Investment, replacement and scrapping in a vintage capital model with embodied technological change

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By

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Abstract

This paper analyzes and compares two alternative policies of determining the service life and replacement demand for vintage equipment under embodied technological change. The policies are the *infinite-horizon* replacement and the *transitory replacement* ending with scrapping. The corresponding vintage capital models are formulated in the dynamic optimization framework. These two approaches lead to different estimates of the duration of replacements and the impact of technological change on the equipment service life.

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Keywords: vintage capital equipment, embodied technological change, service life, replacement, scrapping

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1. Introduction

In order to secure the services of durables at minimum cost, producers and consumers confront invariably the question: How frequently should a stock of old durables be replaced by a stock of new ones? Clearly, the old durables should not be replaced too soon because the cost of acquiring them will occur too frequently and this will raise the unit cost of their services. However, the durables should not be replaced too late either, because their rising operating costs and the higher productivity of durables of newer vintages render them economically inferior. So, to tackle the issues involved in determining the optimal service life of durables, researchers in the fields of management and economics have adopted over the years various approaches.

Preinreich (1940) was the first to show how the optimal life of durables can be determined. More specifically, according to his theorem, the optimal economic life of a single machine should be computed together with the economic life of each machine in the chain of future replacements extending as far into the future as the owner's profit horizon[1]. However, the theorem was formulated under two crucial assumptions. The first of them abstracted from a technological progress and postulated that newer machines of identical type replaced older machines (like-for-like). This assumption contradicted casual observations and was ultimately relaxed by Smith (1961) who generalized the above result to the case where the older machines were replaced by more productive machines embodying the most recent advances in science and technology. The second assumption concerned the horizon of the reinvestment process and required the owner of the machine to choose its duration on the basis of their perception on how long the investment opportunity might remain profitable. Later, depending on the specification of the owner's profit horizon, different models emerged for the determination of the optimal lifetime of assets.

In particular, by limiting the owner's profit horizon to a single investment cycle, researchers in the field of capital budgeting obtained the so-called "abandonment" class of models and used it to derive strict rules regarding the optimal asset life. Initially, Robichek and Van Horne (1967) suggested that an asset should be abandoned during any period, in which the present value of future cash flows did not exceed its abandonment value. Then, based on the possibility that the function of cash flows might not have a single peak, Dyl and Long (1969) argued that abandonment should not occur at the earliest possible date that the above abandonment condition was satisfied, but rather at the date that yielded the highest net present value over all future

[1] The terms "capital", "equipment", and "machine" have been employed frequently in the relevant literature to indicate that the good under consideration has the properties of a producer's durable. In this paper, we use these terms interchangeably. The same comment holds also for "economic life, "service life", "life", and "lifetime".

abandonment opportunities. Later, Howe and McCabe (1983) highlighted the patterns of cash flows and scrap values under which the "abandonment" model led to a unique global optimum of the abandonment time. They also characterized the complete range of models that could be obtained by varying the owner's profit horizon and clarified the practical guidelines for the choice between "abandonment" and "replacement" models.

Theoretical economists, on the other hand, continued to work in the tradition of Terborgh (1949) and Smith (1961) by assuming invariably that the owner's *profit horizon is infinite*. This in turn led them to concentrating on a single class of replacement models, all of which presumed that the infinite reinvestments *took place at equal time intervals*. This pervasive conceptualization was adopted in all significant contributions in the area from Brems (1968) to Nickel (1975), Rust (1987) and to Mauer and Ott (1995) [2]. More recently, Van Hilten (1991), Hritonenko and Yatsenko (1996b, 2004, 2005, 2006), Regnier, Sharp, and Tovey (2004) relaxed these assumptions and considered both the variable replacement period and the finite profit horizon.

The practical importance of the finite-horizon replacement problem was highlighted by Hartman and Murphy (2006), who explored the replacement policy which occurs when companies only require an asset for a specified length of time, usually to fulfill a specific contract and identify when this policy deviates significantly from optimal. Bitros and Flytzanis (2005) demonstrated that the policies of *infinite-horizon* replacements and *transitory replacements* ending with scrapping lead to different results regarding the profit horizon, the duration of replacements, the timing of scrapping, and the impact of output and market structure on service lives. In doing so, they assumed that the technological progress had the form of random breakthroughs, which at the time of their occurrence rendered all existing equipment inoperable.

In the real world, there are two fundamentally different modes of technical progress (e.g., Simpson, Toman, and Ayres 2005, p.144): a "*normal mode*", in which technological improvements occur incrementally and more or less automatically as a result of accumulated experience and learning, and the *radical innovations (technological breakthroughs)*. The normal mode is characterized by a simple positive feedback between increasing consumption, increasing investment, increasing scale, and learning-by-doing (or experience). The production technology itself may also gradually become more efficient. It results in gradually declining costs and prices, stimulating further increases in consumption and, hence, economic growth. The second mode involves multiple competing and evolving production technologies

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[2] Probably, the persistence of this approach was encouraged by the proof that Elton and Gruber (1976) provided regarding the optimality of an equal life policy for a machine subject to technological improvement. However, subsequent research has established that the equal life policy is only a special case of a much more general set of variable life replacement policies. We will say more on this later on in this paper.

