The optimal lifetime of assets under uncertainty in the rate of embodied technical change

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Abstract

This paper investigates the effects of uncertainty emanating from technological improvements on the optimal lifetime of assets. In does so in a dynamic model in which: a) technological change increases continuously the productivity of producers' durables, b) potential competition induces firms to price their output in a way that passes all benefits to final consumers, and c) the mean and the variance are considered sufficient statistics to describe the probability distribution of technological change. From the analysis it turns out that in general this type of uncertainty shortens the optimal lifetime of assets. More specifically, the analysis shows that: *replacement under uncertainty* leads to optimal lifetimes of assets that are shorter than in any other mode of operation; depending on the mean rate of technological progress, $\bar{\mu}$, and the price elasticity of demand, η, *scrapping under uncertainty* yields lifetimes that may be shorter or longer than those determined by *replacement under certainty*; and, irrespective of the values of these parameters, the optimal lifetime of assets from a policy of s*crapping under uncertainty* is always shorter than that from *scrapping under certainty*. However, the robustness of these results under alternative specifications of the probability distribution of technological change remains an open question.

JEL Classification: D81, E22, O22, O33
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1. Introduction

In Bitros and Flytzanis (2005) we investigated the problem of determining the optimal lifetime of assets under a rather reasonable set of assumptions regarding the process and the effects of technological change. In particular, drawing on their impact on the value of existing assets, we assumed that innovations are either *minor* or *major*. With respect to the former, we adopted the view that their effects are completely counterbalanced by maintenance investment of the upgrading type. This, we thought, was in line with the spirit of McFadden and Fuss (1978) since it implied that equipment is designed in an *ex ante* flexible way so as to permit the incorporation of the limited improvements that appear in the few years that follow its construction. As to the effects of *major* innovations or technological *breakthroughs* we stipulated that they are allowed for by raising the discount factor on two grounds: first, that the probability of their occurrence increases exponentially with time, and, second, that when they occur they render existing equipment totally obsolete, by eliminating *a la* Kamien and Schwartz (1971) all revenue and scrap value thereafter. However, these assumptions abstracted from those innovations that can neither be embedded into the original design of the equipment nor be allowed for via the channel of the discount factor.

Traditionally the effects of these *intermediate* innovations on the value of existing assets have been modeled as a continuous trend grafted into the time dependence of the functions involved. At the time it was introduced by Terborgh (1949) and reinforced by Smith (1961), most likely, this approximation was justified because such innovations were slower and more predictable. But in recent years their pace and variability have increased significantly. As a result they are much less predictable and introduce a lot of uncertainly into the process of decision-making regarding the optimal lifetime of assets. Therefore, the time is quite ripe to investigate how the uncertainty that surrounds the rate of embodied technical change affects the optimal lifetime of assets.

In pursuing this objective the next section of the paper presents the model. From Bitros (2005) and Bitros and Flytzanis (2005) we know that the optimal lifetime of assets varies according to whether their owners follow a replacement or a scrapping policy. In particular, we know that under the above-mentioned dichotomy of *minor-major* innovations and certainty in the predicted rate of embodied technical change the optimal lifetime of assets is always lower under replacement than under scrapping. But this result may not hold under the uncertainty that accompanies the production and adoption of *intermediate* innovations. So, Section 3 is devoted to the solution and analysis of the model under certainty, whereas Section 4 does the same under uncertainty in the rate of embodied technical change. Moreover, the latter section compares the results from the two solutions and comments on their differences. Lastly, Section 5 provides a synopsis of the main findings and conclusions.

2. The model[1]

Consider an economy with a representative firm, which is characterized by the following specifications:

2.1 Supply

During year υ the representative firm uses $K(\upsilon)$ units of capital, all of which are equally productive because they embody the same technology. Hence, let its production function takes the form:[2]

$$b(\upsilon) = \frac{K(\upsilon)}{X(\upsilon)}, \tag{1}$$

$b(\upsilon)$ being the capital-output coefficient.

Next, let embodied technological change advance at the rate μ, a random variable. Drawing on the arguments and derivations in Appendix A, its impact on the output-capital coefficient is presumed to proceed as follows:

$$b(t) = e^{\frac{\bar{\mu}}{f(\sigma)}(t-\upsilon)} b(\upsilon), \;\; \upsilon < t < \upsilon + T, \tag{2}$$

with the function $f(\sigma)$ taking on the following values:

$$0 < f(\sigma) \leq 1, f(0) = 1, f(\infty) \cong 0 \tag{3}$$

where t stands for time, T denotes the useful life of capital, $\bar{\mu} < 0$ is the mean rate of technological change, and σ represents the standard deviation of the probability distribution involved as a proxy for the degree of uncertainty. From this specification we may surmise that, *ceteris paribus*, an increase (decrease) in σ would give rise to an increase (decrease) in the mean rate of technological change per unit of uncertainty. As a result, the capital-output coefficient would decline (increase) exponentially, and this in turn would induce an equally fast increase (decline) in the productivity of equipment. So the question that arises is why the uncertainty regarding technological change and the productivity of equipment should move in this fashion? The answer is because the ever-calculating representative firm is risk-averse in that it requires equipment with faster increasing (decreasing) rates of productivity to compensate for higher (lower) uncertainty.

