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26 January 2012

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January 26, 2012

Abstract

The aim of this paper is to explore the structure of cities as a function of labor differentiation, gains to trade, a fixed cost for constructing the transportation network, a variable cost of commodity transport, and the commuting costs of consumers. Firms use different types of labor to produce different outputs. Locations of all agents are endogenous as are prices and quantities. This is among the first papers to apply smooth economy techniques to urban economics. Existence of equilibrium and its determinacy properties depend crucially on the relative numbers of outputs, types of labor and firms. More differentiated labor implies more equilibria. We provide tight lower bounds on labor differentiation for existence of equilibrium. If these sufficient conditions are satisfied, then generically there is a continuum of equilibria for given parameter values. Finally, an equilibrium allocation is not necessarily Pareto optimal in this model.

Keywords: city structure, heterogeneous labor, transportation network, general equilibrium.

JEL Classification: D51, R14

*We thank Masahisa Fujita, Roger Guesnerie, Hideo Konishi, Fan-Chin Kung, Val Lambson, Guy Laroque, Thijs ten Raa, Adam Rose, Jacques-François Thisse and Ping Wang for valuable comments. The first draft of this paper was written while the first author was visiting CORE in *1995*. Their financial support and hospitality as well as support provided by National Science Foundation grant SBR-9319994 are gratefully acknowledged. The authors retain full responsibility for any errors.

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1 Introduction

This work will attempt to address some of the classic, but relatively unexplored, questions raised in urban economics that deal with the economics of cities. Our questions include the following. Why do cities form where they do? What are the driving forces behind the formation of cities? What roles do increasing returns, gains to trade and the location of marketplaces play? What role does the location of firms play? Is market failure necessary for agglomeration? Is perfect competition consistent with spatial modeling? Why do some cities grow faster than others? This set of questions frames our line of research. The answers to these questions have important policy implications, since the predictability of the effects of government actions rests on an understanding of the mechanisms driving the urban economy. The ability of policy makers to make informed decisions about contemporary issues such as government policy pertaining to migration to and from cities, or social policy directed at revitalizing cities, relies on information provided by urban economic theory.

In order to have a theory explaining city formation and structure, it is necessary to construct a class of models in which the locations of all agents are endogenous, including both consumers and firms. As explained in Berliant and ten Raa (1994), the nature of most of the existing literature is partial rather than general equilibrium in the sense that either the locations of consumers or firms are fixed. The literature reviewed here is distinct since the locations of all agents are endogenous, and thus these models have the potential to answer the questions we have posed.

There are many approaches to answering the questions addressed by our model that generally suggest different economic causes for city formation and growth, and consequently different modeling strategies. Each theory relies on different forces to explain agglomeration and therefore has different consequences in terms of welfare. So it is important to generate testable hypotheses to distinguish among the theories. We categorize the literature on city formation into four groups, using the Spatial Impossibility Theorem of Starrett (1978), as interpreted in Fujita (1986) and Fujita and Thisse (2002), which states that there is no spatial equilibrium with agglomeration if the following conditions are met: (i) no relocation cost, (ii) consumers' preferences and firms' technologies are independent of location, (iii) the economy is closed, (iv) each location has complete competitive markets. Each of the four groups explains agglomeration by relaxing at least one of the hypotheses of the Starrett Theorem.

In the first group, city formation is explained using increasing returns to scale. This group violates (iv) since it assumes imperfect competition. Indeed, Fujita and Krugman (1995, 2000), Fujita, Krugman and Venables (1999) and Krugman (1991, 1993a, 1993b) use a Dixit-Stiglitz framework and increasing returns to generate city formation in a monopolistic competition context.[1] This work was preceded by Fujita (1988), Abdel-Rahman (1988, 1990) and Abdel-Rahman and Fujita (1990).

The second group of models uses spatial externalities to explain city formation (Beckmann (1977), Fujita and Ogawa (1982), Papageorgiou and Smith (1983), ten Raa (1984)). These models violate (ii), (iv) or both, since agent utilities or production functions depend on the locations of the other agents and these externalities are not priced.

In the third group, agglomeration is explained by strategic interactions between firms (spatial competition a la Hotelling; for surveys see Gabszewicz and Thisse (1986, 1992)). This group also violates (iv) since it assumes imperfect competition.

The fourth group of models (Berliant and Konishi (2000), Berliant and Wang (1993), Wang (1990)) uses gains to trade and setup costs of marketplaces and transportation networks to generate agglomeration. This group violates (ii) and (iv), since there may be no marketplace (and consequently no market) at some locations, and therefore agents care about location.

Another explanation for city formation is the differentiation of labor. For example, Rochester, NY, has a highly specialized labor force that serves companies such as Kodak, Xerox, and Bausch and Lomb. These companies employ workers who know about optics and engineering. Another example is Silicon Valley, that has a concentration of labor specialized in the production of semiconductors. This idea has not been modelled formally, although some attempts have been made to model the worker choice of human capital investment using very crude spatial structures (Baumgardner (1988), Benabou (1993), Kim (1991)). Zenou (2009) surveys this literature and studies labor markets in various imperfect competition or search contexts.

The specific question we address in the present paper is: *How does labor differentiation affect city structure*?

In our approach (which can be considered as a fifth group), the formation

[1] Kim (1995) provides some empirical evidence on regional concentration or specialization of industrial production in the United States. For the data from the 1920's to the present, this evidence seem to contradict the empirical implications of the monopolistic competition models.

of cities is explained by labor differentiation, gains to trade, a fixed cost for the transportation network, a variable cost of commodity transport, and the commuting costs of consumers. Since firms are used as marketplaces, our model violates (ii) and (iv) of the Starrett Theorem, similar to the fourth group above.

Observe that in all models of general equilibrium with endogenous locations (including ours), the main problem is to show the existence of a spatial equilibrium and to find its determinacy properties. Why is this a problem? When one introduces space in a general equilibrium framework with a continuum of consumers (and locations) and a finite number of firms, there are non-convexities for both consumers (the consumption set and preferences are non-convex due to the discrete choice of one location of residence) and firms (firm reaction correspondences are not convex valued since given prices, a firm's profit could be maximized at two different locations), and therefore the usual fixed point theorems relevant for proving existence of an equilibrium do not work.

On the consumer side, we can easily convexify the aggregate demand correspondence with a large number of individuals (see Hildenbrand (1974)). Schweizer, Varaiya and Hartwick (1976), Ellickson (1979), and Grimaud and Laffont (1989) have used this technique to prove the existence of a spatial equilibrium. Observe that the number of locations in all of this work is finite. In contrast, our model as well as the standard models of location theory use a continuum of locations. In this case, Hildenbrand's type of argument does not necessarily apply. As is common in the literature (see Fujita (1989)), we use the bid rent approach, imposing the condition that consumers can choose only one location, to solve the non-convexity problem; this technique does not require a fixed point argument or convexity, but relies on direct calculations.

How can we solve the firm's non-convexity problem? To the best of our knowledge, there are two techniques that work. Both of them fix producer locations and solutions are computed for non-location variables given these locations. The two techniques use different methods of fixing producer locations. The first uses Negishi's (1960) method, that fixes firms at the Pareto optimal locations and more generally fixes the allocation at the Pareto optimum. Then one has to decentralize, and find a price system that supports the optimum (see Wang (1990), Berliant and Wang (1993)). Obviously, this method does not work if there is a market failure. The second technique consists of characterizing the spatial equilibrium. However, since there is in general a lot of endogenous variables (prices and quantities as usual, but also locations of producers and consumers), the characterization of equilibrium is difficult. This method was used by Fujita and Krugman (1995, 2000) and in order to characterize the equilibrium they resort to specific functional forms (e.g. CES utility functions and a Samuelsonian 'iceberg' transport technology). This approach suffers from a lack of robustness in the specification of functional forms as well as from a large indeterminacy in the number and qualitative properties of equilibrium. They select one equilibrium. Moreover, consumer commuting is not allowed by this type of model. In our approach, we also use the (second) technique of calculating the equilibrium directly, but we avoid the problems encountered by Fujita and Krugman (1995, 2000). In essence, due to the interaction of firms, considered to be marketplaces as well for exchanging goods, we are able to characterize their equilibrium locations in general, and therefore to prove existence of equilibrium by employing a fixed point argument for all of the non-locational variables.

A natural question that comes to mind is why, in a model of general equilibrium with endogenous locations (such as ours), does one not use *randomized strategies* to alleviate the non-convexity problem and to prove existence of an equilibrium? Basically, there are two reasons not to use mixed strategies. First, if one studies what happens after a randomization is realized, one can easily end up at an infeasible allocation. For example, each firm randomizes over all possible parcels ex ante, but ex post, firms might happen to pile up on one interval, an ex post infeasible allocation. Mixed strategies also involve precommitment of firms to the parcels they get after randomization, and when the random draw is realized, they might have more profits with a different strategy. The latter effect is common to games allowing mixed strategies. The second, and perhaps more important reason, is that the obvious equilibrium with mixed strategies is where everyone is spread out uniformly by randomizing over all locations: it is the Starrett theorem in the context of randomization. Such an allocation clearly minimizes transport costs. There is no agglomeration, and it is uninteresting just as in the standard Starrett theorem. Equilibrium is an artefact of randomization. This is very similar to what is proposed in Koopmans and Beckmann (1957) as the solution where there is no equilibrium in their quadratic assignment model, and should be rejected for the same reasons.

To be more precise, our model uses a very general setting, allowing a multidimensional location space and multiple firms using different types of labor to produce different output commodities. Locations of all agents are endogenous as are prices and quantities. Firms anticipate the relocation of consumers, but influence land prices through their location decisions. As we shall explain shortly, we stick as close to perfect competition as possible while retaining existence of equilibrium. Firms use compact, convex sets of land while consumers buy densities of land. It is assumed that each firm can use only one type of labor and produce only one type of output. With reference to the labor economics literature, labor differentiation can be viewed either as general or specific human capital depending on the number of firms using a particular type of labor. Within this framework, we characterize the spatial configuration of firms in equilibrium. Firms are adjacent to each other at any equilibrium allocation. This is very similar to the principle of minimum differentiation although the force that is driving agglomeration in our model is the transportation cost of outputs. This result is used to prove existence and to examine the determinacy of equilibrium. The proof of existence is unusual in that it uses a mixture of bid rent and fixed point techniques. Concerning the determinacy analysis, we use differential topology techniques in combination with the bid rent approach. This paper can be counted among the first applications of smooth economy techniques to urban economics. To our knowledge, Berliant and Kung (2006, 2009) are the only predecessors, and they apply these techniques to New Economic Geography models.

We show that whether or not equilibrium exists and whether or not it is locally unique depend crucially on the relative numbers of outputs, types of labor and firms. The multiplicity of equilibria is positively associated with the degree of labor differentiation. Finally, an equilibrium is not necessarily Pareto optimal in this model.

Loosely speaking, our main conclusion is as follows. When labor is not completely differentiated, in the sense that more than one firm is drawing from the same pool of labor, equilibrium might not exist; a counterexample (Example 4) is provided in the Appendix. (This does not exclude existence of equilibrium for other examples.) Once one has sufficient conditions on labor differentiation for existence of equilibrium, generically there is a continuum of equilibria. This derives from the classical idea that indivisibilities, particularly in the order and location of firms in our model, both inhibits existence of equilibrium and, once equilibrium is found, allows variation in continuous endogenous parameters without altering the order of firms and without destroying equilibrium.

The remainder of the paper is organized as follows. Section 2 sets up the model and provides the notation and basic definitions. In section 3, we characterize the locations of firms and consumers in equilibrium. It is unusual to be able to do this analytically. Section 4 examines the determinacy of equilibrium in this model. For given values of the exogenous parameters, we find that, depending on the number of labor types, firms and output goods, the set of equilibria might be empty, might be finite or might form a continuum. Section 5 provides sufficient conditions for existence of an equilibrium in the model. These conditions, in turn, imply that generically the set of equilibria forms a continuum. In section 6, we give an example showing that both welfare theorems can fail. Section 7 concludes. An Appendix, available at the first author's web site, contains complementary illustrative examples and all proofs.

2 The general setting

The location space Z is a compact, convex subset of the Euclidean space $\Re^K$.[2] Land of density 1 is available in all locations in Z. There are V types of labor, $v = 1, ..., V$. Each consumer is classified in one category of labor. This could, for example, be derived from the endogenous choice of human capital by consumers as in Rosen (1983), Kim (1989) or Baumgardner (1988). However, here we assume that labor heterogeneity is exogenous. There are $s = 1, ..., S$ goods and each firm produces only one type of good s. Each worker of a given type has the same utility function and is endowed with one unit of labor but no consumption commodities. N^v is the measure of consumers who supply labor of type v. Each individual[3] of type v chooses location $z \in Z$, land consumption $q^v(z)$ (where $q(z) = [q^1(z), ..., q^V(z)]$), consumption good quantities $X_s^v(z)$ in order to maximize utility $U^v(q^v(z), X^v(z))$, where $X^v(z)$ is the vector $[X_1^v(z), ..., X_S^v(z)]$ and $X(z) = [X^1(z), ..., X^V(z)]$, under a budget constraint that will be given later. The consumption set of each consumer is the positive orthant of $\Re^{s+1}$. We assume that utility is strictly monotonic in all arguments and represents C^r differentiably strictly convex preferences (see Mas-Colell (1985, ch. 2.6)). We also assume that for any $q > 0$, $X > 0$, $U^v(q, X) > 0 = U^v(0, X) = U^v(q, 0) = U^v(0, 0)$, so any consumption (q, X) with q and X strictly positive yields higher utility than any consumption on the boundary of $\Re_+^{s+1}$. For simplicity, there is no disutility from labor.

Each firm uses only one type of labor, along with land, to produce exactly

[2] Typically, $K = 1$ or 2. With multi-story buildings, $K = 3$.

