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Parental altruism, life expectancy and dynamically inefficient equilibria*

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Abstract: Macrodynamic models with finite lifetime and selfish individuals may feature (dynamically) inefficient equilibria, while models with infinite lifetime and altruistic individuals cannot. Do strong intergenerational altruism and high life expectancy prevent the occurence of inefficient equilibria? To answer this question, we present a continuous time OLG model which generalizes the Blanchard-Buiter-Weil model. Our main innovation relies on the introduction of parental altruism, whose intensity is variable. We show that parental altruism and life expectancy actually favor overaccumulation. Theoretical results are illustrated by a parametrization from US data. Our numerical exercises suggest that the US economy is dynamically inefficient, mainly because life expectancy is sufficiently short.

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Keywords: Overlapping generations model; Productive capital; Dynamic (in)efficiency; Intergenerational altruism

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1 Introduction

Dynamic economies, although perfectly competitive, may yield inefficient outcomes. The purpose of this paper is to revisit the roles played by intergenerational altruism and life expectancy in such failure of the first welfare theorem. It is well-known that macrodynamic models with finite-lived and selfish individuals are compatible with (dynamically) inefficient equilibria, while models with infinite-lived and dynastically altruistic individuals are not. This suggests that strong intergenerational altruism within the population and long life expectancy prevent the occurence of inefficient equilibria. We challenge this view: we claim that parental altruism and life expectancy actually favorize the emergence of dynamically inefficient allocations. We proceed in two steps. First, we demonstrate our claim in a text-book model whose demographic side has been especially enriched to account for parental altruism. Second, we consider a parametrization of our model on US data, and highlight the role played by the strength of intergenerational links and the length of planning horizons. Our simulations suggest that the US economy is dynamically efficient, mainly because life expectancy is sufficiently short.

In a path-breaking paper, Diamond (1965) shows that an overlapping-generation (OLG) economy with productive capital can reach a dynamically inefficient steady state. Dynamic inefficiency means oversavings/overaccumulation[1]: gross investment exceeds gross capital income and it is possible to increase the consumption of at least one generation without reducing the consumption of any other. Overaccumulation consists of a major market failure, since it gives some relevance to the use of public debt, or the creation of pay-as-you-go retirement schemes. Economists have for long questioned the causes of overaccumulation. In particular, they have pointed out the role played by intergenerational links and the length of planning horizons.

Given that Ramsey-type economies never oversave, one may wonder which structural parameter is responsible for dynamic inefficiency. A wide consensus in the literature emerged after the cutting-edge paper of Weil (1989). Weil shows that some 'disconnectedness' between families is required to obtain dynamically inefficient steady states[2]. By disconnectedness, he means that all agents do not belong to the same representative family. In our paper, we aim to answer the two following questions: how do parental al-

[1] The first studies of inefficient capital accumulation are due to Phelps (1961, 1965) and Koopmans (1965). Major developments were subsequently proposed by Cass (1972) and, concerning OLG economies, by Shell (1971), Balasko and Shell (1980), Okono and Zilcha (1980) and Saint-Paul (1992). Galor and Ryder (1991) provide sufficient conditions on technological parameters that rule out dynamically inefficient steady states in the Diamond model. Note that a complete resolution of the problem when agents live for two periods can be found in De la Croix and Michel (2002).

[2] Weil (1989) does so in an exchange economy, but generalizes his result to capital accumulation in the appendix.

truism and life expectancy relate to dynamic efficiency? Are real economies dynamically inefficient?

We propose a generalization of the continuous time OLG models of Blanchard (1985), Buiter (1988) and Weil (1989). This generalization introduces a variable degree of family altruism. There are productive capital, capital depreciation, labor-augmenting technical progress, and CRRA preferences. The model's key innovation resides in its demographic side. We make four major assumptions. First, each individual is *perpetually young* (Blanchard, 1985): he/she may die with age-independent risk δ at each instant. Second, there is *intra-family growth*: children are born at rate m. Third, there is *inter-family growth*: there is a continuous inflow of new families, and such inflow grows at rate n. This is due to immigration. Finally, there is *selective altruism* (Abel, 1988) within families: only a proportion λ of children benefits from the love of their parents, while the remaining $1-\lambda$ are unloved, disinherited children. We shall refer to λ as the intensity of parental altruism, or parental altruism for short. These different assumptions have several immediate implications. On the one hand, the rate of disconnectedness is $n_E = (1-\lambda)\,m + n$. It is decreasing in the intensity λ of parental altruism. On the other hand, the previous models studied in the literature are special cases of ours. In particular, the Ramsey (1928) model with population growth corresponds to $1-\lambda = \delta = n = 0$: there are no unloved children, no risk of dying, and no entries from outside the economy.

We prove the uniqueness of a saddle-path stable steady state before studying its dynamic efficiency. There is dynamic efficiency whenever the stationary differential between the interest rate and the output growth rate – we call it the *interest-growth spread* – is nonnegative. We then turn to the two questions marked above.

To answer the first question, we analyse the effects of parental altruism λ and risk of dying δ on the occurrence of dynamically inefficient steady-states. Each of these parameters is likely to alter the interest-growth spread through its impact upon the family-saving behavior, or its effect upon the composition by age of the population. Our results do not suffer from any ambiguity. Parental altruism reduces the interest-growth spread. Indeed, parental altruism does not alter output growth, while it does increase each family's propensity to save. Similarly, the risk of dying increases the interest-growth spread – why should one save for a very unlikely future? Our main conclusion follows: altruism and life expectancy favor dynamic inefficiency.

These results lead us to the following question: if altruism and the length of horizons favor dynamic inefficiency, why are Ramsey-type economies – where agents are infinite-lived and love all their children – dynamically efficient? The answer can be found in Shell (1971). Shell noticed that the present value at competitive price of society's wealth is infinite in dynamically inefficient situations. But, in Ramsey-type economies, the family wealth is equivalent to the economywide wealth. This explains why overaccumulation cannot happen in such economies: overaccumulation, defined as a negative interest growth spread, implies that the present value of family wealth is infinite. This is not compatible with the existence of a competitive equilibrium. Hence, a typical necessary condition for the existence of a stationary state forbids dynamic inefficiency. Unlike Ramsey-type economies, OLG economies are compatible with dynamic inefficiency. Indeed, the present value of family wealth is finite if the interest rate is greater than the family wealth growth rate. Given that unloved children and migrants enter the economy at constant rate, the family wealth growth rate is lower than the economy growth rate, which gives rise to overaccumulation.

