n^{th} order approximation of L in (2.13):

$$L_n = L_0 + \sum_{k=1}^{n} \alpha_k (x-\bar{x})^k \left(\partial_{xx} - \partial_x\right) \tag{4.50}$$

Then we have the following result.

Theorem 4.1. *For any $k\geq 1$ and $(t,x)\in\mathbb{R}^2$, the function $G^k(t,x;\cdot,\cdot)$ in (2.16) is the solution of the following dual Cauchy problem on $]t,+\infty[\times\mathbb{R}$*

$$\begin{cases}\widetilde{L}_0^{(T,y)}G^k(t,x;T,y) = -\sum\limits_{h=1}^{k}\left(\widetilde{L}_h^{(T,y)} - \widetilde{L}_{h-1}^{(T,y)}\right)G^{k-h}(t,x;T,y),\\ G^k(t,x;t,y) = 0, \qquad y\in\mathbb{R},\end{cases} \tag{4.51}$$

where

$$\begin{aligned}\widetilde{L}_h^{(T,y)} - \widetilde{L}_{h-1}^{(T,y)} &= \alpha_h (y-\bar{x})^{h-2}\Big((y-\bar{x})^2\partial_{yy} + (y-\bar{x})\left(2h + (y-\bar{x})\right)\partial_y\\ &\quad + h\left(h-1+y-\bar{x}\right)\Big).\end{aligned}$$

Proof. By the standard representation formula for the solutions of the *backward* parabolic Cauchy problem (2.16), for $k\geq 1$ we have

$$G^k(t,x;T,y) = \sum_{h=1}^{k}\int_t^T\int_{\mathbb{R}} G^0(t,x;s,\eta)M_h^{(s,\eta)}G^{k-h}(s,\eta;T,y)d\eta ds, \tag{4.52}$$

where to shorten notation we have set

$$M_h^{(t,x)} = L_h^{(t,x)} - L_{h-1}^{(t,x)}.$$

By (4.49) and since

$$\widetilde{M}_h^{(T,y)} = \widetilde{L}_h^{(T,y)} - \widetilde{L}_{h-1}^{(T,y)}.$$

the thesis is equivalent to

$$G^k(t,x;T,y) = \sum_{h=1}^{k} \int_t^T \int_{\mathbb{R}} G^0(s,\eta;T,y) \widetilde{M}_h^{(s,\eta)} G^{k-h}(t,x;s,\eta) d\eta ds, \quad (4.53)$$

where here we have used the representation formula for the solutions of the *forward* Cauchy problem (4.51) with $k \geq 1$.

We proceed by induction and first prove (4.53) for $k = 1$. By (4.52) we have

$$\begin{aligned} G^1(t,x;T,y) &= \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_1^{(s,\eta)} G^0(s,\eta;T,y) d\eta ds \\ &= \int_t^T \int_{\mathbb{R}} G^0(s,\eta;T,y) \widetilde{M}_1^{(s,\eta)} G^0(t,x;s,\eta) d\eta ds, \end{aligned}$$

and this proves (4.53) for $k = 1$.

Next we assume that (4.53) holds for a generic $k > 1$ and we prove the thesis for $k+1$. Again, by (4.52) we have

$$\begin{aligned} G^{k+1}(t,x;T,y) &= \sum_{j=1}^{k+1} \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_j^{(s,\eta)} G^{k+1-j}(s,\eta;T,y) d\eta ds \\ &= \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_{k+1}^{(s,\eta)} G^0(s,\eta;T,y) d\eta ds \\ &\quad + \sum_{j=1}^{k} \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_j^{(s,\eta)} G^{k+1-j}(s,\eta;T,y) d\eta ds = \end{aligned}$$

(by the inductive hypothesis)

$$\begin{aligned} &= \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_{k+1}^{(s,\eta)} G^0(s,\eta;T,y) d\eta ds \\ &\quad + \sum_{j=1}^{k} \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_j^{(s,\eta)} \cdot \\ &\quad \cdot \sum_{h=1}^{k+1-j} \int_s^T \int_{\mathbb{R}} G^0(\tau,\zeta;T,y) \widetilde{M}_h^{(\tau,\zeta)} G^{k+1-j-h}(s,\eta;\tau,\zeta) d\zeta d\tau d\eta ds \\ &= \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta) M_{k+1}^{(s,\eta)} G^0(s,\eta;T,y) ds d\eta \\ &\quad + \sum_{h=1}^{k} \sum_{j=1}^{k+1-h} \int_t^T \int_t^\tau \int_{\mathbb{R}^2} G^0(t,x;s,\eta) G^0(\tau,\zeta;T,y) \cdot \\ &\quad \cdot M_j^{(s,\eta)} \widetilde{M}_h^{(\tau,\zeta)} G^{k+1-j-h}(s,\eta;\tau,\zeta) d\eta d\zeta ds d\tau \end{aligned}$$

$$
\begin{aligned}
&= \int_t^T \int_{\mathbb{R}} G^0(s,\eta;T,y)\widetilde{M}_{k+1}^{(s,\eta)} G^0(t,x;s,\eta)dsd\eta \\
&\quad + \sum_{h=1}^{k} \int_t^T \int_{\mathbb{R}} G^0(\tau,\zeta;T,y)\widetilde{M}_h^{(\tau,\zeta)} \cdot \\
&\quad \cdot \left(\sum_{j=1}^{k+1-h} \int_t^\tau \int_{\mathbb{R}} G^0(t,x;s,\eta) M_j^{(s,\eta)} G^{k+1-h-j}(s,\eta;\tau,\zeta)d\eta ds \right) d\zeta d\tau =
\end{aligned}
$$