Finally, denoting by β the minimum amount of labor required to construct a unit of capital, assume that:

$$\beta = M[b(0)]^{\gamma}, \tag{4}$$

where M is a positive multiplicative constant and $\gamma < -1$. This implies that the minimum labor required to build a unit of capital embodying the new technology exceeds that required to build a unit of capital from older vintages.

2.2 Demand

Assume that the representative firm faces a demand curve of the constant elasticity type:

$$X(v) = N[P(v)]^{\eta}. \tag{5}$$

In this equation $X(v)$ stands for output in year v, N denotes a multiplicative constant, $P(v)$ is the price of output, and $\eta < -1$, $X(v) > 0$, $N > 0$, $P(v) > 0$.

2.3 Pricing and market structure

Lastly, recall that since the service life of capital is T, $K(v)$ is kept in operation for the time interval $v < t < v + T$. During this time interval other firms may enter the market by purchasing newer, and hence more productive, capital. So to deter potential competition the firm reduces the price of its output according to the rule:[3]

$$P(t) = e^{\frac{\bar{\mu}}{f(\sigma)}(t-v)} P(v), \tag{6}$$

where $P(v)$ denotes the price of consumer goods produced by capital of vintage v.

At this point one may ask: how do we know that this pricing rule does deter new entrants? To ascertain that it does, divide (6) by (2) and set $v = 0$ to obtain:

$$\frac{P(t)}{b(t)} = \frac{P(0)}{b(0)}. \tag{7}$$

What this equation signifies is that by following (6) the representative firm prices the outputs produced by the various vintages of equipment so as to equate the value of their marginal products to that of the initial vintage. But according to the proof in Appendix B the price of output from the initial vintage is calculated to reduce the unit net worth of equipment to zero. Consequently, the same must hold for every vintage up to t, and hence no potential competitor should have an incentive to enter.

From (6) it follows that if the firm were able to predict the mean rate of embodied technological change with certainty, it would set $\sigma = 0$. In that event we would have $f(0) = 1$ and the firm would become a completely contestable monopolist maintaining its position ad infinitum by applying the pricing rule as follows:

$$P(t) = e^{\bar{\mu}(t-v)} P(v). \tag{8}$$

Thus, since under (8) the firm would protect itself from such entry, its normal profit opportunities would repeat indefinitely and the proper model to determine the optimal lifetime of capital would be the one of infinite replacements at equal time intervals. This yields the *replacement under certainty* solution of the model.

However, its solution for *scrapping under certainty* is less easy to rationalize. The reason is that in the above setting it is not obvious why a monopolist would wish to plan for exit from, and possibly re-entry into business at the end of the current investment cycle. One likely explanation is that in this case the monopolist may prefer the strategy of scrapping, which entails lower profits than that of infinite equidistant replacements, because it affords him two advantages: first, to postpone until the end of the present investment cycle the decision of whether to exit from or re-enter into the current business, and, second, to avoid having to regret in case he had to absorb an unforeseen cost in terms of foregone profits, if he chose the strategy of infinite equidistant replacements and he were forced to stop it for some reason. Moreover, as we have shown in Bitros and Flytzanis (2005), where we solve for the re-investment horizon as an endogenous variable, if the firm finds it optimal to end the re-investment process at some intermediate stage, the first few investment cycles yield most of the overall profits.[4] Hence, here I adopt the simplification that solution of the optimization problem involved leads the monopolist to choose a single investment cycle, which ends with scrapping.

Next, suppose that $\sigma \to \infty$. In this case the firm would be completely uncertain about its predictions regarding the mean rate of embodied technical change and the effect of this uncertainty would lead it to set:

$$P(t) = 0, \tag{9}$$

because $\dfrac{\bar{\mu}}{f(\infty)} \to -\infty$. In turn, this implies that all profit opportunities would vanish and the firm would have to shut down immediately, if it was already in operation, or it would not enter into business, if it were in the planning process.