(plastics and synthetic fibers substituting iron and steel, automobiles replacing horses and carriages, air transport displacing railways, and so on). The substitution of major general-purpose technologies can be both very productive and very traumatic.

The goal of this paper is to compare the “abandonment” and “replacement” management policies when the technological change is in the normal mode and of the embodied type. For this reason, we adopt the so-called *Vintage Capital Model* (*VCM*), in which the units of equipment brought into operation are more productive than the ones already in place because they embody the most recent advances in science and technology. We expect to demonstrate that this setting poses essential challenges because the optimal lifetime of equipment in each vintage depends on the horizon of the reinvestment opportunity as well as on the date of its introduction into operations. It requires the solution of a non-linear control problem.

The rest of the paper is organized as follows. In the next section, we set up the model. Unlike the equipment replacement models considered above, its specification allows for improvements in the productivity of consecutive vintages of equipment. This, in turn, leads to a new non-linear optimisation problem that involves the optimal control of the equipment lifetime in each vintage. Then, in the following two sections, we investigate the implications of two different approaches to the administration of equipment. Section 3 focuses on the strategy of *infinite-horizon replacements,* implying that the equipment is being replaced indefinitely, whereas Section 4 concentrates on the strategy of *transitory replacements*, where the equipment is replaced a finite number of times, ending with abandonment or scrapping. Lastly, in Section 5, we conclude with a synopsis of main findings.

2. The model and the optimization problem

At the end of the service life of equipment, there are always two options: to replace it and continue doing so up to some profit horizon, or to abandon or scrap it and terminate operations. To examine their implications, we assume that during year τ the representative firm acquires $K(\tau)$ units of new capital, which possess the same efficiency $b(\tau)$ because they belong to the same vintage and embody the same technology. The output of the new capital $K(\tau)$ is denoted as $X(\tau)=b(\tau)K(\tau)$. The units of capital built later in the year $t>\tau$ are more productive because they embody the latest advances of science and technology. To describe this process, we assume that the capital efficiency (output-capital ratio) is:

$$b(t)=b(\tau)e^{\mu(t-\tau)}, \qquad (1)$$

where $\mu > 0$ is a constant and exogenous rate of technological change. Thus, the efficiency of capital in each vintage depends on the date ν of its construction. Following Malcomson (1975), van Hilten (1991), Hritonenko and Yatsenko (1996, 199, 2003), Boucekkine et al. (1997, 1998), Greenwood et al. (2000), and others, we assume that the representative firm acquires only the newest vintage of equipment and removes from service the oldest equipment that has become obsolete. Then, the total output produced in year *t* is described as:

$$X(t) = \int_{a(t)}^{t} X(\tau)d\tau = b(0)\int_{a(t)}^{t} e^{\mu\tau} K(\tau)d\tau, \qquad a(t) < t, \tag{2}$$

where $a(t)$ is the purchasing time of the equipment scrapped at time *t* (known as the *capital scrapping time* in Boucekkine et al. (1998)). Namely, the capital bought at time $a(t)$ is scrapped at the current time t. Then $t - a(t)$ is the lifetime of equipment bought at time $a(t)$. The integral over $[a(t), t]$ in (2) implies that at time t the firm uses only the equipment units placed into service between $a(t)$ and t. Introducing the market price $p(t)$ of output $X(t)$, we can represent the net operating revenue of the firm as:

$$Q(t) = p(t)b(0)\int_{a(t)}^{t} e^{\mu\tau} K(\tau)d\tau - w(t)L(t), \tag{3}$$

where $L(t)$ is the total labor employed in year *t* and $w(t)$ is the wage rate. In this paper, we restrict ourselves to the labor expenses only, although other operating costs can be also considered. Assuming that $m(\tau)$ units of labor operate each equipment unit introduced at time τ, the total labor demand of the firm is described as:

$$\bar{L}(t) = \int_{a(t)}^{t} K(\tau)m(\tau)d\tau. \tag{4}$$

We shall notice that this resource constraint can be imposed on any other critical resource of a firm such as energy, finances, operating space, or even repair facilities. For example, in energy production, a crucial restriction is set by the environment contamination limits.

Compared to other models reviewed in the previous section, the *VCM* (1)-(4) provides a convenient tool to consider the optimal lifetime of equipment as an unknown (endogenous) variable. To determine this endogenous variable, we formulate an optimization problem by assuming that the present value of total profits over the planning horizon $[t_0, T_{\max}]$

$$\Pi = \int_{t_0}^{T\max} e^{-rt}[Q(t) - q(t)K(t)]dt \qquad (5)$$

is maximized under the given labor resource constraint (4). Here $q(t)$ is the acquisition price of the new equipment unit, $[t_0, T_{\max}]$ is the planning horizon, and r is the discount rate. We assume that the residual (salvage) value of the scrapped equipment is negligible compared with its acquisition price. The dynamics of $q(t)$ is also determined by technological change and, together with the output-capital coefficient described in (1), appears to be critical for determining the optimal service life of equipment. In this paper, we assume dynamics of $q(t)$ and $p(t)$ are different:

$$q(t) = q(0)e^{\eta t}, \; p(t) = p(0)e^{\zeta t}, \qquad (6)$$

where the constants η and ζ may be positive or negative.