[3] In our notation, superscripts represent agents and subscripts represent commodities.

one output commodity. For each $s = 1, ..., S$ and $v = 1, ..., V$, there are M_s^v firms producing type s commodity with type v labor, where $L_s^{i,v}$, $Q_s^{i,v}$ and $Y_s^{i,v}$ are respectively the labor, the land input and the output of firm i producing good s using type v labor. Each firm is thus characterized by a triple (i, s, v). Here the index i runs from 1 to M_s^v. Define the following vectors: $L_s^v = [L_s^{1,v}, ..., L_s^{M_s^v,v}]$, $L_s = [L_s^1, ..., L_s^V]$, $L = [L_1, ..., L_S]$, $Q_s^v = [Q_s^{1,v}, ..., Q_s^{M_s^v,v}]$, $Q_s = [Q_s^1, ..., Q_s^V]$, $Q = [Q_1, ..., Q_S]$, $Y_s^v = [Y_s^{1,v}, ..., Y_s^{M_s^v,v}]$, $Y_s = [Y_s^1, ..., Y_s^V]$, $Y = [Y_1, ..., Y_S]$. The production function for firm (i, s, v) is given by $f_s^v(L_s^{i,v}, Q_s^{i,v}) = Y_s^{i,v}$. *Firms using the same type of labor to produce the same commodity are identical.* We assume that $f_s^v(\cdot)$ is C^r with strictly positive derivatives on $\Re^2_{++}$, that $D^2 f_s^v$ is negative definite, and that $f_s^v(L_s^{i,v}, 0) = 0$.

Assuming constant returns to scale would not change the basic results but would change supply functions to correspondences, and this would make the differential topology argument used for the determinacy of equilibrium much more difficult.

In our model, absentee landlords are endowed with all of the land and all of the profit shares but consume only produced goods. It is standard in the literature of urban economics to employ absentee landlords to insulate the model from income effects in order to use a bid rent approach (see e.g. Fujita (1989)). Our results are easily generalized to allow consumers to be endowed with land or profit shares.[4] We can also allow 'absentee' shareholders (as distinct from landlords) to have all the profit shares.

We assume that *the market for all goods (including labor) except land are competitive*. This assumption of perfectly competitive markets will be made formal in the statements of the consumer and producer optimization problems, and is important in order to prove that equilibrium exists. As we will see later, whatever the assumption on the transportation cost of goods, it is this assumption of perfect competition that makes good prices location-independent. We will discuss later the assumption that the land market is not competitive.

Consumers of type v purchase a density of land $q^v(z)$, whereas firms will use an area of land. Let μ be Lebesgue measure on measurable subsets of $\Re^k$. All measure theoretic statements (such as "almost surely") are made with respect to this measure. Let d be a metric on $\Re^k$. For any Lebesgue measurable A

[4] For that, one must include this extra revenue in the consumer budget constraint and skip the analysis of the absentee landlord. This is called the *public ownership model* (see Fujita (1989, ch.3)).

and any $\epsilon > 0$, define

$$\begin{aligned} B_\epsilon(A) &= \{z \in Z \mid \inf_{y \in A} d(y, z) \leq \epsilon\} \text{ and} \\ g(A, \epsilon) &= \mu(B_\epsilon(A)). \end{aligned}$$

Firms use Lebesgue measurable subsets of Z as inputs for production. Let $\mathbf{C}$ be any collection of Lebesgue measurable subsets of Z satisfying the following conditions:

(a) For each $A \in \mathbf{C}$ there is a $\delta > 0$ such that $B_\epsilon(A) \in \mathbf{C}$ $\forall \epsilon \in [0, \delta)$ and $g(A, \cdot)$ is twice continuously differentiable on $[0, \delta)$.[5]

(b) Fix any $C_1^{1,1}, ..., C_S^{M_S^v,V} \in \mathbf{C}$. If some firm is using a positive quantity of land, then there exists a (possibly different) firm (i, s, v) with $\mu(C_s^{i,v}) > 0$ such that for every $\delta' > 0$ there is an $A \in \mathbf{C}$, $A \subseteq C_s^{i,v}$ a.s. with $\mu(C_s^{i,v}) - \delta' < \mu(A) < \mu(C_s^{i,v})$ and

$$\inf_{\substack{x \in C_s^{i,v} \\ y \in C_{s'}^{j,v'}}} \|x - y\| = \inf_{\substack{x \in A \\ y \in C_{s'}^{j,v'}}} \|x - y\|$$

$$\text{for } s' = 1, .., S; v' = 1, ..., V; j = 1, ..., M_s^v; \forall (j, s', v') \neq (i, s, v)$$

For instance, if $Z = [0, 1]^K$, the hyper-rectangles of the form $[\underline{\theta}, \overline{\theta}] \times [0, 1]^{K-1}$ aligned parallel to the axes in $\Re^K$ used below will satisfy these conditions, as will many other collections. For assumption (b), the firm parcel that contracts is one at the edge of the economy. The idea behind this assumption is that small expansions and contractions of parcels are possible, so calculus can be used. Cantor sets violate this assumption.

If firm i producing good s and using type v labor chooses a $C_s^{i,v} \in \mathbf{C}$, then $Q_s^{i,v} = \mu(C_s^{i,v})$.

Consumers pay a commuting cost to travel to work. For simplicity, commuting costs are only monetary costs and therefore there is no time cost of commuting. Let t denote the vector of input requirements for one unit of commuting distance, so the cost per unit is $p(i, s, v) \cdot t$ (where $p(i, s, v)$ is the vector of prices of the output goods, $p(i, s, v) \in \Re_+^s$, $t \in \Re^s$). Observe that $p(i, s, v)$ is a price vector at a firm-market place *so prices are firm (and hence location) dependent*. Consumers of type v are employed only by firms using type v labor.

[5]It is easy to verify that this assumption holds for intervals when $K = 1$. It also holds when $K = 2$ and a parcel is a disk, and when $K = 3$ and a parcel is a sphere. Many other shapes can be accommodated as well. General metrics are employed here because, for instance, the sup norm metric is useful for squares.

We assume that *every pair of firms must be connected by a transport network* (the cost of connecting any pair of firms is shared equally by each member of the pair) and that *all consumers purchase their consumption goods at the location of a firm, i.e. firms are marketplaces.*[6] These assumptions mean that transportation of goods takes place only between firms, where consumers purchase all their consumption commodities. The transportation network for commodities will connect all firms, even those producing the same good, since local demand and supply of consumption commodities might not be balanced. In the model, all markets are spatially global, but labor markets can differ due to labor heterogeneity. Variable transport cost can create differences in the price of a consumption good across locations and firms. That is the reason $p(i,s,v)$ is indexed by firm. Later, we shall argue that competition forces p to be constant across firms and hence locations. Observe that *the reason that goods must be transported among firms, and the reason that firms must pay a fixed and variable transportation cost, is that consumers use the firms as marketplaces and, due to the boundary condition on preferences, want some of all of the goods.*

More formally, let $\tau \in \Re^s$ denote the vector of marginal physical requirements for setting up *half* of a transportation network between two firms, so the cost per unit of distance to firm (i,s,v) is $p(i,s,v) \cdot \tau$. This is a fixed cost independent of volume. Next we specify variable costs.

Let $g_{(j,s',v')}^{(i,s,v)}$ represent the vector of net shipments of consumption goods between two firms and let T be a non-negative $S \times S$ matrix. Row s of the matrix T gives the cost in terms of the S consumption goods (per unit of distance) of shipping one unit of consumption good s to or from a firm. A diagonal non-negative matrix with entries less than 1 would be a Samuelsonian 'iceberg' transport technology.

Our assumption about the cost sharing rule means that this rule is essentially equal division of the network transportation cost for connecting each firm to another. However, our results can be extended to arbitrary monotonic and lower semi-continuous functions of this fixed cost. Relocation costs for both firms and consumers are zero.

We could employ a more general transportation technology as in Berliant and Konishi (2000) which would specify a set of input requirements, each element of which can produce one unit of transportation services. For simplicity,

[6] An alternative is to model marketplace structure as a public good for consumers; see Berliant and Konishi (2000).

we use the special case given above. The use of a more general technology will not affect the main results, as we shall explain in section 3 below. Let us now define two important concepts.

Definition 1 *Firm (i,s,v) and firm (j,s',v') are* ***adjacent*** *if*

$$\inf_{\substack{x\in C_s^{i,v}\\ y\in C_{s'}^{j,v'}}} \|x-y\| = 0.$$

Definition 2 *Firm (i,s,v) and firm (j,s',v') are* ***connected*** *if there is a list of firms such that firm i is the first on the list and firm j is the last on the list and each firm is adjacent to its predecessor and successor on the list.*

We assume no transport cost between adjacent firms. This implies no transport costs between connected firms, because otherwise firms would simply ship to adjacent firms, and those firms would re-ship to the next firm. Thus, *zero transport costs between adjacent firms actually implies zero transport costs among connected firms.* Hence, we state the total cost for firm (i,s,v) of a transportation network between firms as follows:

$$\sum_{s'=1}^{S}\sum_{v'=1}^{V}\sum_{j=1}^{M_{s'}^{v'}} p(i,s,v)\cdot\left[\tau+T\cdot g_{(j,s',v')}^{(i,s,v)}\right] \inf_{\substack{x\in C_s^{i,v}\\ y\in C_{s'}^{j,v'}}} \|x-y\|\cdot \mathbf{I}_{X_{(j,s',v')}^{(i,s,v)}}$$

where $\mathbf{I}_X$ is the indicator function of the event X and $X_{(j,s',v')}^{(i,s,v)} = \{\text{Firm } (i,s,v)$ is not connected to firm $(j,s',v')\}$. Notice that we separate the last two parts of this expression, since firms that are connected are not necessarily adjacent. Observe that even when firms are connected, commuting distance and thus commuting cost of a worker to various firms is not the same.

Let $\overline{\Psi}$ be the constant (over location) unit price of land to firms. The idea that this must be constant will be explained after the statement of the firm's optimization problem. Firm (i,s,v) maximizes its profit π_s^v (firms using the same input v and output s are identical) by solving the following problem, taking as given the locations of other firms. To give meaning to other firms' choices, let $C_s^{-(i,s,v)} = \left[C_1^{1,1},...,C_s^{i-1,v},C_s^{i+1,v},...,C_S^{M_S^V,V}\right]$. $C_s^{i,v}\in\mathbf{C}$ represents the choice of parcel. For now, we take good shipments $g_{(j,s',v')}^{(i,s,v)}$ to be exogenous, but shortly we shall see that T can be taken to be zero without loss of generality, so $g_{(j,s',v')}^{(i,s,v)}$ is immaterial.

$$\max_{L_s^{i,v},C_s^{i,v}} \pi_s^v(L_s^{i,v},C_s^{i,v},C_s^{-(i,s,v)}) \tag{1}$$

where

$$\pi_s^v(L_s^{i,v}, C_s^{i,v}, C_s^{-(i,s,v)}) = p_s(i,s,v)\, f_s^v(L_s^{i,v}, \mu(C_s^{i,v})) - w^v L_s^{i,v} - \mu(C_s^{i,v})\overline{\Psi}$$
$$-\sum_{s'=1}^{S}\sum_{v'=1}^{V}\sum_{j=1}^{M_{s'}^{v'}} p(i,s,v)\cdot[\tau + T\cdot g_{(j,s',v')}^{(i,s,v)}] \inf_{\substack{x\in C_s^{i,v} \\ y\in C_{s'}^{j,v'}}} \|x-y\| \cdot \mathbf{I}_{X_{(j,s',v')}^{(i,s,v)}}.$$

Notice that a firm does not think that changing its location will affect its unit price of land $\overline{\Psi}$ or any other price.

Observe that the profit function depends on the location of firm (i,s,v) and *the locations of all other firms* through the fourth term of the RHS of (1). This is the Nash assumption about equilibrium. When a firm moves, it takes as given the locations of other firms. Here, in terms of location, *consumers are behaving competitively whereas firm behavior is Nash.* The concept of equilibrium we have here is a sort of a combination of the ones introduced by Fujita and Thisse (1986) and Anderson and Engers (1994). Indeed, in the first paper a land market is considered in the Hotelling model and the authors introduce the possibility of workers' relocation in reaction to firms' location decisions. Thus, consumers are the followers in a Stackelberg game where firms are the leaders. In the second paper, firms are price-taking but have Nash behavior in location; consumer locations are fixed. It is important to highlight here that *our concept of equilibrium is Nash in location but firms take all prices (good prices as well as wages) as given.* In other words, each firm moves, taking prices and the location of other firms as given.

Observe moreover that, when a firm changes its location and locates farther away from other firms, this firm might think that its variable transportation cost will go up but, since it is competitive (in the product market), it thinks its output price won't change. Therefore, even with the assumption of positive variable transportation costs, good prices are not location dependent.

This implies location-independent (good and labor) prices. Thus, we have a constant wage gradient, which is just a special case of allowing wages to vary across locations. If wages are allowed to vary across firm locations but agents take the wage gradient as parametric, we would simply have more equilibria. Furthermore, if wages vary with residential distance from a firm, this is a form of price discrimination and seems to be incompatible with perfect competition. In this case, it is unclear what firms think wages would be if they move. Moreover, it seems unrealistic to assume that firms wage-discriminate based on the residence locations of worker/consumers.

There are several alternative equilibrium concepts that could be used with this model. We use the simplest, namely perfect competition, taking prices and the actions of other agents as given, with one small deviation. We assume that the price of land faced by firms is constant across locations. An implication is that it is independent of consumer bid-rent or the unit price of land paid by consumers. Another implication is that, *in equilibrium*, the price of land faced by firms will be at least as high as the maximal bid-rent of consumers.

There are two reasons we make this assumption. First, the firms know that if they relocate, consumers will follow and bid up the price of land nearby. Although it would be best to model this process explicitly, the resulting model would be very complex and difficult to analyze. In particular, non-convexities in reaction correspondences of firms would make the analysis attempted here impossible. Instead, we model this process implicitly through a constant price of land for firms.

Second, it is clear that at any possible equilibrium allocation, the bid-rent curves of consumers are highest near a firm and decline with distance from a firm. If firms pay the price given by the highest bid rent for parcels and have location-independent production functions, they will always want to locate where rent is lowest, namely as far away from their current location as possible. Such behavior is unlikely to lead to an equilibrium.

Perhaps the most natural alternative equilibrium concept is as follows. There are two stages, and the solution concept is pure strategy subgame perfect Nash equilibrium. The second stage is an equilibrium under perfect competition given fixed firm parcels. The first stage is a simultaneous move Nash equilibrium with firms choosing the parcels that they will use in production and will rent in the second stage. The firms know perfectly how their choice of parcel in the first stage will affect equilibrium in the second stage, including their land rent and the value of their output.