(again by (4.52))

$$
\begin{aligned}
&= \int_t^T \int_{\mathbb{R}} G^0(t,\eta;T,y)\widetilde{M}_{k+1}^{(s,\eta)} G^0(t,x;s,\eta)dsd\eta \\
&\quad + \sum_{h=1}^{k} \int_t^T \int_{\mathbb{R}} G^0(\tau,\zeta;T,y)\widetilde{M}_h^{(\tau,\zeta)} G^{k+1-h}(t,x;\tau,\zeta)d\zeta d\tau \\
&= \sum_{h=1}^{k+1} \int_t^T \int_{\mathbb{R}} G^0(\tau,\zeta;T,y)\widetilde{M}_h^{(\tau,\zeta)} G^{k+1-h}(t,x;\tau,\zeta)d\zeta d\tau.
\end{aligned}
$$

$\square$

Next we solve problems (4.49)-(4.51) by applying the Fourier transform in the variable y and using the identity

$$
\mathcal{F}_y\left(\widetilde{L}_0^{(T,y)} u(T,y)\right)(\xi) = \psi(\xi)\hat{u}(T,\xi) - \partial_T \hat{u}(T,\xi), \tag{4.54}
$$

where

$$
\psi(\xi) = -\frac{\alpha_0}{2}(\xi^2 + i\xi) + i\bar{r}\xi + \int_{\mathbb{R}} \left(e^{iz\xi} - 1 - iz\xi \mathbb{1}_{\{|z|<1\}}\right)\nu(dz). \tag{4.55}
$$

We remark explicitly that ψ is the characteristic exponent of the Lévy process

$$
dX_t^0 = \left(\bar{r} - \frac{\alpha_0}{2}\right)dt + \sqrt{\alpha_0}dW_t + dJ_t, \tag{4.56}
$$

whose Kolmogorov operator is L^0 in (4.47). Then:

- from (4.49) we obtain the ordinary differential equation

$$
\begin{cases}
\partial_T \hat{G}^0(t,x;T,\xi) = \psi(\xi)\hat{G}^0(t,x;T,\xi), & T > t, \\
\hat{G}^0(t,x;t,\xi) = e^{i\xi x}.
\end{cases} \tag{4.57}
$$

  with solution

$$
\hat{G}^0(t,x;T,\xi) = e^{i\xi x + (T-t)\psi(\xi)} \tag{4.58}
$$

  which is the 0^{th} order approximation of the characteristic function $\varphi_{X_T^{t,x}}$.

- from (4.51) with $k=1$, we have

$$\begin{cases} \partial_T \hat{G}^1(t,x;T,\xi) = \psi(\xi)\hat{G}^1(t,x;T,\xi) \\ \qquad\qquad\qquad\qquad +\alpha_1\left((i\partial_\xi+\bar{x})(\xi^2+i\xi)-2i\xi+1\right)\hat{G}^0(t,x;T,\xi) \\ \quad \hat{G}^1(t,x;t,\xi)=0, \end{cases}$$

with solution

$$\hat{G}^1(t,x;T,\xi) = \int_t^T e^{\psi(\xi)(T-s)}\alpha_1\left((i\partial_\xi+\bar{x})(\xi^2+i\xi)-2i\xi+1\right)\hat{G}^0(t,x;s,\xi)ds =$$

(by (4.58))

$$\begin{aligned} &= -e^{ix\xi+\psi(\xi)(T-t)}\alpha_1\int_t^T(\xi^2+i\xi)\left(x-\bar{x}-i(s-t)\psi'(\xi)\right)ds \\ &= -\hat{G}^0(t,x;T,\xi)\alpha_1(T-t)(\xi^2+i\xi)\left(x-\bar{x}-\frac{i}{2}(T-t)\psi'(\xi)\right), \end{aligned} \tag{4.59}$$

which is the first order term in the expansion (4.45).

- regarding (4.51) with $k=2$, a straightforward computation based on analogous arguments shows that the second order term in the expansion (4.45) is given by

$$\hat{G}^2(t,x;T,\xi) = \hat{G}^0(t,x;T,\xi)\sum_{j=0}^{2} g_j(T-t,\xi)(x-\bar{x})^j \tag{4.60}$$

where

$$\begin{aligned} g_0(s,\xi) &= \frac{1}{2}s^2\alpha_2\xi(i+\xi)\psi''(\xi) \\ &\quad -\frac{1}{6}s^3\xi(i+\xi)\psi''(\xi)\left(\alpha_1^2(i+2\xi)-2\alpha_2\psi''(\xi)+\alpha_1^2\xi(i+\xi)\right) \\ &\quad -\frac{1}{8}s^4\alpha_1^2\xi^2(i+\xi)^2\psi''(\xi)^2, \\ g_1(s,\xi) &= \frac{1}{2}s^2\xi(i+\xi)\left(\alpha_1^2(1-2i\xi)+2i\alpha_2\psi''(\xi)\right) \\ &\quad -\frac{1}{2}s^3 i\alpha_1^2\xi^2(i+\xi)^2\psi''(\xi), \\ g_2(s,\xi) &= -\alpha_2 s\xi(i+\xi)+\frac{1}{2}s^2\alpha_1^2\xi^2(i+\xi)^2. \end{aligned}$$

Plugging (4.58)-(4.59)-(4.60) into (4.45), we finally get the second order approximation of the characteristic function of X. In Subsection 4.1, we also provide the expression of $\hat{G}^k(t,x;T,\xi)$ for $k=3,4$, appearing in the 4^{th} order approximation.