We address the second question through a parametrization of our model on US data. This may seem strange: given the simplicity of the condition of dynamic efficiency, that is a negative interest-growth spread, why not directly test dynamic efficiency on adequate time series data? In fact, this strategy of research has been pursued on several occasions (see Abel et al, 1989), and does not seem too costly. Indeed, choosing the true rate of return on savings seems a priori easier than determining the fair value of parameters related to individual preferences. Yet there are two strong reasons to proceed as we do. On the one hand, direct tests of overaccumulation mainly reveal themselves to be inconclusive. The results dramatically depend on the proxy of the real rate of return on savings. If one chooses the short-term real interest rate on government debt, a risk-free asset, then overaccumulation results. Such rates have been below one percent over the previous half century in most OECD countries, while the output growth rate averaged three percent. If one chooses some market based value of the rate of return on capital, then dynamic efficiency obtains. Such rates of return are usually much higher than output growth rates. On the other hand, Tirole (1985) has shown that dynamically inefficient steady states are likely to produce bubbles on artificial or real assets. In such cases, excess savings are captured by the bubble, up to the Golden Rule level. Hence, following Tirole's argument, we should never observe a negative interest-growth spread, even though the steady state is dynamically inefficient.

Our numerical experiments question the plausibility of the set of pairs (intensity of parental altruism, risk of dying) leading to dynamically inefficient equilibria. We use US demographic data from the 1990s to set the birth rate and the immigration rate. The technological parameters – elasticity of output with respect to capital, capital depreciation rate, and stationary productivity growth rate – are set to standard values. The preference parameters are much less consensual. We start by setting the pure rate of time preference at one percent per year, and the elasticity of intertemporal substitution at one. Dynamic inefficiency always occurs when the agents are infinite-lived, but it is ruled out for plausible values of life expectancy. A sensitivity analysis reveals that less conservative choices for the preference parameters can yield more ambiguous results. However, they are typically associated with very low values of the pure rate of time preference[3], or very large values of elasticity of intertemporal substitution. We thus claim that sufficiently short lifetimes ($\delta >> 0$) and imperfect parental altruism ($\lambda << 1$) prevent the occurence of dynamically inefficient equilibria in the US.

The paper is organized as follows. Section 2 presents the model and solves its steady state. Our main theoretical results are exhibited in section 3, while section 4 considers a broad calibration of the US economy. All proofs are set forth in the Appendix.

2 The model

We describe a continuous time OLG model à la Blanchard-Buiter-Weil with productive capital, CRRA preferences, exogenous productivity growth and both intra- and inter-family demographic growth. The main innovations relate to the demographic and family assumptions.

2.1 Assumptions

Demographics. Time is continuous and goes from 0 onward. The population consists of individuals who belong to different families. Each individual faces the constant risk of dying $\delta \geq 0$. He/she may also give birth to a child at the constant rate $m \geq 0$. However, some of these children are born unloved – they are 'disinherited children' in the terminology of Abel (1988). As a family is defined with respect to the notion of dynastic altruism – see Barro (1974) – unloved children do not belong to existing families, and thus generate new families. Let $\lambda \in [0, 1]$ be the proportion of loved children. The parameter λ is the intensity of parental altruism, or parental altruism for short. In addition, there is a constant inflow of external families entering the economy at rate $n \geq 0$. Typically, n is the rate of immigration.

Define:

$$n_I = \lambda m - \delta \tag{1}$$

$$n_E = n + (1 - \lambda) m \tag{2}$$

[3]There are strong reasons to calibrate the pure rate of time preference this low. As reported by Bullard and Russell (1999) (who actually defend negative rates of time preference), it is probably the only way to reconcile the fairly low observed real interest rates on risk-free assets, and the strong growth of individual/aggregate consumption.

where n_I is the internal rate of growth of existing families, while the stock of families grows at external rate n_E. We shall refer to n_E as the degree of disconnectedness, or disconnectedness for short. Disconnectedness unambiguously decreases with parental altruism.

The demographic structure of our model encompasses several earlier models. In the Ramsey case – the optimal growth framework – there is no disconnectedness in the economy, which corresponds to $n = \delta = 0$ and $\lambda = 1$. The Blanchard (1985) case of selfish agents with "finite horizons" and constant population corresponds to $\lambda = m = 0$ and $n = \delta > 0$. The length of the residual life-span follows a Poisson process, of which δ is the instantaneous death rate. Hence, each individual/family has an age-independent residual life expectancy worth $1/\delta$. The Weil (1989) case of overlapping families with infinitely-lived agents is obtained when $\lambda = \delta = 0$. Finally, the Buiter (1988) model corresponds to $\delta > 0$, $\lambda = 0$ and $n \geq 0$.

Family behavior. Let $N(\tau, t)$ denotes the size at time t of the family who entered the economy at time $\tau \in [0, t]$. We have:

$$N(\tau, t) = N(\tau, \tau) \exp[n_I (t - \tau)] \tag{3}$$

Let $c(\tau, t)$ denote the consumption at time t of an individual who belongs to the family who entered the economy at time τ. The objective of the representative individual of such a family is:

$$V(\tau, t) = \int_t^{+\infty} \exp[-(\rho - n_I)(z - t)] \, u(c(\tau, z)) \, dz \tag{4}$$

where ρ is the pure rate of time preference. Instantaneous utility u only depends on consumption. To make aggregation easier, we limit our study to the class of CRRA functions. Hence,

$$u(c) = \frac{c^{1-1/\sigma} - 1}{1 - 1/\sigma} \tag{5}$$

where $\sigma > 0$ is the elasticity of intertemporal substitution (EIS). At this stage, we must provide a restriction ensuring that the integral in equation (4) converges. For this purpose, let $A(t) \equiv e^{gt}$ be the level of technological progress embodied in labor with $g \geq 0$. To ensure that (4) is always finite the pure rate of time preference ρ must satisfy:

$$\rho > \rho_{\lim} \equiv n_I + g(1 - 1/\sigma) \tag{6}$$

There is a single asset which yields a risk-free interest rate $r(t)$. Each individual supplies a unit of labor and receives an age independent wage $w(t)$. Let $x(\tau, t)$ denote the financial wealth at time t of an individual who belongs to the family having arrived at time τ. The per capita budget constraint is:

$$\frac{\partial x(\tau, z)}{\partial z} = [r(z) - n_I] \, x(\tau, z) + w(z) - c(\tau, z) \tag{7}$$

Per capita financial wealth increases at rate r minus the growth rate of the family, plus the spread of wage income over per capita consumption. As we highlight below, our formalization encompasses several earlier contributions. However, models with selfish individuals of uncertain lifetimes require a slightly different interpretation of the budget constraint (7). In such models where $n_I = -\delta$, individuals have no concern for their heirs. As they may die with the positive (flow) probability δ, they face the perspective of leaving unintended bequests to their children. There is thus a demand for insurance which takes the following form (see Yaari, 1965 and Blanchard, 1985): insurance companies make premium payments to the living in exchange for the receipt of their estates in the event they die. Assuming perfect competition on the insurance market and abstracting from transaction costs in the insurance industry, the risk premium paid by the insurance company is δ.

