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Halkos, George and Kevork, Ilias

University of Thessaly, Department of Economics

January 2012

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By

George E. Halkos and Ilias S. Kevork

Department of Economics, University of Thessaly

ABSTRACT

This paper considers the classical Newsvendor model, also known as the Newsboy problem, with the demand to be fully observed and to follow in successive inventory cycles one of the Exponential, Rayleigh, and Log-Normal distributions. For each distribution, appropriate estimators for the optimal order quantity are considered, and their sampling distributions are derived. Then, through Monte-Carlo simulations, we evaluate the performance of corresponding exact and asymptotic confidence intervals for the true optimal order quantity. The case where normality for demand is erroneously assumed is also investigated. Asymptotic confidence intervals produce higher precision, but to attain equality between their actual and nominal confidence level, samples of at least a certain size should be available. This size depends upon the coefficients of variation, skewness and kurtosis. The paper concludes that having available data on the skewed demand for enough inventory cycles enables (i) to trace non-normality, and (ii) to use the right asymptotic confidence intervals in order the estimates for the optimal order quantity to be valid and precise.

Keywords: Inventory Control; Newsboy Problem; Skewed Demand; Exact and Asymptotic Confidence Intervals; Monte-Carlo Simulations.

JEL Codes: C13; C15; C44; D24; M11.

1. INTRODUCTION

Inventory management is a crucial task in the operation of firms and enterprises. For products whose life-cycle of demand lasts a relatively short period (daily and weekly newspapers and magazines, seasonal goods etc), newsvendor (or alternatively newsboy) models offer quantitative tools to form effective inventory policies. The short period where demand refers to and inventory decisions should be made represents an inventory cycle. In general, applications of such models are found in fashion industry, airline seats pricing, and management of perishable food supplies in supermarkets.

Among the alternative forms of newsvendor models, the classical version refers to the purchasing inventory problem where newsvendors decide on a one-time basis and their decisions are followed by a stochastic sales outcome. In such cases, newsvendors have to predict order quantities in the beginning of each inventory cycle (or period) and products cannot be sold in the next time period if the actual demand is greater than the order quantity, as any excess inventory is disposed of by buyback arrangements. At the same time, there is an opportunity cost of lost profit in the opposite situation (Chen and Chen, 2009), where at the end of the inventory cycle an excess demand is observed.

During the last decades, a number of researchers have explored the issue of the optimal order quantity for cases of uncertainty in demand. Alternative extensions of newsvendor models have been published in the literature, and Khouja (1999) provides an extensive search of these works till 1999. Since then, various papers have explored the newsboy-type inventory problem like Schweitzer and Cachon (2000), Casimir (2002), Dutta et al. (2005), Salazar-Ibarra (2005), Matsuyama (2006), Benzion et al. (2008), Wang and Webster (2009), Chen and Chen (2010), Huang et al. (2011), Lee and Hsu (2011), and Jiang et al. (2012). However, the crucial condition of applying these models in practice is that parameters of demand distributions should be known, something that does not hold. And, unfortunately, the extent of applicability of newsvendor models in inventory management to determine the level of customer service depends upon the estimation of demand parameters. The problem of uncertainty becomes even more severe for certain types of product, like seasonal clothing, for which data on demand are available only for few inventory cycles (ensuring that market conditions do not change), and this makes estimation procedures to be under question.

Research on studying the effects of demand estimation on optimal inventory policies is limited (Conrad 1976; Nahmias, 1994; Agrawal and Smith, 1996; Hill, 1997; Bell, 2000).

Besides, none of these works addressed the problem of how sampling variability of estimated values of demand parameters influences the quality of estimation concerning optimal ordering policies. And recently, it has been recognized that effective applications of newsvendor models to form reliable inventory policies depend upon the variability of estimates for the parameters of probabilistic laws which generate demand in successive inventory cycles. Assuming that demand follows the normal distribution, for the classical newsvendor model, Kevork (2010) developed appropriate estimators to explore the variability of estimates for the optimal order quantity and the maximum expected profit. His analysis showed that the weak point of applying this model to real life situations is the significant reductions in precision and stability of confidence intervals for the true maximum expected profit when high shortage costs occur. Su and Pearn (2011) developed a statistical hypothesis testing methodology to compare two newsboy-type products and to select the one that has a higher probability of achieving a target profit under the optimal ordering policy. The authors provided tables with critical values of the test and the sample sizes which are required to attain designated type I and II errors. Prior to these two works, Olivares et al. (2008) presented a structural estimation framework to disentangle whether specific factors affect the observed order quantity either through the distribution of demand or through the overage/underage cost ratio.

When demand follows specific non-symmetric patterns, this paper explores the extent of applicability of the classical newsvendor model to form effective ordering policies on the basis of studying quality and precision of estimates for the optimal order quantity. To the authors' knowledge, this is performed for the first time for skewed distributions of demand. Assuming that demand is fully observed and is formed independently in successive inventory cycles according to the Exponential, Rayleigh, and Log-Normal distributions, appropriate estimators for the optimal order quantity are established, whose form incorporates known estimators for the parameter(s) of each distribution. To measure the variability of different estimates for the optimal order quantity, exact and asymptotic confidence intervals for the true optimal order quantity are derived. Organizing appropriate Monte-Carlo simulations, the properties of confidence intervals are traced by evaluating at different sample sizes (a) the coverage, namely, the estimated actual confidence level attained by the intervals, and (b) their relative precision regarding the true optimal order quantity.

The choice of Exponential and Rayleigh distributions was made as their coefficients of variation, skewness, and excess kurtosis remain unaltered in changes of their parameter. The Log-Normal distribution was also selected to study effects of changing coefficients of variation, skewness and excess kurtosis on quality and precision of estimates for the optimal order quantity. For this reason, two sets of distributions are defined. The first set includes the Rayleigh and Log-Normal, and the second set the Exponential and Log-Nornal again, but the values of mean and variance for the two Log-Normal distributions were specified in such a way that the distributions of each set to have the same coefficient of variation. Comparing also the two sets, (a) in each set the Log-Normal displays greater skewness and higher kurtosis than the Rayleigh or Exponential respectively, and (b) the second set including the Exponential distribution presents as a whole larger coefficients of variation, skewness and excess kurtosis than the first set having the Rayleigh distribution.

The Exponential, Weibull (a special case of which is the Rayleigh) and Log-Normal distributions have been extensively used in various papers related to inventory management, but with different aims than those which are considered in the current work. For the Newsboy problem, Khouja (1996) used an exponential distributed demand to illustrate the effect of an emergency supply, and Lau (1997) offered closed-form formulas for computing the expected cost and the optimal expected cost when demand follows the exponential distribution. Hill (1997) used the exponential distribution to perform analytical and numerical comparisons between the Frequentist and Bayesian approach for demand estimation. The Log-Normal distribution was used in the work of Ridder et al. (1998), who illustrated for the newsvendor model that a reduction of demand uncertainty will not result in the desired cost reduction. Lau and Lau (2002) used both the exponential and Weibull distributions in a manufacturer-retailer channel to study the effects of retail-market demand uncertainty on revenues, order quantities and expected profits. For the multi-product Newsboy model with constraints, Areeratchakul and Abdel-Malek (2006) modeled demand as Exponential, Log-Normal, and Weibull to provide a solution methodology, which was based on quadratic programming and a triangular presentation of the area under the cumulative distribution function of demand. Considering that demand follows the Weibull distribution and including risk preferences of the inventory manager to the classical newsvendor problem, Jammernegg and Kischka (2009) showed that robust ordering decisions can be derived from assumptions on stochastic dominance. For the multi-product competitive newsboy problem, Huang et al. (2011) used the exponential distribution to test the validity of a static service-rate approximation for the dynamic and stochastic availability of each product. Apart from the previous indicative list of papers, the Exponential and the Log-Normal distributions have been also used to many other works like Geng et al. (2010), Grubbstrom (2010), Dominey and Hill (2004), Mostard et al. (2005), Choi and Ruszczynski (2008).

The experimental framework of the current work also allows the study of effects of assuming a normal demand when in fact demand distribution follows a non-symmetric pattern. Under a normal demand, exact confidence intervals are given, while for large samples, Kevork's (2010) confidence interval form is followed. Assuming erroneously normality, we investigate, analytically and through Monte-Carlo simulations, the performance of exact and asymptotic confidence interval methods held under a normal demand, when in fact demand follows one of the three skewed distributions under consideration. The two criteria are again the coverage and the relative precision regarding the true optimal order quantity which is obtained from the true demand distribution.

The aforementioned arguments and remarks lead the rest of the paper to be structured as follows. Section 2 establishes the experimental framework for the adopted skewed distributions and develops alternative estimators for the optimal order quantity. Section 3 derives exact and asymptotic distributions of the estimators and presents Monte-Carlo simulation results for the coverage and precision that the corresponding confidence intervals attain under different combinations of finite samples and probabilities of not observing stock-outs at the end of any inventory cycles. Under both an analytic way and through Monte-Carlo simulations, section 4 explores the validity of applying confidence intervals held under a normal demand to different sample sizes from the three specific skewed distributions. Finally, the last section summarizes the most important findings of the current work.