Remark 4.2. *The basepoint $\bar{x}$ is a parameter which can be freely chosen in order to sharpen the accuracy of the approximation. In general, the simplest choice $\bar{x} = x$ seems to be sufficient to get very accurate results.*

Remark 4.3. *To overcome the use of the adjoint operators, it would be interesting to investigate an alternative approach to the approximation of the characteristic function based of the following remarkable symmetry relation valid for time-homogeneous diffusions*

$$m(x)\Gamma(0, x; t, y) = m(y)\Gamma(0, y; t, x) \tag{4.61}$$

where m is the so-called density of the speed measure

$$m(x) = \frac{2}{\sigma^2(x)} \exp\left(\int_1^x \left(\frac{2r}{\sigma^2(z)} - 1\right) dz\right).$$

Relation (4.61) *is stated in [19] and a complete proof can be found in [10].*

For completeness, we close this section by stating an integral pricing formula for European options proved by Lewis [21]; the formula is given in terms of the characteristic function of the underlying log-price process. Formula below (and other Fourier-inversion methods such as the standard, fractional FFT algorithm or the recent COS method [11]) can be combined with the expansion (4.45) to price and hedge efficiently hybrid LV models with Lévy jumps.

We consider a risky asset $S_t = e^{X_t}$ where X is the process whose risk-neutral dynamics under a martingale measure Q is given by (1.1). We denote by $H(t, S_t)$ the price at time $t < T$, of a European option with underlying asset S, maturity T and payoff $f = f(x)$ (given as a function of the log-price): to fix ideas, for a Call option with strike K we have

$$f^{\mathrm{Call}}(x) = (e^x - K)^+.$$

The following theorem is a classical result which can be found in several textbooks (see, for instance, [26]).

Theorem 4.4. *Let*

$$f_\gamma(x) = e^{-\gamma x} f(x)$$

and assume that there exists $\gamma \in \mathbb{R}$ such that

i) $f_\gamma, \hat{f}_\gamma \in L^1(\mathbb{R})$*;*

ii) $E^Q\left[S_T^\gamma\right]$ *is finite.*

Then, the following pricing formula holds:

$$H(t, S_t) = \frac{e^{-r(T-t)}}{\pi} \int_0^\infty \hat{f}(\xi + i\gamma)\varphi_{X_T^{t,\log S_t}}(-(\xi + i\gamma))d\xi.$$

For example, f^{Call} verifies the assumptions of Theorem 4.4 for any $\gamma > 1$ and we have

$$\hat{f}^{\text{Call}}(\xi + i\gamma) = \frac{K^{1-\gamma} e^{i\xi \log K}}{(i\xi - \gamma)\,(i\xi - \gamma + 1)}.$$

Other examples of typical payoff functions and the related Greeks can be found in [26].

4.1 High order approximations

The analysis of Section 4 can be carried out to get approximations of arbitrarily high order. Below we give the more accurate (but more complicated) formulae up to the 4^{th} order that we used in the numerical section. In particular we give the expression of $\hat{G}^k(t,x;T,\xi)$ in (4.45) for $k = 3, 4$. For simplicity, we only consider the case of time-homogeneous coefficients and $\bar{x} = x$.

We have

$$\hat{G}^3(t,x;T,\xi) = \hat{G}^0(t,x;T,\xi) \sum_{j=3}^{7} g_j(\xi)(T-t)^j$$

where

$$\begin{aligned}
g_3(\xi) &= \frac{1}{2}\alpha_3(1 - i\xi)\xi\psi^{(3)}(\xi), \\
g_4(\xi) &= \frac{1}{6} i\xi(i+\xi)\Big(2\psi'(\xi)\left(\alpha_1\alpha_2 - 3\alpha_3\psi''(\xi)\right) \\
&\quad + \alpha_1\alpha_2\left(3(i+2\xi)\psi''(\xi) + 2\xi(i+\xi)\psi^{(3)}(\xi)\right)\Big), \\
g_5(\xi) &= \frac{1}{24}(1 - i\xi)\xi\Big(-8\alpha_1\alpha_2(i+2\xi)\psi'(\xi)^2 + 6\alpha_3\psi'(\xi)^3 \\
&\quad + \alpha_1\psi'(\xi)\left(\alpha_1^2(-1 + 6\xi(i+\xi)) - 16\alpha_2\xi(i+\xi)\psi''(\xi)\right) \\
&\quad + \alpha_1^3\xi(i+\xi)\left(3(i+2\xi)\psi''(\xi) + \xi(i+\xi)\psi^{(3)}(\xi)\right)\Big), \\
g_6(\xi) &= -\frac{1}{12} i\alpha_1\xi^2(i+\xi)^2\psi'(\xi)\Big(\alpha_1^2(i+2\xi)\psi'(\xi) \\
&\quad - 2\alpha_2\psi'(\xi)^2 + \alpha_1^2\xi(i+\xi)\psi''(\xi)\Big), \\
g_7(\xi) &= -\frac{1}{48} i\left(\alpha_1\xi(i+\xi)\psi'(\xi)\right)^3.
\end{aligned}$$