Families enter the economy with no financial assets, but the families present at time $t = 0$ endowed with positive nonhuman wealth. Hence,

$$x(\tau, \tau) = \begin{cases} 0 \text{ if } \tau > 0 \\ x_0 > 0 \text{ if } \tau = 0 \end{cases} \tag{8}$$

A No-Ponzi game condition is assumed such that:

$$\lim_{z \to \infty} x(\tau, z)\, \beta(t, z) \geq 0 \tag{9}$$

where $\beta(t, z) \equiv \exp\left(-\int_t^z [r(s) - n_I]\, ds\right)$ is the discount factor prevailing between dates t and z.

Aggregation. Let $N(t)$ denote the size of the population at time t. We have $N(\tau, \tau) = (n_I + n_E)\, N(\tau)$. To obtain the population size at time t, we sum the different families available at each time. Using (3),

$$N(t) = N(0) \exp[n_I t] + \int_0^t N(\tau, t)\, d\tau = N(0) \exp[(n_I + n_E)\, t] \tag{10}$$

where $n_I + n_E \equiv n + m - \delta$ is the total population growth rate. Without loss of generality, $N(0) \equiv 1$ is assumed.

Due to population and productivity growth, the model is nonstationary. We therefore consider the following standard transformation. For any private variable $y(\tau, t)$, its aggregate 'productivity detrended' per capita counterpart is denoted by:

$$y(t) = A(t)^{-1} \left\{ y(0, t) \exp[-n_E t] + \int_0^t n_E \exp[-n_E (t - \tau)]\, y(\tau, t)\, d\tau \right\} \tag{11}$$

In equation (11), $n_E \exp[-n_E (t - \tau)]$ is the weight in the population as of time t of the families who entered the economy at time τ. Per capita (detrended) variables are therefore altered by changes in the rate of entry of new families through a composition effect. Note that the intra-family growth rate n_I does not create such a composition effect. Demographic parameters thus affect aggregate variables through two distinct channels: intra-family growth modifies the microeconomic behavior of each family, while inter-family growth affects the composition by age of families.

Technology. Output is produced by a neoclassical technology with capital depreciation. Let k be the capital stock per efficient unit of labor, $\mu \geq 0$ be the depreciation rate, and $A(t) f(k)$ be the production function in intensive form. The function $f : [0, \infty) \rightarrow [0, \infty)$ is such that $f'(k) > 0$ and $f''(k) < 0$ for all $k > 0$, and $f(0) \geq 0$, $\lim_{k \rightarrow \infty} f(k) = \infty$. Two points should be noticed concerning the generality of our assumptions upon the technology. First, we allow for capital depreciation. Second, the function f is compatible with a large set of production functions, including CES technologies. These restrictions do not guarantee the existence of equilibrium. We shall provide additional constraints on the function f later.

2.2 Dynamics and the stationary equilibrium

Let k and c denote respectively the aggregate capital and consumption per efficient unit of labor and p the propensity to consume out financial and human wealth.

Lemma 1 *The intertemporal equilibrium is the solution of the following system:*

$$\dot{k} = f(k) - (\mu + n_I + n_E + g) k - c \tag{12}$$

$$\dot{c} = \{\sigma [f'(k) - \mu - \rho] - g\} c - n_E p k \tag{13}$$

$$\dot{p} = \left[1 - \frac{(1-\sigma)[f'(k) - \mu] - n_I + \sigma\rho}{p}\right] p^2 \tag{14}$$

with $k_0 \equiv x_0$ *given,*

$$\lim_{z \rightarrow \infty} \beta(t, z) \omega(z) = 0 \tag{15}$$

and

$$\lim_{z \rightarrow \infty} [(1-\sigma)(f'(k(z)) - \mu) - n_I + \sigma\rho] > 0 \tag{16}$$

The presence of the rate of disconnectedness n_E in the consumption equation (13) is the main departure from the optimal growth model induced by this type of OLG structure. Note that conditions (15) and (16) guarantee that human wealth and the propensity to consume out of total family wealth are both positive and finite.

A non-trivial stationary equilibrium is a strictly positive vector (k^*, c^*, p^*) that solves $\dot{k} = c = \dot{p} = 0$ and satisfies:

$$\lim_{z \rightarrow \infty} [f(k^*) - k^* f'(k^*)] \exp[-(f'(k^*) - \mu - n_I) z] = 0 \tag{17}$$

$$(1-\sigma)(f'(k^*) - \mu) - n_I + \sigma\rho > 0 \tag{18}$$

The following result summarizes our knowledge of the stationary equilibrium.

Proposition 1 EXISTENCE AND PROPERTIES OF STATIONARY EQUILIBRIUM.

(i) There exists a unique stationary equilibrium if

$$f'(0) - \mu > \max\left\{\frac{n_I - \sigma\rho}{1-\sigma}, \rho + \frac{n_E + g}{\sigma}\right\}. \tag{19}$$

(ii) If there exists an equilibrium, it satisfies

$$\rho + g/\sigma \leq f'(k^*) - \mu \leq \rho + (n_E + g)/\sigma. \tag{20}$$

(iii) The equilibrium is saddle-path stable.

Point *(i)* provides a sufficient condition for the existence and uniqueness of a stationary equilibrium with positive capital and consumption while Point *(iii)* recalls that the stationary equilibrium has the saddle-path property, which implies that the perfect foresight trajectory is uniquely determined.

Point *(ii)* provides bounds for the stationary interest rate r^*. To understand these bounds, assume that the economy is in steady state and let $c(a)$ and $x(a)$ denote 'productivity detrended' consumption and assets of an individual who belongs to the representative aged-a family. Using (30) and (31), one has:

$$\begin{aligned} c(a) &= c(0)\exp\left[\sigma(r^* - \rho)a\right] && (21) \\ x(a) &= \frac{w(a)}{r^* - n_I - g}\left\{\exp\left[(\sigma(r^* - \rho) - g)a\right] - 1\right\} && (22) \end{aligned}$$

Consider first the lower bound $\rho + g/\sigma$. Indeed, $r^* > \rho + g/\sigma$ means that individual consumption grows at a higher rate than wages. As $\rho > \rho_{\lim}$, equation (22) implies that financial wealth is positive at all ages and increases with age. Since individuals are endowed with zero wealth at birth, they never accumulate if $r^* = \rho + g/\sigma$ and are always indebted if $r^* < \rho + g/\sigma$. In both cases, the 'initial' families are the only savers, and their savings are allocated between productive capital and loans to other families. This is not compatible with a constant stationary k^*, unless external growth $n_E = 0$. In this latter case, $r^* = \rho + g/\sigma$.