2. ESTIMATORS FOR OPTIMAL ORDERING POLICIES SPECIFICATIONS

Regarding the classical form of the newsvendor model, the aim is to determine at the start of the cycle the order quantity, Q, which maximizes the expected value of profit function,

$$\xi = \begin{cases} (p-c)Q - (p-v)(Q-X) & \text{if} \quad X \leq Q \\ (p-c)Q + s(Q-X) & \text{if} \quad X > Q \end{cases}, \tag{1}$$

where, X is a random variable representing size of demand at any inventory cycle, p the selling price, c the purchase (or production) cost, v the salvage value, and s the shortage cost per unit. A vital assumption of the model is that any excess inventory at the end of the inventory cycle is disposed of by buyback arrangements which are carried through by using the salvage value. In case where at the end of the cycle excess demand is observed, shortage cost per unit incorporates current losses and present value of future payoffs expected to be lost from present unsatisfied customers.

Taking first and second derivatives of $E(\xi)$ by using Leibniz's rule, the optimal order quantity maximizing expected profit function satisfies the equation,

$$\Pr(X \le Q^*) = F(Q^*) = \frac{p - c + s}{p - v + s} = R, \tag{2}$$

where R is a critical fractile whose value identifies the product as low profit (R<0,5) or high profit (R>0,5) according to the principle stated by Schweitzer and Cachon (2000).

For the Exponential, Rayleigh, and Log-Normal distributions, the specifications of (2), and the optimal order quantities, Q_j^* (j = EX, RY, LN), are given as follows:

Exponential (λ): $F(Q^*) = 1 - e^{-\frac{Q^*}{\lambda}}$ and

$$Q_{EX}^* = \lambda \cdot |\ln(1 - R)| \tag{3a}$$

Rayleigh (σ) = Weibull $\boldsymbol{(2, \sigma\sqrt{2})}$***:*** $F(Q^*) = 1 - e^{-\frac{1}{2}\left(\frac{Q^*}{\sigma}\right)^2}$ and

$$Q_{RY}^* = \sigma \cdot \sqrt{2|\ln(1 - R)|} \tag{3b}$$

Log-Normal $\boldsymbol{(\mu_{LN}, \sigma_{LN}^2)}$: $F(Q^*) = \Pr\left(Z \le \frac{\ln Q^* - \mu_{LN}}{\sigma_{LN}}\right) = \Pr(Z \le z_R) = R$ and

$$Q_{LN}^* = e^{\mu_{LN} + z_R \cdot \sigma_{LN}} \tag{3c}$$

where z_R is the inverse function of the standard normal evaluated at R.

To study changes of the optimal order quantity, Q_j^*, at different levels of R, initially, for the three distributions, we set a common expected demand, $E(X) = 300$. To attain this size of expected demand for the Exponential and Rayleigh distributions, the values of their parameters should be defined respectively at $\lambda = 300$ and $\sigma = 300\sqrt{2/\pi}$. Regarding the Log-Normal distribution, we consider two specifications, coded as $(\text{log-normal})_1$ and $(\text{log-normal})_2$, which have common $\mu_{LN} = \ln 300 - \sigma_{LN}^2/2$. In the first specification, by setting $\sigma_{LN}^2 = \ln(4/\pi)$, we make the coefficient of variation, $CV = sd(X)/E(X)$, to be equal among Log-Normal and Rayleigh, while in the second specification, to equate CV's of Log-Normal and Exponential, we set $\sigma_{LN}^2 = \ln 2$. Table 1, summarizes the characteristics of each distribution, (note that the coefficients of skewness and kurtosis were computed from formulae available in Johnson et al, 1994), and figure 1 presents their probability density functions. From figure 2, Q_j^* is an increasing function of R, and given R, how large Q_j^* should be for each distribution to maximize expected profits depends upon the size of CV. So, for low-profit products (R is relatively small), the lower the CV of demand distribution we observe, the higher the quantity the newsvendor orders. Contrary to that, for R relatively high (high-profit products), the newsvendor will order higher quantities with demand distributions characterized by larger CV's.

Skewness and kurtosis also affect the optimal order quantity when demand follows distributions belonging to the same set and having the same CV. For such cases, when R takes on values relatively close to zero or relatively close to unity, greater skewness and higher kurtosis lead to larger optimal order quantities. However, how far R lies from zero or how close R is to unity in order the previous remark to hold depends upon the size of CV. For the pair of Rayleigh and (Log-Normal)$_1$, greater skewness and higher kurtosis give larger Q_j^* when R is closer to zero or R lies further away from unity, compared to the other pair of Exponential and (Log-Normal)$_2$ which has a larger CV.

<u>Table 1:</u> Summary of parameters for the three distributions, $E(X) = 300$

	Demand Parameters	**CV**	**Skewness**	**Excess Kurtosis**
Rayleigh	$\sigma = 239{,}3654$	0,5227	0,6311	0,2451
(Log-normal)$_1$	$\mu_{LN} = 5{,}5830$, $\sigma^2_{LN} = 0{,}241564$	0,5227	1,7110	5,6197
Exponential	$\lambda = 300$	1	2	6
(Log-normal)$_2$	$\mu_{LN} = 5{,}3572$, $\sigma^2_{LN} = 0{,}693147$	1	4	38

To estimate now the optimal order quantity for the next inventory cycle (period), let $X_1, X_2, ..., X_n$ be a sequence of independent random variables representing demand for a sample of n consecutive inventory cycles. Modeling demand in each period by each one of the three distributions under consideration, the following consistent estimators for the optimal order quantity of the next inventory cycle, n+1, are defined on the basis of forms (3a)-(3c):

Exponential (λ):

$$\hat{Q}^*_{EX} = \hat{\lambda} \cdot |\ln(1-R)| = \frac{|\ln(1-R)|}{n} \sum_{t=1}^{n} X_t \,, \tag{4a}$$

where $\hat{\lambda} = \sum_{t=1}^{n} X_t \Big/ n$ is an unbiased estimator for λ. From the Weak Law of Large Numbers,

$p\lim \hat{\lambda} = \lambda$, and thus $p\lim \hat{Q}^*_{EX} = |\ln(1-R)| \cdot p\lim \hat{\lambda} = Q^*_{EX}$.

Figure 1: Plots of probability density functions

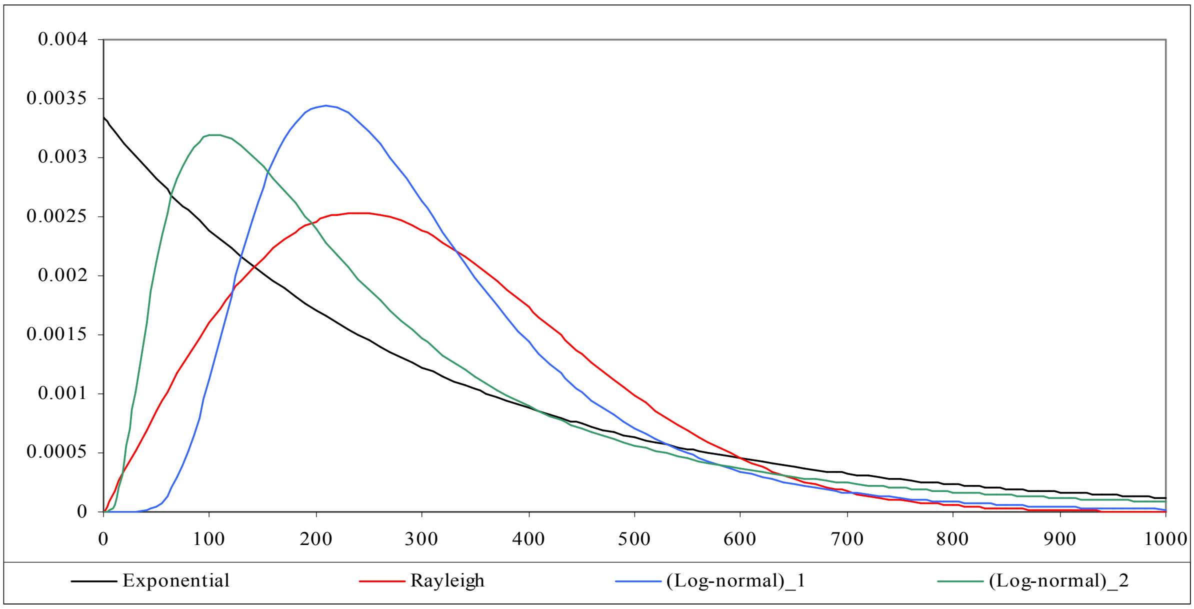
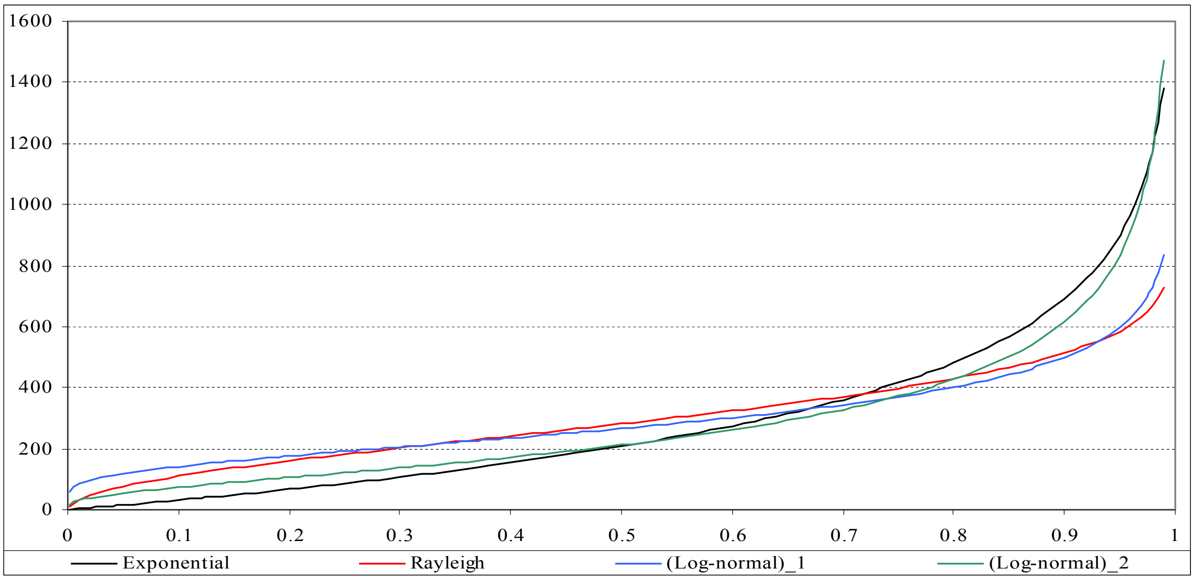