In the formulated optimization problem, the unknown controls are the investment $K(t)$ and the scrapping time $\alpha(t)$. In contrast to the simple model of infinite equal time replacements employed by Bitros (2005) to compare the two policies, the *VCM* (1)-(6) considers the variable equipment lifetime (service life) $T(t) = t - a(t)$. The output-capital ratio $b(\tau)$, the labor-capital ratio $m(\tau)$, the total labor $L(t)$, the acquisition price of capital $q(t)$, and the product price $p(t)$ are given on $t \in [t_0, T_{\max}]$. It is convenient to assume that one man operates one unit of equipment. Then, $m(\tau)$=1 in (4), $b(\tau)$ in (1) is the output-labor coefficient, and $q(t)$ in (5) is the *relative price of a labor unit of equipment* as in Greenwood et al. (2000). To simplify the optimization analysis, we also assume that $\overline{L}(t)$=const.

Remark 1.
Bitros and Flytzanis (2005) considered a similar problem in a simpler time-invariant framework with an impatience assumption in the absence of embodied technological change.

Let us impose some necessary restrictions on the unknown variables. First of all, we set $0 \le K(t) \le K_{\max}(t)$. This implies that the *maximal possible investment* $K_{\max}(t)$ is determined by certain financial constraints faced by the representative firm. It is also natural to assume that the scrapped equipment cannot be used again, i.e. $a'(t) \ge 0$. Finally, as the model (1)-(6) is defined on the future interval $[t_0, T_{\max}]$, a specific vintage structure of the equipment should be known at the initial time t_0. This structure is defined by the given investments $K(\tau) = K_0(\tau)$ undertaken throughout the pre-history interval $[a(t_0),t_0]$.

Thus, the optimization problem is to find the functions $K(t)$ and $a(t)$, $t \in [t_0, T_{\max}]$), $T_{\max} \leq \infty$, which maximize the objective functional (5) under the constraint-equality (4), the constraint-inequalities:

$$0 \leq K(t) \leq K_{\max}(t), \tag{7}$$

$$a'(t) \geq 0, \; a(t) < t \;, \tag{8}$$

and the initial conditions:

$$a(t_0) = a_0 < t_0, \; K(\tau) = K_0(\tau), \; \tau \in [a_0, t_0]. \tag{9}$$

Malcomson (1975) was first to introduce the *VCM* (2)-(3) to find the optimal capital replacement policy of an individual firm with vintage technology under the embodied technological progress. Silverberg (1988) used the *VCM* (2)-(3) to describe the rational equipment replacement of a firm in an evolutionary model of self-regulatory market. In the same framework, van Hilten (1991) investigated a finite–horizon optimization problem where the constant lifetime of capital was optimal and showed the importance of the *zero-investment period*. Hritonenko and Yatsenko (1996b, 2004) provided a qualitative analysis of the optimization problem (2)-(9).

Remark 2.
It would be interesting to assume that the representative firm faces a demand curve of the constant elasticity type. However, this assumption introduces a scale effect and makes the solution of the problem (1)-(9) considerably more difficult. In particular, the optimal lifetime of equipment would depend on the amount of output produced. So, we leave this specification for future research.

We turn now to the investigation of the possible differences between the two approaches to the management of capital. In the optimization problem (1)-(9), the policy of *infinite-horizon replacements* corresponds to the case $T_{\max} = \infty$, whereas the policy of *transitory replacements* ending with scrapping corresponds to the case $T_{\max} < \infty$. The structure of the solutions $(K^*(t), a^*(t))$ appears to be quite different under $T_{\max} = \infty$ and $T_{\max} < \infty$. Section 3 below is devoted to the analysis of the *infinite horizon replacement* policy, whereas the investigation of the policy of *transitory replacements* ending with scrapping is relegated to Section 4.

3. Infinite-horizon replacements

The optimization problem (1)-(9) is meaningful at $T_{\max}=\infty$ if the value of the improper integral in (5) is finite (otherwise, there is no sense to maximize it). Let us assume that

$$r>\mu \quad \& \quad r>\mu+\zeta. \tag{10}$$

Conditions (10) reflect the natural requirements that the discount factor needs to be greater than the technological progress rate in order for (5) to yield a finite value of profits in the infinite horizon.