Although this equilibrium concept has intuitive appeal, there are three serious technical problems associated with it. First, the second stage competitive equilibrium will not be unique in general. This presents problems for both results concerning existence of equilibrium and determinacy of equilibrium. Of course, multiple equilibria in the second stage can cause non-convexities in (or an ill-defined) first stage reaction correspondence. Possible remedies include using mixed strategies in the first stage, refining second stage equilibria so that they are unique, or imposing conditions on the economy so that second stage equilibria are unique. Any of these solutions will either complicate matters or limit the robustness of results substantially.

The second technical problem is that firms might not be connected in an equilibrium. Prices of mobile goods in the second stage can differ across locations, rendering the equilibrium concept very complex. Non-convexities in first stage firm reaction correspondences can result. Moreover, the arguments we use for existence of an equilibrium and determinacy properties would no longer work, since they take as given the locations of firms.

The third technical problem with the alternative equilibrium concept is that if firms in the first stage choose a disequilibrium configuration, for example if two firms decide to use the same parcel of land, then payoffs are not well-defined. The reason is that given such firm location choices, the second stage configuration of firms is not feasible, so there is no competitive equilibrium associated with it.

For these reasons, we use the first equilibrium concept, employing (virtually) perfect competition. This can be seen as a special case of the alternative equilibrium concept, where the transport cost faced by firms is much larger than the commuting cost faced by consumers, so that firms will always choose to form a connected set in equilibrium.

Let us turn to the consumer's problem. A consumer of type v performs the following optimization program taking as given prices, $p = [p_1, ..., p_S]$, the wage rate, w^v, the integrable land rent, $\Psi : Z \rightarrow \Re_+$, and the location of all firms where $C_s^v = [C_s^{1,v}, ..., C_s^{M_s^v,v}]$, $C_s = [C_s^1, ..., C_s^V]$, $C^v = [C_1^v, ..., C_S^v]$ and $C = [C_1, ..., C_S]$.

$$\max_{X^v, q^v, z} U^v(q^v, X^v) \tag{2}$$

$$s.t. \qquad p \cdot X^v + q^v \Psi(z) + p \cdot t \inf_{\substack{y \in C_s^{i,v} \\ i=1,...,M_s^v \\ s=1,...,S}} \|z - y\| = w^v$$

Observe that there is only one wage for each labor type since consumers are mobile and one unit of labor is supplied inelastically by each consumer. Define $w = [w^1, ..., w^V]$.

Let us denote by $R_s^{i,v} = \{z \in Z \mid i, s \in \arg\min_{\substack{i'=1,...,M_s^v \\ s'=1,...,S}} (\inf_{y \in C_{s'}^{i',v}} \|z - y\|)\}$, the set of locations from which firm (i, s, v) draws its labor.

Let's now consider the absentee landlord's problem. Let $X^L = [X_1^L, ..., X_S^L]$, $\pi^v = [\pi_1^v, ..., \pi_S^v]$ and $\pi = [\pi^1, ..., \pi^V]$. So, the absentee landlord utility function is $U^L(X^L)$, and we assume that it has the same standard properties as the consumer utility function. The absentee landlord program is therefore:

$$\max_{X^L} U^L(X^L) \quad s.t. \quad p \cdot X^L = \int_Z \Psi(z)dz + \sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v \pi_s^v \tag{3}$$

where $X^L(\int_Z \Psi(r)dr, \pi)$ is the 'argmax' of (3).

The following are standard definitions.

If D is a Lebesgue measurable subset of Z, then $\mathbf{1}_D$ is the indicator function of the set D, namely $\mathbf{1}_D(x) = 1$ if $x \in D$ and $\mathbf{1}_D(x) = 0$ otherwise. Let $n^v(z)$ be the density of population of type v at location z, with $n = [n^1, ..., n^V]$. For the matrix T and the vector g, let $T \cdot g = [(T \cdot g)_1, ..., (T \cdot g)_S]$.

Definition 3 *A **feasible allocation** is a vector* $\left(C, L, X(\cdot), q(\cdot), Y, X^L, n(\cdot)\right)$ *such that* $\forall i, s, v$, $C_s^{i,v} \in \mathbf{C}$, $L_s^{i,v} \geq 0$, $Y_s^{i,v} \geq 0$; $X^L \in \Re_+^S$; *and* $X : Z \rightarrow \Re_+^S$, $q : Z \rightarrow \Re_+$ *and* $n : Z \rightarrow \Re_+^V$ *are measurable functions, such that:*

$$\sum_{s=1}^{S}\sum_{v=1}^{V}\sum_{i=1}^{M_s^v} \mathbf{1}_{C_s^{i,v}}(z) + \sum_{v=1}^{V} n^v(z)q^v(z) = 1 \text{ almost surely for } z \in Z \tag{4}$$

$$\int_Z n^v(z)dz = N^v \qquad v = 1, ..., V \tag{5}$$

$$L_s^{i,v} = \int_{R_s^{i,v}} n^v(z)dz \qquad s = 1, ..., S, \ v = 1, ..., V, \ i = 1, ..., M_s^v \tag{6}$$

$$Y_s^{i,v} = f_s^v(L_s^{i,v}, \mu(C_s^{i,v})) \qquad s = 1, ..., S, \ v = 1, ..., V, \ i = 1, ..., M_s^v \tag{7}$$

$$\sum_{s=1}^{S}\sum_{s'=1}^{S}\sum_{v=1}^{V}\sum_{v'=1}^{V}\sum_{i=1}^{M_s^v}\sum_{j=1}^{M_{s'}^{v'}} [\tau_s + (T \cdot g)_s] \inf_{\substack{x \in C_s^{i,v} \\ y \in C_{s'}^{j,v'}}} \|x - y\| \cdot \mathbf{I}_{X_{(j,s',v')}^{(i,s,v)}} \tag{8}$$

$$+\int_Z X_s^v(z)n^v(z)dz + X_s^L + \sum_{v=1}^{V}\int_Z n^v(z)t_s \left(\inf_{\substack{y \in C_s^{i,v} \\ i=1,...,M_s^v \\ s=1,...,S}} \|z - y\|\right)dz = \sum_{v=1}^{V}\sum_{i=1}^{M_s^v} Y_s^{i,v} \qquad s = 1, ..., S$$

Equation (4) is the material balance condition for land. Notice that one unit of land is available at each point in Z and all land is used. This is due to the fact that we have assumed that both preferences and production functions are monotonic, which implies that land is productive. In one dimension, this is a linear city. Our model can easily be generalized to an arbitrary supply density for land. Equation (5) is the population balance condition. Equation

(6) is the material balance condition for local labor markets.[7] Equation (7) is production feasibility and equation (8) is material balance for the product markets. At a feasible allocation, firm (i, s, v) is (almost surely) the only employer of workers in $R_s^{i,v}$.

Definition 4 *An **equilibrium** is a vector*

$$(p^*, \overline{\Psi}^*, \Psi^*(\cdot), w^*, C^*, L^*, X^*(\cdot), q^*(\cdot), Y^*, X^{L^*}, n^*(\cdot))$$

such that $\left(C^*, L^*, X^*(\cdot), q^*(\cdot), Y^*, X^{L^*}, n^*(\cdot)\right)$ *is a feasible allocation,* $p^* \in \Re_+^S$, $\overline{\Psi}^* \in \Re_+$, $\Psi^* : Z \to \Re_+$ *is measurable,* $w^* \in \Re_+^V$, $\Psi^*(z) = \overline{\Psi}^*$ *almost surely for* $z \in \bigcup_{s=1}^{S} \bigcup_{v=1}^{V} \bigcup_{i=1}^{M_s^v} C_s^{i,v*}$, *and such that, for each* s, *for each* v, *for each* i, L_s^{i,v^*} *and* C_s^{i,v^*} *solve (1), and such that, for each* v, *and almost every* $z \in Z$, $X^{v^*}(z)$, $q^{v^*}(z)$ *solve (2), and finally such that* X^{L^*} *solves (3).*

Definition 5 *An **equal treatment Pareto optimum** is a feasible allocation*

$$\left(C, L, X(\cdot), q(\cdot), Y, X^L, n(\cdot)\right)$$

such that for each v, *for each* z *and* z' *with* $n^v(z) > 0$ *and* $n^v(z') > 0$, $u^v = U^v(q^v(z), X^v(z)) = U^v(q^v(z'), X^v(z'))$, *and such that there is no feasible allocation* $\left(C', L', X'(\cdot), q'(\cdot), Y', X^{L'}, n'(\cdot)\right)$ *such that* $U^L(X^{L'}) \geq U^L(X^L)$ *and for all* v *and all* z *with* $n^{v'}(z) > 0$, $U^v(q^{v'}(z), X^{v'}(z)) \geq u^v$, *with strict inequality holding for a set of consumers of positive measure or for the absentee landlord.*

3 What does equilibrium look like?

Example 1, found in the Appendix, motivates our first result and gives us good intuition about equilibrium in the case of two firms and two types of workers. We state the general result.[8]

[7] In general, it is possible that there are i, i', s, s', v such that $\mu(R_s^{i,v} \cap R_{s'}^{i',v}) > 0$, namely that two firms using the same type of labor draw it from the same locations, and these locations have non-negligible overlap. If this were to happen, there would be a serious problem in the part of the definition of a feasible allocation related to labor, necessitating a more elaborate definition. However, we note that the definition of a feasible allocation is not used in Theorem 1, while all remaining results in the paper employ the special case where firms use hyper-rectangles. In that special case, the set of workers with no unique closest employer is of measure zero.

[8] All proofs are given in the Appendix. This Theorem is anything but trivial for the case $K > 1$.

Theorem 1 *Fix any equilibrium land allocation for firms, C. Then it has the property that all firms using positive Lebesgue measure of land are connected (in the sense of Definition 2).*[9] *Thus, without loss of generality, we can take equilibrium inter-firm transportation costs to be zero.*

The following comments are in order. First, in one dimension ($K = 1$, a linear city), the city is monocentric since the firms are connected. However, contrary to the standard urban economics literature (Fujita (1989)), the Central Business District (CBD) is not a point but a set of firms interacting with each other. It is in this sense the same kind of result as that obtained by Fujita and Ogawa (1982). In two dimensions, ($K = 2$, a circular city for example), the city can be monocentric or the firms can form any connected set, including an annulus or rectangle. We can have, for example, a circular city where all firms occupy all the locations on the fringe whereas the consumers reside inside the ring.

Second, *since a firm takes everything as given except for transport costs (variable or fixed, or both), it will seek to minimize them.* Thus Theorem 1 will work even with positive variable transportation costs, provided that these costs are zero when two firms are connected. Thus variable costs can be taken to be zero without loss of generality, since firms will be connected in equilibrium in any case. Now consider the case where variable transportation costs are non-zero even when firms are connected. Positive variable costs make no difference, in equilibrium, where they are absorbed by the producer. In this case, they simply are drawn from profits, and can only have the effect of reinforcing the agglomeration of firms. An alternative assumption is that they are passed on to consumers in the form of higher prices for consumption goods. In this case, variable transport costs are irrelevant to a firm's decision problem, since the firm does not pay them. Once again, Theorem 1 applies and firms are connected in equilibrium due to the fixed set-up cost of the transport network. Of course, intermediate cases, where variable transport costs are partially passed through to consumers via higher prices and partially absorbed by producers, are possible. Theorem 1 is robust to these variations since variable transport costs only serve to reinforce the agglomeration of firms. Variable costs will only drive firms together or, if costs are passed to consumers, be neutral.

Third, the assumption of zero transport costs between connected firms is not necessary to obtain Theorem 1. We simply must assume that total

[9] Any firm using zero Lebesgue measure of land is shut down.

transport costs for a firm are minimized when the firm is connected to every other. For example, we could add a constant to the transport costs between firms, independent of location. If two firms are connected, then the transport cost is just this constant. If the two firms are not connected, then it is the constant plus the cost defined in section 2. This obviously leads to no alteration in the result and proof of Theorem 1. It will also be obvious that results in sections 4, 5 and 6 will still be valid, provided that the constant does not exceed a firm's profits.

More generally, the result given in Theorem 1 holds when the transportation cost function for a firm is any monotonic increasing function of the pairwise (closest point) distances between the firm in question and every other firm, except that it is assumed that the minimum of the function is attained when all firms are connected. In other words, transportation cost is a function of the entire configuration (specifically, when all firms are connected), not just pairwise distances. This function is lower semi-continuous, but it might not be continuous. The addition of a constant to the cost function given in section 2 is a special case of this, as specified just above. The result can be extended even further. For example, consider the case when the location space is 2 dimensional. Assume that the transportation cost of a firm is equal to a monotonic function of pairwise distances between it and every other firm except in the case when the configuration of firm locations (i.e. the land they use) is convex; in the latter case we assume that the function reaches its minimum (as in the case of connected firms above). Then the same proof gives us that in any equilibrium (if it exists), the configuration of firms is convex. Of course the question of existence is then more difficult, since fewer configurations of firms are possible in equilibrium, so (as we shall see) labor must be more differentiated to give enough freedom in endogenous variables to prove that such an equilibrium exists. What we are saying, in essence, is that the equilibrium configuration is driven by assumptions about the transport costs faced by firms. The assumptions, in turn, are informed by our understanding of real world cost functions as well as common sense, but variations are certainly compatible with the model and the techniques. Therefore, we can force any configuration of firms we choose as cost-minimizing to be a necessary condition for an equilibrium using the ideas above.

Last, at an equilibrium in which all firms are connected, the land price of any firm inside the interval (in one dimension) is constant and at least as large as the bid rents at the boundaries of the two extreme firms. In equilibrium, it is indeed not possible that any firm has a lower land price since the one next to it will be induced to move into the land area of the firm in question if the land price is lower. To see this, consider an 'equilibrium' where land prices are not constant across connected firms. If land price is decreasing at the boundary of the land used by a particular firm, and if this firm expands its land use in the direction where the price is decreasing, the firm is not maximizing its profit since the marginal revenue product of land exceeds its cost. Although the first order conditions for profit maximization of this particular firm might be satisfied at the 'equilibrium' allocation, a marginal expansion of the firm land usage will yield greater profit since the second order conditions for profit maximization will not be satisfied when land price decreases. Therefore, equilibrium land price is constant locally around this boundary of the firm's land parcel. Moreover, our definition of equilibrium requires that the price of land is constant within a firm's parcel. In fact, if it relocates it thinks the land price is constant and the same as at its current location.