Figure 2: Plot of Optimal Order Quantities, Q_j^*, against critical fractile, R



Rayleigh (σ):

$$\hat{Q}_{RY}^* = \hat{\sigma} \cdot \sqrt{2\left|\ln(1-R)\right|}, \tag{4b}$$

where $\hat{\sigma} = \left(\sum_{t=1}^{n} X_t^2 \Big/ 2n \right)^{0,5}$ (Johnson et al., 1994), and $2n\hat{\sigma}^2/\sigma^2 \sim \chi_{2n}^2$ (Balakrishnan and Cohen, 1991). Using the last distributional result, $E(\hat{\sigma}^2) = \sigma^2$, and $Var(\hat{\sigma}^2) = \sigma^4/n \to 0$, as $n \to \infty$. Thus $p\lim \hat{\sigma}^2 = \sigma^2$, and $p\lim \hat{Q}_{RY}^* = \sqrt{2\left|\ln(1-R)\right|} \cdot \left(p\lim \hat{\sigma}^2\right)^{0,5} = Q_{RY}^*$.

Log-Normal $\left(\mu_{LN}, \sigma^2_{LN}\right)$:

$$\ln \hat{Q}^*_{LN} = \hat{\mu}_{LN} + z_R \cdot s_{LN}, \tag{4c}$$

or alternatively

$$\hat{Q}^*_{LN} = e^{\hat{\mu}_{LN} + z_R \cdot \hat{\sigma}_{LN}}, \tag{4d}$$

where $\hat{\mu}_{LN} = \sum_{t=1}^{n} \ln X_t \Big/ n$, $\hat{\sigma}^2_{LN} = \sum_{t=1}^{n} \left(\ln X_t - \hat{\mu}_{LN}\right) \Big/ n$, and $s^2_{LN} = \frac{n}{n-1}\hat{\sigma}^2_{LN}$. Since $\ln X_t \sim N\left(\mu_{LN}, \sigma^2_{LN}\right)$, $p\lim \hat{Q}^*_{LN} = e^{p\lim \hat{\mu}_{LN} + z_R \cdot \left(p\lim \sigma^2_{LN}\right)^{0,5}} = Q^*_{LN}$ and $p\lim \ln \hat{Q}^*_{LN} = \ln Q^*_{LN}$. Estimators (4c) and (4d) will be used in the next section to derive respectively exact and asymptotic confidence intervals for Q^*_{LN}.

3. CONFIDENCE INTERVALS FOR THE OPTIMAL ORDER QUANTITY

In this section, first we derive exact and asymptotic distributions for $\hat{Q}^*_j$, and then we evaluate the performance of the corresponding extracted confidence intervals for the true optimal order quantity through Monte-Carlo simulations.

Exponential (λ): From (4a), as $\sum_{t=1}^{n} X_t \sim \text{Gamma}(n, \lambda)$, it is easily deduced that

$$\hat{Q}^*_{EX} \sim \text{Gamma}\left(n, \lambda \frac{\left|\ln(1-R)\right|}{n}\right).$$

The last result makes the construction of exact confidence intervals for Q^*_{EX} impossible since the required critical values of the gamma distribution cannot be obtained as they depend upon the unknown λ. The alternative solution, therefore, is to derive the asymptotic distribution of Q^*_{EX}. By defining $\kappa^2 = \text{Var}(X)$, the central limit theorem, $\sqrt{n}\left(\hat{\lambda} - \lambda\right) \xrightarrow{D} N\left(0, \kappa^2\right)$, and the consistency of $\hat{Q}^*_{EX}$ allow the application of univariate delta method, which leads to,

$$\sqrt{n}\left(\hat{Q}^*_{EX} - Q^*_{EX}\right) \xrightarrow{D} N\left(0, \kappa^2 [\ln(1-R)]^2\right). \tag{5}$$

From the last asymptotic distributional result, an approximate $(1-\alpha)100\%$ confidence intervals for Q^*_{EX} is given by

$$\hat{Q}^*_{EX} \pm z_{\alpha/2} \frac{\hat{\kappa}}{\sqrt{n}} \left|\ln(1-R)\right|, \tag{6}$$

where $\hat{\kappa}^2 = \sum_{t=1}^{n}\left(X_t - \hat{\lambda}\right)^2 \Big/ (n-1)$.

Rayleigh (σ)

Regarding the estimator of σ, $\hat{\sigma} = \left(\sum_{t=1}^{n} X_t^2 \Big/ 2n\right)^{0,5}$, we stated in the previous section that the statistic $2n\hat{\sigma}^2/\sigma^2$ follows the Chi-squared distribution with 2n degrees of freedom. Using also the property that the product of a chi-squared variable with v degrees of freedom by a constant b follows the gamma distribution $\Gamma(v/2, 2b)$, we obtain the following distributional result

$$\frac{1}{2\left|\ln(1-R)\right|} \cdot \frac{2n\hat{\sigma}^2}{\sigma^2} = \frac{2n\hat{\sigma}^2}{\left(Q^*_{RY}\right)^2} \sim \Gamma\left(n \, , \frac{1}{\left|\ln(1-R)\right|}\right),$$

which leads to the $(1-\alpha)100\%$ exact confidence interval

$$\hat{\sigma}\sqrt{\frac{2n}{\Gamma_{1-\frac{\alpha}{2}}\left(n, \frac{1}{\left|\ln(1-R)\right|}\right)}} \le Q^*_{RY} \le \hat{\sigma}\sqrt{\frac{2n}{\Gamma_{\frac{\alpha}{2}}\left(n, \frac{1}{\left|\ln(1-R)\right|}\right)}} \, . \tag{7}$$

Further, in Appendix we prove that,

$$\sqrt{n}\left[\hat{Q}^*_{RY} - Q^*_{RY}\right] \xrightarrow{D} N\left(0 \, , \sigma^2 \frac{\left|\ln(1-R)\right|}{2}\right). \tag{8}$$

Convergence to normality, as it is stated in (8), enables us to use also in finite samples the approximate $(1-\alpha)100\%$ confidence interval

$$\hat{Q}^*_{RY} \pm z_{\alpha/2} \frac{\hat{\sigma}}{\sqrt{n}} \sqrt{\frac{\left|\ln(1-R)\right|}{2}} \, . \tag{9}$$

Log-Normal $\left(\boldsymbol{\mu_{LN}} \, , \boldsymbol{\sigma^2_{LN}}\right)$

For the estimator defined in (4c), we show (see Appendix) that for small samples and R<0,5, the $(1-\alpha)100\%$ exact confidence interval for Q^*_{LN} is constructed from

$$\exp\left(\hat{\mu}_{LN} - t'_{n-1,1-\frac{\alpha}{2}}(\delta) \cdot \frac{s_{LN}}{\sqrt{n}} \right) \le Q^*_{LN} \le \exp\left(\hat{\mu}_{LN} - t'_{n-1,\frac{\alpha}{2}}(\delta) \cdot \frac{s_{LN}}{\sqrt{n}} \right), \tag{10a}$$

while for R>0,5, from

$$\exp\left(\hat{\mu}_{LN} + t'_{n-1,\frac{\alpha}{2}}(\delta)\cdot\frac{s_{LN}}{\sqrt{n}}\right) \le Q^*_{LN} \le \exp\left(\hat{\mu}_{LN} + t'_{n-1,1-\frac{\alpha}{2}}(\delta)\cdot\frac{s_{LN}}{\sqrt{n}}\right), \tag{10b}$$

where $t'_{n-1}(\delta)$ is the non-central t-student distribution with non-centrality parameter $\delta = |z_R|\sqrt{n}$ for R<0,5, and $\delta = z_R\sqrt{n}$ for R>0,5.

Additionally to (10a) and (10b), for the estimator defined in (4d), we derive in Appendix the asymptotic distributional result,

$$\sqrt{n}\left[\hat{Q}^*_{LN} - Q^*_{LN}\right] \xrightarrow{D} N\left(0, \left(1 + \frac{z_R^2}{2}\right)\cdot\left(Q^*_{LN}\right)^2 \cdot \sigma^2_{LN}\right), \tag{11}$$

which leads to the following approximate $(1-\alpha)100\%$ confidence interval for finite samples,

$$\hat{Q}^*_{LN} \pm z_{\alpha/2}\cdot\hat{Q}^*_{LN}\cdot\sqrt{1+\frac{z_R^2}{2}}\cdot\frac{\sigma_{LN}}{\sqrt{n}}. \tag{12}$$

To study in finite samples the performance of confidence intervals given in (6), (7), (9), (10) and (12), 10000 replications of 300 observations each were generated from each distribution, by using the parameter(s) values of table 1, and applying traditional inverse-transform algorithms to 10000 sequences of random numbers uniformly distributed on (0,1). The algorithms can be found in Law (2007), on pages 448, 452, and 454 respectively for the Exponential distribution, the Rayleigh (setting $\beta = \sigma\sqrt{2}$), and the Log-Normal. For the latter distribution, the required sequences of random numbers for the standard normal were generated using the method of Box and Muller (Law, 2007, p. 453). Details for the random number generator and its validity can be found in Kevork (2010).