As shown in Hritonenko and Yatsenko (2004), the optimization problem (1)-(9) at $T_{\max}=\infty$ has a unique solution $(K^*(t),a^*(t),t\in[t_0,\infty))$ such that the optimal scrapping time $a^*(t)$ coincides with a special trajectory (*turnpike*) $a^{\sim}(t),t\in[t_0,\infty)$ on the planning horizon $[t_0,\infty)$ except for an initial (transition) period $[t_0,\mu]$. The *turnpike* $a^{\sim}(t)$ is determined from the following nonlinear integral equation:

$$\int_t^{a^{-1}(t)} e^{(-r+\zeta)\tau}[1-e^{-\mu(t-a(\tau))}]d\tau = Ce^{(-r-\mu+\eta)t}, \quad t\in[t_0,\infty), \tag{11}$$

where $a^{-1}(t)$ is the inverse function of $a(t)$ and $C=q(0)/p(0)b(0)$. The inverse function $a^{-1}(t)$ exists because $a'(t)\geq 0$ by (8) and is discontinuous when $a'(t)=0$. Equation (11) demonstrates that the profit from introducing a new equipment unit and using it during its future lifetime with the simultaneous retirement of an older unit must be equal to the acquisition price of the new equipment unit.

Proposition 1.

If (10) holds and μ>0, then equation (11) has the unique solution $a^{\sim}(t)$ at least in the following cases:

- If μ = r-ζ and $C\mu$ <1, then $a^{\sim}(t)\equiv t-T, t\in[t_0,\infty)$, where the constant T is determined by the following non-linear equation:

  $$(r-\zeta)e^{-\mu T}-\mu e^{-(r-\zeta)T}=(r-\zeta-\mu)(1-C(r-\zeta)). \tag{12}$$

  At small μ, the constant T is approximately equal to $\sqrt{2C/\mu}$.
- If μ≠r-ζ, then $T^{\sim}(t)=t-a^{\sim}(t)>0$ exists only if

$C(r-\zeta)(r-\eta)e^{(\eta-\zeta-\mu)t} < r-\zeta-\mu$

- If $\mu > r\text{-}\zeta$, then equation (11) has the unique monotonically increasing solution $a^{\sim}(t)$ in an interval (t_{cr},∞), such that $T^{\sim}(t)\to 0$ at $t\to\infty$, and $T^{\sim}(t)\to\infty$ at $t\to t_{cr}$. The critical time t_{cr} can be estimated as:

$$t_{cr} \approx \frac{1}{\eta-\zeta-\mu}\ln\frac{r-\zeta-\mu}{C(r-\eta)} \tag{13}$$

- If $\mu < r\text{-}\zeta$, then equation (11) has the unique solution $a^{\sim}(t)$ on an interval $(-\infty,\ t_{cr})$, such that the solution $a^{\sim}(t)$ increases on $(-\infty,\ t_{cr})$, decreases on $(t_c,\ t_{cr})$ at some $t_c < t_{cr}$, and $a^{\sim}(t)\to\infty, T^{\sim}(t)\to\infty$ at $t\to t_{cr}$.

  The proof of Proposition 1 is provided in the Appendix.

Proposition 1 can be used to estimate the optimal equipment replacement strategies at the individual plant or firm level. In particular, the following statements about the firm's equipment replacement are valid for the problem (1)-(9):

- Except for a possible initial (transitory) period, the optimal lifetime of equipment $T^{\sim}(t) = t - a^{\sim}(t)$ depends neither on the quantity of produced output nor on the initial equipment structure and is determined only by the rates of technological change, prices, and discount rate.
- The optimal lifetime of equipment may be finite only in the presence of the embodied technological progress, i.e., when the vintage productivity $b(t)$ increases ($\mu > 0$ in (1)).
- The proportion between the value of productivity $b(t)p(t)$ and the equipment price $q(t)$ determines the dynamics of the optimal equipment lifetime. If $\mu+\zeta>\eta$ (i.e., the revenue $b(t)p(t)/q(t)$ per unit of new equipment acquisition price increases), then the optimal lifetime decreases. If $\mu+\zeta<\eta$ (that is, $b(t)p(t)/q(t)$ decreases), then the optimal lifetime increases and becomes infinite at some finite instant t_{cr}.
- If the sum of the rates of technological change and output price is equal to the rate of price change of new equipment, $\mu+\zeta=\eta$, then the ratio $b(t)p(t)/q(t)$ is constant and the optimal equipment lifetime is also constant. This constant depends only on the discount rate and the ratio between the value of productivity and the acquisition price of equipment.

We shall notice that the last two properties imply that the well-known equidistant equipment replacement is a sub-case in our more flexible model, which appears to be optimal only in the case $\mu+\zeta=\eta$.

Bitros and Flytzanis (2005) introduced some special terminology to classify the various types of equipment on a scale of replaceability. Following them, we say that the equipment is:

- *Finitely replaceable*, if it has a finite number $N>0$ of profitable replacements;
- *Infinitely replaceable*, if $N=\infty$;
- *Non-replaceable,* if no replacement is profitable.

Then, in the setting of the optimization problem (1)-(9), the vintage equipment is:

- *Infinitely replaceable*, if $\mu+\zeta\geq\eta$,
- *Non-replaceable*, if $\mu+\zeta<\eta$ and $t_{cr}<t_0$,
- *Finitely replaceable* if $\mu+\zeta<\eta$ and $t_{cr}>t_0$. Then, the exact number *N* of profitable replacements depends on the proportion between the horizon length T_0-t_0 and $T^{\sim}(t)$, where $T^{\sim}(t)$ is determined from equation (11).