4 Determinacy of equilibrium

Examples 2 and 3, contained in the Appendix, illustrate the determinacy properties of equilibrium. We now study the general case by examining the dimension of the set of equilibria (the equilibrium manifold) for a fixed set of exogenous parameters. First, let us define the bid rent functions.

The bid-rent approach is a well-known technique in urban economics using duality theory to find the maximal willingness to pay for land at each location (see Fujita (1989)). The consumer with maximal willingness to pay at a location will live there in equilibrium. Our innovations are to combine it with differential topology methods for use in determinacy analysis, and to combine it with fixed point methods for use in proving existence of equilibrium. Moreover, we extend bid-rent from a one dimensional to a multi-dimensional tool.

Let $Z = [0,1]^K$. Let firm (i,s,v) use the hyper-rectangle $C_s^{i,v} \equiv [\omega^h, \omega^{h+1}] \times [0,1]^{K-1}$ where $h = \sum_{s'<s} \sum_{v'<v} M_{s'}^{v'} + i$. In this way, firms are always connected. Then $Q_s^{i,v} \equiv \omega^{h+1} - \omega^h$, and let $\overline{\omega} = (\omega^{[\sum_{s=1}^{S} \sum_{v=1}^{V} M_s^v]+1} + \omega^1)/2$ be the midpoint of firm land use.

For this subsection only, we must alter our commuting cost function. The reason is that, as specified, it is not C^1 at zero distance. Thus, for this subsection, we will take it to be a general C^1 function $t : \Re_+ \to \Re^s_+$ that is non-decreasing. For example, $t(\cdot)$ could be quadratic.

Definition 6 *The **bid rent** function for a type v worker at a given location z is defined to be:*

$$\begin{aligned} & \Xi^v(z, p, w^v, C^v, u^v) \qquad (9) \\ = & \max_{X^v, q^v} \{[w^v - p \cdot X^v - p \cdot t(\inf_{\substack{y \in C^{i,v}_s \\ i=1,\ldots,M^v_s \\ s=1,\ldots,S}} \|z - y\|)]/q^v \mid U^v(q^v, X^v) = u^v\} \end{aligned}$$

Define $X^v(z, p, w^v, C^v, u^v)$ and $q^v(z, p, w^v, C^v, u^v)$ to be the 'argmax' of (9). Of course, there might not be a solution to (9).[10] This can happen in several different ways, generally at boundaries. For example, u^v might specify a level of utility above the supremum utility in the range of the utility function. In that case, one would want to set $\Xi^v = 0$, $q^v = \infty$ and $X^v = \infty$. Another possibility is a negative value for Ξ^v, which does not cause problems and is thus permissible. If the price of some consumption goods are zero, then one would want to set consumption of those goods to infinity, consumption of all other goods (including land) to zero, and bid-rent at infinity. Finally, if utility were set at its infimum, then one would want to define all consumption levels to be zero and bid-rent to be infinite.

These boundary problems have no effect on determinacy analysis, which is next, but can affect the proof of existence of equilibrium. The reason there is no impact on determinacy analysis is that we will restrict our analysis to interior (non-zero) prices and utility levels. We only prove that the system is smooth on this domain; equilibria must reside in this region, so determinacy analysis can proceed where variables do not hit boundaries. Obviously, to prove that an equilibrium exists, behavior of the system on boundaries matters. However, rather than worry about defining bid rent at all of these boundaries, it is easier simply to define excess demand correspondences directly at these boundaries, and be sure that they are upper-hemicontinuous. We shall do this in the proof of Theorem 4.

Since firms are connected in the spatial arrangement we have postulated, transportation (but not commuting) cost can be neglected. This simplifies matters substantially. The optimization problem (1) of firm (i, s, v) becomes:

$$\max_{L^{i,v}_s, Q^{i,v}_s} \pi^v_s = p_s\, f_s(L^{i,v}_s, Q^{i,v}_s) - w^v L^{i,v}_s - Q^{i,v}_s \overline{\Psi} \qquad (10)$$

[10]This problem is usually not addressed by models using the bid-rent approach, but should be. We are grateful to Guy Laroque for pointing this out.

where $\overline{\Psi}$ will be the (uniform) price of land for firms. Define $\pi_s^v(p_s, w^v, \overline{\Psi})$ to be the maximum and define $L_s^{i,v}(p_s, w^v, \overline{\Psi})$ and $Q_s^{i,v}(p_s, w^v, \overline{\Psi})$ to be the 'argmax' of (10), all of which exist if w^v and $\overline{\Psi}$ are positive. Given our assumptions on production, if a solution to (10) exists, it is unique. Given $Q_s^{i,v}(p_s, w^v, \overline{\Psi})$, define $C_s^{i,v}(p_s, w^v, \overline{\Psi})$ to be the hyper-rectangle given above Definition 6.

Of course, as in the case of bid rent (9), there might not be a solution to (10) when the price of an input is zero. Once again, these boundary problems have no effect on determinacy analysis, which is next, but can affect the proof of existence of equilibrium. Thus, we shall take up this issue in the proof of Theorem 4.

We begin by restating the set of equations that define equilibrium. The goal, as in Berliant and Kung (2006), is to reduce the number of equations and endogenous variables from a continuum to a finite number. That is why we use bid rent. Substituting demands and supplies into the feasibility conditions (4)$-$(8) and using (9) and (10), we obtain:

$$\sum_{s=1}^{S}\sum_{v=1}^{V}\sum_{i=1}^{M_s^v}\mathbf{1}_{C_s^{i,v}(p_s,w^v,\overline{\Psi})}(z) + \sum_{v=1}^{V} n^v(z) q^v(z, p, w^v, C^v, u^v) - 1 = 0 \qquad (11)$$

almost surely for $z \in Z$

$$\int_Z n^v(z)dz - N^v = 0 \qquad v = 1, ..., V \qquad (12)$$

$$L_s^{i,v}(p_s, w^v, \Psi^*) - \int_{R_s^{i,v}} n^v(z)dz = 0 \qquad s = 1, ..., S,\ v = 1, ..., V,\ i = 1, ..., M_s^v \qquad (13)$$

$$Y_s^{i,v} = f_s^v\left(L_s^{i,v}(p_s, w^v, \overline{\Psi}), \mu(C_s^{i,v}(p_s, w^v, \overline{\Psi}))\right) \qquad s = 1, ..., S,\ v = 1, ..., V,\ i = 1, ..., M_s^v \qquad (14)$$

$$\sum_{v=1}^{V}\int_Z X_s^v(z, p, w^v, C^v, u^v) n^v(z)dz + X_s^L(z, p, w^v, C^v, u^v) \qquad (15)$$

$$+\sum_{v=1}^{V}\int_Z n^v(z) t_s \left(\inf_{\substack{y\in C_s^{i,v} \\ i=1,...,M_s^v \\ s=1,...,S}} \|z-y\|\right)dz = \sum_{v=1}^{V}\sum_{i=1}^{M_s^v} Y_s^{i,v} \qquad s = 1, ..., S$$

We assume that for fixed output prices, fixed wages, fixed locations of firms, and fixed utility levels, all the consumer bid rent functions are well-behaved in the sense of Fujita (1989, definition 4.1, p.99) and that each pair of bid rent functions crosses on a set of measure zero. A related assumption is that bid rent functions can be ordered by relative steepness (see assumption 4.3, p.102 in Fujita (1989)). Our definition is different from Fujita's since consumers work in and thus commute to different places while in Fujita they all work in the same place (the CBD). Formally, the assumption is:

$$\forall v \neq v', \quad \forall p, w^v, w^{v'}, C^v, C^{v'}, u^v, u^{v'} \text{ almost surely for } z \in Z \setminus \bigcup_{s=1}^{S} \bigcup_{i=1}^{M_s^v} C_s^{i,v}$$

$$\frac{\partial \Xi^v(z, p, w^v, C^v, u^v)}{\partial z} \neq \frac{\partial \Xi^{v'}(z, p, w^{v'}, C^{v'}, u^{v'})}{\partial z}$$

We use this assumption in order to be sure that the bid-rent functions of consumers do not coincide on an open set of locations. For if they were to coincide, the bid rent approach would not lead to a unique distribution of population. In this case, we would have to deal with aggregate demand correspondences instead of functions and both determinacy and equilibrium analysis would be much more complicated, but the results would likely be similar. Our assumption would follow, for instance, if consumers have Cobb-Douglas utilities where different types of workers have different parameters attached to land consumption. We conjecture that our assumption is generic in utilities, though that idea is far removed from the point of this work.

We also want land consumption for workers and firms, composite good consumption and firm labor demand to be well-defined as functions of profit levels, utility levels, wages and prices. That is the next step, and we will obtain this result in Lemma 1 below.

We reformulate the equilibrium conditions using a bid rent approach (see e.g., Fujita and Ogawa (1982) or Fujita (1989)).

Let $\alpha : Z \times \Re_+^S \times \Re_+^V \times \mathbf{C}^{\sum_{s=1}^{S} \sum_{v=1}^{V} M_s^v} \times \Re^V \rightarrow \Re$ be an arbitrary smooth function (we write $\alpha(z, p, w, C, u)$, where $u = [u^1, ..., u^V]$ are utility levels) so that for all (p, w, C, u), $\partial\alpha(z, p, w, C, u)/\partial z_1 < 0 \forall\ z_1 < \overline{\omega}, \partial\alpha(z, p, w, C, u)/\partial z_1 > 0 \forall z_1 > \overline{\omega}$, and $\inf_{z \in Z \setminus \bigcup_{s=1}^{S} \bigcup_{v=1}^{V} \bigcup_{i=1}^{M_s^v} C_s^{i,v}} \alpha(z, p, w, C, u) \geq \sup_{z \in \bigcup_{s=1}^{S} \bigcup_{v=1}^{V} \bigcup_{i=1}^{M_s^v} C_s^{i,v}} \max_{v=1,...,V} [\Xi^v(z, p, w^v, C^v, u^v)]$.

The purpose of the function α is to stand in for the maximal bid-rent function, since the latter function is not smooth.[11] The restriction on the slope of α will ensure that when the price of land paid by the firms, $\overline{\Psi}$, is too low, there is excess demand for land. If it is too high, there is excess supply. We let $\gamma \geq 1$ be a variable that will scale α. Once we have some function α satisfying

[11] An example is $\alpha(z, p, w, C, u) = (z_1 - \overline{\omega})^2 + \beta(p, w, C, u)$ where β is smooth and $\beta(p, w, C, u) \geq \sup_{z \in \bigcup_{s=1}^{S} \bigcup_{v=1}^{V} \bigcup_{i=1}^{M_s^v} C_s^{i,v}} \max_{v=1,...,V} [\Xi^v(z, p, w^v, C^v, u^v)]$ pointwise on its domain.

the requirements just above, $\gamma \cdot \alpha$ will also satisfy these requirements, leading to another degree of indeterminacy.

Let

$$\begin{aligned}
Z^v &= \{z \in Z \setminus \bigcup_{s=1}^{S} \bigcup_{v=1}^{V} \bigcup_{i=1}^{M_s^v} C_s^{i,v} \mid \gamma \cdot \alpha(z,p,w,C,u) \geq \overline{\Psi} \\
\text{and } 0 &\leq \Xi^v(z,p,w^v,C^v,u^v) = \max_{v=1,\ldots,V} [\Xi^v(z,p,w^v,C^v,u^v)]\} \\
\cup\{z &\in \bigcup_{s=1}^{S} \bigcup_{v=1}^{V} \bigcup_{i=1}^{M_s^v} C_s^{i,v} \mid \gamma \cdot \alpha(z,p,w,C,u) > \overline{\Psi} \\
\text{and } 0 &\leq \Xi^v(z,p,w^v,C^v,u^v) = \max_{v=1,\ldots,V} [\Xi^v(z,p,w^v,C^v,u^v)]\}
\end{aligned}$$

Since we shall employ Walras' law in the context of the bid-rent approach, we must define (out of equilibrium) rent collections:

$$R(p,w,C,u,\overline{\Psi}) \equiv \sum_{v=1}^{V} \int_{Z^v} \Xi^v(z,p,w^v,C^v,u^v)dz + \overline{\Psi} \cdot \mu(\bigcup_{s=1}^{S} \bigcup_{v=1}^{V} \bigcup_{i=1}^{M_s^v} C_s^{i,v})$$

This is consistent with (3) in equilibrium.

The remaining equilibrium equations are

$$\int_{Z^v} \frac{1}{q^v(z,p,w^v,C^v(p_s,w^v,\overline{\Psi}),u^v)} dz - N^v = 0 \qquad v = 1,...,V \tag{16}$$

$$L_s^{i,v}(p_s,w^v,\overline{\Psi}) - \int_{R_s^{i,v} \cap Z^v} \frac{1}{q^v(z,p,w^v,C^v(p_s,w^v,\overline{\Psi}),u^v)} dz = 0 \tag{17}$$

$$s = 1,...,S, \ v = 1,...,V, \ i = 1,...,M_s^v$$

$$\begin{aligned}
&\sum_{v=1}^{V} \int_{Z} \frac{X_s^v(z,p,w^v,C^v(p_s,w^v,\overline{\Psi}),u^v)}{q^v(z,p,w^v,C^v(p_s,w^v,\overline{\Psi}),u^v)} dz + X_s^L(R,\pi) \qquad (18) \\
&+\sum_{v=1}^{V} \int_{Z} \frac{1}{q^v(z,p,w^v,C^v(p_s,w^v,\overline{\Psi}),u^v)} t_s \Big(\inf_{\substack{y \in C_s^{i,v} \\ i=1,\ldots,M_s^v \\ s=1,\ldots,S}} \|z-y\|\Big) dz \\
&-\sum_{v=1}^{V} \sum_{i=1}^{M_s^v} f_s^v \left(L_s^{i,v}(p_s,w^v,\overline{\Psi}), \mu(C_s^{i,v}(p_s,w^v,\overline{\Psi}))\right) = 0 \qquad s = 1,...,S
\end{aligned}$$

$$\mu\left(\left\{z \in \bigcup_{s=1}^{S} \bigcup_{v=1}^{V} \bigcup_{i=1}^{M_s^v} C_s^{i,v}(p_s,w^v,\overline{\Psi}) \mid \gamma \cdot \alpha(z,p,w,C,u) > \overline{\Psi}\right\}\right) \tag{19}$$

$$-\mu\left(\left\{z \in Z \backslash [\bigcup_{s=1}^{S} \bigcup_{v=1}^{V} \bigcup_{i=1}^{M_s^v} C_s^{i,v}(p_s, w^v, \overline{\Psi})] \mid \gamma \cdot \alpha(z, p, w, C, u) < \overline{\Psi})\right\}\right) = 0$$

where equation (19) is the market clearing condition for land under the bid rent approach.