For each distribution, using each replication, estimates for the corresponding parameter(s) were taken ($\hat{\lambda}$, $\hat{\sigma}$, and $\hat{\mu}_{LN},\hat{\sigma}_{LN}$) at different sample sizes. Then, for each sample size, n, corresponding estimates for the optimal order quantity were computed for different values of the critical fractile, R, using (4a) for the Exponential Distribution, (4b) for the Rayleigh and (4c), (4d) for the Log-Normal. Finally, for each distribution, and for each combination of n and R, 10000 different confidence intervals were computed using the exact forms (7), (10) and the asymptotic ones (6), (9) and (12). The critical values for the non-central t and the gamma distributions were computed using the statistical package MINITAB.

Having available for each distribution 10000 different confidence intervals for each combination of method (exact or asymptotic), n, and R, two statistical measures have been computed: the coverage (COV) and the relative average half-length (RAHL). The first measure, COV, is the percentage of the 10000 confidence intervals containing the true optimal order quantity, Q_j^*, estimating in that way the actual confidence level that the corresponding methods attains. RAHL refers to the precision of confidence intervals that the method produces and is computed by dividing the average half-length from the 10000 confidence intervals by Q_j^*. Tables 2 and 3 display respectively the values for COV and RAHL, at 95% nominal confidence level.

A first important remark concerns the Exponential and Rayleigh distributions, for which COV and RAHL are the same regardless of the value of R. This is deduced by considering first the asymptotic confidence intervals of these two distributions given respectively in (6) and (9), whose COV estimates in finite samples the following probabilities:

$$\Pr\left(-z_{\alpha/2} \le \frac{\sqrt{n}\left(\hat{Q}_{EX}^* - Q_{EX}^*\right)}{\hat{\kappa}\left|\ln(1-R)\right|} \le z_{\alpha/2}\right) = \Pr\left(-z_{\alpha/2} \le \frac{\sqrt{n}\left(\hat{\lambda} - \lambda\right)}{\hat{\kappa}} \le z_{\alpha/2}\right),$$

$$\Pr\left(-z_{\alpha/2} \le \frac{\sqrt{n}\left(\hat{Q}_{RY}^* - Q_{RY}^*\right)}{\hat{\sigma}\sqrt{\dfrac{\left|\ln(1-R)\right|}{2}}} \le z_{\alpha/2}\right) = \Pr\left(-\frac{z_{\alpha/2}}{2} \le \frac{\sqrt{n}(\hat{\sigma} - \sigma)}{\hat{\sigma}} \frac{z_{\alpha/2}}{2}\right).$$

In each one of these two equalities, the second probability does not depend on R, verifying in that way the previous remark Similarly, for the Rayleigh distribution, using (3b) and the scaling property of the Gamma distribution, the exact confidence interval given in (7) takes the following final form,

$$\hat{\sigma}\sqrt{\frac{n}{\Gamma_{1-\frac{\alpha}{2}}(n\,,1)}} \le \sigma \le \hat{\sigma}\sqrt{\frac{n}{\Gamma_{\frac{\alpha}{2}}(n\,,1)}}.$$

which again does not contain R.

From Table 2 we verify that exact confidence interval methods attain coverage (rounded at two decimal digits) equal to the nominal confidence level at any sample under consideration. We also observe that for the asymptotic methods, the convergence rate of coverage to the nominal confidence level depends upon the coefficients of variation (CV), skewness and kurtosis. For the Log-Normal distribution, the rate of convergence depends also on the cririical fractile, R. So, given CV, skewness and kurtosis, the larger the R we take, the slower the rate of convergence we face. For example, for $(\text{Log-Normal})_1$, when R=0,4 an approximate coverage of 93% is attained with approximately 20 observations, while with R=0,95 to observe the same coverage, we need a sample size of at least 50 observations.

Comparing the coverage of asymptotic methods among the two sets of distributions, the simultaneous increase of CV, skewness and kurtosis, produces slower convergence rates to the nominal confidence level. Among the Rayleigh and Exponential, coverage of 93% is attained for the first distribution with a sample of at least 15 observations, while for the second with a sample of 40 observations or more. The same remark holds for the Log-Normal distribution. For $(\text{Log-Normal})_1$, to take COV=93% for R=0,8, we need samples of approximately 40 observations, whereas for $(\text{Log-Normal})_2$, samples over 50 observations. On the other hand, given CV, the extent where skewness and kurtosis affect convergence rates depends upon the size of CV itself, and the value of R. So for relatively low CV's, the increase of coefficients of skewness and kurtosis leads to slower rate of convergence even for moderate or low values of R. For large CV's, slower rates of convergence are observed only for large R's. To indicate this remark, consider the first set of distributions having the lowest CV. To attain COV=93%, we need a larger sample for the $(\text{Log-Normal})_1$, compared to the Rayleigh distribution, even with a moderate value of R=0,4. For the second set, larger samples are needed for $(\text{Log-Normal})_2$, compared to the Exponential distribution, only when R is large, and especially close to unity.

Although exact confidence interval methods attain the required coverage at any sample size, we find out in Table 3 for rather small samples that these methods give very low precision, especially for high coefficients of variation, skewness and kurtosis. As an indicative example we mention the case of $(\text{Log-Normal})_2$ where, for R=0,95, a sample of 10 observations gives confidence intervals with an average length of 1290 units. Such intervals offer almost zero information at the stage of decision making in inventory management practices, when in fact the average demand is 300 units and the true optimal order quantity 834 units. On the contrary, given R, CV, skewness and kurtosis, asymptotic confidence interval methods produce on average higher precisions compared to the exact methods. Therefore, for sample sizes where asymptotic methods attain acceptable coverage, they should be preferred against the corresponding exact ones. Finally, we identify from the cases of $(\text{Log-Normal})_1$ και $(\text{Log-Normal})_2$, that average half lengths are increasing as R is getting larger values.

Table 2: Coverage of 95% confidence interval methods for Q_j^*

n	Rayleigh		(Log-Normal)$_1$						Exponential	(Log-Normal)$_2$					
			Exact			Asymptotic				Exact			Asymptotic		
	Exact	Asymptotic	R=0,4	R=0,8	R=0,95	R=0,4	R=0,8	R=0,95		R=0,4	R=0,8	R=0,95	R=0,4	R=0,8	R=0,95
5	0,95	0,90	0,95	0,95	0,95	0,85	0,80	0,75	0,81	0,95	0,95	0,95	0,85	0,78	0,72
10	0,95	0,92	0,95	0,95	0,95	0,90	0,87	0,84	0,87	0,95	0,95	0,95	0,91	0,85	0,81
15	0,95	0,93	0,95	0,95	0,95	0,92	0,90	0,87	0,90	0,95	0,95	0,95	0,92	0,89	0,85
20	0,95	0,94	0,95	0,95	0,95	0,93	0,91	0,89	0,91	0,95	0,95	0,95	0,93	0,90	0,88
25	0,95	0,94	0,95	0,95	0,95	0,93	0,92	0,90	0,92	0,95	0,95	0,95	0,93	0,91	0,89
30	0,95	0,94	0,95	0,95	0,95	0,93	0,92	0,91	0,92	0,95	0,95	0,95	0,94	0,92	0,90
40	0,95	0,95	0,95	0,95	0,95	0,94	0,93	0,92	0,93	0,95	0,95	0,95	0,94	0,92	0,91
50	0,95	0,94	0,95	0,95	0,95	0,94	0,93	0,92	0,93	0,95	0,95	0,95	0,94	0,92	0,91
100	0,95	0,95	0,95	0,95	0,95	0,94	0,94	0,94	0,94	0,95	0,95	0,95	0,94	0,94	0,93
300	0,95	0,95	0,95	0,95	0,95	0,95	0,94	0,95	0,95	0,95	0,95	0,95	0,95	0,94	0,95

Table 3: Precision of 95% confidence intervals for Q_j^*

n	Rayleigh		(Log-Normal)$_1$						Exponential	(Log-Normal)$_2$					
			Exact			Asymptotic				Exact			Asymptotic		
	Exact	Asymptotic	R=0,4	R=0,8	R=0,95	R=0,4	R=0,8	R=0,95		R=0,4	R=0,8	R=0,95	R=0,4	R=0,8	R=0,95
5	0,517	0,429	0,584	1,281	3,407	0,382	0,430	0,570	0,761	1,102	4,260	27,209	0,682	0,766	1,051
10	0,335	0,306	0,348	0,495	0,805	0,291	0,331	0,437	0,573	0,613	1,022	2,014	0,505	0,574	0,773
15	0,267	0,251	0,272	0,354	0,528	0,243	0,276	0,365	0,479	0,472	0,673	1,114	0,418	0,476	0,636
20	0,228	0,218	0,231	0,290	0,416	0,212	0,242	0,319	0,421	0,397	0,531	0,820	0,364	0,415	0,552
25	0,202	0,195	0,204	0,251	0,354	0,191	0,218	0,288	0,380	0,351	0,453	0,674	0,327	0,373	0,496
30	0,184	0,178	0,185	0,225	0,313	0,175	0,200	0,264	0,349	0,317	0,401	0,583	0,299	0,342	0,454
40	0,158	0,155	0,159	0,190	0,260	0,153	0,174	0,230	0,303	0,271	0,334	0,472	0,260	0,297	0,393
50	0,141	0,138	0,141	0,167	0,227	0,137	0,156	0,206	0,272	0,241	0,291	0,406	0,233	0,266	0,351
100	0,099	0,098	0,099	0,115	0,154	0,097	0,111	0,146	0,194	0,168	0,197	0,267	0,165	0,189	0,249
300	0,057	0,057	0,057	0,065	0,086	0,056	0,064	0,085	0,113	0,096	0,111	0,147	0,095	0,109	0,144

4. EFFECTS OF NON-NORMALITY

Traditional textbooks on inventory management (see for example Silver et al, 1998) illustrate how to apply optimal ordering policies with a normal demand. Assuming normality for simplification reasons, some practitioners apply such policies even when enough data on demand are available. Others apply traditional normality tests to either small or large samples before they use such optimal ordering rules based on a normal demand. Either assuming or testing normality for demand, such actions from practitioners raise certain issues which should be addressed: First, to what extent non-normality can be identified, especially when data on demand are available for few past inventory cycles, and, if we erroneously accept normality, how accurate and precise the estimates of the optimal order quantity held under a normal demand are.