Now, let us describe the optimal dynamics of the corresponding *investment* $K^*(t)$. During the initial transitory period $[t_0,\mu]$, the investment is maximum possible $K^*(t)=K_{\max}(t)$ if $a_0<a^{\sim}(t_0)$ or minimum possible $K^*(t)=0$ if $a_0>a^{\sim}(t_0)$. Differentiating (4), we get $K(a(t))a'(t)=K(t)-\overline{L}'(t)$. Hence, under our constant labor condition $\overline{L}(t)=\text{const}$, the *minimum replacement investment regime* is $K^*(t)=0$, $a^{*\prime}(t)=0$, $a^*(t)=\text{const}$ (no working equipment is scrapped and no new equipment is acquired).

So, in the general case, the optimal investment trajectory $K^*(t)$ is boundary (minimum or maximum) at a beginning part $[t_0,\mu]$ of the planning horizon. After that, $K^*(t)$ is found from (4) as $K^*(t)=K(a^{\sim}(t))da^{\sim}(t)/dt$. The last formula shows that the initial boundary-valued section of $K^*(t)$ is reproduced throughout the whole horizon $[t_0,T]$. In particular, in the case $\mu+\zeta=\eta$, the constant lifetime $a^{\sim}(t)=t-T$ and the strictly periodic investment $K^*(t)=K(t-T)$ represent the optimal policy. The repetition pattern with bursts and slumps in $K^*(t)$ can be observed in Figure 1. These "spikes" (*replacement echoes* by Boucekkine et al. (1997)) were experimentally discovered in recent years by researchers studying investment at the plant level.

4. Transitory replacements ending with scrapping

Here, we assume that the equipment is managed optimally for a finite number of operating periods ending with scrapping, i.e., $T_{\max}<\infty$. Then the structure of the solutions of the optimization problem is more complicated as compared with the case $T_{\max}=\infty$.

The key new feature is the existence of the ***"zero-investment period"*** $[\Theta, T_{\max}]$, $t_0 \le \Theta < T_{\max}$, at the end of the planning horizon $[t_0, T_{\max}]$ (see van Hilten (1991)). During the $[\Theta, T_{\max}]$ period, the investment is minimum possible since there is no sense in investing in new capital given that the firm quits production at $T_{\max}$. This effect is well known in various economic optimization models. Under our condition of constant labor, $L(t) = \text{const}$, the minimum investment regime is $K^*(t) = 0$ and $da^*(t)/dt = 0$, $a^*(t) = a^*(\Theta) = \text{const}$ at $t \in [\Theta, T_{\max}]$, i.e., no new investment is made and no equipment is scrapped. The optimal lifetime $T^*(T_{\max}) = T_{\max} - a^*(T_{\max}) = T^*(\Theta)$ of the oldest equipment at the horizon end $t = T_{\max}$ is always larger than the optimal lifetime $T^{\sim}(T_{\max})$ in the indefinite-replacement case from Section 3.

The presence of the *zero-investment period* causes essential changes in the behavior of the optimal trajectories $K^*(t)$ and $a^*(t)$ on the whole horizon $[t_0,\ T_{\max}]$. However, the impact of the zero-investment period weakens when the duration $T_{\max} - t_0$ becomes larger. In particular, the optimal lifetime $T^*(t) = t - a^*(t)$ strives to the "indefinite-replacement" optimal lifetime $T^{\sim}(t) = t - a^{\sim}(t)$ as $T_{\max} - t \to \infty$. Mathematically (see Hritonenko and Yatsenko (2004)), at $\mu + \zeta \ge \eta$ the finite–horizon of the optimization problem (1)-(9) has the unique solution $K^*(t), a^*(t)$,

$$a^*(t) = \begin{cases} a_\mu(t), & t \in [t_0, \mu], \\ a_i(\alpha_i), & t \in [\alpha_i, \beta_i] \quad i = 1,2,..., \quad t \in [t_0,\ T_{\max}], \\ a_i(t), & t \in [\beta_{i+1}, \alpha_i] \end{cases} \tag{14}$$

where the trajectories a_i, $i = 1,2,3...$ are

$$a_1(t) = t + \frac{1}{c}\ln\left\{1 - \lambda_0(c_3 - c) + \frac{c}{c_3}\left[e^{-c_3(T_{\max} - t)} - 1\right]\right\} \tag{15}$$

$$a_{i+1}(t) = t + \frac{1}{c}\ln\left\{1 - \lambda_0(c_3 - c) + \frac{c}{c_3}\left[e^{-c_3(a_i^{-1}(t) - t)} - 1\right]\right\}, \qquad i = 1,2,3,...,$$

and the parameters $\alpha_i, \beta_i, i = 1,2,3...,$ are uniquely determined, $\beta_1 = T_{\max}$, $\alpha_1 = \Theta$, $\alpha_i < \beta_i$, $\beta_{i+1} < \alpha_i$.