Lastly, let us consider the more realistic case in which σ takes on an intermediate value such that $0 < f(\sigma) = f(\sigma_1) < 1$. In this case the firm will set:

$$P_1(t) = e^{\frac{\bar{\mu}}{f(\sigma_1)}(t-v)} P(v) \tag{10}$$

But now it cannot be certain that competition will be held in abeyance. The reason being that, if there are competitors that can afford more uncertainty about the mean rate of technological change, the standard deviation characterizing their expectations will be higher, i.e. $\sigma_2 > \sigma_1$, and the output prices at which they will be willing to enter will be lower because:

$$P_1(t) = e^{\frac{\bar{\mu}}{f(\sigma_1)}(t-v)} P(v) > P_2(t) = e^{\frac{\bar{\mu}}{f(\sigma_2)}(t-v)} P(v). \tag{11}$$

Consequently, the incumbent firm would be reckless to apply the *replacement under cer-*

tainty solution of the model by planning its reinvestments as if profit opportunities will last forever. Rather, in light of its inability to predict exactly the mean rate of technological progress, it will chose to exit, if another less uncertain firm enters into the market, or plan to exit and re-enter, if none enters during the useful life of capital. This behavior is consistent with the *scrapping under uncertainty* solution of the model.

The question that arises now is whether a policy of *replacement under uncertainty* is conceivable. Clearly, given that the rate of embodied technological change cannot be predicted with any accuracy, particularly as we look into the distant future, it would be quite farfetched to assume that a monopolist would be prepared to precommit to a perpetual series of reinvestments at equal time intervals. Thus the solution of the model for such a policy cannot be expected to yield anything more than a pure mathematical result. Yet, even at the risk of conveying the impression that it constitutes something more than that, the replacement under uncertainty policy will be derived just for purposes of reference and comparison. So what I intend to do in the following two sections is to solve the model for replacement and scrapping under certainty and uncertainty and compare the differences of the solutions on the optimal lifetime of capital.

3. Solution under certainty, $\sigma = 0$

If at $\upsilon = 0$ the salvage value of capital on retirement is zero, following the demonstration in Appendix C, the net worth of a unit of new capital under certainty, $n(0)$, is given by:

$$n(0) = \int_0^T [\frac{P(0)}{b(0)} e^{\bar{\mu}t} - w] e^{-\rho t} dt - \beta w = \frac{P(0)}{b(0)} \frac{1 - e^{(\bar{\mu}-\rho)T}}{\rho - \bar{\mu}} - w \frac{1 - e^{-\rho T} + \beta\rho}{\rho}, \qquad (12)$$

where the symbols w and ρ denote the economy wide rates of wages and interest.

Consider first the policy of *replacement under certainty*. As mentioned above the firm will behave as if its monopoly will last forever. This implies that at any period it must have no more and no less than the necessary capital to meet the demand for its output. For if it has less it will be losing sales and if it has more it will be wasting resources. As a result, the firm will be led to maximize the present value of profits from an infinite series of equidistant replacements given by:

$$\Pi(T,P(0)) = \frac{b(0)n(0)X(0)}{1-e^{-(\bar{\mu}-\rho)T}} = b(0)N[P(0)]^{\eta}\left[\frac{P(0)}{b(0)}\frac{1}{\rho-\bar{\mu}} - \frac{w}{\rho}\frac{1-e^{-\rho T}+\beta\rho}{1-e^{(\bar{\mu}-\rho)T}}\right]. \quad (13)$$

From $\partial\Pi(T,P(0))/\partial T = 0$ we obtain:

$$g_c(T) = \rho e^{-\bar{\mu}T} - \bar{\mu}e^{-\rho T} = (1+\beta\rho)(\rho-\bar{\mu}). \quad (14)$$

This equation does not permit an explicit solution for T. However, it can be established that one and only one positive solution for T exists. To sketch the proof, consider Figure 1 below. Setting $T = 0$, we see that the left-hand side of (14) reduces to $\rho - \bar{\mu}$. Next, letting T rise above zero and taking the derivative, we can ascertain that the left-hand side of (13) rises without bound with rising T. These findings are depicted by the upward sloping curve $g_c(T)$. Finally, looking at the right-hand side of (14), observe that it defines a horizontal line, F_c, which cuts the vertical axis above the value $\rho - \bar{\mu}$. Therefore, the upward sloping curve $g_c(T)$ is bound to cut the horizontal line just once, giving the optimal service life $T^*_{r,\sigma=0}$.