Using again for each one of the four distributions under consideration the 10000 replications for demand, the Jarque-Bera statistic for testing normality (Judge et al., 1982) was computed in each replication for different sample sizes. Table 4 displays the percentage of those replications where the null hypothesis could not be rejected at level of significance 1% and 5%. These percentages estimate the type II error of not rejecting normality, when in fact demand is generated by the Exponential, Rayleigh and Log-Normal Distributions. Depending upon the size of coefficients of variation, skewness and kurtosis, there is a certain range of relatively small samples for which non-normality cannot be traced. For the Rayleigh distribution (which has the lowest CV, skewness and kurtosis), even with n=100, the probability not to reject normality at α=0,05 is almost a half, while for the $(\text{Log-Normal})_2$, even with samples of 20 observations this probability is approximately 40%. For levels of significance 1%, the corresponding probabilities are much higher, namely, more than 70% for the Rayleigh distribution and approximately 50% for $(\text{Log-Normal})_2$.

Table 4: Estimated probabilities of not rejecting normality

	Rayleigh		$(\text{Log-Normal})_1$		Exponential		$(\text{Log-Normal})_2$	
n	α = 0,01	α = 0,05	α = 0,01	α = 0,05	α = 0,01	α = 0,05	α = 0,01	α = 0,05
10	0,9958	0,9834	0,9566	0,911	0,9192	0,8513	0,8603	0,7868
15	0,9762	0,9526	0,8574	0,7895	0,7747	0,6723	0,6612	0,5619
20	0,9614	0,9301	0,7662	0,6801	0,639	0,5162	0,4927	0,3846
25	0,9473	0,9077	0,6755	0,5724	0,52	0,3808	0,3613	0,2536
30	0,9328	0,88	0,5964	0,484	0,4129	0,2749	0,2527	0,1578
40	0,9032	0,8337	0,4559	0,3295	0,2451	0,1245	0,1161	0,0571
50	0,8767	0,7869	0,334	0,2083	0,1291	0,0488	0,0477	0,015
100	0,7107	0,493	0,0425	0,0118	0,0008	0	0,0001	0
300	0,0395	0,0026	0	0	0	0	0	0

Suppose now that the sample $X_1, X_2, ..., X_n$ of demand for the past n inventory cycles has been generated from a normal distribution. Following the proofs of (10a) and (10b) in Appendix for the Log-Normal distribution, the exact confidence intervals held under normality for the true optimal order quantity are derived in a similar manner and are given by

$$\overline{X}_n - t'_{n-1,1-\frac{\alpha}{2}}(\delta) \cdot \frac{S_n}{\sqrt{n}} \le Q^*_{NM} \le \overline{X}_n - t'_{n-1,\frac{\alpha}{2}}(\delta) \cdot \frac{S_n}{\sqrt{n}} \tag{13a}$$

for R<0,5, and

$$\overline{X}_n + t'_{n-1,\frac{\alpha}{2}}(\delta) \cdot \frac{S_n}{\sqrt{n}} \le Q^*_{NM} \le \overline{X}_n + t'_{n-1,1-\frac{\alpha}{2}}(\delta) \cdot \frac{S_n}{\sqrt{n}} \tag{13b}$$

for R>0,5, where $\overline{X}_n = \sum_{t=1}^{n} X_t \Big/ n$ and $S_n^2 = \sum_{t=1}^{n} \left(X_t - \overline{X}_n\right) \Big/ (n-1)$.

If n is sufficiently large, the asymptotic confidence interval for Q^*_{NM} is given in Kevork (2010), as

$$\hat{Q}^*_{NM} \pm z_{\alpha/2} \frac{S_n}{\sqrt{n}} \sqrt{1 + \frac{z_R^2}{2}}\,, \tag{14}$$

where $\hat{Q}^*_{NM} = \overline{X}_n + z_R \cdot S_n$.

Using the available 10000 replications of demand, for each one of the three skewed distributions under consideration, we computed the coverage which is attained by (13) and (14). From figures (3a) and (3b), exact confidence intervals held under normality attain acceptable coverage, when demand is in fact skewed, only for small samples and when R<0,5 (low profit products). For high-profit products, and especially when R is close to unity, the coverage is lower than the nominal confidence level even for samples of small size.

The coverage, which is attained by (14), is illustrated in figures (4a) and (4b). In this case, for both moderate and large R's, the coverage is lower that 95% even for small samples. Besides, the coverage tends to zero as the sample is getting larger and larger. The last result holds for every R, as for the Rayleigh, Exponential and Log-Normal distributions, we show in Appendix that the following probability,

$$\Pr\left(-z_{\alpha/2} \le \frac{\sqrt{n}\left(\hat{Q}^*_{NM} - \hat{Q}^*_j\right)}{S_n \sqrt{1 + \frac{z_R^2}{2}}} \le \right) \to 0 \tag{15}$$

when $n \to \infty$.

Figure 3a: Coverage of 95% exact confidence intervals assuming normality, R=0,4

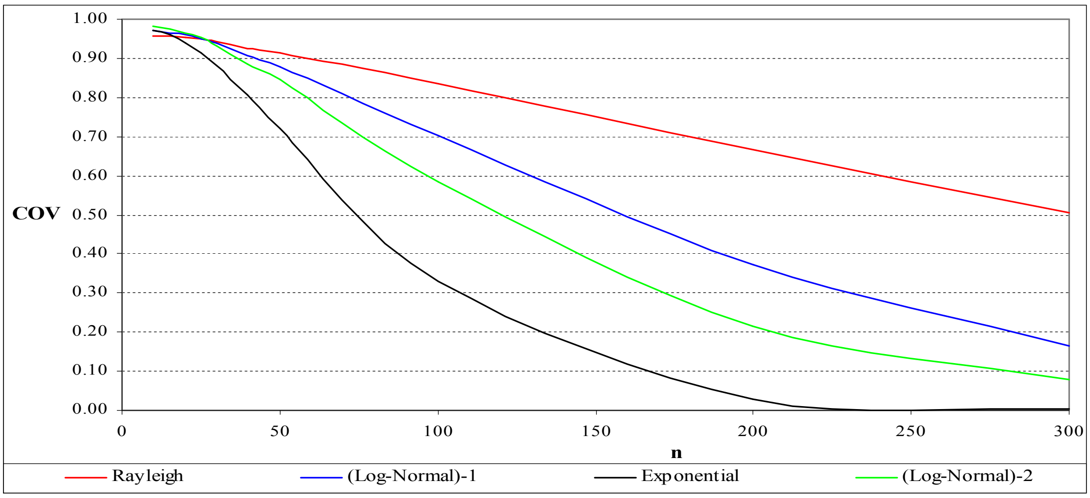


Figure 3b: Coverage of 95% exact confidence intervals assuming normality, R=0,95

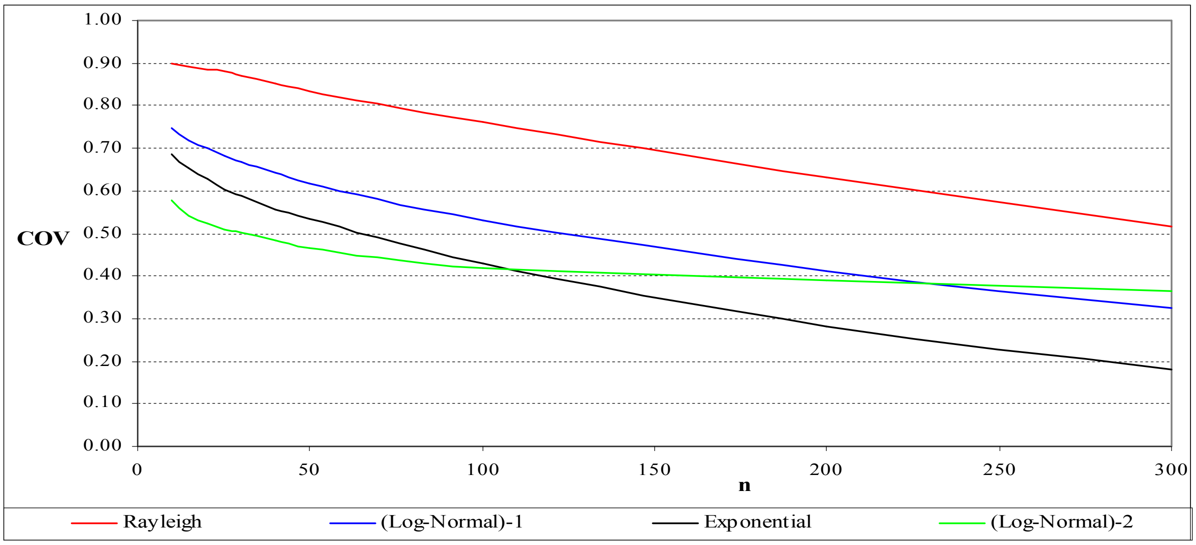


Figure 4a: Coverage of 95% asymptotic confidence intervals assuming normality, R=0,4