The solution of the optimization problem $K^*(t), a^*(t)$ is shown in Figure 1. The *infinite-horizon* replacement solution is indicated with the dotted line (by Section 3, $a^{\sim}(t) = t - T$ is the turnpike in the case $c_2 = c_1$). The finite-horizon equipment lifetime $a^*(t)$ tends to the infinite-horizon turnpike when the time *T-t* left to the horizon end *T* increases. One can notice that the finite-horizon optimal policy possesses sharper changes at certain "critical" instants $\alpha_i, \beta_i, i = 1, 2, 3...,$, which depend on the length of the planning horizon $[t_0,\ T_{\max}]$. These changes can be referred to as the *zero–investment echoes*, because they are caused by the zero–investment period $[\Theta,\ T_{\max}]$ and propagate backward through the entire planning horizon $[t_0,\ T_{\max}]$.

The beginning moment Θ of the zero-investment period $[\Theta,\ T_{\max}]$ is found from the condition:

$$\int_{\theta}^{T_{\max}} e^{-c_3 \tau}[e^{c_1 \theta} - e^{c_1 a_1(\theta)}]d\tau - Ce^{(-c_3 + c_2)\theta} = 0 \ .$$

Obviously, it is smaller than $T_{\max}$. If the calculated Θ appears to be less than t_0, then the equipment is *finitely non-replaceable* because the planning horizon $[t_0,\ T_{\max}]$ is too small. If $\Theta > t_0$, then the corresponding equipment is *finitely replaceable* in the interval $[t_0,\ T_{\max}]$. If the equipment is *finitely replaceable* on the interval $[t_0,\ T_{\max}]$, then it is *infinitely replaceable* (see Section 3). Conversely, if the equipment is *infinitely replaceable* on the interval $[t_0, \infty)$ then it is *finitely replaceable* on large enough planning horizons $[t_0,\ T_{\max}]$. Therefore, as in Bitros (2005), the profitability condition for the *infinite-horizon* replacement and *transitory* replacement on large planning horizons is the same.

Remark 3.
As in Bitros (2005), the profit horizon for the *transitory replacement* ending with scrapping can be determined endogenously by the equipment parameters and the external market environment (it is obviously infinite in the *infinite-horizon* case). Namely, taking into account the given initial equipment distribution on the pre-history interval $[a(t_0),\ t_0]$, it may be more profitable to extend (or decrease) slightly the interval $[t_0,\ T_{\max}]$. For example, in capital budgeting it will allow us to consider the endogenous influence of profit horizon on the selection of projects (and inversely). So, the *transitory* approach to replacement is more flexible. Mathematically, it requires adding the value $T_{\max}$ as the additional control variable to the optimization problem. The authors are going to explore this idea in later research.

5. Summary

Our objective in this paper was to compare the differences between the policy of *infinite-horizon* replacements and that of *transitory replacements* ending with scrapping in the presence of embodied technological change. To accomplish it, we have adopted the *vintage capital model*, in which the new units of equipment brought into operation are more productive than those already in place due to advances in science and technology. Our main findings show that, if the vintage equipment is *finite-horizon replaceable*, it is always *infinite-horizon replaceable*. The *infinite-horizon* policy predicts shorter replacement durations than the *transitory* replacement policy does. This difference is significant at the end of the planning horizon in the case of the *transitory* replacement policy and becomes smaller when the planning horizon ends in the more distant future. So, if equipment is *infinite-horizon replaceable*, it is *finite-horizon replaceable* at *large planning horizons*.

As in Bitros and Flytzanis (2005), the replacement period in the *infinite-horizon* case adjusts gradually to possible changes in economic parameters. In the case of the *transitory* replacement policy, in addition to this smooth change, we have also sharp changes in optimal policies when the parameters (such as the planning horizon length) cross certain critical values. These changes are caused by the existence of the *zero–investment period* at the end of the planning horizon, when no new investment is made and no working equipment is scrapped. Because of the zero–investment period, the optimal replacement possesses the *zero–investment echoes* that propagate backward through the entire planning horizon. This fact demonstrates that the *multi-step transitory* replacement is much more complex and flexible management policy, which is often overlooked in economic theory and management practice.

In the new settings, a new question arises about possible effects that appear if, instead of treating the owner's profit horizon as given, it is considered as an endogenous variable along with all other variables in the optimization process. The authors are going to explore it in the future.

Appendix

Proof of Proposition 1

The proof uses the technique developed in Yatsenko and Hritonenko (2005) for similar integral equations. Let us adopt the following symbol simplifications:

$$c_1 = \mu,\ \ c_2 = \eta - \zeta \text{ and } c_3 = r - \zeta .$$

The differentiation of equation (11) in the text with the respect to the unknown function $T(t) = t - a(t) > 0$ leads to

$$c_3 e^{-c_1 T(t)} - c_1 e^{-c_3 T(a^{-1}(t))} = c_3 - c_1 - c_3 C(c_3 - c_2) e^{(c_2 - c_1)t} . \quad (16)$$

So, if a solution of (11) exists, it satisfies (16). The case $c_2 = c_1$ was analysed in Hritonenko and Yatsenko (2004) and is easily verified by direct substitution into (16). The proof in the case $c_2 \neq c_1$ includes the following steps.