Now let us turn to the policy of *scrapping under certainty*. Assuming again that the value of capital on retirement is zero, the firm maximizes (15) with respect to T and $P(0)$:

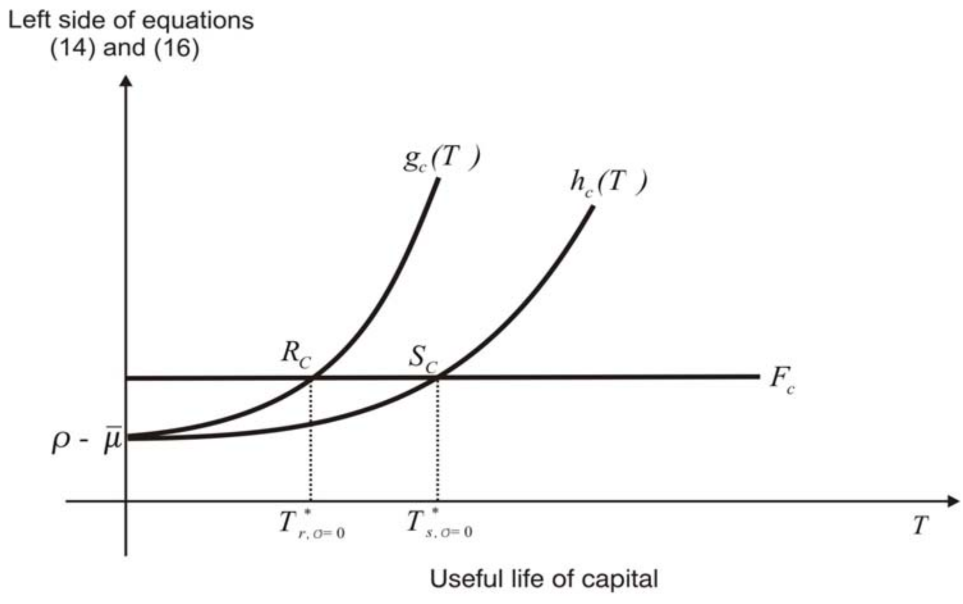

Figure 1

$$\Pi(T,P(0)) = b(0)N[P(0)]^{\eta}\left[\frac{P(0)}{b(0)}\frac{1-e^{(\bar{\mu}-\rho)T}}{\rho-\bar{\mu}} - w\frac{1-e^{-\rho T}+\beta\rho}{\rho}\right]. \tag{15}$$

From the first order conditions we obtain for *T*:

$$\frac{1+\eta}{\eta}\rho e^{-\bar{\mu}T} - \frac{\eta\bar{\mu}+\rho}{\eta}e^{-\rho T} = (1+\beta\rho)(\rho-\bar{\mu}) \ . \tag{16}$$

Denoting by $h_c(T)$ the left-hand side of equation (15) we observe that, as T goes to zero, $h_c(T)$ turns into $\rho-\bar{\mu}$. Hence, both $g_c(T)$ and $h_c(T)$ start from the same point on the vertical axis in Figure 1. Next let T rise above zero and take the derivative of $h_c(T)$. As T rises without bound, this derivative remains positive, which means that $h_c(T)$ always rises. Then the question is whether $h_c(T)$ rises to the left or to the right of $g_c(T)$. Comparing the derivatives of the left-hand sides of equations (14) and (16) we can establish that the $h_c(T)$ curve rises always to the right of $g_c(T)$. This implies in turn that $h_c(T)$ will cut the horizontal line F_c to the right of $T^*_{r,\sigma=0}$, say at $T^*_{s,\sigma=0}$. So what has been established is that:

> ***Remark 1***
> *Under conditions of certainty regarding the mean rate of embodied technical change, replacement leads to an optimal lifetime of assets, which is always lower than that from scrapping.*

3. Solution under uncertainty, $0<\sigma<1$

Let us turn first to replacement. Assuming that the mean rate and standard deviation of the embodied technological change remain fixed for all indefinite future, the firm is expected to behave according to the solution derived from the maximization of:

$$\Pi(T,P(0)) = \frac{b(0)n(0)X(0)}{1-e^{-(\frac{\bar{\mu}}{f(\sigma)}-\rho)T}} = b(0)N[P(0)]^{\eta}\left[\frac{P(0)}{b(0)}\frac{1}{\rho-\dfrac{\bar{\mu}}{f(\sigma)}} - \frac{w}{\rho}\frac{1-e^{-\rho T}+\beta\rho}{1-e^{(\frac{\bar{\mu}}{f(\sigma)}-\rho)T}}\right], \tag{17}$$

with respect to T and $P(0)$. The first order condition for T yields:

$$g_u(T) = \rho e^{-\frac{\overline{\mu}}{f(\sigma)}T} - \frac{\overline{\mu}}{f(\sigma)} e^{-\rho T} = (1+\beta\rho)(\rho - \frac{\overline{\mu}}{f(\sigma)}). \tag{18}$$