We now turn to the upper bound $\rho + (n_E + g)/\sigma$. Observe that 'productivity detrended' stationary average consumption is worth:

$$c = \lim_{a\to\infty} c(a)\exp\left[-(n_E + g)a\right] + \int_0^\infty n_E \exp\left[-(n_E + g)\alpha\right]c(\alpha)\,d\alpha \tag{23}$$

The term $n_E \exp(-n_E\alpha)$ is the weight of aged-α families in the population. Consequently, the term $n_E \exp\left[-(n_E + g)\alpha\right]c(\alpha)$ is the weight of aged-α family consumption in average consumption. It declines with age at rate $n_E + g$. Suppose that $r^* > \rho + (n_E + g)/\sigma$. The consumption growth rate of each member of a given family would be higher than the rate at which the relative weight of the family declines. Aggregate consumption per capita would therefore not be finite – which is impossible by construction. The particular case $r^* = \rho + (n_E + g)/\sigma$ leads to the same result, except in the case where $n_E = 0$. This explains why r^* is lower than $\rho + (n_E + g)/\sigma$. In the Ramsey case, external growth is nil, i.e. $n_E = 0$. Consequently, the lower bound equals the upper bound and the stationary interest rate is $r^* = \rho + g/\sigma$.

3 Dynamic efficiency, parental altruism and life expectancy

This section contains our main theoretical results. We proceed in four steps. First, we define dynamically (in)efficient stationary equilibria. Second, we focus on the role played by parental altruism λ. We show that increasing λ raises the likelihood of dynamic inefficiency. Third, we concentrate on the risk of dying δ. We show that decreasing δ raises the likelihood of dynamic inefficiency. Finally, we offer additional results concerning the remaining parameters of the model.

3.1 Dynamic (in)efficiency with productive capital

Let us first recall the condition for dynamic (in)efficiency: a competitive allocation is not a Pareto allocation whenever it is possible to increase the consumption of at least one generation without decreasing that of any other. In an economy with capital accumulation and an exogenous productivity trend, this leads to the following definition: a competitive path is dynamically inefficient if and only if detrended per capita capital is higher than that which maximizes detrended per capita consumption, the so-called Golden Rule of capital accumulation (Phelps, 1961).

Using equation (12), stationary per capita consumption is maximized for $k = \widehat{k}$ if and only if $\mu + n_E + n_I + g > 0$ and $f'\left(\widehat{k}\right) = \mu + n_E + n_I + g$, or $\mu + n_E + n_I + g \leq 0$ and $\widehat{k} = +\infty$. This leads to the following property.

Property 1 *Let $\gamma \equiv r^* - (n + m - \delta + g)$ be the interest-growth spread. The stationary equilibrium is dynamically efficient if and only if (i) $\mu + n + m - \delta + g > 0$ and $\gamma \geq 0$, or (ii) $\mu + n + m - \delta + g \leq 0$.*

The interest-growth spread γ is the actual discount rate used to compute the present value of the economy's wealth – the 'wage bill' in the words of de la Croix and Michel

(2002). Dynamic efficiency means that the present value of the economy's wealth is finite: there is nothing to redistribute at the end. Conversely, overaccumulation means that the present value of the economy's wealth tends towards infinity.

Let us write down the interest-growth spread γ in a convenient way. Let

$$\alpha(k) \equiv \frac{f'(k) - (\mu + n_E + g)}{f(k)/k - (\mu + n_E + g)} \tag{24}$$

Since $f'(k^*) \geq \rho + \mu + g/\sigma$ and $\rho > \rho_{\lim}$, we have $f'(k^*) > \mu + n_E + g$. Moreover, the strict concavity of function f guarantees that $f(k)/k > f'(k)$ for all $k > 0$. Consequently, $0 < \alpha(k^*) < 1$ for all $k > 0$.

Simple manipulations at steady state using (12), (13) and (14) yield:

$$\gamma = \rho - \rho_{\lim} + \frac{n_E}{\sigma}\left[\alpha(k^*) - \sigma\right] \tag{25}$$

We can deduce three properties from this equation.

First, $\rho \geq \rho_{\lim}$ means that dynamic inefficiency cannot occur in Ramsey models, in which $n_E = 0$. Indeed, Point (ii) of Proposition 1 provides a lower bound for the interest rate: $r^* \geq \rho + g/\sigma$. Together with $\rho > \rho_{\lim}$, this condition implies that $r^* > n_I + g$. The intuition for this result is very simple. Dynamic inefficiency means that the resources of the whole economy are infinite. But, in Ramsey models, family wealth coincides with the economy's wealth. Thus, family wealth in a dynamically inefficient economy is infinite. This implies the nonexistence of a competitive intertemporal equilibrium, a contradiction.

Second, dynamic inefficiency requires some disconnectedness, that is n_E must be positive. This result is well-known since Weil (1989), who shows in a model with $m = \delta = 0$ that $n > 0$ is a *necessary* condition for dynamic inefficiency. It is important to understand the exact role played by disconnectedness. Increasing the degree of disconnectedness does not necessarily raise the likelihood of reaching an inefficient steady state (see sub-section 3.4). Disconnectedness creates a wedge between the growth rate of family wealth and the growth rate of the economy's wealth. Consequently, family wealth can be finite although there is overaccumulation.

Third, the occurrence of dynamic inefficiency depends on the value of the EIS σ. Overaccumulation may occur if the EIS σ is larger than $\alpha(k)$ for all $k \geq 0$, which is obviously the case[4] when $\sigma > 1$. Conversely, overaccumulation can always be ruled out when σ is low.

[4] The condition $\sigma > 1$ is not a necessary condition for dynamic inefficiency, as our calibrations of Section 4 leading to dynamic inefficiency show.

3.2 The role of parental altruism

In this sub-section, we consider the relationship between dynamic efficiency and parental altruism λ. We study the marginal impact of λ upon the interest-growth spread γ.

Proposition 2 DYNASTIC ALTRUISM FAVORS DYNAMIC INEFFICIENCY.