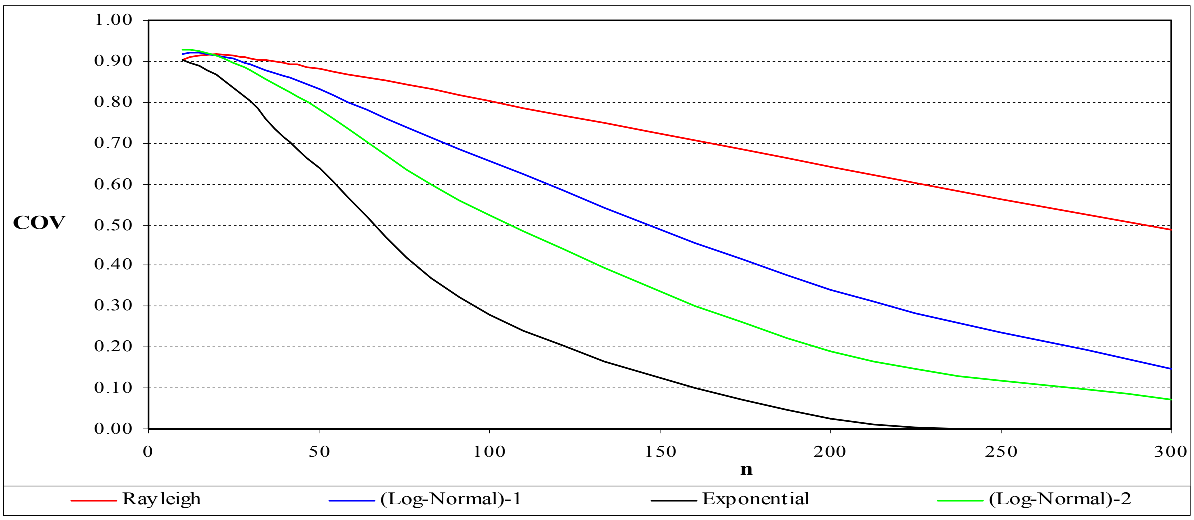
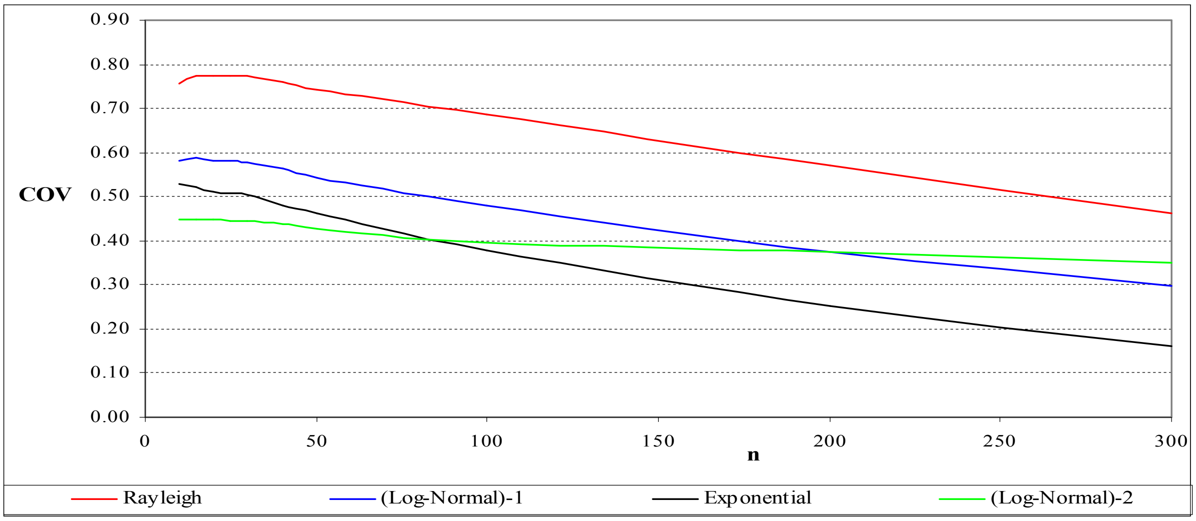


Figure 4b: Coverage of 95% asymptotic confidence intervals assuming normality, R=0,95



Finally, table 5 displays for R=0,4 the RAHL of (13a) only for those samples for which coverage is greater than 90%. RAHL values of table 5 are larger compared to those ones of table 3. This indicates that for R=0,4, applying exact confidence interval held under normality to small samples from the three skewed distribution under study, although we attain satisfactory coverage, the penalty of not rejecting normality is higher sampling errors which we get for the estimated values of the true optimal order quantity.

Table 5: RAHL of 95% exact confidence intervals for Q_j^* assuming normality, R=0,4

n	Rayleigh	Exponential	$(\text{Log-Normal})_1$	$(\text{Log-Normal})_2$
10	0,4587	1,3194	0,4558	1,0879
15	0,3583	1,0453	0,3595	0,8729
20	0,3044	0,8951	0,3064	0,7521
25	0,2693	0,7960	0,2723	0,6743
30	0,2442		0,2478	0,6183
40	0,2093		0,2133	
50	0,1862			

5. CONCLUSIONS

The current paper explores the effectiveness of applying optimal ordering policies to newsvendor type of products, when demand is skewed and follows the Exponential, Rayleigh and Log-Normal distributions. The coefficients of variation, skewness, and kurtosis of the Exponential and Rayleigh remain the same regardless the parameter values of each distribution. Among the two, the Exponential distribution has the highest coefficients. Regarding the Log-Normal distribution, two specifications are defined. Assigning appropriate values to its parameters, the first specification has the same coefficient of variation with the Rayleigh distribution and the second with the Exponential. In each pair, Rayleigh-Log-Normal, or Exponential-Log-Normal, the Log-Normal distribution has higher coefficients of skewness and excess kurtosis.

At a theoretical level, the optimal order quantity depends upon the size of the three coefficients, as well as, the value of a critical fractile, R, expressing the probability of not experiencing stock-outs in successive inventory cycles. Given R, for the pair of Exponential-Log-Normal, optimal order quantities are larger (smaller) compared to those ones of the second pair Rayleigh-Log-Normal, only when R is relatively high (low). Besides, within each pair, higher coefficients of skewness and kurtosis result in larger optimal order quantities only when R is relatively close to zero or to unity.

Theoretical analysis relies on the assumption that the parameter(s) of demand distribution are known. In real-life inventory problems, unfortunately, this is not true. Practitioners have as the only alternative to estimate the parameters from available data on demand in samples of past successive inventory cycles. Then these estimates are used to take corresponding estimates for the optimal order quantity. To offer useful guidelines to such an estimation process, the current paper determines the validity and precision of estimates for the optimal order quantity, when demand follows one of the aforementioned three skewed distributions. Exact and asymptotic confidence intervals for the true optimal order quantity are derived, and their coverage (as an estimate of the actual confidence level) and precision are estimated in finite samples using appropriate Monte-Carlo simulations. For different sample sizes and different combinations of values of the critical fractile and values of the coefficients of variation, skeweness and excess kurtosis, the coverage and the average half-length (as a percentage of the true optimal order quantity) are reported for both the exact and asymptotic confidence intervals.

The two tables of section 3 displaying all previous information facilitate researchers to evaluate their estimates for the optimal order quantity when they face empirical demand distributions displaying similar characteristics with the Exponential, Rayleigh and Log-Normal. To provide a summary of prons and cons of exact versus asymptotic confidence intervals methods, we report that exact methods attain the required coverage at any sample size, but their precision is on average lower compared to the corresponding asymptotic methods. Regarding the latter ones, the rate of convergence of coverage to the nominal confidence level depends upon the coefficients of variation, skewness and kurtosis. The larger the values these coefficients can take on, the slower the rates of convergence the coverage attain.

This paper also investigates the validity of confidence intervals held under a normal demand, when in fact the true demand follows the Exponential, Rayleigh and Log-Normal distributions. When data on demand from these three distributions are available only for few inventory cycles, it is very likely to accept normality applying the classical Jarque-Bera test. Nonetheless, using the exact confidence intervals held under normality in such small samples, we finally get acceptable coverage. On the other hand, the use of the asymptotic confidence intervals held under normality would lead to acceptable coverage only for small or moderate values of R. However, applying either the exact or the asymptotic confidence intervals held under normality, we shall experience lower precision compared to that we would have with the true skewed demand distribution. For the three distributions under consideration, we also show that when the sample is sufficiently large, the actual confidence level of the asymptotic intervals held under normality is close to zero. Further using either the exact or the asymptotic confidence interval method held under normality, we find out that convergence rates of coverage to zero are differentiated not only between different values of coefficients of variation, skewness and kurtosis of the true demand distribution, but also between different values of the critical fractile.