From this equation we observe that, as T goes to zero, the left-hand side turns into $\rho - \frac{\overline{\mu}}{f(\sigma)}$. This implies that the $g_u(T)$ curve starts off from a point on the vertical axis that lies above the point $\rho - \frac{\overline{\mu}}{f(\sigma)}$, but below the point at which the horizontal line F_u, defined by the right-hand side of this equation, cuts the vertical axis. Moreover, the derivative of $g_u(T)$ with respect to T rises continuously. Hence the question that arises is at what point $g_u(T)$ will cut the horizontal line F_u relative to the points R_c and S_c shown in Figure 2. To answer this question we can compare the derivatives of $g_u(T)$ and $g_c(T)$. If at each and every T the derivative of the former curve is higher than that on the latter curve, then $g_u(T)$ will lie to the left of $g_c(T)$ throughout and the optimal lifetime of capital from replacement under uncertainty will be lower than that from replacement *under certainty*. In other words, the situation will be as shown by point R_u in Figure 2.

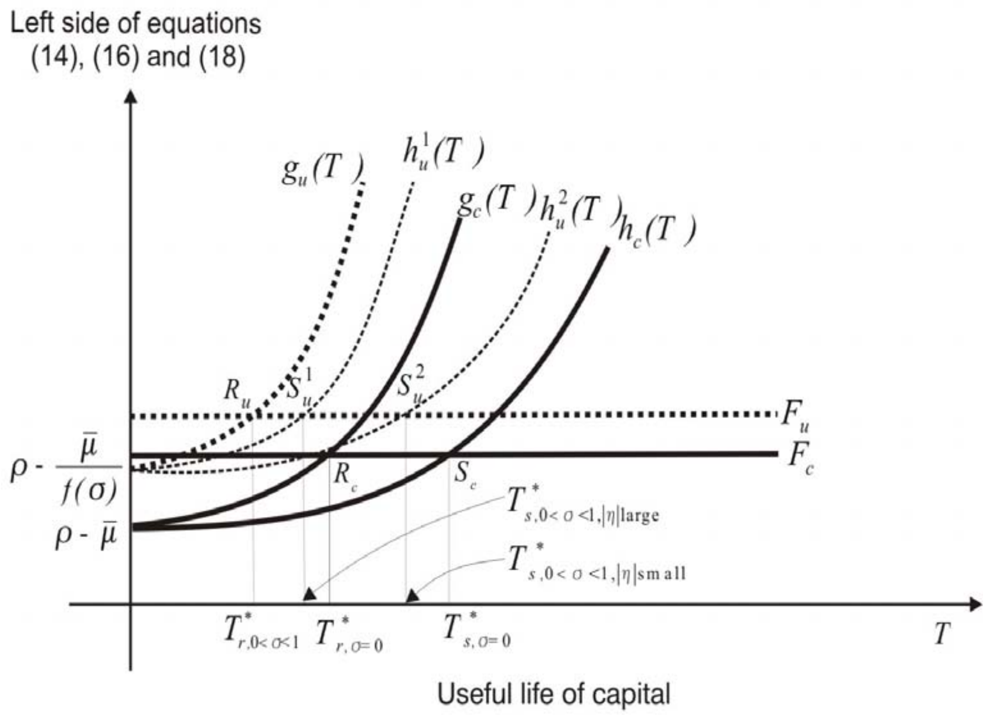

Figure 2

Proof

From the left-hand side of (18) we have:

$$g_u(T) = \rho e^{-\frac{\bar{\mu}}{f(\sigma)}T} - \frac{\bar{\mu}}{f(\sigma)} e^{-\rho T} \qquad (19.1)$$

The derivative of this function with respect to *T* is to be compared with the derivative of:

$$g_c(T) = \rho e^{-\bar{\mu}T} - \bar{\mu} e^{-\rho T} \ . \qquad (19.2)$$

The derivative of the latter function is:

$$\frac{\partial(\rho e^{-\bar{\mu}T} - \bar{\mu} e^{-\rho T})}{\partial T} = -\bar{\mu}\rho e^{-\rho T}(e^{(\rho-\bar{\mu})T} - 1). \qquad (19.3)$$

For $\rho > 0, \bar{\mu} < 0$ and $T > 0$ this derivative is positive, proving that the $g_c(T)$ curve rises always with rising *T*. On the other hand the derivative of $g_u(T)$ is:

$$\frac{\partial(\rho e^{-\frac{\bar{\mu}}{f(\sigma)}T} - \frac{\bar{\mu}}{f(\sigma)} e^{-\rho T})}{\partial T} = -\frac{\bar{\mu}}{f(\sigma)}\rho e^{-\rho T}(e^{(\rho-\frac{\bar{\mu}}{f(\sigma)})} - 1). \qquad (19.4)$$

This is positive and higher than that in (19.3) for every *T*. Therefore $g_u(T)$ lies throughout above $g_c(T)$, and this proves that *replacement under uncertainty* yields an optimal lifetime that is shorter than that of *replacement under certainty*.