Assume that there exists a stationary equilibrium.

(i) The interest growth spread γ is strictly decreasing in λ. Consequently,

(ii) If there is dynamic inefficiency in the Buiter model, i.e. $\lambda = 0$, then there is dynamic inefficiency for all λ.

(iii) If there is dynamic efficiency when $\lambda = 0$, and dynamic inefficiency when $\lambda = 1$, then there exists a critical intensity of parental altruism below (above) which there is dynamic (in)efficiency.

The likelihood of dynamically inefficient steady states rises with parental altruism. Indeed, take the derivative of the interest-growth spread γ defined in Property 1 with respect to λ:

$$\frac{d\gamma}{d\lambda} = \frac{dr^*}{d\lambda} < 0 \tag{26}$$

Parental altruism modifies the savings behavior of the family, while keeping intact the growth rate of the economy. An increase in λ raises the family growth rate. Consequently, it also increases the propensity to save of currently living individuals, who wish to maintain the consumption level of each future member of the family. This reduces the interest rate. Hence, a dynamically inefficient equilibrium is more likely to occur when the family growth rate is high.

Points *(ii)* and *(iii)* of Proposition 2 draw simple corollaries of Point (i). They suggest a methodology to analyze whether there is dynamic efficiency in a parametrized model or not. One should first consider the case where there is no parental altruism, i.e. $\lambda = 0$. If overaccumulation results, then there is overaccumulation for all intensities of parental altruism. If the equilibrium is dynamically efficient, then one should consider the polar case where $\lambda = 1$ – or the highest λ compatible with the existence of a stationary equilibrium. If dynamic efficiency prevails once again, then the equilibrium is dynamically efficient for all λ. If there is overaccumulation, then there exists a unique intensity of parental altruism that separates dynamic efficiency and dynamic inefficiency. Our numerical experiments in the next section obey such a methodology.

3.3 The role of life expectancy

In this sub-section, we consider the relationship between dynamic efficiency and the risk of dying δ.

Proposition 3 Long horizons favor dynamic inefficiency.

Assume that there exists a stationary equilibrium.

(i) The interest growth spread γ is strictly increasing in δ. Consequently,

(ii) If there is dynamic efficiency in the Weil model, i.e. when $\delta = \lambda = 0$, then there is dynamic efficiency in the Buiter model where $\lambda = 0$ and $\delta > 0$.

(iii) If there is dynamic inefficiency in the Buiter model, i.e. $\lambda = 0$ and $\delta > 0$, then there is dynamic inefficiency in the Weil model, i.e. $\lambda = \delta = 0$.

(iv) If there is dynamic inefficiency when $\delta = 0$, there exists a critical risk of dying above (below) which there is dynamic (in)efficiency.

Overaccumulation is more likely to occur more likely when life expectancy is long. To understand this result, take the derivative of the interest-growth spread γ defined in Property 1 with respect to δ:

$$\frac{d\gamma}{d\delta} = \frac{dr^*}{d\delta} - \frac{d\left(n + m - \delta + g\right)}{d\delta} \tag{27}$$

Life expectancy has two effects.

– On the one hand, *it decreases the stationary interest rate*. Other things being equal, an increase in life expectancy raises the growth rate of the labor force. This leads to a dilution effect, whereby per capita capital is lower. The dilution effect acts so as to raise the marginal return on capital. However, the dilution effect is offset within each family by a rise in the propensity to save, the concentration effect. In the Ramsey case where there is a unique family, the two effects are exactly balanced and the stationary per capita capital and interest rate do not depend on life expectancy. In the general case where disconnected families coexist, the concentration effect is stronger than the dilution effect and the interest rate decreases.

– On the other hand, *the fall in the risk of dying raises population growth.* This increases aggregate wealth. Both effects strengthen each other and reduce the interest-growth spread.

Proposition 3 sheds new light on Weil (1989). In the peculiar case where $\delta = \lambda = 0$, Weil (1989) shows that the equilibrium can be inefficient. He concludes that the length of planning horizons is not significant regarding the issue of overaccumulation. In fact, Points *(ii)* and *(iii)* make clear that Weil's configuration maximizes the likelihood of dynamically inefficient steady states. Moreover, Point *(iv)* tells that the equilibrium is always dynamically efficient when the risk of dying is high.

3.4 Other parameters

In this sub-section, we offer additional results concerning the remaining parameters of the model economy. We focus on the immigration rate n, the EIS σ, and the growth rate g of labor productivity.

Proposition 4 EIS, PRODUCTIVITY GROWTH AND IMMIGRATION.

(i) The interest-growth spread is strictly decreasing in σ.

(ii) There exists $\bar{g}$ such that there is dynamic (in)efficiency if $g > \bar{g}$ and $\sigma > 1$ (≤ 1).

(iii) There exists $\bar{n} > 0$ such that there is dynamic (in)efficiency if $n > \bar{n}$ and $\lim_{k \to 0} \alpha(k) \geq \sigma$ $(< \sigma)$.

The likelihood of dynamically inefficient steady states increases with the EIS σ. The reason is straightforward. Indeed, σ has no direct impact on output growth, while it raises the propensity to save out of total wealth. Capital, therefore, is more abundant. The interest rate is lower, and so is the interest-growth spread.

Unlike the EIS, it is difficult to establish an unambiguous relationship between the growth rate g of labor productivity and the interest-growth spread. On the one hand, g increases output growth, which tends to reduce the interest-growth spread. On the other hand, g increases future wage earnings, which reduces the propensity to save. This makes capital scarcer, and, therefore, the interest rate is higher. In Ramsey-type models, the global effect on the interest-growth spread simply depends on the EIS: the global effect is negative (positive) provided that $\sigma > 1$ $(\sigma < 1)$. Point *(ii)* shows that this latter property holds in our model for sufficiently large growth rates.

Increasing disconnectedness through immigration can work against dynamic inefficiency. To understand this point, note that both the interest rate and the growth rate of the population are increasing in n. Unlike intra-family growth, immigration does not directly influence saving behavior – see equation (22). However, it involves a composition effect that increases the weight of the population with low financial assets. Therefore, it reduces aggregate per capita capital, which in turn raises the stationary interest rate. The relationship between dynamic efficiency and the rate of immigration entry n is thus ambiguous. Point *(iii)* shows that asymptotically, the effect of n upon the occurrence of dynamic efficiency crucially depends on parameter $\alpha(0)$.