Step 1. Let us show that, if $c_2 \neq c_1$, then (16) cannot have a positive solution $T(t)$ on the infinite interval $(-\infty,\infty)$. Indeed, we can rewrite (16) as

$$G(T(t), T(a^{-1}(t))) = f(t)$$

where $G(x, y) = c_3 e^{-c_1 x} - c_1 e^{-c_3 y}$, $f(t) = c_3 - c_1 - c_3 C(c_3 - c_2) e^{(c_2 - c_1)t}$

One can see that $-c_1 < G(x, y) < c_3$ for any $x, y>0$, whereas $f(t) \to \infty$ at $t \to \infty$ or $t \to -\infty$. Hence, for some values of t, there exist no positive $T(t)$ and $T^{-1}(t)$, that satisfy (16).

For certainty, let us consider the case $c_2 > c_1$. Then $f(t) < -c_1$, hence, (16) has no solution at $t \geq$

$$t_f = \frac{1}{c_2 - c_1} \ln \frac{c_3}{c_3 C(c_3 - c_2)} .$$

Step 2. Let us analyse the possibility of a solution $a(t)$ on an interval $(-\infty, t_{cr})$, $t_{cr} < t_f$. If such solution $a(t)$ exists, it satisfies $a(t) < t-d < t_{cr}-d$, d=const>0, at t close to t_{cr}. It means that the replacement will never happen for the equipment purchased after $t_{cr}-d$. Hence, $a^{-1}(t)=\infty$ in the upper limit of (11) at $t_{cr}-d \leq t < t_{cr}$. Then, differentiating (11) gives

$$c_3 e^{-c_1 T(t)} = c_3 - c_1 - c_3 C(c_3 - c_2) e^{(c_2 - c_1)t} . \quad (17)$$

Equation (17) has the unique solution $T(t)$ on $[t_{cr}-d, t_{cr})$ such that $T(t) \to \infty$ and $a(t) \tilde{\to} \infty$ at $t \to t_{cr}$. Differentiating (17) gives

$$-c_1 (1 - x'(t)) e^{-c_1 T(t)} = -c_3 C(c_2 - c_1)(c_3 - c_2) e^{(c_2 - c_1)t}$$

The substitution of $e^{-c_1 T(t)}$ from (16) into the last formula shows that the function $a(t)=t-T(t)$

is such that $a'(t)=0$ at $t=t_d$, $a'(t)>0$ at $t<t_d$, and $a'(t)<0$ at $t_d<t<t_{cr}$, where $t_d = \frac{1}{c_2-c_1}\ln\frac{c_1(c_3-c_1)}{c_2c_3C(c_3-c_2)}$

Step 3. In continuous time t, the delay equation (16) connects $T(t)=t-a(t)$ and $T(a^{-1}(t))$. It can be iteratively solved forward or backward. By analogy with the previous step, we consider the backward solution. Choosing any fixed instant u, $a^{-1}(u)<t_{cr}$, and a continuous initial function $T(t)=T_0(t)$ at $t\in[u, a^{-1}(u)]$, which satisfies (11) at $t=u$, we obtain from (16) the continuous solution of (11)

$$T(t) = -\frac{1}{c_1}\ln\left\{1 - C(c_3-c_2)e^{(c_2-c_1)t} + \frac{c_1}{c_3}[e^{-c_3T(a^{-1}(t))}-1]\right\} \quad (18)$$

for $t\in[a(u), u]$. To continue this process to $[a(a(u)), a(u)]$ and further up to $-\infty$, we need to prove that the backward iterative formula (18) is convergent. To do that, let us give a small variation $\delta a(t)$ to the (16) solution $a(t)$. By (16) and $c_3>c_1$, we obtain that $|\delta a(t)| = e^{c_1T(t)-c_3T(a^{-1}(t))}|\delta a(x^{-1}(t))| < |\delta a(a^{-1}(t))|$ for small values of $\delta a(t)$. Hence, the iterations converge. If we choose the solution $T(t)$ of (17) from Step 2 as the initial function $T(t)=T_0(t)$ at $t\in[t_d, t_{cr})$, then the recurrent formula (19) produces the unique solution $T(t)$ at $t\in(-\infty, t_{cr})$.