There remains the case of *scrapping under uncertainty*. As it was stated above the firm in this case plans to exit at the end of the useful life of capital and re-enter, if market conditions warrant it. Thus, assuming again that the scrap value of capital on retirement is zero, the firm maximizes:

$$\Pi(T, P(0)) = b(0)N[P(0)]^{\eta}\left[\frac{P(0)}{b(0)}\frac{1 - e^{(\frac{\bar{\mu}}{f(\sigma)}-\rho)T}}{\rho - \frac{\bar{\mu}}{f(\sigma)}} - w\frac{1 - e^{-\rho T} + \beta\rho}{\rho}\right], \qquad (20)$$

with respect to T and $P(0)$.

From the first order conditions for maximization of (20) we obtain for *T*:

$$h_u(T) = \frac{1+\eta}{\eta}\rho e^{-\frac{\bar{\mu}}{f(\sigma)}T} - \frac{\eta\frac{\bar{\mu}}{f(\sigma)} + \rho}{\eta} e^{-\rho T} = (1 + \beta\rho)(\rho - \frac{\bar{\mu}}{f(\sigma)}) \ . \qquad (21)$$

At T=0 the left-hand side of this equation reduces to $\rho - \frac{\overline{\mu}}{f(\sigma)}$. So the $h_u(T)$ curve departs from the same point on the vertical axis as the $g_u(T)$ curve. Moreover, as $\eta \rightarrow -\infty$, $h_u(T)$ converges to $g_u(T)$. Thus in nearly perfectly competitive markets *scrapping under uncertainty* will give a longer optimal lifetime than *replacement under uncertainty* but shorter than *replacement under certainty*. This finding is shown in Figure 2 by S_u^1. The last issue is what happens to the optimal lifetime in markets in which the price elasticities of demand are closer to -1. To highlight it, consider the following derivative of $h_u(T)$ with respect to η:

$$\frac{\partial\left(\frac{1+\eta}{\eta}\rho e^{-\frac{\overline{\mu}}{f(\sigma)}T} - \frac{\eta\frac{\overline{\mu}}{f(\sigma)}+\rho}{\eta}e^{-\rho T}\right)}{\partial\eta} = \frac{1}{\eta^2}\rho e^{-\rho T}\left(e^{T(\rho-\frac{\overline{\mu}}{f(\sigma)})}-1\right) \tag{22}$$

This derivative is positive and its magnitude becomes equal to that of (19.3) when $-\overline{\mu} = 1/\eta^2$. Consequently, for values of η that lead to $-\overline{\mu} < 1/\eta^2$, the $h_u(T)$ curve lies above $g_c(T)$, whereas for values for which $-\overline{\mu} > 1/\eta^2$, the $h_u(T)$ curve lies below $g_c(T)$. This is the reason why in Figure 2 $h_u^2(T)$ crosses $g_c(T)$ and cuts the horizontal line F_u at the point S_u^2. On account of these results, the preceding analysis has established that:

Remark 2

A. Under uncertainty regarding the rate of embodied technological change:

1. Replacement leads to optimal lives of assets that are:

(i) Always shorter than replacement under certainty.

(ii) Always shorter than scrapping under uncertainty

2. Scrapping leads to optimal lives of assets that are:

(i) Shorter than replacement under certainty for large values of $|\eta|$.

(ii) Longer than replacement under certainty for small values of $|\eta|$.

(iii) Always shorter than scrapping under certainty.

B. Under certainty regarding the rate of embodied technological change:

1. Replacement leads always to optimal lives of assets that are shorter than scrapping under certainty.

Their importance can hardly be overstressed. For up to the present day what is emphasized in the relevant literature is the proof by Elton and Gruber (1976) regarding the optimality of an equal life policy for equipment subject to technological improvement. However, in the environment we live now, in which technological improvements are faster and less predictable than earlier decades, this may no longer be the case.

5. Conclusions

Nowadays technological progress embodied in the form of innovations in the consecutive vintages of capital is fast and unpredictable. So the objective in this paper was to investigate the effects of uncertainty that emanates from technological improvements on the optimal lifetime of assets. In doing so I adopted a dynamic model in which technical change increases continuously the productivity of producers' durables and potential competition induces firms to price their output in a way that passes all benefits to final consumers. From its analysis it turned out that in general the type of uncertainty considered shortens the optimal lifetime of assets. In particular, the analysis established that *replacement under uncertainty* leads to optimal lifetimes of assets that are shorter than in any other mode of operation, whereas *scrapping under uncertainty* yields optimal lifetimes of assets that may be shorter or longer than those determined by *replacement under certainty*, depending on the mean rate of technological progress, $\bar{\mu}$, and the price elasticity of demand, η. But irrespective of the values of these parameters, the optimal lifetime of assets computed from a policy of s*crapping under uncertainty* is always shorter than that from *scrapping under certainty*.