We need to show that the LHS of equations (16)−(19) are continuous functions for the proof of existence and smooth functions for determinacy analysis. Define

$$\Omega^\circ = \left\{(p, w, \overline{\Psi}) \mid \sum_{s=1}^{S} p_s + \sum_{v=1}^{V} w^v + \overline{\Psi} = 1, p_s > 0, w^v > 0, \overline{\Psi} > 0, s = 1, ..., S, v = 1, ..., V\right\}$$

and $u^\circ = \{(u^1, ..., u^V) \in \Re_+^V \mid u^v > 0 \text{ for } v = 1, .., V\}$.

We recapitulate our key assumptions here for completeness.

Lemma 1 *Suppose that $r \geq 1$ and for each v, U^v is C^{r+1} differentiably strictly convex (and thus has no critical point). Suppose that for each s and v, f_s^v is C^{r+1} and that $D^2 f_s^v$ is negative definite (thus f_s^v is strictly concave). Suppose that for each (z, p, w^v, C^v, u^v) and $w^{v'}, C^{v'}, u^{v'}$, $\frac{\partial \Xi^v(z,p,w^v,C^v,u^v)}{\partial z} \neq \frac{\partial \Xi^v(z,p,w^{v'},C^{v'},u^{v'})}{\partial z}$. Suppose that commuting cost as a function of distance to a firm, $t : \Re_+ \to \Re_+^s$, is C^r with $\frac{\partial t_s}{\partial r} \geq 0$ for all s. Finally, restrict attention to those allocations where firms using each type of labor v are connected. Then the equation system (16)−(19) is C^r on domain $\Omega^\circ \times u^\circ$.*[12]

To examine determinacy of equilibrium we use the implicit function theorem (see Mas-Colell (1985, theorem C.3.2, p.20)). If zero is a regular value of the set of functions defined by the LHS of equations (16)−(19), then by the implicit function theorem the set of equilibria forms a manifold of dimension equal to the number of unknowns minus the number of equations. Any regular parameterization (see Mas-Colell (1985, definition 5.8.12, p.226)) of the economies defined by the LHS of equations (16)−(19) will imply that zero is a regular value generically in the parameters; see Mas-Colell (1985, proposition 8.3.1, p.320). *An example of a regular parameterization of our system can be found in our Appendix.*

This is the point at which the results for models with location, such as this one, begin to diverge from those more standard models without location. First,

[12] For the usual reasons, equilibria will never lie on the boundary of $\Omega^\circ \times u^\circ$, so we do not need to examine behavior of our system on the boundary of $\Omega^\circ \times u^\circ$ for the equilibrium determinacy analysis that follows.

notice that there will be a different equilibrium manifold for each order of firms. In essence, the set of all equilibria will be the union of these manifolds. Second, there will be a distinction between the dimension of the equilibrium manifold under symmetry of the configuration of firms as opposed to asymmetry. This latter subject requires a definition.

Definition 7 *If for all $v = 1, ..., V$ and $s = 1, ..., S$, except for one (v', s'), $M_s^v = 2$ or $M_s^v = 0$,*[13] *and either $M_{s'}^{v'} = 1$ or $M_{s'}^{v'} = 0$, then we call the production sector* ***symmetrizable****. An* ***order*** *of firms is a list of all firms by type (v, s) in the economy. An order is called* ***symmetric*** *if it reads the same from left to right as from right to left.*

The idea behind this definition is that a production sector is symmetrizable if and only if one can create a symmetric distribution of firms, placing the exceptional firm (v', s') (if it is present) in the middle of the interval and allocating each pair of identical firms on either side of the exceptional firm so as to create a symmetric distribution. When this is possible, determinacy properties are different.

Theorem 2 *Let $K = 1$, $Z = [0, 1]$ and suppose that the production sector* ***is not*** *symmetrizable. Let $r > \min(0, V - \sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v + 2)$.*[14] *Take any C^r regular parameterization of the economies defined by (16)–(19).*[15] *Fix any order of firms (from left to right). Then generically in parameters, the set of equilibria forms a manifold of dimension:*

$$V - \sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v + 2 \tag{20}$$

This theorem is in accordance with Examples 2 and 3, found in the Appendix. Indeed, in Example 3 where $S = 2, V = 2$ and $\sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v = 2$, we obtain two dimensions of indeterminacy, which is consistent with formula (20) (generically). Now consider Example 2 with $S = 2, V = 1$ and two different firms. It is easy to check by using formula (20) that there is one dimension of indeterminacy (generically).

[13] This particular kind of economy follows the biblical story of Noah's ark.

[14] Since $V \leq \sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v$, $r \geq 3$ is sufficient.

[15] For example, use the lemma to obtain such a C^r system.

This means that existence and determinacy of equilibrium are very sensitive to both the degree of labor differentiation and the number of firms. What is the intuition about why the degree of indeterminacy is expressed according to formula (20)? Consider the firms with the same labor input and same output. These firms must have the same profit level in equilibrium. It is the number of such equal profit constraints that affects the formula (20). So *as labor becomes more differentiated, there are fewer such constraints. Hence, the more differentiated the labor, the larger the variety of equilibria.* Another way of interpreting the intuition behind formula (20) is that identical firms must offer the same wages to the same workers and this cannot compensate for locational and hence commuting cost differences between firms.

Theorem 3 *Let $K = 1$, $Z = [0,1]$ and suppose that the production sector* ***is*** *symmetrizable. Fix any symmetric order of firms. Let $r > \min(0, V - [\sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v]/2+1)$ if the number of firms is even, and $r > \min(0, V-[\sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v + 1]/2 + 1)$ if the number of firms is odd. Take any C^r regular parameterization of the economies defined by (16)−(19). Then generically in parameters, the set of equilibria forms a manifold of dimension:*

$$V - [\sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v]/2 + 1 \tag{21}$$

if the number of firms is even and

$$V - [\sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v + 1]/2 + 1 \tag{22}$$

if the number of firms is odd.

What distinguishes symmetrizable production sectors is that we only need to deal with half (possibly plus one) the number of firms. This means that the number of constraints in a symmetric equilibrium, particularly in (17), can be dropped by half (modulo the central firm). The reason the last term in the indeterminacy formula is 1 for this theorem whereas it's 2 for the previous theorem is that under symmetry, $\overline{\omega}$, the midpoint of firm land use, is fixed, whereas under asymmetry, it's not.

In the general case when Z is not necessarily $[0, 1]$, the analysis above applies, but it is very difficult to parameterize the location of firms. When $Z = [0, 1]^K$, formula (20) gives only a lower bound on the dimension of the equilibrium manifold, since firms might not be configured in the linear fashion that we have postulated. Configurations that are not rectangular are also possible. At the end of section 3, we discussed various configurations of firms. In the most general case when Z is an arbitrary compact, convex subset of $\Re^k$, Theorem 1 tells us that firms will be connected but the configuration can vary in a multitude of ways. Hence, it is difficult to draw general conclusions about the determinacy properties of equilibrium when location is multidimensional, except for noting that the general expression for the dimension of the equilibrium manifold becomes larger as labor differentiation increases.

5 Existence of equilibrium

Putting aside the problem of multiplicity of equilibria, we next provide sufficient conditions for the set of equilibria to be non-empty. In the Appendix we provide Example 4 where formula (22) tells us that equilibria should be locally unique (as the dimension of the equilibrium manifold is equal to zero), but the set of equilibria is in fact empty. This might seem paradoxical given formula (22), but the resolution, of course, is that the empty set is a manifold of any dimension.

Theorem 4 *Fix K, a positive integer. Let $Z = [0,1]^K$. Suppose that labor is completely differentiated, that is $V = \sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v$. Suppose that the assumptions of Lemma 1 hold with $r = 1$. Then there exists an equilibrium.*

We have an analogous result when the production sector is symmetrizable, since in that case only half the firms matter.

Theorem 5 *Fix K, a positive integer. Let $Z = [0,1]^K$. Suppose that the production sector is symmetrizable, and that labor is completely differentiated in each half-economy, that is*

$$V = [\sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v]/2 \tag{23}$$

if the number of firms is even and

$$V = [\sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v + 1]/2 \tag{24}$$

if the number of firms is odd. Suppose that the assumptions of Lemma 1 hold with $r = 1$. Then there exists an equilibrium.

In cases not covered by these two Theorems, it is easy to generate counterexamples like the one in the Appendix. If two firms are drawing their labor supply from the same pool of commuters, one of the firms is going to be farther away from the pool and will never be able to hire the labor it demands at any wage. Example 4 shows that the assumptions on labor differentiation in Theorems 4 and 5 are tight. When these conditions are satisfied, Theorems 2 and 3 imply (generically) that there is a continuum of equilibria.

The method of proof can be extended to more general settings. For instance, Z can be more general. In more general settings, one must be careful with land parcels used by firms since they can hit the boundary of Z; that is the reason we use the particular connected spatial configuration of firms in this theorem.

Observe also that the assumption that markets for all goods (including labor) except land are competitive is used to prove Theorem 1. Indeed, our equilibrium concept is such that each firm takes all prices as given. If product and labor markets were not competitive, i.e. firms do not take all prices as given, then firm reaction correspondences would not be convex valued since given prices, a firm's profit could be maximized at two different locations. In this case, the existence of equilibrium could not be proved in the way we have done it.

6 Welfare properties of equilibrium

Theorem 6 *An equilibrium allocation might not be Pareto optimal.*

The proof (found in the Appendix) proceeds by presenting an example where the equilibrium allocation is not Pareto optimal. This is accomplished by taking an equilibrium where the only two firms in the economy are adjacent, pulling them apart a little bit, and putting some consumers in between the firms, thereby reducing commuting costs. Although transportation costs rise, parameters are taken so that this is more than offset by the drop in commuting cost.

Notice that, if a Pareto optimum exists, it will not have adjacent firms (for the reason given in the proof of Theorem 6), and thus will not be an equilibrium allocation. Therefore, *the second welfare theorem will also fail for this model and this example, provided that a Pareto optimum exists for this example.* We do not prove that a Pareto optimum (equal treatment or otherwise) exists for this example since such a proof is both technical and peripheral to our work here.

Notice also that even though firms and consumers are price takers in all markets, we still have a market failure. More precisely, when a firm moves it anticipates the relocation of its workers, but it does not take into account that it affects workers' commuting cost and hence utility levels. In a certain sense, there is an externality that causes a market failure.

7 Conclusion

This article has explained city structure driven by labor differentiation and transportation costs. We have examined the characteristics, welfare properties, existence and determinacy of equilibrium.

It is immediately apparent from section 6 that there is a role for the government in improving consumer welfare relative to an equilibrium allocation, since the two welfare theorems can fail. In the one dimensional example, the government might be able to improve welfare by separating firms and reducing consumer commuting cost. It would be interesting to examine this further in a more general setting.

Theorem 1 provides a testable implication of the model: that firms are connected in equilibrium. Comparative static properties of the model are within reach but messy. One can see that the more differentiated the labor, the larger the variety of equilibria. For example, in the homogeneous labor case for $K = 1$, $V = 1$, $S = 2$, supposing that there are 3 firms producing each of the two products, then there is no equilibrium. When labor is differentiated, $K = 1$, $V = 6$, $S = 2$ so that each firm uses a unique type of labor, then generically there is a continuum of equilibria. Thus, the variety of city types is positively related to the degree of labor differentiation, a second testable implication of the model. The collection of essays in Pereira and Mata (1996) provide interesting data on this subject for Portugal.

Our results provide an illustration of the manifestations of indivisibilities in location models. Due to the discreteness with which firms must be ordered in one dimension, the set of equilibria for fixed values of other exogenous parameters can skip from emptiness to a continuum as labor differentiation is increased.

Our model can be extended in the following way. We have taken as exogenous the general and specific human capital of workers. We can make the choice of human capital endogenous by adding a first stage to the model in which each worker chooses an investment in human capital, with perfect foresight, that will determine their skill and type in the second stage. The first stage can be handled in a standard way, for example as in Rosen (1983). The second stage is the model examined in the present paper where the populations of the various types of workers are endogenous and determined in the first stage. We intend to examine this more elaborate model in the future.

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Labor Differentiation and Agglomeration in General Equilibrium

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Appendix

Not for Publication

8 Appendix

8.1 Examples

Example 1 *Two different firms and two different types of workers: equilibrium with adjacent firms.*

Take $K = 1$, $Z = [0, 1]$. There are two firms; firm 1 produces good 1 using type 1 labor and firm 2 produces good 2 using type 2 labor. Hence, we have: $S = 2$, $V = 2$, $M_1^1 = 1$, $M_2^2 = 1$, $M_1^2 = 0$ and $M_2^1 = 0$. We will simplify the notation and omit some of the subscripts and superscripts. The transportation technologies are given by $[t, 0]$ (for consumers) and $[\tau, 0]$ (for firms), where good 1 is taken to be the numéraire. There is a fixed transport cost but no variable cost. Assume the utility and the production functions are respectively equal to:

$$U^v(q^v, X^v) = (q^v)^{1/3}(X_1^v)^{1/3}(X_2^v)^{1/3} \qquad v = 1, 2$$

$$f_s(L_s, Q_s) = (L_s)^{1/4}(Q_s)^{1/4} \qquad s = 1, 2$$

We omit the superscript since there is one firm of each type and $s = v$. In this example, the profit function of firm 1 reduces to:

$$\pi_1 = L_1^{1/4}(z_1^+ - z_1^-)^{1/4} - w_1 L_1 - \int_{z_1^-}^{z_1^+} \Psi(z)dz - \tau(z_2^- - z_1^+) \tag{25}$$

where $C_v = [z_v^-, z_v^+]$, $v = 1, 2$ and without loss of generality we take firm 1 on the left and firm 2 on the right. Observe that we use here a general formulation of the rent function $\Psi(z)$; we will impose our assumption concerning the firm's view of the price of land shortly, as we wish to explain it in the context of this example.