Condition (iii) is very convenient in the CES case. Let $f(k) = \left[\pi k^{\phi} + 1 - \pi\right]^{1/\phi}$, for all $k \geq 0$, with $\pi \in (0,1)$ and $\phi < 1$. Let $a(k) \equiv kf'(k)/f(k)$ denotes the capital share in gross output. Hence, condition (iii) becomes:

$$\lim_{k \to 0} \left[\frac{a(k) - (\mu + m) k/f(k)}{1 - (\mu + m) k/f(k)} - \sigma\right] \geq 0 \tag{28}$$

which reduces to $\lim_{k\to 0}[a(k)-\sigma]\geq 0$ since $\lim_{k\to 0} k/f(k)=0$. Now, observe that:

$$\lim_{k\to 0} a(k)=\begin{cases} 0 \text{ if } \phi>0 \\ \pi \text{ if } \phi=0 \\ 1 \text{ if } \phi<0 \end{cases} \tag{29}$$

At large n, the elasticity of substitution between capital and labor has a crucial importance regarding dynamic efficiency. Substitutability between capital and labor tends to generate dynamically inefficient steady states. In the Cobb-Douglas case, the question simply depends on the π/σ ratio.

4 Is the US economy dynamically inefficient?

Our theoretical analysis shows that parental altruism and life expectancy increase the likelihood of dynamic inefficiency. The purpose of this section is to question whether such theoretical points have any relevance from an empirical perspective. Our answer is positive: other things being equal, our model suggests that the US economy would be inefficient if agents were infinite-lived. However, plausible values of the risk of dying and the intensity of parental altruism are associated with dynamic efficiency.

4.1 Parameter values

We use US demographic data for the 1990s to discuss the relevance of dynamic efficiency in the US economy. The US economy is especially interesting as immigration strongly contributes to demographic growth. The requirement of 'some degree of disconnectedness' is therefore satisfied, whatever the strength of altruistic links within families.

Average annual data presented in Table 1 show that two thirds of the population growth is explained by natural growth, while the remaining one third is due to immigration. Of course, these data are not representative of a stable population: cohorts of the 'baby boom' were below ages of high mortality rate during the 1990s. Therefore, the crude death rate underestimates its long-run value.

Table 1: US demographics

Crude birth rate m	1.55%
Crude death rate δ	.87%
Natural increase growth $m-\delta$	.68%
Immigration rate n	.35%
Population growth $m+n-\delta$	1.03%

For the production side of the model, we refer to the literature – e.g. see Prescott (1986). The capital share in output is worth $a = .36$. Concerning capital depreciation, we take $\mu = 10\%$, which corresponds to an average capital life-span of ten years. Finally, the output growth rate is set at $g = 2\%$. Preference parameters are much less consensual and crucial for our results. Clearly, the stationary equilibrium is dynamically efficient if the pure rate of time preference is larger than the growth rate of output. Hence, many values that are regarded as empirically plausible rule out steady states with over-accumulation. However, many studies have reported even negative estimates of the rate of time preference[5]. The elasticity of intertemporal substitution (EIS) also plays an important role – as condition (6) and Proposition 4 show. In our baseline simulation, we choose the EIS to be $\sigma = 1$ and the pure rate of time preference to be $\rho = 1\%$. This is rather different from Bullard and Russel (1999) who set $\rho = -1\%$ and $\sigma = .59$. Consequently, we provide a sensitivity analysis with $\sigma \in [0.5, 2]$ and $\rho \in [-1\%, 3\%]$ that therefore includes Bullard's and Russel's choice of parameters. Table 2 recalls the values of the parameters in the baseline simulation[6].

Table 2: Parameters

Capital share, a	.36
Capital depreciation, μ	10%
Output growth, g	2%
EIS, σ	1
Time preference, ρ	1%

We now discuss the occurrence of dynamic (in)efficiency relative to parental altruism and life expectancy.

4.2 Results

The results of the baseline simulation are presented in Figure 1, which shows the pairs (λ, δ) leading to dynamic efficiency and dynamic inefficiency. There are two lines, and a set of plausible values. We now explain these various elements.

There exists a competitive equilibrium if and only if the pairs (λ, δ) are located above the line of finite achievements. Such a line is obtained from equations (1) and (6) by computing the risk of dying δ below which the objective of a dynasty diverges at stationary

[5]See Bullard and Russel (1999) and the references therein – especially in sub-section 3.3.4. As Bullard and Russel point out, such estimates are not surprising given the very low real interest rates and very strong growth rates of individual/aggregate consumption.

[6]Note, however, that the wage and the rate of time preference are age-independent in our framework. This implicitly means that seniority effects balance retirement effects. The question of the life-cycle path of earnings is not so important in a model that accounts for parental altruism.

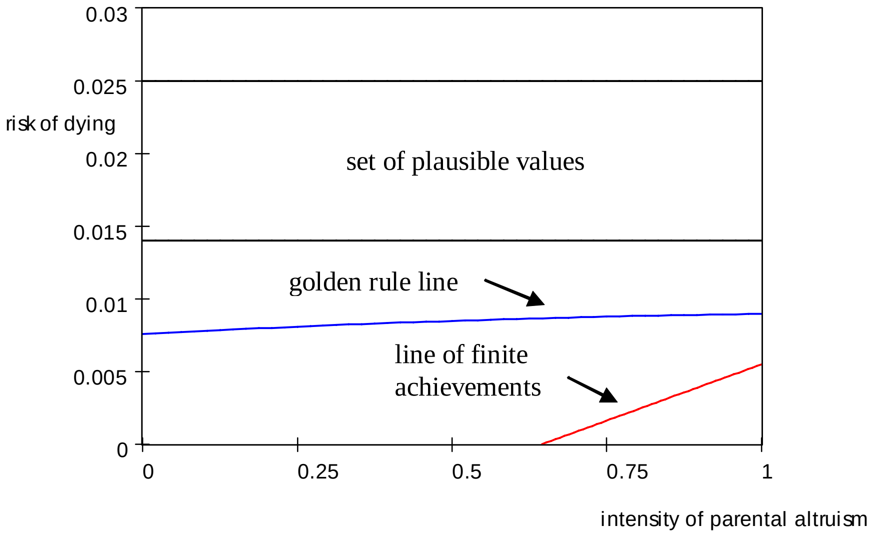

Figure 1: Baseline simulation: $\sigma = 1$, $\rho = 1\%$. There exists a competitive equilibrium above the line of finite achievements, and it is dynamically efficient above the golden rule line.

prices. It is positively sloped, because an increase in parental altruism raises the internal growth rate of the dynasty, that can only be balanced by a corresponding increase in the risk of dying.