These results were obtained on the presumption that two measures (mean and variance) of the probability distribution governing the process of technological change can be usefully summarized with one, i.e. the expected rate of embodied technological change per unit of uncertainty. But this may not be always a satisfactory approximation. For example, such might be the case if *intermediate* innovations are correlated over time. Therefore, the robustness of the results under alternative specifications of the probability distribution of μ remains an open question.

Appendix A

To model uncertainty I draw on Markowitz' (1991) mean-variance paradigm, which in the present context assumes that the mean and standard deviation are sufficient statistics to describe the probability distribution that governs the process of embodied technological change. Clearly, by adopting this approach I take into consideration neither the implications that may derive from the probability distribution itself, nor the possibility of significant differences hidden in higher moments or in distributions across states of nature that may be associated with different levels of utility on the part of the representative firm. When such considerations are especially important, the mean and the standard deviation may not suffice, requiring the use of additional or substitute measures. However, my objective in this paper is simply to examine the situations in which two measures (mean and variance) can be summarized usefully with one, i.e. the expected rate of embodied technological change per unit of uncertainty.

Aside from the above, the analysis is limited also in another respect. This has to do with the impact of technological change. Clearly, there are two possibilities. Technological change may affect the capital-output coefficient either positively or negatively. Normally new production techniques take the form of innovations that reduce the capital-output coefficient. But one cannot preclude isolated episodes of technological regression or of some heavily capital-intensive innovation, which for some period may increase the capital-output coefficient above its downward trend. In this paper I focus only on the innovations that as a rule and on the average reduce the capital-output coefficient.

In light of these delimitations, suppose that embodied technological progress is exogenous. Next, let its rate μ be a random variable with mean $\bar{\mu} < 0$ and standard deviation σ. Finally, define a function f such that as σ increases (decreases) the $f(\sigma)$ takes declining (increasing) values in the interval $0 < f(\sigma) \leq 1$ with $f(0) = 1$ and $f(\infty) \cong 0$. Then an uncertainty-weighted measure of the mean rate of technological change is given by $\bar{\mu}/f(\sigma)$. As in the case of Sharpe's Ratio, this measure scales with time because both the mean and the standard deviation depend on the interval over which they are considered. So suppose that we are at time t and that a vintage of capital which was built at υ will live for T years. That is, $\upsilon < t \leq \upsilon + T$. On average, from υ to t the capital coefficient must have declined as follows:

$$\frac{b(t)-b(\upsilon)}{b(\upsilon)} = \frac{\bar{\mu}}{f(\sigma)}(t-\upsilon)\,. \tag{A.1}$$

From this we get:

$$b(t) = b(\upsilon)[1+\frac{\bar{\mu}}{f(\sigma)}(t-\upsilon)]\,. \tag{A.2}$$

Now assume that the process of technological progress begins at $t=\upsilon$. At that moment the capital coefficient will be equal to $b(\upsilon)$. Next divide the interval $t-\upsilon$ into m equal timesteps. At the end of the first timestep the capital coefficient will be:

$$b(\upsilon+\frac{t-\upsilon}{m}) = b(\upsilon)[1+\frac{\bar{\mu}}{f(\sigma)}\frac{t-\upsilon}{m}]. \tag{A.3}$$

At the end of the second timestep the capital coefficient will be:

$$b(\upsilon+2\frac{t-\upsilon}{m}) = b(\upsilon+\frac{t-\upsilon}{m})[1+\frac{\bar{\mu}}{f(\sigma)}\frac{t-\upsilon}{m}] = b(\upsilon)[1+\frac{\bar{\mu}}{f(\sigma)}\frac{t-\upsilon}{m}]^2. \tag{A.4}$$

.

And at the end of the m timestep the capital coefficient will have declined to:

$$b(t) = b(\upsilon+(m-1)\frac{t-\upsilon}{m})[1+\frac{\bar{\mu}}{f(\sigma)}\frac{t-\upsilon}{m}]^m\,. \tag{A.5}$$

Now let $m \to \infty$. Then, the timestep $(t-\upsilon)/m$ will go to zero and we will have:

$$\begin{gathered} b(t) = b(\upsilon)[1+\frac{\bar{\mu}}{f(\sigma)}\frac{t-\upsilon}{m}]^m = b(\upsilon)e^{\frac{\bar{\mu}}{f(\sigma)}(t-\upsilon)},\ \upsilon < t \le T, \\ 0 < f(\sigma) \le 1,\ f(0)=1,\ f(\infty) \cong 0 \end{gathered} \tag{A.6}$$

which is equation (2) in the text.