Since a consumer/worker consumes both goods but can work only in the firm using labor type v to produce good $s = v$, we can study the optimization problem of type v workers and the optimization problem of the firm where they work, denoted by firm v. Thus, the budget constraint for a type v worker working in a firm v reduces to:

$$X_1^v + p_2 X_2^v + q^v \Psi(z) + t\left[\min\{\left|z_v^+ - z\right|, \left|z_v^- - z\right|\}\right] = w^v \qquad v = 1, 2 \tag{26}$$

By combining the first order conditions of the worker's program, we obtain:

$$q^v(z) = \frac{1}{3}\left[\frac{w^v - t\left[\min\{|z_v^+ - z|, |z_v^- - z|\}\right]}{\Psi(z)}\right] \qquad v = 1, 2 \tag{27}$$

Workers of type v are indifferent between locations z_v^+ and z_v^- so if they live at both locations the rents must be the same. We have therefore:

$$\Psi(z_v^+) = \Psi(z_v^-) \qquad v = 1, 2 \tag{28}$$

It is a necessary condition if consumers of type v are to live on both sides of firm $s = v$ when solving (2).

Maximizing π_v with respect to z_v^+, z_v^- and L_v yields:

$$\frac{1}{4} L_v^{1/4} (z_v^+ - z_v^-)^{-3/4} - \Psi(z_v^+) + \tau = 0 \qquad v = 1, 2 \tag{29}$$

$$\frac{1}{4} L_v^{1/4} (z_v^+ - z_v^-)^{-3/4} - \Psi(z_v^-) = 0 \qquad v = 1, 2 \tag{30}$$

$$\frac{1}{4} L_v^{-3/4} (z_v^+ - z_v^-)^{1/4} - w^v = 0 \qquad v = 1, 2 \tag{31}$$

Equation (28) implies that, independent of the location of the firm, in equilibrium land rent at the endpoints are equal to the same constant. We assume that land price is constant in the interval $C_v = [z_v^-, z_v^+]$, $v = 1, 2$. As we will see, this will be consistent with equilibrium. Under this assumption, the firm perceives that, no matter where it locates, and accounting for the movement of consumers (who behave competitively) the unit price of land is always the same. Combining (29) and (30) yields:

$$\Psi(z_v^-) + \tau = \Psi(z_v^+) \qquad v = 1, 2 \tag{32}$$

Actually, (32) is a necessary condition for $C_v = [z_v^-, z_v^+]$ to be profit maximizing, that is, to prevent a deviation by firm v. This condition first appeared in Berliant and Fujita (1992) in the context of Alonso's *discrete* model of consumer location. The intuition and reasons for its presence are the same in this model and Alonso's model. Consider altering firm 1's parcel by removing a marginal piece of land from the side of the parcel further away from firm 2 and adding a marginal piece of land to the side of the parcel closer to firm 2; this will not affect output or revenue of firm 1. Due to the fixed cost of the transportation of goods, the fixed transportation cost will decrease by τ. So, at a profit maximum, the absolute difference between the price of land at either end of the parcel of firm 1 must be τ.

Now, suppose that type v workers locate on both sides of firm v, i.e., at z_v^+ and z_v^-. Then equations (28) and (32) are contradictory. What is going on here? The implication is that workers of type 1 can only reside on one side of firm 1 and similarly for firm 2.

Recalling equation (25) and keeping the size of the land parcel used by firm v constant, all the terms of the profit function are constant except the fixed transportation cost of commodities. Since firms are maximizing profit, in Nash equilibrium each firm will choose to be adjacent to the other.

In the next two examples, we use Cobb-Douglas utility and production functions, a one dimensional location space ($K = 1$ and $Z = [0, 1]$), and transportation technology given by $[t, 0]$ (for consumers) and $[\tau, 0]$ (for firms), so that transport and commuting cost are paid in terms of good 1 only. Good 1 is taken to be the numéraire. There are no variable costs. Contrary to example 1, we do not want to examine what equilibrium looks like (since by Theorem 1 we know that all firms are connected in equilibrium), but instead we want to examine its determinacy properties. For that we consider two examples corresponding to two different cases: a one dimensional continuum of equilibria and a two dimensional continuum of equilibria.

Example 2 *Two different firms and homogeneous workers: the case of a one dimensional continuum of equilibria.*

There are two different firms using the same type of labor $V = 1$ to produce different outputs $s = 1, 2$. In equilibrium, the two firms are connected (Theorem 1) and the transport cost of goods is zero. We have therefore: $S = 2$, $V = 1$, $M_1^1 = 1$, $M_2^1 = 1$, $M_2^2 = 0$ and $M_1^2 = 0$. As in example 1, land parcels used by each firm $i = 1, 2$ are denoted by $C_1 = [z_1^-, z_1^+]$ and $C_2 = [z_2^-, z_2^+]$. Without loss of generality, we take firm 1 on the left and firm 2 on the right so that the location of each firm is denoted by z_i, $i = 1, 2$ with $z_i = z_1^-$ or $z_i = z_2^+$.

In this context, each consumer located at z and working at firm i solves the following program:

$$\max_{q, X_1, X_2} U(q, X_1, X_2) = q^{1/3} X_1^{1/3} X_2^{1/3} \qquad s.t. \quad X_1 + p_2 X_2 + q\Psi(z) + t\,|z - z_i| = w$$

By differentiating the Lagrangian of this program with respect to q, X_1, X_2 and λ (the Lagrange multiplier), and by combining the four resulting equations, we obtain the following Marshallian demands:

$$q^* = \frac{1}{3\Psi(z)}\,[w - t\,|z - z_i|] \qquad X_1^* = \frac{1}{3}\,[w - t\,|z - z_i|] \qquad X_2^* = \frac{1}{3p_2}\,[w - t\,|z - z_i|]$$

The indirect utility function is thus equal to:

$$U(q^*, X_1^*, X_2^*) = \frac{1}{3}\,[w - t\,|z - z_i|]\,(p_2\Psi(z))^{-1/3} \equiv u^*$$

where u^* is the equilibrium utility level for all (homogeneous) workers in the city. The bid rent function, $\Xi(\cdot)$, which is the inverse of the indirect utility function with respect to $\Psi(z)$, is equal to:

$$\Xi(z, z_i, u^*) = \frac{(u^*)^{-3}}{9p_2} \left[w - t\,|z - z_i|\right]^3 .$$

In equilibrium, workers' bid rent $\Xi(z, z_i, u^*)$ is equal to the equilibrium land rent $\Psi(z)$ at each *residential* location $z \in Z$. Moreover, in equilibrium, it must be that all workers reach utility level u^* and that $\Psi(z_1^-) = \Psi(z_2^+) = \frac{1}{9p_2}\left(\frac{w}{u^*}\right)^3$, which implies that all workers in the city earn the same wage w and consume the same amount of q^*, X_1^* and X_2^* . We have therefore:

$$\overline{\Psi} = \frac{1}{9p_2}\left(\frac{w}{u^*}\right)^3 \cdot \gamma \tag{33}$$

where $\gamma \geq 1$ is a scaling factor that fixes, in equilibrium, how much above the maximal consumer bid-rent the producer land price will be set. There will be a one dimensional continuum of equilibria in this example, indexed by γ.

Let us now focus on firm i's program. It solves:

$$\max_{L_i, z_i^+, z_i^-} \pi_i = p_i\, L_i^{1/4}(z_i^+ - z_i^-)^{1/4} - wL_i - \overline{\Psi}(z_i^+ - z_i^-) \qquad i = 1, 2$$

where $p_1 = 1$ and $p_2 > 0$. By combining the first order conditions and by using (33), we easily obtain:

$$(z_1^+ - z_1^-)^* = 3^3 2^{-4} p_2^{3/2} w^{-5} u^{9/2} \gamma^{-3/2}$$

$$(z_2^+ - z_2^-)^* = 3^3 2^{-4} p_2^{3} w^{-5} u^{9/2} \gamma^{-3/2}$$

$$L_1^* = 2^{-4} 3\, p_2^{1/2} w^{-3} u^{3/2} \gamma^{-1/2}$$

$$L_2^* = 2^{-4} 3\, p_2^{7/3} w^{-3} u^{3/2} \gamma^{-1/2}$$

Using these values, the equilibrium profit is equal to:

$$\pi_1^* = 2^{-3} 3\, p_2^{1/2} w^{-2} u^{3/2} \gamma^{-1/2}$$

$$\pi_2^* = 2^{-3} 3\, p_2^{2} w^{-2} u^{3/2} \gamma^{-1/2} \left(3p_2^{1/3} - 1\right)$$

So with this example, we have closed form solutions and there is a one dimensional continuum of equilibria indexed by γ. The intuition is as follows. When there are two firms connected to each other, there is no inducement for them to change location. Consumers have the same level of utility whatever firm they choose as employer. The different equilibria are indexed by the price differential between firm land and consumer land.

There is one interesting feature of this example that we wish to note. There are two market clearing conditions for labor (one for each firm) but only one wage, since there is only one type of worker. In order to get enough freedom in endogenous variables to generate an equilibrium, the midpoint of firm land $\overline{\omega}$ is used. By varying this to the left or right, the supply of labor commuting to one of the two firms can be equated to demand. This gives us two endogenous variables to satisfy the two labor market clearing conditions. Since technologies are symmetric, the equilibrium $\overline{\omega}$ will be 1/2.

Example 3 *Two different firms and two different types of workers: the case of a two dimensional continuum of equilibria.*

We use exactly the same hypotheses as in example 1 but we focus on the determinacy properties of the equilibrium. In equilibrium all (two) firms are connected so that the good transportation costs are zero (Theorem 1). By combining the first order conditions for both firms and workers and equilibrium conditions, we obtain:

$$\left(\frac{w^1}{w^2}\right)^{5/8}\left(\frac{N^1}{N^2}\right)^{-1/2}\left(\frac{a}{b}\right)^{1/4}\frac{\gamma}{p_2}=1$$

and

$$\Psi(z)=a\left[\alpha\left(w^1-t(z_1^- - z)\right)\right]^3=b\left[\alpha\left(w^2-t(z-z_2^+)\right)\right]^3$$

where a and b are constants of integration.

By further manipulating the equations defining equilibrium, one can see that equilibrium exists, and in fact, there is a one dimensional family of equilibria parameterized by a constant of integration or by the location of a firm. There is another one dimensional family parameterized by γ, as in the previous example. This means that there is a two dimensional continuum of equilibria. The main difference with the previous example is that since workers are heterogeneous they will not necessarily reach the same utility level and it is possible that $u^{1*}\neq u^{2*}$. Thus, there is one less equation that must hold in equilibrium, compared to example 2.

Moreover, from these equations, it appears that prices (and consumption and production) are genuinely different in these equilibria. Indeed, the equilibria that form this two dimensional continuum cannot be obtained from one another by simple translation of the location of agents. There are substantial differences in both prices and allocations of different equilibria, and thus there is genuine indeterminacy.

Example 4 *Three identical firms and homogeneous workers: no equilibrium.*

There are three firms using the same type of labor $V = 1$ to produce the same type of output $S = 1$, which is taken as the numéraire. We have therefore: $S = 1$, $V = 1$, $M_1^1 = 3$. In equilibrium all (three) firms are connected so that the good transportation costs are zero (Theorem 1). Without loss of generality, we take firm 1 on the left, firm 3 on the right, and firm 2 in between so that the location of each firm is denoted by z_i, $i = 1, 2, 3$ with $z_i = z_1^-$ or $z_i = z_2^-$ or $z_i = z_2^+ (= z_3^-)$ or $z_i = z_3^+$. Moreover, land parcels consumed by each firm $i = 1, 2, 3$ are denoted by $C_i = [z_i^-, z_i^+]$. Each individual residing at z and working at firm i solves the following program:

$$\max_{q,X} U(q, X) = q^{1/2} X^{1/2} \qquad s.t. \quad X + q\Psi(z) + t\,|z - z_i| = w$$

By combining the first order conditions, we easily obtain:

$$q^* = \frac{w - t\,|z - z_i|}{2\Psi(z)} \qquad X^* = \frac{w - t\,|z - z_i|}{2}$$

Let us denote by u^* the equilibrium utility level for all (homogeneous) workers in the city. Then, the bid rent $\Xi(\cdot)$, which is the inverse of the indirect utility function with respect to $\Psi(z)$, is equal to

$$\Xi(z, z_i, u^*) = \left[\frac{w - t\,|z - z_i|}{2u^*}\right]^2$$

In equilibrium, where workers' bid rent coincides with the equilibrium land rent at each location, we must have that all workers reach the utility level u^* and that:

$$\Psi(z_1^-) = \Psi(z_1^+) = \Psi(z_2^-) = \Psi(z_2^+) = \Psi(z_3^-) = \Psi(z_3^+) \tag{34}$$

Now consider the comparison of commuting cost for a consumer commuting to firms 1 and 3 as opposed to firm 2. We have:

$$\begin{aligned} \Psi(z_1^-) &= \Psi(z_3^+) = \left(\frac{w}{2u^*}\right)^2 > \Psi(z_2^-) = \left[\frac{w - t\,(z_2^- - z_1^-)}{2u^*}\right]^2 \\ &= \Psi(z_2^+) = \left[\frac{w - t\,(z_3^+ - z_2^+)}{2u^*}\right]^2 \end{aligned} \tag{35}$$

which contradicts (34), if and only if $z_3^+ - z_2^+ = z_2^- - z_1^- > 0$. We now show that if firms maximize their profit, this is always true. The profit function of firm i is equal to:

$$\pi_i = L^{1/4}(z_i^+ - z_i^-) - wL - \gamma\,\Psi(z_i^+)(z_i^+ - z_i^-) \qquad i = 1,2,3$$

By combining the first order equations, we obtain:

$$z_i^+ - z_i^- = \frac{wL}{\gamma\,\Psi(z_i^+)} > 0 \qquad i = 1,2,3$$

Therefore, when firms maximize their profit, inequality (35) always holds and no equilibrium with connected firms can exist since it contradicts the equilibrium condition (34). The intuition is quite simple. Since all three firms are using the same input to produce the same output and since in equilibrium, one of the firms has to be in between the other two, the commuting cost of all consumers to that firm is larger than to the others. So nobody would work there and there is no equilibrium. Obviously for firm 2 located in between firms 1 and 3, u^* can never be reached by any worker commuting to firm 2 since the commuting distance and cost is greater and the competitive wage w must be the same for all workers in any firm. So why doesn't firm 2 simply shut down, and why isn't there an equilibrium with only firms 1 and 3 operating? Notice that in this example, there is a decreasing returns to scale technology for each firm. So in such a configuration, firms 1 and 3 earn positive profits, while firm 2 is shut down and earns zero profits. Thus, firm 2 will try to mimic the production plan (including land usage) of another firm, resulting in disequilibrium.