Moreover, we have

$$\hat{G}^4(t,x;T,\xi) = \hat{G}^0(t,x;T,\xi) \sum_{j=3}^{9} g_j(\xi)(T-t)^j$$

where

$$
\begin{aligned}
g_3(\xi) &= -\frac{1}{2}\alpha_4\xi(i+\xi)\psi^{(4)}(\xi),\\
g_4(\xi) &= \frac{1}{6}\xi(i+\xi)\Big(2\psi''(\xi)\left(\alpha_2^2+3\alpha_1\alpha_3-3\alpha_4\psi''(\xi)\right)\\
&\quad+2\left(\left(\alpha_2^2+2\alpha_1\alpha_3\right)(i+2\xi)-4\alpha_4\psi'(\xi)\right)\psi^{(3)}(\xi)\\
&\quad+\left(\alpha_2^2+2\alpha_1\alpha_3\right)\xi(i+\xi)\psi^{(4)}(\xi)\Big),\\
g_5(\xi) &= -\frac{1}{24}\xi(i+\xi)\Big(\alpha_1^2\alpha_2(-7+44\xi(i+\xi))\psi''(\xi)\\
&\quad-\left(7\alpha_2^2+15\alpha_1\alpha_3\right)\xi(i+\xi)\psi''(\xi)^2\\
&\quad-2\psi'(\xi)^2\left(2\alpha_2^2+9\alpha_1\alpha_3-18\alpha_4\psi''(\xi)\right)\\
&\quad+\psi'(\xi)\Big((i+2\xi)\left(8\alpha_1^2\alpha_2-\left(14\alpha_2^2+33\alpha_1\alpha_3\right)\psi''(\xi)\right)\\
&\quad-\left(10\alpha_2^2+21\alpha_1\alpha_3\right)\xi(i+\xi)\psi^{(3)}(\xi)\Big)\\
&\quad+3\alpha_1^2\alpha_2\xi(i+\xi)\left(4(i+2\xi)\psi^{(3)}(\xi)+\xi(i+\xi)\psi^{(4)}(\xi)\right)\Big),\\
g_6(\xi) &= \frac{1}{120}\xi(i+\xi)\Big(2\left(8\alpha_2^2+21\alpha_1\alpha_3\right)(i+2\xi)\psi'(\xi)^3-24\alpha_4\psi'(\xi)^4\\
&\quad+2\psi'(\xi)^2\left(\alpha_1^2\alpha_2(11-70\xi(i+\xi))+\left(26\alpha_2^2+57\alpha_1\alpha_3\right)\xi(i+\xi)\psi''(\xi)\right)\\
&\quad+\alpha_1^2\psi'(\xi)\Big((i+2\xi)\left(\alpha_1^2(-1+12\xi(i+\xi))-112\alpha_2\xi(i+\xi)\psi''(\xi)\right)\\
&\quad-38\alpha_2\xi^2(i+\xi)^2\psi^{(3)}(\xi)\Big)+\alpha_1^2\xi(i+\xi)\Big(\alpha_1^2(-7+36\xi(i+\xi))\psi''(\xi)\\
&\quad-26\alpha_2\xi(i+\xi)\psi''(\xi)^2+\alpha_1^2\xi(i+\xi)\left(6(i+2\xi)\psi^{(3)}(\xi)+\xi(i+\xi)\psi^4(\xi)\right)\Big)\Big),\\
g_7(\xi) &= \frac{1}{144}\xi^2(i+\xi)^2\Big(-32\alpha_1^2\alpha_2(i+2\xi)\psi'(\xi)^3+2\left(4\alpha_2^2+9\alpha_1\alpha_3\right)\psi'(\xi)^4\\
&\quad+2\alpha_1^4\xi^2(i+\xi)^2\psi''(\xi)^2\\
&\quad+\alpha_1^2\psi'(\xi)^2\left(\alpha_1^2(-5+26\xi(i+\xi))-47\alpha_2\xi(i+\xi)\psi''(\xi)\right)\\
&\quad+\alpha_1^4\xi(i+\xi)\psi'(\xi)\left(13(i+2\xi)\psi''(\xi)+3\xi(i+\xi)\psi^{(3)}(\xi)\right)\Big),\\
g_8(\xi) &= \frac{1}{48}\alpha_1^2\xi^3(i+\xi)^3\psi'(\xi)^2\Big(\alpha_1^2(i+2\xi)\psi'(\xi)\\
&\quad-2\alpha_2\psi'(\xi)^2+\alpha_1^2\xi(i+\xi)\psi''(\xi)\Big),\\
g_9(\xi) &= \frac{1}{384}\alpha_1^4\xi^4(i+\xi)^4\psi'(\xi)^4.
\end{aligned}
$$

5 Numerical tests

In this section our approximation formulae (4.45) are tested and compared with a standard Monte Carlo method. We consider up to the 4^{th} order expansion (i.e. $n=4$ in (4.45)) even if in most cases the 2^{nd} order seems to be sufficient to get very accurate results. We analyze the case of a constant elasticity of variance (CEV) volatility function with Lévy jumps of Gaussian or Variance-Gamma type. Thus, we consider the log-price dynamics (1.1) with

$$\sigma(t, x) = \sigma_0 e^{(\beta-1)x}, \quad \beta \in [0, 1], \ \sigma_0 > 0,$$

and J as in Examples 2.1 and 2.2 respectively. In our experiments we assume that the initial stock price is $S_0 = 1$, the risk-free rate is $r = 5\%$, the CEV volatility parameter is $\sigma_0 = 20\%$ and the CEV exponent is $\beta = \frac{1}{2}$. Moreover we use an Euler Monte Carlo method with 200 time-steps per year and 500 000 replications.

5.1 Tests under CEV-Merton dynamics

In order to assess the performance of our approximations for pricing Call options in the CEV-Merton model, we consider the following set of parameters: the jump intensity is $\lambda = 30\%$, the average jump size is $m = -10\%$ and the jump volatility is $\delta = 40\%$.