The equilibrium is dynamically efficient above the golden rule line. To obtain this line, one must set the interest-growth spread to zero. Along the line, the risk of dying is an increasing function of parental altruism. Indeed, parental altruism boosts the saving motive, thereby decreasing the stationary interest rate. This must be compensated by an increase in the risk of dying, which reduces savings and lowers the growth rate.

Finally, the set of plausible values corresponds to the pairs (λ, δ) that seem empirically relevant. The risk of dying δ must belong to $[0.014, 0.025]$. The lower bound corresponds to an expected lifetime of 70 years, while the upper bound corresponds to an expected (active) lifetime of 40 years. Given that the golden rule line depends much more on δ than on λ, we have decided to let λ free[7].

Figure 1 displays two main results. First, dynamic inefficiency is compatible with the US fundamentals. It is a typical result of our simulations when the risk of dying is very low. Second, dynamic inefficiency is ruled out for plausible values of the risk of dying. Hence, the actual life expectancy is sufficiently low to prevent the occurrence of

[7] The parameter λ only affects the interest-growth spread through its impact on individual savings. By contrast, the parameter δ alters both individual behavior and the growth rate of output.

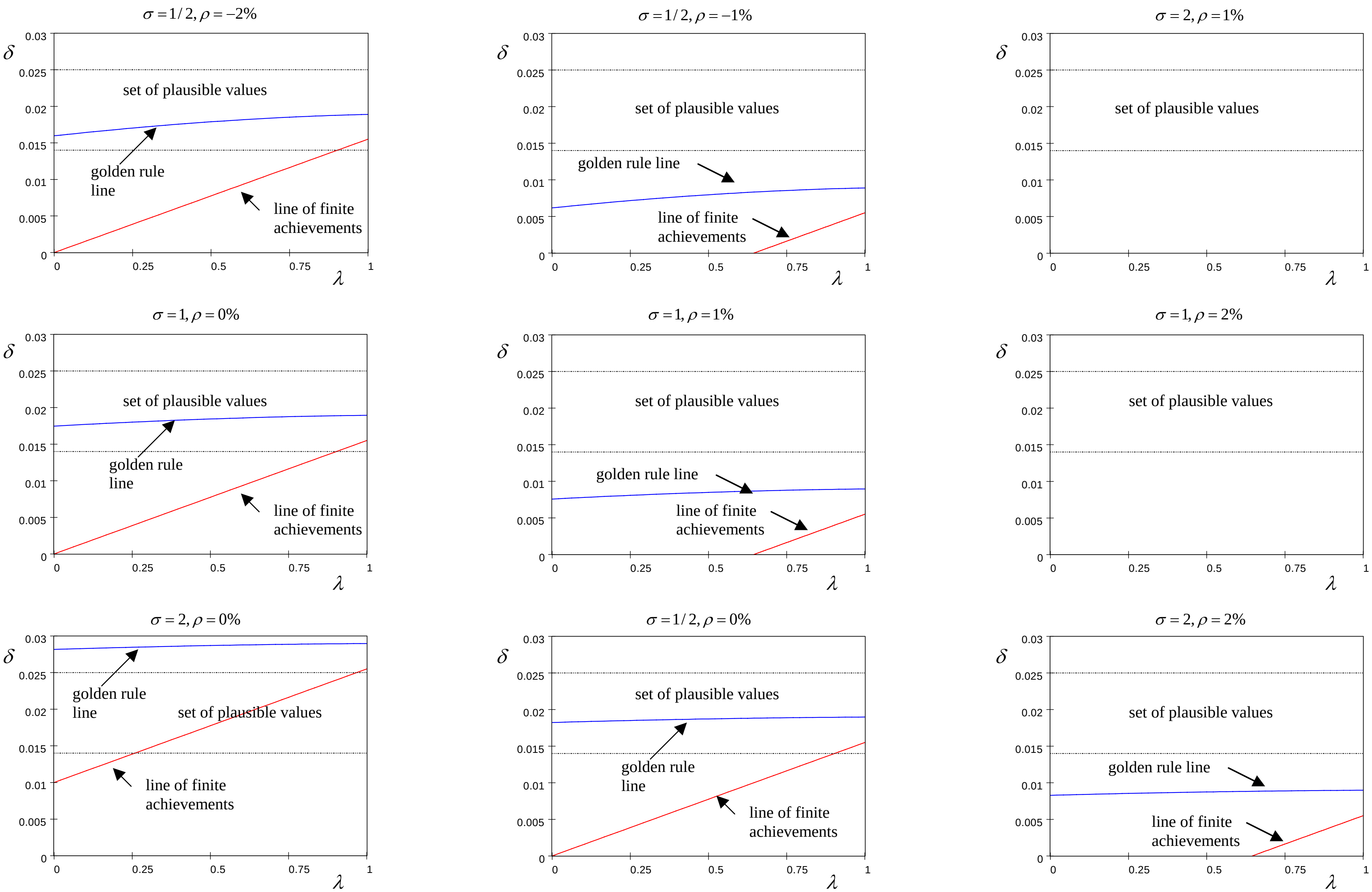

Fig.2 : Sensitivity analysis

There exists a competitive equilibrium above the line of finite achievements, and it is dynamically efficient above the golden rule line.

dynamically inefficient equilibria. Remarkably, the steady state would be inefficient if δ were set at its crude value given by Table 1. Such value of the risk of dying would involve an expected lifetime of 115 years.

The panel of Figure 2 proposes a sensitivity analysis[8]. We have tried various combinations of rate of time preference ρ and EIS σ. When the EIS is one half, dynamic inefficiency can occur for reasonable values of the risk of dying. However, the pure rate of time preference must be very low, i.e. $\rho = -2\%$. In such a case, the golden rule line falls within the set of plausible pairs (λ, δ). Assuming that the real (λ, δ) is uniformly drawn from the set of plausible values, diminished from the subset of parameters' values located below the line of finite achievements, one can compute the probability that the actual steady state is dynamically efficient[9]. This probability is about 2/3. When the EIS is one, dynamic inefficiency requires that the pure rate of time preference is worth $\rho = 0$. In such case, the probability that the steady state is dynamically efficient is about 60%. Finally, dynamic inefficiency is more likely when the EIS is two. The steady state is dynamically inefficient when $\rho = 0$, while it may be inefficient with a probability equal to 40% when $\rho = 1\%$.

The main lesson derived from such exercises is that dynamic inefficiency cannot be postulated unless one is willing to consider a very low pure rate of time preference ($\rho \leq 0$), or a very large EIS ($\sigma = 2$). For instance, when we consider the parameter values used in Bullard and Russel (1999) – $\rho = -1\%$ and $\sigma = 1/2$ –, our simulations yield a dynamically efficient equilibrium. Reaching a conclusion similar to theirs requires lowering the rate of time preference down to $\rho = -2\%$.