Closing this last section we summarize the drawbacks of having limited past data on demand. First, it is very likely to accept that the true distribution is normal. But even if this will not happen and we use the right exact or asymptotic confidence interval methods, we experience low precision for the exact methods, and coverage not close enough to the nominal confidence level for the asymptotic methods. We conclude, therefore, that a large sample is necessary no matter how expensive the process of its collection might be. The large sample will enable researchers to trace non-normality, and to use the right asymptotic confidence interval method in order to attain the required coverage and precision. Unfortunately, for certain types of newsvendor products, due to their nature and market conditions, past history of demand is limited. Seasonal clothing belongs to this category of products. In such cases, we would not recommend the application of newsvendor models with parametric estimation as this might lead to ordering policies which will be far away from the real optimal ones, especially when the true demand distribution is characterized by large coefficients of variation, skewness, and kurtosis.

APPENDIX

Proof of (8):

For n sufficiently large $\dfrac{\dfrac{2n\hat{\sigma}^2}{\sigma^2} - 2n}{\sqrt{4n}} \xrightarrow{D} N(0,1)$, or $\sqrt{n}\left[\hat{\sigma}^2 - \sigma^2\right] \xrightarrow{D} N(0, \sigma^4)$.

Setting $\hat{Q}^*_{RY} = h(\hat{\sigma}^2) = (\hat{\sigma}^2)^{1/2}\sqrt{2|\ln(1-R)|}$, and since $\hat{Q}^*_{RY}$ is a consistent estimator for Q^*_{RY}, the application of the univariate delta method gives:

$$\sqrt{n}\left[h(\hat{\sigma}^2) - h(\sigma^2)\right] \xrightarrow{D} N\left(0, \sigma^4 \left\{ \left.\frac{dh}{d\hat{\sigma}^2}\right|_{\hat{\sigma}^2=\sigma^2} \right\}^2 \right), \tag{A1}$$

The result follows from (A1) since $\left.\dfrac{dh}{d\hat{\sigma}^2}\right|_{\hat{\sigma}^2=\sigma^2} = \dfrac{1}{\sigma}\sqrt{\dfrac{|\ln(1-R)|}{2}}$.

Proof of (10a):

As $\ln X_t \sim N(\mu_{LN}, \sigma^2_{LN})$, the exact distributional results, $\dfrac{\sqrt{n}[\hat{\mu}_{LN} - \mu_{LN}]}{\sigma_{LN}} \sim N(0,1)$ and $\dfrac{(n-1)s^2_{LN}}{\sigma^2_{LN}} \sim \chi^2_{n-1}$ hold. Then the statistic,

$$\frac{\dfrac{\sqrt{n}[\hat{\mu}_{LN} - \mu_{LN}]}{\sigma_{LN}} + \delta}{\dfrac{s_{LN}}{\sigma_{LN}}} \tag{A2}$$

follows the non-central t-student distribution with n-1 degrees of freedom and non-centrality parameter equal to δ. Setting $\delta = |z_R|\sqrt{n}$ in (A2), we take

$$\frac{\sqrt{n}\{\hat{\mu}_{LN} - (\mu_{LN} - |z_R|\sigma_{LN})\}}{s_{LN}} = \frac{\sqrt{n}\left(\hat{\mu}_{LN} - \ln Q^*_{LN}\right)}{s_{LN}} \sim t'_{n-1}\left[|z_R|\sqrt{n}\right], \tag{A3}$$

and

$$\Pr\left(t'_{n-1,\frac{\alpha}{2}}\left[|z_R|\sqrt{n}\right] \leq \frac{\sqrt{n}\left(\hat{\mu}_{LN} - \ln Q^*_{LN}\right)}{s_{LN}} \leq t'_{n-1,1-\frac{\alpha}{2}}\left[|z_R|\sqrt{n}\right] \right) = 1 - \alpha . \tag{A4}$$

From (A4), the $(1-\alpha)100\%$ confidence interval for $\ln \mathrm{Q}_{\mathrm{LN}}^{*}$ will be

$$\hat{\mu}_{\mathrm{LN}} - t'_{n-1,1-\frac{\alpha}{2}}(\delta)\cdot\frac{s_{\mathrm{LN}}}{\sqrt{n}} \le \ln \mathrm{Q}_{\mathrm{LN}}^{*} \le \hat{\mu}_{\mathrm{LN}} - t'_{n-1,\frac{\alpha}{2}}(\delta)\cdot\frac{s_{\mathrm{LN}}}{\sqrt{n}},$$

from which the result follows.

Proof of (10b):

The result follows from (A2), (A3) and (A4), setting $\delta = -z_R\sqrt{n}$, and noting that

$$t'_{n-1,\frac{\alpha}{2}}\left(-z_R\sqrt{n}\right) = -t'_{n-1,1-\frac{\alpha}{2}}\left(z_R\sqrt{n}\right),$$

$$t'_{n-1,1-\frac{\alpha}{2}}\left(-z_R\sqrt{n}\right) = -t'_{n-1,\frac{\alpha}{2}}\left(z_R\sqrt{n}\right).$$

Proof of (11):

Knight (1999, p. 258) states that $\sqrt{n}\begin{bmatrix}\hat{\mu}_{\mathrm{LN}} - \mu_{\mathrm{LN}} \\ \hat{\sigma}_{\mathrm{LN}} - \sigma_{\mathrm{LN}}\end{bmatrix} \xrightarrow{D} N_2[\mathbf{0},\mathbf{\Sigma}]$,

where $\mathbf{\Sigma} = \begin{bmatrix}\sigma_{\mathrm{LN}}^2 & 0 \\ 0 & \frac{\sigma_{\mathrm{LN}}^4}{2}\end{bmatrix}$.

Rewriting $\hat{\mathrm{Q}}_{\mathrm{LN}}^{*} = h(\hat{\mu}_{\mathrm{LN}},\hat{\sigma}_{\mathrm{LN}}) = \exp(\hat{\mu}_{\mathrm{LN}} + z_R\hat{\sigma}_{\mathrm{LN}})$, and since $\hat{\mathrm{Q}}_{\mathrm{LN}}^{*}$ is a consistent estimator for $\mathrm{Q}_{\mathrm{LN}}^{*}$, the application of multivariate delta method (knight, 1999, p. 148) results in,

$$\sqrt{n}\left(\hat{\mathrm{Q}}_{\mathrm{LN}}^{*} - \mathrm{Q}_{\mathrm{LN}}^{*}\right) \xrightarrow{D} N\left(0,\mathbf{L}'\cdot\mathbf{\Sigma}\cdot\mathbf{L}\right) \tag{A5}$$

where $\mathbf{L}' = \begin{bmatrix}\frac{\partial h}{\partial\hat{\mu}_{\mathrm{LN}}} & \frac{\partial h}{\partial\hat{\sigma}_{\mathrm{LN}}}\end{bmatrix}$, and with the partial derivatives to be evaluated at $\hat{\mu}_{\mathrm{LN}} = \mu_{\mathrm{LN}}$ and $\hat{\sigma}_{\mathrm{LN}} = \sigma_{\mathrm{LN}}$.

Evaluating the partial derivatives at $\hat{\mu}_{\mathrm{LN}} = \mu_{\mathrm{LN}}$ and $\hat{\sigma}_{\mathrm{LN}} = \sigma_{\mathrm{LN}}$, we take $\frac{\partial h}{\partial\hat{\mu}_{\mathrm{LN}}} = \mathrm{Q}_{\mathrm{LN}}^{*}$, $\frac{\partial h}{\partial\hat{\sigma}_{\mathrm{LN}}} = z_R\cdot\mathrm{Q}_{\mathrm{LN}}^{*}$, and

$$\mathbf{L}'\cdot\boldsymbol{\Sigma}\cdot\mathbf{L}=\left(1+\frac{z_R^2}{2}\right)\cdot\left(Q_{LN}^*\right)^2\cdot\sigma_{LN}^2. \tag{A6}$$

The result follows after replacing (A6) to (A5).

<u>Proof of (15):</u>

For the Exponential (j=EX), Rayleigh (j=RY), and Log-Normal (j=LN) distributions rewrite the probability given in (14) as

$$\Pr\left(\frac{-z_{\alpha/2}}{\left|\ln(1-R)\right|}\cdot\frac{\hat{Q}_{EX}^*-Q_{EX}^*}{\hat{Q}_{NM}^*-Q_{EX}^*}\le\frac{\sqrt{n}\left(\hat{Q}_{EX}^*-Q_{EX}^*\right)}{S_n\cdot\left|\ln(1-R)\right|\cdot\sqrt{1+\frac{z_R^2}{2}}}\le\frac{z_{\alpha/2}}{\left|\ln(1-R)\right|}\cdot\frac{\hat{Q}_{EX}^*-Q_{EX}^*}{\hat{Q}_{NM}^*-Q_{EX}^*}\right), \tag{A7.1}$$

$$\Pr\left(\frac{-z_{\alpha/2}}{\sqrt{\frac{\left|\ln(1-R)\right|}{2}}}\frac{\hat{Q}_{RY}^*-Q_{RY}^*}{\hat{Q}_{NM}^*-Q_{RY}^*}\le\frac{\sqrt{n}\left(\hat{Q}_{RY}^*-Q_{RY}^*\right)}{S_n\sqrt{\frac{\left|\ln(1-R)\right|}{2}}\sqrt{1+\frac{z_R^2}{2}}}\le\frac{z_{\alpha/2}}{\sqrt{\frac{\left|\ln(1-R)\right|}{2}}}\frac{\hat{Q}_{RY}^*-Q_{RY}^*}{\hat{Q}_{NM}^*-Q_{RY}^*}\right), \tag{A7.2}$$

$$\Pr\left(\frac{-z_{\alpha/2}}{\sigma_{LN}\cdot Q_{LN}^*}\cdot\frac{\hat{Q}_{LN}^*-Q_{LN}^*}{\hat{Q}_{NM}^*-Q_{LN}^*}\le\frac{\sqrt{n}\left(\hat{Q}_{LN}^*-Q_{LN}^*\right)}{S_n\cdot\sigma_{LN}\cdot Q_{LN}^*\cdot\sqrt{1+\frac{z_R^2}{2}}}\le\frac{z_{\alpha/2}}{\sigma_{LN}\cdot Q_{LN}^*}\cdot\frac{\hat{Q}_{LN}^*-Q_{LN}^*}{\hat{Q}_{NM}^*-Q_{LN}^*}\right). \tag{A7.3}$$