Figures 1, 2 and 3 show the performance of the approximations against the Monte Carlo 95% and 99% confidence intervals, marked in dark and light gray respectively. In particular, Figure 1 shows the cross-sections of absolute (left) and relative (right) errors of the 1^{st} (dotted line), 2^{nd} (dashed line), 3^{rd} (solid line) order approximations for the price of a Call with short-term maturity $T = 0.25$ and strike K ranging from 0.5 to 1.5. The relative error is defined as

$$\frac{\text{Call}^{\text{approx}} - \text{Call}^{\text{MC}}}{\text{Call}^{\text{MC}}}$$

where $\text{Call}^{\text{approx}}$ and Call^{MC} are the approximated and Monte Carlo prices respectively.

In Figure 2 we repeat the test for the medium-term maturity $T = 1$ and the strike K ranging from 0.5 to 2.5. Finally in Figure 3 we consider the long-term maturity $T = 10$ and the strike K ranging from 0.5 to 4.

Other experiments that are not reported here, show that the 2^{nd} order expansion (3.36), which is valid only in the case of Gaussian jumps, gives the same results as formula (4.45) with $n = 2$, at least if the truncation index M is suitable large, namely $M \geq 8$ under standard parameter regimes. For this reason we have only used formula (4.45) for our tests.

5.2 Tests under CEV-Variance-Gamma dynamics

In this subsection we repeat the previous tests in the case of the CEV-Variance-Gamma model. Specifically, we consider the following set of parameters: the variance of the Gamma subordinator is $\kappa = 15\%$, the drift and the volatility of the Brownian motion are $\theta = -10\%$ and $\sigma = 20\%$ respectively. The results are reported in Figures 4, 5 and 6. Notice that, for

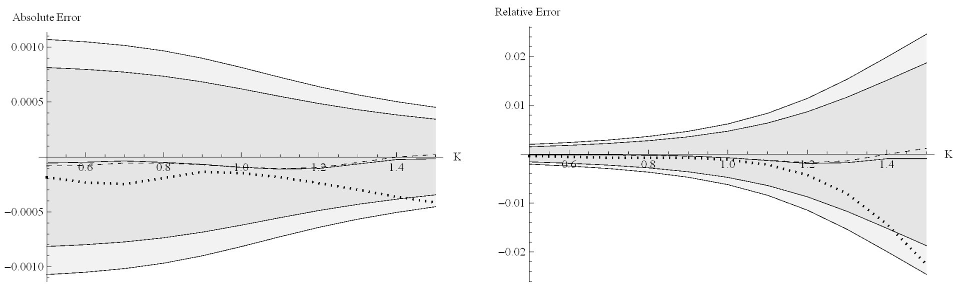

Figure 1: Absolute (left) and relative (right) errors of the 1st (dotted line), 2nd (dashed line), 3rd (solid line) order approximations of a Call price in the **CEV-Merton** model with maturity $\mathbf{T = 0.25}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{1.5}]$. The shaded bands show the 95% (dark gray) and 99% (light gray) Monte Carlo confidence regions

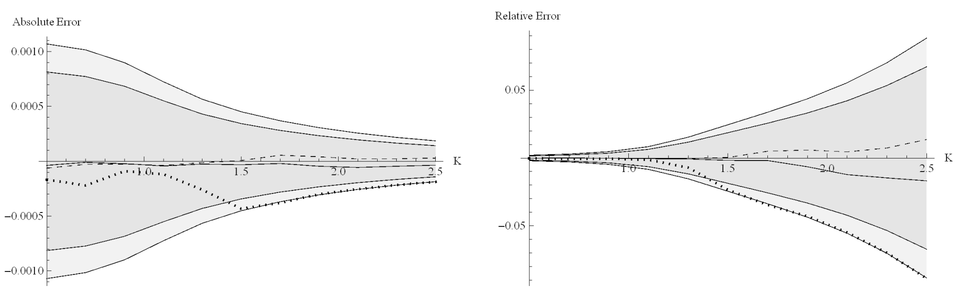

Figure 2: Absolute (left) and relative (right) errors of the 1st (dotted line), 2nd (dashed line), 3rd (solid line) order approximations of a Call price in the **CEV-Merton** model with maturity $\mathbf{T = 1}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{2.5}]$

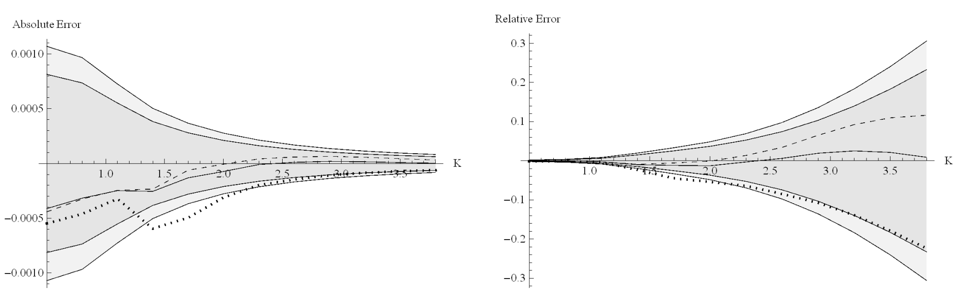

Figure 3: Absolute (left) and relative (right) errors of the 1st (dotted line), 2nd (dashed line), 3rd (solid line) order approximations of a Call price in the **CEV-Merton** model with maturity $\mathbf{T = 10}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{4}]$

longer maturities and deep out-of-the-money options, the lower order approximations give good results in terms of absolute errors but only the 4^{th} order approximation lies inside the confidence regions. For a more detailed comparison, in Figures 5 and 6 we plot the 2^{nd} (dotted line), 3^{rd} (dashed line), 4^{th} (solid line) order approximations. Similar results are obtained for a wide range of parameter values.