Observe that the argument depends crucially on the dimension of the location space Z. If Z were two dimensional, then it would be possible to construct an equilibrium for this example since the three firms can be connected and at the same time workers do not have to cross one firm to work at another.

8.2 Proofs

Proof of Theorem 1:

Suppose that in an equilibrium, not all firms are connected. Then the fixed transport cost for *every* firm is positive. Let us focus on one firm, the firm (i,s,v) given in (b) of the assumption on the collection $\mathbf{C}$. Define

$$\Pi(\epsilon) = p_s(i,s,v)\, f_s^v(L_s^{i,v}, \mu(B_\epsilon(C_s^{i,v}))) - w^v L_s^{i,v} - \mu(B_\epsilon(C_s^{i,v}))\overline{\Psi}$$

$$-\sum_{s'=1}^{S}\sum_{v'=1}^{V}\sum_{j=1}^{M_{s'}^{v'}} p(i,s,v)\cdot[\tau+T\cdot g_{(j,s',v')}^{(i,s,v)}] \inf_{\substack{x\in B_\epsilon(C_s^{i,v})\\ y\in C_{s'}^{j,v'}}} \|x-y\|\cdot \mathbf{I}_{X_{(j,s',v')}^{(i,s,v)}}.$$

We will show that under the assumptions of the Theorem, the equilibrium production plan of firm (i,s,v) is not profit optimizing, leading to a contradiction. In order to accomplish this, we use part (a) of the assumption on the collection $\mathbf{C}$ and a one term Taylor's series expansion of Π at $\epsilon=0$.

$$\Pi(\epsilon)=\Pi(0)+\epsilon\cdot p_s(i,s,v)\cdot d\,f_s^v(L_s^{i,v},\mu(C_s^{i,v}))/dQ\cdot dg(C_s^{i,v},\epsilon)/d\epsilon\mid_{\epsilon=0}$$

$$-\epsilon\cdot\overline{\Psi}\cdot dg(C_s^{i,v},\epsilon)/d\epsilon\mid_{\epsilon=0}$$

$$+\epsilon\cdot\sum_{s'=1}^{S}\sum_{v'=1}^{V}\sum_{j=1}^{M_{s'}^{v'}} p(i,s,v)\cdot[\tau+T\cdot g_{(j,s',v')}^{(i,s,v)}] \inf_{\substack{x\in C_s^{i,v}\\ y\in C_{s'}^{j,v'}}} \|x-y\|\cdot \mathbf{I}_{X_{(j,s',v')}^{(i,s,v)}}+R(\epsilon),$$

$$\text{where }\lim_{\epsilon\to 0}\frac{R(\epsilon)}{\epsilon}=0.$$

(The sign on the transport cost term is reversed because an ϵ expansion of land use by firm (i,s,v) results in a decrease in transport cost.) Now in equilibrium, it must be the case that

$$0\geq\Pi(\epsilon)-\Pi(0)=\epsilon\cdot p_s(i,s,v)\cdot d\,f_s^v(L_s^{i,v},\mu(C_s^{i,v}))/dQ\cdot dg(C_s^{i,v},\epsilon)/d\epsilon\mid_{\epsilon=0}$$

$$-\epsilon\cdot\overline{\Psi}\cdot dg(C_s^{i,v},\epsilon)/d\epsilon\mid_{\epsilon=0}$$

$$+\epsilon\cdot\sum_{s'=1}^{S}\sum_{v'=1}^{V}\sum_{j=1}^{M_{s'}^{v'}} p(i,s,v)\cdot[\tau+T\cdot g_{(j,s',v')}^{(i,s,v)}] \inf_{\substack{x\in C_s^{i,v}\\ y\in C_{s'}^{j,v'}}} \|x-y\|\cdot \mathbf{I}_{X_{(j,s',v')}^{(i,s,v)}}+R(\epsilon),$$

so

$$\epsilon\cdot\overline{\Psi}\cdot dg(C_s^{i,v},\epsilon)/d\epsilon\mid_{\epsilon=0}\geq\epsilon\cdot p_s(i,s,v)\cdot d\,f_s^v(L_s^{i,v},\mu(C_s^{i,v}))/dQ\cdot dg(C_s^{i,v},\epsilon)/d\epsilon\mid_{\epsilon=0}$$

$$+\epsilon\cdot\sum_{s'=1}^{S}\sum_{v'=1}^{V}\sum_{j=1}^{M_{s'}^{v'}} p(i,s,v)\cdot[\tau+T\cdot g_{(j,s',v')}^{(i,s,v)}] \inf_{\substack{x\in C_s^{i,v}\\ y\in C_{s'}^{j,v'}}} \|x-y\|\cdot \mathbf{I}_{X_{(j,s',v')}^{(i,s,v)}}+R(\epsilon).$$

Now since transport costs for this firm are positive, the second to last term is positive and by choosing ϵ small so that the second to last term dominates the last term

$$\epsilon\cdot\overline{\Psi}\cdot dg(C_s^{i,v},\epsilon)/d\epsilon\mid_{\epsilon=0}>\epsilon\cdot p_s(i,s,v)\cdot d\,f_s^v(L_s^{i,v},\mu(C_s^{i,v}))/dQ\cdot dg(C_s^{i,v},\epsilon)/d\epsilon\mid_{\epsilon=0}$$

or

$$\overline{\Psi} \cdot dg(C_s^{i,v}, \epsilon)/d\epsilon \mid_{\epsilon=0} > p_s(i,s,v) \cdot d\, f_s^v(L_s^{i,v}, \mu(C_s^{i,v}))/dQ \cdot dg(C_s^{i,v}, \epsilon)/d\epsilon \mid_{\epsilon=0} .$$

This condition says that the cost of the last unit of land exceeds its marginal revenue product. Next we apply part (b) of the assumption on collection $\mathbf{C}$: for every $\delta' > 0$ there is an $A \in \mathbf{C}$, $A \subseteq C_s^{i,v}$ a.s. with $\mu(C_s^{i,v}) - \delta' < \mu(A) < \mu(C_s^{i,v})$ and

$$\inf_{\substack{x \in C_s^{i,v} \\ y \in C_{s'}^{j,v'}}} \|x - y\| = \inf_{\substack{x \in A \\ y \in C_{s'}^{j,v'}}} \|x - y\|$$

so transport cost is unchanged when replacing $C_s^{i,v}$ with A. So by choosing δ' small, the firm has higher profits using less land A instead of $C_s^{i,v}$, contradicting profit optimization in equilibrium on the part of firm (i,s,v). ■

Proof of Lemma 1:

Our goal is to show that the equation system $(16)-(19)$ is C^r. In order to accomplish this, it is convenient to establish some preliminary results first, namely that supply and demand are C^r. The proofs of these preliminary results parallel the proof of Mas-Colell (1985, Proposition 2.7.2, p.85), but with a subtle twist due to our use of bid rent.

Let us begin with production. Profits are given in equation (10), and the first order conditions are:

$$p_s \frac{\partial f_s^v(L,Q)}{\partial L} - w^v = 0$$

$$p_s \frac{\partial f_s^v(L,Q)}{\partial Q} - \overline{\Psi} = 0$$

This equation system has (L,Q) as endogenous variables as functions of prices, $(p_s, w^v, \overline{\Psi})$, and any other exogenous variables desired. The Jacobian of this equation system (with respect to the endogenous variables) is simply the Hessian of f_s^v. By assumption, $D^2 f_s^v(x)$ is negative definite, so by Mas-Colell, Whinston and Green (1995, Example M.D.1, p.937), the determinant of this Hessian is positive. Using the implicit function theorem (Mas-Colell, 1985, C.32, p.20), the derived demand functions are C^r. Plugging the derived demand functions back into the C^{r+1} production function, a C^r supply function for outputs is obtained.

Turning next to the consumption sector, we use a similar technique. Bid rent is defined in equation (9). From Mas-Colell (1985, D.1, p.22), the first order conditions are:

$$\lambda\left(\frac{-p}{q^v}\right) + \gamma D_{X^v}U^v(q^v, X^v) = 0$$

$$-\lambda[w^v - p\cdot X^v - p\cdot t \ (\inf_{\substack{y\in C_s^{i,v}\\ i=1,\ldots,M_s^v\\ s=1,\ldots,S}} \|z-y\|)]/(q^v)^2 + \gamma D_{q^v}U^v(q^v, X^v) = 0$$

$$-u^v + U^v(q^v, X^v) = 0$$

The first equation implies that both λ and γ are positive, so defining $\eta = \lambda/\gamma > 0$, the first order conditions become:

$$D_{X^v}U^v(q^v, X^v) - \eta\left(\frac{p}{q^v}\right) = 0$$

$$D_{q^v}U^v(q^v, X^v) - \eta[w^v - p\cdot X^v - p\cdot t \ (\inf_{\substack{y\in C_s^{i,v}\\ i=1,\ldots,M_s^v\\ s=1,\ldots,S}} \|z-y\|)]/(q^v)^2 = 0$$

$$-u^v + U^v(q^v, X^v) = 0$$

This equation system has (q^v, X^v, η) as endogenous variables as functions of prices, (p, w^v), and any other exogenous variables desired. The Jacobian of this equation system (with respect to the endogenous variables) is as follows:

$$A = \begin{pmatrix} D^2_{X^v}U^v & D^2_{X^v q^v}U^v + \eta p/(q^v)^2 & -p/q^v \\ D^2_{q^v X^v}U^v + \eta p/(q^v)^2 & D^2_{q^v}U^v + 2\eta\Lambda/(q^v)^3 & -\Lambda/(q^v)^2 \\ D_{X^v}U^v & D_{q^v}U^v & 0 \end{pmatrix}$$

where $\Lambda \equiv w^v - p\cdot X^v - p\cdot t \ (\inf_{\substack{y\in C_s^{i,v}\\ i=1,\ldots,M_s^v\\ s=1,\ldots,S}} \|z-y\|)$.

Substitute the first order conditions into the last column and multiply the last column by $-\eta$; we obtain a new matrix B, where $Det(A) = -\frac{1}{\eta}Det(B)$. Now multiply the last row by $1/q^v$ and subtract it from the next to last row of B. Multiply the last column by $1/q^v$ and subtract it from the next to last column of B. The result is the matrix

$$C = \begin{pmatrix} D^2_{X^v}U^v & D^2_{X^v q^v}U^v & D_{X^v}U^v \\ D^2_{q^v X^v}U^v & D^2_{q^v}U^v & D_{q^v}U^v \\ D_{X^v}U^v & D_{q^v}U^v & 0 \end{pmatrix}$$

where $-\frac{1}{\eta}Det(C) = -\frac{1}{\eta}Det(B) = Det(A)$. By Mas-Colell (1985, 2.5.1, p.76), $Det(C)$ is nothing more than Gaussian curvature (assumed non-zero since U^v

has no critical point) multiplied by a non-zero constant, and thus $Det(A) \neq 0$. Applying the implicit function theorem once again, the demand functions implied by the bid rent calculation (9) are C^r. Plugging these solutions into the definition of bid rent, we obtain that bid rent is C^r provided that commuting cost is a C^r function of z.

Here there is a potential problem if firms using the same type of labor are located on two different sides of a consumer, since in that case commuting cost is continuous but not necessarily differentiable in z. But if we restrict attention to those allocations where firms using the same type of labor are connected (as assumed) and using hyper-rectangles centered at $\overline{\omega}$, as postulated at the beginning of section 4.2, then commuting cost is C^r.

Finally, the assumptions that

$$\forall v \neq v', \quad \forall p, w^v, w^{v'}, C^v, C^{v'}, u^v, u^{v'} \text{ almost surely for } z \in Z \setminus \bigcup_{s=1}^{S} \bigcup_{i=1}^{M_s^v} C_s^{i,v}$$

$$\frac{\partial \Xi^v(z, p, w^v, C^v, u^v)}{\partial z} \neq \frac{\partial \Xi^{v'}(z, p, w^{v'}, C^{v'}, u^{v'})}{\partial z}$$

and $\frac{\partial \alpha(z,p,w,C,u)}{\partial z} \neq 0$ (except at $z = \overline{\omega}$) allow us to apply the implicit function theorem to locations z where $\Xi^v(z, p, w^v, C^v, u^v) = \Xi^{v'}(z, p, w^{v'}, C^{v'}, u^{v'})$ or $\gamma \cdot \alpha(z, p, w, C, u) = \overline{\Psi}$, so that the implicit function parameterizing the intersection of two consumers bid rents or a consumer bid rent with the producer land price is C^r. Thus the equation system (16)$-$(19) is C^r. $\blacksquare$

An Example of a Regular Parameterization:

Let $K = 1$ and $Z = [0, 1]$. Using Walras' law, we take (19) to be the redundant equation in the system. We modify (16)−(18) to incorporate the following exogenous parameters, some of which are not yet in the system. Already appearing in equation (16) only are N^v $(v = 1, ..., V)$. We relax the assumption that production functions for all firms using the same type of labor to produce the same output are the same, and parameterize production functions as follows. For $\lambda_s^{i,v} > 0$,

$$f_s^{i,v}(L_s^{i,v}, \mu(C_s^{i,v}); \lambda_s^{i,v}) = \frac{1}{\lambda_s^{i,v}} \overline{f}_s^{i,v}(L_s^{i,v}) + \widetilde{f}_s^{i,v}(\mu(C_s^{i,v}))$$

where $\overline{f}_s^{i,v}$ is C^2 and $\frac{d\overline{f}_s^{i,v}}{dL} > 0$ and $\frac{d^2\overline{f}_s^{i,v}}{dL^2} < 0$. (This is easily generalized to the case where $f_s^{i,v}$ is not additively separable by using the implicit function theorem in combination with the first order conditions for profit maximization, as in the proof of Lemma 1).