Step 4. Finally, let us analyse the behaviour of the solution $T(t)$, when t tends to $-\infty$. Starting with any $u<t_{cr}$, we introduce the monotonically decreasing sequence $\{t_k\}$, $t_k=a^k(u)$, $k=1,2,3,\ldots$, and the sequence $z_k=T(t_k)=\tilde{t}_k-a(t_k)>0$, $k=1,2,3,\ldots$. By (16), the sequence $\{z_k\}$ satisfies the following non-linear difference equation

$$c_3e^{-c_1z_k} - c_1e^{-c_3z_{k-1}} = -c_3 - c_1 - c_3C(c_3-c_2)e^{(c_2-c_1)t_k}, \text{ or}$$

$$z_k = \varphi(z_{k-1}, t_k) = -\frac{1}{c_1}\ln\left\{1 - C(c_3-c_2)e^{(c_2-c_1)t_k} + \frac{c_1}{c_3}[e^{-c_3z_{k-1}}-1]\right\} \quad (19)$$

Calculating and estimating the derivative of $\varphi(z_{k-1}, t_k)$, we obtain

$$0 < \frac{\partial\varphi(z_{k-1}, t_k)}{\partial z_{k-1}} = e^{c_1z_k - c_3z_{k-1}} < 1$$

Then, as follows from the theory of difference equations, the (19) solution z_k strives to 0 at $k\to-\infty$. Therefore, $T(t)\to 0$ as $t\to-\infty$.

The case $c_1>c_2$ is treated similarly.

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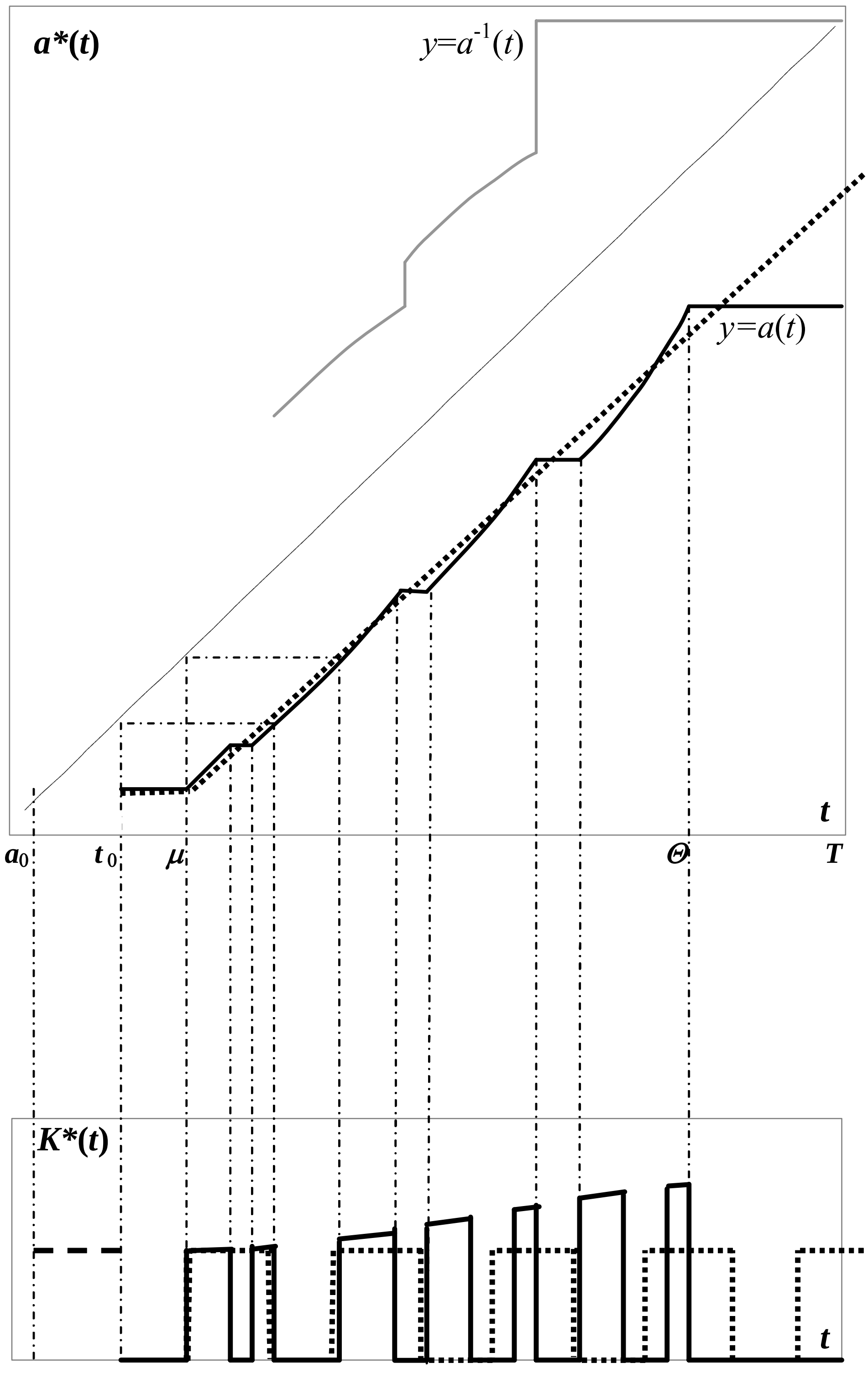
a*(t)
y=a^-1(t)
y=a(t)
t
a0
t0
μ
Θ
T
K*(t)
t