(i) By the Weak Law of Large Numbers,

$$p\lim\overline{X}_n=E(X)=\begin{cases}\lambda & \text{for } j=EX\\ \sigma\sqrt{\pi/2} & \text{for } j=RY\\ e^{\mu_{LN}+\sigma_{LN}^2/2} & \text{for } j=LN\end{cases}. \tag{A8}$$

(ii) Knight (1999, p. 189) proves that S_n^2 is a consistent estimator of $\mathrm{Var}(X)$ for any distribution with finite variance. Thus

$$p\lim S_n=\left(p\lim S_n^2\right)^{1/2}=\begin{cases}\lambda & \text{for } j=EX\\ \sigma\sqrt{(4-\pi)/2} & \text{for } j=RY\\ E(X)\cdot\sqrt{e^{\sigma_{LN}^2}-1} & \text{for } j=LN\end{cases}. \tag{A9}$$

and

(iii) $$\operatorname{plim}\left(\hat{Q}^*_{NM} - Q^*_j\right) = \operatorname{plim}\overline{X}_n + z_R \cdot \operatorname{plim} S_n - Q^*_j . \tag{A10}$$

Using (3a)-(3c) of section 2, the asymptotic distributional results (5), (8), (11) of section 3, and (A8), (A9) and (A10), the three probabilities of (A7), as $n \to \infty$, tend to

$$\Pr\left(-z_{\alpha/2} \cdot \frac{g_R}{Q^*_j} \cdot \operatorname{plim}\left(\hat{Q}^*_j - Q^*_j\right) \le Z \le z_{\alpha/2} \cdot \frac{g_R}{Q^*_j} \cdot \operatorname{plim}\left(\hat{Q}^*_j - Q^*_j\right)\right) = \Pr(Z = 0) = 0$$

as $\operatorname{plim}\hat{Q}^*_j = Q^*_j$, and

$$g_R = \begin{cases} \dfrac{\sqrt{1+\dfrac{z_R^2}{2}}}{1 + z_R - \left|\ln(1-R)\right|} & \text{for } j = EX \\[2ex] \dfrac{2\sqrt{1+\dfrac{z_R^2}{2}}}{\sqrt{\dfrac{\pi}{4-\pi}} + z_R - 2\sqrt{\dfrac{\left|\ln(1-R)\right|}{4-\pi}}} & \text{for } j = RY \;. \\[2ex] \dfrac{\sigma_{LN}}{\sqrt{e^{\sigma_{LN}^2} - 1}}\left(1 - \dfrac{1}{\sqrt{e^{\sigma_{LN}(2z_R - \sigma_{LN})}}}\right) + \sigma_{LN} z_R & \text{for } j = LN \end{cases}$$

References

Agrawal N., Smith, SA. (1996). Estimating Negative Binomial Demand for Retail Inventory Management with Unobservable Lost Sales. *Naval Research Logistics* **43:** 839–861.

Areeratchakul N, Abdel-Malek L. (2006). An approach for solving the Multi-product Newsboy Problem. *International Journal of Operations Research*, **3**: 219-227.

Balakrishnan N, Cohen AC. (1991). Order Statistics and Inference Estimation Methods, San Diego: Academic Press.

Benzion U, Cohen Y, Peled R, Sharit T. (2008). Decision Making and the newsvendor problem: an experimental study. *Journal of the Operational Research Society* **59**: 1281-1287.

Bell PC. (2000). Forecasting Demand Variation when there are Stockouts. *Journal of the Operational Research Society* **51:** 358–363.

Casimir RJ. (2002). The value of information in the multi-item newsboy problem. *Omega* **30**: 45–50.

Chen LH, Chen YC. (2009). A newsboy problem with a simple reservation arrangement. *Computers and Industrial Engineering* **56**: 157-160

Chen LH, Chen YC. (2010). A multiple-item budget-constraint newsboy problem with a reservation policy. *Omega* **38**: 431–9.

Choi S, Ruszczynski A. (2008). A Risk-Averse Newsvendor with Law Invariant Coherent Measures of Risk. *Operations Research letters*, **36**: 77-82.

Conrad SA. (1976). Sales Data and the Estimation of Demand. *Operational Research Quarterly*, **27:** 123–127.

Dominey MJG, Hill RM. (2004). Performance of Approximations for Compound Poisson Distributed Demand in the Newsboy Problem. *International Journal of Production Economics*, **92**: 145-155.

Dutta P, Chakraborty D, Roy AR. (2005). A single period inventory model with fuzzy random variable demand. *Mathematical and Computer Modeling,* **41**: 915-922.

Geng W, Zhao X, Gao D. (2010). A Single-Period Inventory System with a General S-Shaped Utility and Exponential Demand. *Journal of Systems Science and Systems Engineering*, **19**: 227-236.

Grubbstrom, RW. (2010). The Newsboy Problem when Customer Demand is a Compound Renewal Process. *European Journal of Operational Research*, **203**: 134-142

Hill RM. (1997). Applying Bayesian Methodology with a Uniform Prior to the Single Period Inventory Model. *European Journal of Operational Research* **98:** 555–562.

Huang D, Zhou H, Zhao QH. (2011). A competitive multiple-product newsboy problem with partial product substitution. *Omega* **39:** 302-312.

Jammernegg W, Kischka P. (2009). Risk Preferences and Robust Inventory Decisions. *International Journal of Production Economics*, **118**: 269-274.

Jiang H, Netessine S, Savin S. (2012). Robust newsvendor competition under asymmetric information. *Forthcoming in Operations Research, DOI 10.128.*

Johnson, NL., Kotz, S., and Bakakrishnan, N., 1994. *Continuous Univariate Distributions.* 2nd Edition. Wiley New York.

Judge GG, Hill RC, Griffiths WE, Lutkerpohl H, Lee TC. (1988). Introduction to the Theory and Practice of Econometrics. 2nd Edition. John Wiley & Sons.

Kevork IS. (2010). Estimating the optimal order quantity and the maximum expected profit for single-period inventory decisions. *Omega* **38**: 218–27.

Khouja M. (1996). A note on the Newsboy Problem with an Emergency Supply Option. *Journal of the Operational Research Society* **47:** 1530-1534.

Khouja M. (1999). The single-period (news-vendor) problem: literature review and suggestions for future research. *Omega* **27**:537–53.

Knight, K., 1999. *Mathematical Statistics*. Taylor & Francis Ltd.

Law AM. (2007). Simulation Modeling and Analysis, 4th Edition, McGraw Hill.

Lau H. (1997). Simple Formulas for the Expected Costs in the Newsboy Problem: an educational note. *European Journal of Operational Research* **100:** 557–61.

Lau AH, Lau H. (2002). The effects of reducing demand uncertainty in a manufacturer-retailer channel for single-periods products. *Computers & Operations Research*, **29**: 1583-1602.

Lee CM, Hsu SL, (2011). The effect of advertising on the distribution-free newsboy problem. *International Journal of Production Economics* **129**: 217-224.

Matsuyama K. (2006). The multi-period newsboy problem. *European Journal of Operational Research* **171**: 170-188.

Mostard J, De Koster R, Teunter R. (2005). The Distribution-Free Newsboy Problem with Resalable Returns. *International Journal of Production Economics*, **97**: 329-342.

Nahmias S. (1994). Demand Estimation in Lost Sales Inventory Systems. *Naval Research Logistics* **41:** 739–757.

Olivares M, Terwiesch C, Cassorla L. (2008). Structural Estimation of the Newsvendor Model: An Application to Reserving Operating Room Time. *Management Science*, **54:** 41–55.

Rider AD, Van der Laan E, Salomon M. (1998). How Larger Demand Variability May Lead to Lower Costs in the Newsvendor Problem. *Operations Research* **46**: 934-936.

Salazar-Ibarra J. (2005). The newsboy Model: Change in risk and price. *The Geneva Risk and Insurance Review* **30**: 99-109.

Schweitzer ME., Cachon GP. (2000). Decision Bias in the Newsvendor Problem with a Known Demand Distribution: Experimental Evidence. *Management Science*, **46:**. 404–420.

Silver, E.A., D.F Pyke, and R. Peterson (1998). Inventory management and Production Planning and Scheduling (3rd ed). John Wiley & Sons, New York, NY

Su RH, Pearn WL (2011). Product selection for newsboy-type products with normal demands and unequal costs. *International Journal of Production Economics* **132**: 214-222.

Wang CX and Webster S. (2009). The loss-averse newsvendor problem. *Omega* **37**: 93–105.