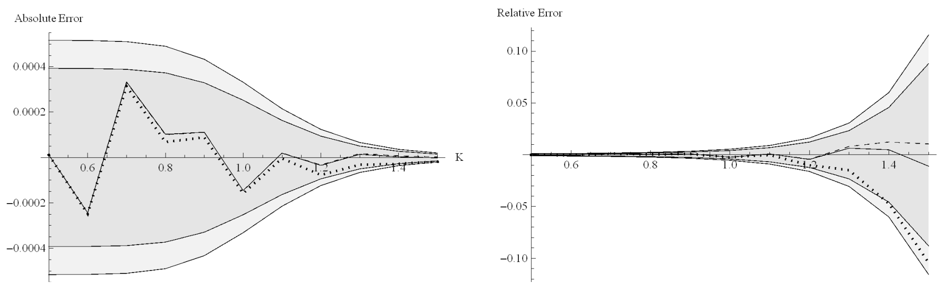

Figure 4: Absolute (left) and relative (right) errors of the 1st (dotted line), 2nd (dashed line), 3rd (solid line) order approximations of a Call price in the **CEV-Variance-Gamma** model with maturity $\mathbf{T = 0.25}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{1.5}]$. The shaded bands show the 95% (dark gray) and 99% (light gray) Monte Carlo confidence regions

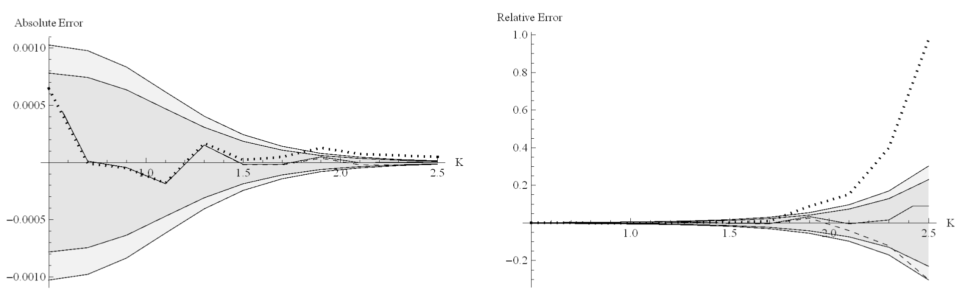

Figure 5: Absolute (left) and relative (right) errors of the 2nd (dotted line), 3rd (dashed line), 4th (solid line) order approximations of a Call price in the **CEV-Variance-Gamma** model with maturity $\mathbf{T = 1}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{2.5}]$

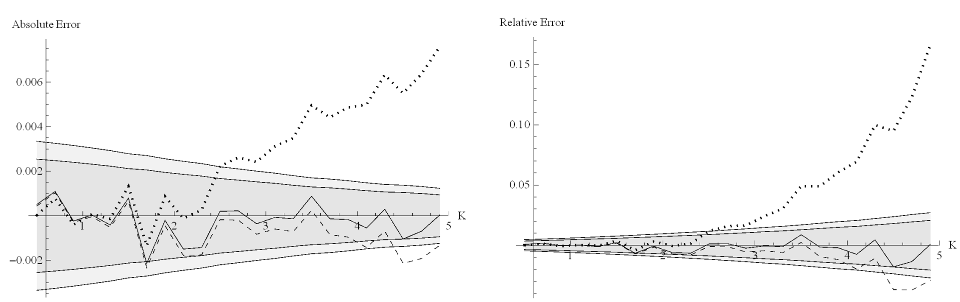

Figure 6: Absolute (left) and relative (right) errors of the 2nd (dotted line), 3rd (dashed line), 4th (solid line) order approximations of a Call price in the **CEV-Variance-Gamma** model with maturity $\mathbf{T = 10}$ and strike $\mathbf{K} \in [\mathbf{0.5}, \mathbf{5}]$

6 Appendix

In this appendix we prove Theorem 2.3 under Assumption A_{N+1} where $N \in \mathbb{N}$ is fixed. For simplicity we only consider the case of $r = 0$ and time-homogeneous coefficients. Recalling notation (2.12), we put

$$L_0 = \frac{\alpha_0}{2} (\partial_{xx} - \partial_x) + \partial_t \tag{6.62}$$

and

$$L_n = L_0 + \sum_{k=1}^{n} \alpha_k (x - \bar{x})^k (\partial_{xx} - \partial_x), \qquad n \le N. \tag{6.63}$$

Our idea is to modify and adapt the standard characterization of the fundamental solution given by the parametrix method originally introduced by Levi [20]. The parametrix method is a constructive technique that allows to prove the existence of the fundamental solution Γ of a parabolic operator with variable coefficients of the form

$$Lu(t,x) = \frac{a(x)}{2} (\partial_{xx} - \partial_x) u(t,x) + \partial_t u(t,x).$$

In the standard parametrix method, for any fixed $\xi \in \mathbb{R}$, the fundamental solution Γ_ξ of the frozen operator

$$L_\xi u(t,x) = \frac{a(\xi)}{2} (\partial_{xx} - \partial_x) u(t,x) + \partial_t u(t,x)$$

is called a *parametrix* for L. A fundamental solution $\Gamma(t,x;T,y)$ for L can be constructed starting from $\Gamma_y(t,x;T,y)$ by means of an iterative argument and by suitably controlling the errors of the approximation.