Appendix B

From the maximization of (13) with respect to $P(0)$ and using (14) we obtain:

$$P(0) = \frac{\eta}{1+\eta} e^{-\bar{\mu}T} b(0) w \,. \tag{B.1}$$

Inserting this into (12) yields:

$$n(0) = -\frac{1}{1+\eta} \frac{1 - e^{\bar{\mu}T} + \beta\rho}{\rho} w \,. \tag{B.2}$$

Now, since the representative firm is presumed to behave as a completely contestable monopolist it will let the price elasticity of demand η approach to minus infinity and use the approximation:

$$\frac{\eta}{1+\eta} = 1 \,. \tag{B.3}$$

So, if we deduct 1 from both sides of this equation and insert the result into (B.2) we obtain:

$$n(0) = 0 \tag{B.4}$$

Hence, under the adopted pricing rule the unit net worth of equipment in the initial vintage is set equal to zero and this establishes the claim in the text.

Appendix C

At υ the worth of revenue minus the labor cost of operating a unit of equipment per small fraction dt of a year located at time t is given by:

$$\left[\frac{P(t)}{b(\upsilon)} - w\right] e^{-\rho(t-\upsilon)} dt\,. \tag{C.1}$$

Assuming $\sigma = 0$ and inserting (2) into the above expression gives:

$$\left[\frac{P(\upsilon)}{b(\upsilon)} e^{\bar{\mu}(t-\upsilon)} - w\right] e^{-\rho(t-\upsilon)} dt\,. \tag{C.2}$$

Consequently at υ the worth of the sum total of revenue minus operating labor cost of a unit of equipment over its entire useful life T is:

$$\int_{\upsilon}^{\upsilon+T} \left[\frac{P(\upsilon)}{b(\upsilon)} e^{\bar{\mu}(t-\upsilon)} - w\right] e^{-\rho(t-\upsilon)} dt = \frac{P(\upsilon)}{b(\upsilon)} \frac{1 - e^{(\bar{\mu}-\rho)T}}{\rho - \mu} - w \frac{1 - e^{-\rho T}}{\rho}\,. \tag{C.3}$$

Now let p be the purchase price of a new unit of equipment. Assuming the salvage value of the unit of equipment when retired is zero, the net worth of the acquisition of the new unit of equipment of vintage υ is:

$$n(\upsilon) = \frac{P(\upsilon)}{b(\upsilon)} \frac{1 - e^{(\bar{\mu}-\rho)T}}{\rho - \mu} - w \frac{1 - e^{-\rho T}}{\rho} - p\,. \tag{C.4}$$

Finally, since by (4) the minimum labor required to build a new unit of equipment is β, under perfect competition in the capital-producing sector of the economy, plus the assumption that new equipment is produced solely by means of labor, it will hold that:

$$p = \beta w\,. \tag{C.5}$$

Thus substituting (C.5) into (C.4) gives (12).

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Endnotes

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[1] To my knowledge the core of the model has its origins in Brems (1968). For this reason, and in recognition of due intellectual debt, I dedicate this paper to his memory.

[2] The production of final goods is presumed to take place by combining one unit of capital with one unit of labour, whereas capital goods are produced solely by means of labour. Hence the addition of an employment equation would turn the model into one of general equilibrium in labour and capital. However, as the ensuing analysis would not be affected, the employment of labour sector of the model is ignored.

[3] This pricing rule can be derived by arguments similar to those described in Appendix A.

[4] Under the conceptualisation adopted in this paper, the investor is presumed to choose the number of replacements as well as their durations. In doing so he decides about the profit horizon of the investment process on the basis of the parameters of the various functions involved, including those that are determined by current and future market conditions. Unlike this approach, researchers in economics, finance and other related areas are content to assume that the profit horizon is infinite and that replacements take place at uniform time intervals. Even though the reasons for adopting these assumptions are not spelled out, the rationale may be traced back to Blackwell's theorem, a suitable form of which in the infinite horizon problem yields an optimal replacement policy with equal lifetimes. But following the arguments by Lehrer and Shmaya (2006) the latter may not be a better approach than the one we have proposed, since once the process of infinite equidistant replacements has started, it does not afford the investor a costless option to stop re-investing. For, if for some reason the investor has to stop re-investing, he may have to absorb such a high cost in terms of foregone profits that the policy of infinite equidistant replacements may turn out to be inferior to scrapping. Of course, in any case, if initial expectations about market conditions prove erroneous at some future date, the investor can always revise his perpetual equidistant replacement policy or even decide to quit altogether because this policy is not an indissoluble contract.