Finally, we replace the single landlord with two, where the first landlord receives ϕ% of rents plus profits and the second landlord receives $(1 - \phi)$% of rents plus profits. (The first landlord will be denoted by bars over variables and the second landlord will be represented by tildes over variables.) Moreover, we endow the landlords with consumption goods $\overline{W}$, $\widetilde{W} \in \Re^s_{++}$. Formally,

$$X^L(R, \pi) = \overline{X}^L(\phi R, \phi\pi) - \overline{W} + \widetilde{X}^L((1 - \phi)R, (1 - \phi)\pi) - \widetilde{W}$$

We claim that the parameterization $\left\{(N^v)_{v=1}^V \; ; \; \left(\lambda_s^{i,v}\right)_{i=1}^{M_s^v}{}_{v=1}^{V}{}_{s=1}^{S} \; ; \phi \; ; \overline{W}, \widetilde{W}\right\}$ is regular. To prove this, we examine the derivative of (16)−(18) with respect to the parameters (at equilibrium). The Jacobian of (16) with respect to $(N^v)_{v=1}^V$ is the negative of the identity matrix, and N^v appears nowhere else in the system. Turning next to the parameters $\lambda_s^{i,v}$ and equation (17), notice that from the first order conditions for profit optimization, $L_s^{i,v} = \overline{f}_s^{\prime i,v-1}\left(\frac{\lambda_s^{i,v} w^v}{p_s}\right)$. The Jacobian of (17) with respect to $\left(\lambda_s^{i,v}\right)_{i=1}^{M_s^v}{}_{v=1}^{V}{}_{s=1}^{S}$ yields a non-singular submatrix of rank $\sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v$. Notice that the Jacobian of (18) with respect to $\lambda_s^{i,v}$ yields non-zero elements, but these can be row-eliminated.

The last part of the proof is similar to Mas-Colell (1985, Example 5.8.5, p.227). For an arbitrary small change in $\overline{W}$, we can compensate with a change in ϕ (for the first landlord) and $\widetilde{W}$ (for the second landlord) so that wealth, and hence gross demand, is unchanged for both landlords. This yields a Jacobian of (18) with respect to $\phi, \overline{W}$ and $\widetilde{W}$ of rank s, and completes the proof.

Proof of Theorem 2:

We need to ascertain the dimensions of the domain and range of the functions given by the LHS of the equations (16)-(19). If we take one of the consumption goods as the numéraire, the following tables describe the number of unknowns and equations.

Unknown	Dimension
p	$S-1$
w	V
$\overline{u}$	V
$\overline{\omega}$	1
Ψ^*	1
γ	1

Equation number	Dimension
(16)	V
(17)	$\sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v$
(18)	S
(19)	1

Observe that one of the s commodity equations in (18) is dependent on the other equations due to Walras law. After subtracting the number of equations from the number of unknowns, we obtain: $V - \sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v + 2$. In order to apply the proposition of Mas-Colell (1985, proposition 8.3.1, p.320), we assume that utilities, production functions and their parameterization (see Mas-Colell (1985, definition 5.8.11, p.226)) are C^r with $r > \min(0, V - \sum_{s=1}^{S}\sum_{v=1}^{V} M_s^v + 2)$ and Theorem 2 follows. ■

Proof of Theorem 3:

The proof proceeds in much the same way as the proof of Theorem 2. Fix some symmetric order of firms. By symmetry, we limit attention to $[0, 1/2] \times [0, 1]^{K-1}$. Fix $\overline{\omega} = 1/2$ so that it is no longer endogenous. (This reduces the number of unknowns by 1.) For the firm in the middle of the order (if any), called (i', s', v'), change its production function to $\widetilde{f}_{s'}^{v'}(L_{s'}^{i',v'}, Q_{s'}^{i',v'}) = (1/2) \cdot f_{s'}^{v'}(2 \cdot L_{s'}^{i',v'}, 2 \cdot Q_{s'}^{i',v'})$. Performing the same analysis contained in the proof of Theorem 2 on this half-economy in $[0, 1/2] \times [0, 1]^{K-1}$, with only firms to the left of (i', s', v') in the order participating and including the modified middle firm (with the right endpoint of its parcel anchored at $1/2$), we obtain the result. Notice that when we add back the firms to the right of the middle firm, the equilibrium is unchanged by symmetry, and the middle firm can return to its original production function without altering the allocation. ■

Proof of Theorem 4:

Let $Z = [0,1]^K$. Let firm (i,s,v) use the hyper-rectangle $C_s^{i,v} \equiv [\omega^h, \omega^{h+1}] \times [0,1]^{K-1}$, where $h = \sum_{s'<s} \sum_{v'<v} M_{s'}^{v'} + i$. In this way, firms are always connected. Fix $\overline{\omega}$ the midpoint of firm land use, to be $1/2$. As mentioned in section 4.2, we must alter the firm's problem (10) slightly to ensure that solutions exist. Instead of asking them to solve (10) unconstrained, we impose the following constraints, and define $\pi_s^v(p_s, w^v, \overline{\Psi})$, $L_s^{i,v}(p_s, w^v, \overline{\Psi})$ and $Q_s^{i,v}(p_s, w^v, \overline{\Psi})$ to be solutions to (10) subject to the following constraints:

$$0 \le L_s^{i,v} \le N^v + 1 \tag{36}$$

$$0 \le Q_s^{i,v} \le 2 \tag{37}$$

The solution to (10) subject to (36) and (37) always exists and is unique.

Define $\overline{u} = [\overline{u}^1, ..., \overline{u}^V]$ to be utility levels. For nonzero prices and utility levels, the LHS of equations (16)$-$(19) are continuous functions on $\Omega^\circ \times u^\circ$ (see Lemma 1) - we will deal with the boundary cases momentarily - and are respectively denoted by:

$$h_1(p, w, \overline{u}, \overline{\Psi}) = 0 \tag{38}$$

$$h_2(p, w, \overline{u}, \overline{\Psi}) = 0 \tag{39}$$

$$h_3(p, w, \overline{u}, \overline{\Psi}) = 0 \tag{40}$$

$$h_4(p, w, \overline{u}, \overline{\Psi}) = 0 \tag{41}$$

Let the domain of the prices (p, w and $\overline{\Psi}$) be

$$\Omega = \left\{ (p, w, \overline{\Psi}) \mid \sum_{s=1}^{S} p_s + \sum_{v=1}^{V} w^v + \overline{\Psi} = 1, p_s \ge 0, w^v \ge 0, \overline{\Psi} \ge 0, s = 1, ..., S, v = 1, ..., V \right\}$$

Due to the special nature of production, the asymptotic cones of the individual production sets are positively semi-independent, so by Debreu (1959, p.23(9)), the aggregate production set is closed. Given that the input endowments of land and labor are bounded, the set of feasible allocations is compact. Let the range of u^v (when consumption is bounded by endowments) be contained in $[0, E-1]$, where $E \in \Re_+$. In other words, when type v workers have all the resources, under equal treatment, they will all get exactly the same utility level, which is at most $E-1$. Further, without loss of generality, we assume that E is contained in the range of u^v (when it is not constrained by endowments). This eliminates one of the boundary problems mentioned earlier.

The dual variables $\widetilde{u}$, $\widetilde{w}$, $\widetilde{p}$ and $\widetilde{\Psi}$ are respectively associated with equations (16)−(19) so that they maximize the following functions:

$$g_1(\cdot) = \widetilde{u} = \arg\max_{u \in [0,E]^V} u \cdot h_1(\cdot) \tag{42}$$

$$g_2(\cdot) = (\widetilde{p}, \widetilde{w}, \widetilde{\Psi}) = \arg\max_{(p,w,\widetilde{\Psi}) \in \pi} \sum_{v=1}^{V} w^v \cdot \sum_{s=1}^{S} \sum_{i=1}^{M_s^v} h_2(\cdot) + p \cdot h_3(\cdot) + \widetilde{\Psi} \cdot h_4(\cdot) \tag{43}$$

We are left with defining g_1 and g_2 at boundary cases when bid rent is undefined. Whenever $p_s = 0$, set $q^v = 0$ so $\widetilde{u}^v = E$ for all v, $g_2 \equiv \{\widetilde{p}, \widetilde{w}, \widetilde{\Psi} \mid \widetilde{w}^v = 0 \quad \forall v, \widetilde{\Psi} = 0, \sum_{\{s \mid p_s = 0\}} \widetilde{p}_s = 1\}$. Whenever, $u^v = 0$, set $q^v = 0$ so $\widetilde{u} = E$, $\widetilde{w}^v = 0$, $\sum_{s=1}^{S} \widetilde{p}_s = 1$, $\widetilde{\Psi} = 0$.

$g_1(\cdot)$ and $g_2(\cdot)$ are convex valued and upper-hemicontinuous correspondences. Let ρ be the Cartesian product of $g_1(\cdot)$ and $g_2(\cdot)$. So $\rho : E^V \times \Omega \to E^V \times \Omega$. Now, we have all the elements to apply the *Kakutani fixed point theorem.* Let the fixed point be $(\widehat{p}, \widehat{w}, \widehat{u}, \widehat{\Psi})$. The boundary conditions on both the utility and the production functions will rule out the possibility of excess demands. Next, let us demonstrate that at a fixed point $(\widehat{p}, \widehat{w}, \widehat{u}, \widehat{\Psi})$, $h_1(\cdot)$, $h_2(\cdot)$, $h_3(\cdot)$ and $h_4(\cdot)$ are all equal to zero and hence we have an equilibrium.

First, let us focus on non-locational variables, where the argument is standard. Suppose that $h_2(\cdot)$ for some firm (i, s, v), or $h_3(\cdot)$ for some s, or $h_4(\cdot)$ is strictly positive. Then $\sum_{v=1}^{V} \widehat{w}^v \cdot \sum_{s=1}^{S} \sum_{i=1}^{M_s^v} h_2(\cdot) + \widehat{p} \cdot h_3(\cdot) + \widehat{\Psi} \cdot h_4(\cdot) > 0$. Summing the budget constraints of all consumers and the landlord/shareholder and using the definition of profit for firms, it must be that (43) is non-positive. This contradicts the inequality above. Suppose that $h_2(\cdot)$ for some firm (i, s, v), or $h_3(\cdot)$ for some s, or $h_4(\cdot)$ is strictly negative. Using the summed budget constraints and the definition of profit, some other components of $h_2(\cdot)$, $h_3(\cdot)$, or $h_4(\cdot)$ must be positive. This leads to another contradiction of the inequality above.

So it remains to show that $h_1(\cdot) = 0$. If $h_1(\cdot) > 0$ for some v, then $\widehat{u}^v = E$. This implies that the allocation of goods is infeasible (since the maximum utility attainable at a feasible allocation is $E-1$), contradicting what we have already proved. If $h_1(\cdot) < 0$ for some v, then $\widehat{u}^v = 0$, and this cannot be a fixed point, since its image under g_1 is E. So $h_1(\cdot) = 0$.

Finally, if for some firm (i, s, v) there is an alternative production plan that yields higher profits than the fixed point production plan, then this alternative violates either (36) or (37). Since the fixed point production plan is feasible and the profit function is linear, there is a convex combination of the fixed point production and the alternative that yields higher profits than the fixed point production plan, satisfies (36) and (37), and (since the production function is concave) can be produced. Since the fixed point production plan solves (10) subject to (36) and (37), this is a contradiction. ■

Proof of Theorem 5:

The proof is the same as that of Theorem 4, as modified as in the proof of Theorem 3 to use only a half-economy. Using symmetry, the equilibrium can be extended to the entire economy. ■

Proof of Theorem 6:

To prove Theorem 6, we will find a counter-example to the first fundamental theorem of welfare economies. Let us use Example 1. One equilibrium of interest is when the two firms are located in the middle of $[0, 1]$, i.e., $z_1^+ = z_2^- = 1/2$. We now find a small Pareto improvement, denoted by tildes over the variables. We give absentee landlords exactly the same allocation of goods in the Pareto improvement as in equilibrium. Starting with the equilibrium allocation, move the firms apart by 2ε so that $\widetilde{z}_1^+ = 1/2 - \varepsilon$ and $\widetilde{z}_2^- = 1/2 + \varepsilon$. Let the firms use the same inputs including the same quantities of land in the new allocation. In other words, $\widetilde{z}_1^- = z_1^- - \varepsilon$ and $\widetilde{z}_2^+ = z_2^+ + \varepsilon$. The firms will produce the same outputs at the new allocation. A consumer located in equilibrium at $\varepsilon \leq z \leq z_1^-$ will get exactly the same allocation of goods but will be located at $\widetilde{z} = z - \varepsilon$, slightly to the left. Then this consumer will have the same level of utility (and commuting cost) in this allocation as in the equilibrium allocation. A consumer located in equilibrium at $\widetilde{z}_2^+ \leq z \leq 1 - \varepsilon$ will get exactly the same allocation of goods but will be located at $\widetilde{z} = z + \varepsilon$, slightly to the right. Then this consumer will have the same level of utility (and commuting cost) in this allocation as in the equilibrium allocation. The consumers who were located at $0 \leq z \leq \varepsilon$ in equilibrium are moved to $1/2 - z$ and will have the same allocation of goods. The consumers who were located at $1 - \varepsilon \leq z \leq 1$ in equilibrium are moved to $3/2 - z$ and will have the same allocation of goods. Notice that commuting cost for the last two types of consumers will be lower in the new allocation, so that there will be some surplus good 1 to distribute to any or all agents and raise utility. To be more precise, the savings of these consumers in terms of commuting cost will be $\int_0^{\varepsilon} n(z)t(z_1^- - \varepsilon)dz + \int_{1-\varepsilon}^{1} n(z)t(1 - \varepsilon - z_2^+)dz$ (where $n(z)$ is the equilibrium population density), whereas the additional cost of transporting goods between the two firms is $4\tau\varepsilon$. By choosing t large enough and τ small enough, we have constructed a feasible Pareto improvement. Hence, the initial equilibrium was not Pareto optimal, and in particular was not an equal treatment Pareto optimum. ■

Although we use the concept of equal treatment Pareto optimum for convenience, notice that any equilibrium involves equal treatment (in utility levels) of consumers of the same type. Consequently, for the counter-example to the first welfare theorem, showing that an equilibrium allocation is not an equal treatment Pareto optimum is the same as showing that it is not a Pareto optimum.

Observe that in this example, locating firms at opposite ends of the unit interval (maximum differentiation) is not Pareto optimal. Indeed, at such an allocation, the social planner can flip the allocation between zero and one-half symmetrically about one-quarter, and flip the allocation between one-half and one symmetrically about three-quarters to obtain a Pareto dominating allocation; the allocation of goods is unchanged except firm transportation costs are reduced and these goods are reallocated to consumers.

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