Our main idea is to *use the N^{th}-order approximation $\Gamma^N(t,x;T,y)$ in* (2.15)-(2.16) *(related to L_n in* (6.62)-(6.63)*) as a parametrix.* In order to prove the error bound (2.19), we carefully generalize some Gaussian estimates: in particular, for $N = 0$ we are back into the classical framework, but in general we need accurate estimates of the solutions of the nested Cauchy problems (2.16).

By analogy with the classical approach (see, for instance, [13] or the recent and more general presentation in [9]), we have that Γ takes the form

$$\Gamma(t,x;T,y) = \Gamma^N(t,x;T,y) + \int_t^T \int_{\mathbb{R}} \Gamma^0(t,x;s,\xi)\Phi^N(s,\xi;T,y) d\xi ds$$

where Φ^N is the function in (6.64) below, which is determined by imposing the condition $L\Gamma = 0$. More precisely, we have

$$0 = L\Gamma(z;\zeta) = L\Gamma^N(z;\zeta) + \int_t^T \int_{\mathbb{R}} L\Gamma^0(z;w)\Phi^N(w;\zeta)dw - \Phi^N(z;\zeta),$$

where, to shorten notations, we have set $z = (t, x)$, $w = (s, \xi)$ and $\zeta = (T, y)$. Equivalently, we have

$$\Phi^N(z;\zeta) = L\Gamma^N(z;\zeta) + \int_t^T \int_{\mathbb{R}} L\Gamma^0(z;w)\Phi^N(w;\zeta)dw$$

and therefore by iteration

$$\Phi^N(z;\zeta) = \sum_{n=0}^{\infty} Z_n(z;\zeta) \tag{6.64}$$

where

$$\begin{aligned} Z_0^N(z;\zeta) &= L\Gamma^N(z;\zeta), \\ Z_{n+1}^N(z;\zeta) &= \int_t^T \int_{\mathbb{R}} L\Gamma^0(z;w)Z_n(w;\zeta)(w;\zeta)dw. \end{aligned}$$

The thesis is a consequence of the following lemmas.

Lemma 6.1. *For any $n \leq N$ the solution of* (2.16)*, with L_n as in* (6.62)-(6.63)*, takes the form*

$$G^n(t,x;T,y) = \sum_{\substack{i\leq n,\, j\leq n(n+3),\, k\leq \frac{n(n+5)}{2} \\ i+j-k\geq n}} c_{i,j,k}^n (x-\bar{x})^i(\sqrt{T-t})^j \partial_x^k G^0(t,x;T,y), \tag{6.65}$$

where $c_{i,j,k}^n$ are polynomial functions of $\alpha_0, \alpha_1, \dots, \alpha_n$.

Proof. We proceed by induction on n. For $n = 0$ the thesis is trivial. Next by (2.16) we have $G^{n+1}(t,x;T,y) = I_{n,2} - I_{n,1}$ where

$$I_{n,l} = \sum_{h=1}^{n+1} \alpha_h \int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta)(\eta-\bar{x})^h \partial_\eta^l G^{n+1-h}(s,\eta;T,y)d\eta ds, \quad l = 1,2.$$

We only analyze the case $l = 2$ since the other one is analogous. By the inductive hypothesis (6.65), we have that $I_{n,2}$ is a linear combination of terms of the form

$$\int_t^T \int_{\mathbb{R}} G^0(t,x;s,\eta)(\sqrt{T-s})^j(\eta-\bar{x})^{h+i-p}\partial_\eta^{k+2-p}G^0(s,\eta;T,y)d\eta ds \tag{6.66}$$

for $p = 0, 1, 2$ and $h = 1, \dots, n+1$; moreover we have

$$i + j - k \geq n + 1 - h, \tag{6.67}$$

$$i \leq n + 1 - h, \tag{6.68}$$

$$j \leq (n+1-h)(n+4-h) \leq n(n+3), \tag{6.69}$$

$$k \leq \frac{(n+1-h)(n+6-h)}{2} \leq \frac{n(n+5)}{2}. \tag{6.70}$$

Again we focus only on $p=0$, the other cases being analogous: then by properties (3.25), (3.24) and (3.23), we have that the integral in (6.66) is equal to

$$\int_t^T (\sqrt{T-s})^j V_{t,s,x}^{h+i} ds\, \partial_x^{k+2} G^0(t,x;T,y) \tag{6.71}$$

where $V_{t,T,x} \equiv V_{t,T,x,0}$ is the operator in (3.26). Now we remark that $V_{t,s,x}^n$ is a finite sum of the form

$$V_{t,s,x}^n = \sum_{\substack{0 \leq j_1, \frac{j_2}{2}, j_3 \leq n \\ j_1+j_2-j_3 \geq n}} b_{j_1,j_2,j_3}^n (x-\bar{x})^{j_1} (\sqrt{s-t})^{j_2} \partial_x^{j_3} \tag{6.72}$$

for some constants b_{j_1,j_2,j_3}^n. Thus the integral in (6.71) is a linear combination of terms of the form

$$(x-\bar{x})^{j_1} (\sqrt{T-s})^{j+2+j_2} \partial_x^{k+2+j_3} G^0(t,x;T,y)$$

where

$$0 \leq j_1, \frac{j_2}{2}, j_3 \leq h+i, \tag{6.73}$$

$$j_1+j_2-j_3 \geq h+i. \tag{6.74}$$

Eventually we have

$$j_1+j+j_2+2-(k+2+j_3) \geq$$

(by (6.74))

$$\geq i+j-k+h \geq$$

(by (6.67))

$$\geq n+1.$$

On the other hand, by (6.73) and (6.68) we have

$$j_1 \leq h+i \leq n+1.$$

Moreover, by (6.73), (6.68) and (6.69) we have

$$j+2+j_2 \leq j+2+2(n+1) \leq n(n+3)+2+2(n+1) = (n+1)(n+4).$$

Finally, by (6.73), (6.68) and (6.70) we have

$$\begin{aligned} k+2+j_3 &\leq k+2+h+i \leq k+n+3 \\ &\leq \frac{n(n+5)}{2}+n+3 = \frac{(n+1)(n+6)}{2}. \end{aligned}$$

This concludes the proof. $\square$

Now we set $\bar{x} = y$ and prove the thesis only in this case: to treat the case $\bar{x} = x$, it suffices to proceed in a similar way by using the backward parametrix method introduced in [8].

Lemma 6.2. *For any $\epsilon, \tau > 0$ there exists a positive constant C, only dependent on $\varepsilon, \tau, m, M, N$ and $\max_{k \le N} \|\alpha_k\|_\infty$, such that*

$$|\partial_{xx} G^n(t,x;T,y)| \le C(T-t)^{\frac{n-2}{2}} \bar{\Gamma}^{M+\epsilon}(t,x;T,y), \tag{6.75}$$

for any $n \le N$, $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \le \tau$.

Proof. By Lemma 6.1 with $\bar{x} = y$, we have

$$\begin{aligned} |\partial_{xx} G^n(t,x;T,y)| \le & \sum_{\substack{i \le n,\, j \le n(n+3),\, k \le \frac{n(n+5)}{2} \\ i+j-k \ge n}} \left|c^n_{i,j,k}\right| \left(\sqrt{T-t}\right)^j \cdot \\ & \cdot \left| \partial_{xx} \left((x-y)^i \partial_x^k G^0(t,x;T,y) \right) \right|. \end{aligned}$$

Then the thesis follows from the boundedness of the coefficients α_k, $k \le N$, (cf. Assumption A_N) and the following standard Gaussian estimates (see, for instance, Lemma A.1 and A.2 in [8]):

$$\begin{aligned} \partial_x^k G^0(t,x;T,y) &\le c \left(\sqrt{T-t}\right)^{-k} \bar{\Gamma}^{M+\epsilon}(t,x;T,y), \\ \left(\frac{x-y}{\sqrt{T-t}}\right)^k G^0(t,x;T,y) &\le c\, \bar{\Gamma}^{M+\epsilon}(t,x;T,y), \end{aligned} \tag{6.76}$$

where c is a positive constant which depends on k, m, M, ε and τ. □

Lemma 6.3. *For any $\epsilon, \tau > 0$ there exists a positive constant C, only dependent on $\varepsilon, \tau, m, M, N$ and $\max_{k \le N+1} \|\alpha_k\|_\infty$, such that*

$$\left|Z_n^N(t,x;T,y)\right| \le \kappa_n (T-t)^{\frac{N+n-1}{2}} \bar{\Gamma}^{M+\epsilon}(t,x;T,y), \tag{6.77}$$

for any $n \in \mathbb{N}$, $x, y \in \mathbb{R}$ and $t, T \in \mathbb{R}$ with $0 < T - t \le \tau$, where

$$\kappa_n = C^n \frac{\Gamma_E\left(\frac{1+N}{2}\right)}{\Gamma_E\left(\frac{n+1+N}{2}\right)}$$

and Γ_E denotes the Euler Gamma function.

Proof. On the basis of definitions (2.15) and (2.16), by induction we can prove the following formula:

$$Z_0^N(z;\zeta) = L\Gamma^N(z;\zeta) = \sum_{n=0}^{N} (L - L_n) G^{N-n}(z;\zeta). \tag{6.78}$$

Indeed, for $N = 0$ we have

$$L\Gamma^0(z;\zeta) = (L - L_0)G^0(z;\zeta),$$

because $L_0 G^0(z;\zeta) = 0$ by definition. Then, assuming that (6.78) holds for $N \in \mathbb{N}$, for $N + 1$ we have

$$L\Gamma^{N+1}(z;\zeta) = L\Gamma^N(z;\zeta) + LG^{N+1}(z;\zeta) =$$

(by inductive hypothesis and (2.16))

$$\begin{aligned}
&= \sum_{n=0}^{N}(L - L_n)G^{N-n}(z;\zeta) + (L - L_0)G^{N+1}(z;\zeta) \\
&\quad - \sum_{n=1}^{N+1}(L_n - L_{n-1})G^{N+1-n}(z;\zeta) \\
&= \sum_{n=1}^{N+1}(L - L_{n-1})G^{N-(n-1)}(z;\zeta) + (L - L_0)G^{N+1}(z;\zeta) \\
&\quad - \sum_{n=1}^{N+1}(L_n - L_{n-1})G^{N+1-n}(z;\zeta) \\
&= (L - L_0)G^{N+1} + \sum_{n=1}^{N+1}(L - L_n)G^{N+1-n}(z;\zeta)
\end{aligned}$$

from which (6.78) follows.

Then, by (6.78) and Assumption A_{N+1} we have

$$\left|Z_0^N(z;\zeta)\right| \le \sum_{n=0}^{N} \|\alpha_{n+1}\|_\infty |x - y|^{n+1} \left|(\partial_{xx} - \partial_x)G^{N-n}(z;\zeta)\right| \tag{6.79}$$

and for $n = 0$ the thesis follows from estimates (6.75) and (6.76). In the case $n \ge 1$, proceeding by induction, the thesis follows from the previous estimates by using the arguments in Lemma 4.3 in [9]: therefore the proof is omitted. □

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