5 Conclusion

Macrodynamic models with finite lifetime and selfish individuals are compatible with (dynamically) inefficient equilibria, while models with infinite lifetime and altruistic individuals are not. This suggests that strong intergenerational altruism and high life expectancy prevent the occurrence of inefficient equilibria. This paper proposes to challenge this view. We build a continuous time OLG model with special emphasis on the demographic side. Individuals have finite horizons in the Blanchard (1985) meaning, there is intra-family growth and inter-family growth (immigration), and there is selective altruism: agents only love a proportion of their children. We show that the intensity of

[8]In the third column of Figure 2, for $\sigma = 1$, $\rho = 1\%$, and $\sigma = 2$, $\rho = 2\%$, both the golden rule line and the line of finite achievments are located below the horizontal axis: the equilibrium is always efficient.

[9]This probability is obtained by computing the surface occupied by plausible and dynamically efficient pairs of parameters, and by dividing it by the total surface occupied by plausible pairs of parameters.

parental altruism and the life expectancy actually favor overaccumulation. Theoretical results are illustrated by a parameterization from US demographic data. Our numerical exercises suggest that the actual life expectancy is sufficiently low to guarantee that the US economy is dynamically efficient.

APPENDIX

proof of lemma 1 The derivation of the equilibrium dynamics follows Blanchard (1985).

The representative individual maximizes (4) subject to (7), (8) and (9). Optimal consumption satisfies:

$$\frac{\partial c(\tau,t)}{\partial t} = \sigma\left[r(t)-\rho\right]c(\tau,t) \tag{30}$$

and the intertemporal budget constraint:

$$\int_t^{\infty} c(\tau,z)\,\beta(t,z)\,dz = x(\tau,t)+h(t) \tag{31}$$

where $\beta(t,z) \equiv \exp\left(-\int_t^z \left[r(s)-n_I\right]ds\right)$ and where $h(t)$ is the human wealth of an individual alive at t such that:

$$h(t) = \int_t^{\infty} w(z)\,\beta(t,z)\,dz \tag{32}$$

Computing (30) in (31) yields:

$$c(\tau,z) = p(t)\left[x(\tau,z)+h(t)\right] \tag{33}$$

where

$$p(t) = \left[\int_t^{\infty} \exp\left(-\int_t^z \left[(1-\sigma)r(s)-n_I+\sigma\rho\right]ds\right)dz\right]^{-1} \tag{34}$$

denotes the propensity to consume financial and human wealth.

Let $c(t)$ and $x(t)$ denote, respectively, aggregate (detrended) per capita consumption and financial wealth. Let also $\omega(t) = w(t)/A(t)$, and $\eta(t) = h(t)/A(t)$. We have:

$$x(t) = A(t)^{-1}\left\{x(0,t)\exp\left[-n_E t\right] + \int_0^t n_E \exp\left[-n_E(t-\tau)\right]x(\tau,t)\,d\tau\right\} \tag{35}$$

$$c(t) = p(t)\left[x(t)+\eta(t)\right] \tag{36}$$

Differentiating (34), (35), (36) with respect to time yields:

$$\frac{dx(t)}{dt} = \left[r(t)-(n_I+n_E+g)\right]x(t)+\omega(t)-c(t) \tag{37}$$

$$\frac{dc(t)}{dt} = \left\{\sigma\left[r(t)-\rho\right]-n_E-g\right\}c(t)+n_E p(t)\,\eta(t) \tag{38}$$

$$\frac{dp(t)}{dt} = \left[1-\frac{(1-\sigma)r(t)-n_I+\sigma\rho}{p(t)}\right]p(t)^2 \tag{39}$$

In the closed economy, the equilibrium is such that $x(t) = k(t)$ for all $t \geq 0$. Moreover factor prices equal marginal products:

$$r(t) = f'(k(t))-\mu \tag{40}$$

$$\omega(t) = f(k(t))-k(t)\,f'(k(t)) \tag{41}$$

Replacing (40), (41), and (36) in (37), (38) and (39), and dropping time indexes, yields the system composed by (12), (13) and (14). ■

proof of proposition 1 From equations (12), (13) and (14), a steady-state satisfies:

$$\begin{aligned} c &= \frac{n_E p k}{\sigma\left[f'(k)-\mu-\rho\right]-g} && (42)\\ c &= f(k)-(\mu+n_I+n_E+g)\,k && (43)\\ p &= (1-\sigma)\left[f'(k)-\mu\right]-n_I+\sigma\rho && (44)\end{aligned}$$

Replace (44) in (42) and define the functions $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{p}$ such that:

$$\begin{aligned} \mathbf{c}_1(k) &= k n_E \frac{(1-\sigma)\left[f'(k)-\mu\right]-n_I+\sigma\rho}{\sigma\left[f'(k)-\mu-\rho\right]-g} && (45)\\ \mathbf{c}_2(k) &= f(k)-(\mu+n_I+n_E+g)\,k && (46)\\ \mathbf{p}(k) &= (1-\sigma)\left[f'(k)-\mu\right]-n_I+\sigma\rho && (47)\end{aligned}$$

A non-trivial steady-state is a strictly positive vector (k^*, c^*, p^*) solving $c^* = \mathbf{c}_1(k^*) = \mathbf{c}_2(k^*)$ and $p^* = \mathbf{p}(k^*)$.

Point (i). We proceed in two steps. We give a sufficient condition such that there exists $k^* \in \left(0, \bar{k}\right)$ where $\bar{k}$ satisfies $f'\left(\bar{k}\right) = \mu + \rho + g/\sigma$. We then prove that k^* is unique.

Step 1. *If (19) holds, there exists k^* such that $\mathbf{c}_1(k^*) = \mathbf{c}_2(k^*)$, $c^* > 0$, and $p^* > 0$.*

We first characterize the domains of k such that $\mathbf{p}$ and $\mathbf{c}_1$ are positive. Define $\hat{k}$ such that $\mathbf{p}\left(\hat{k}\right) = 0$. Now observe that provided $f'(0) - \mu > (n_I - \sigma\rho)/(1-\sigma)$, one has $\mathbf{p}(k) > 0$ for (i) $k \in \left[0, \hat{k}\right)$ and $\sigma \in (0,1)$ and (ii) $k \in \mathbb{R}^+$ and $\sigma > 1$. For $\sigma = 1$, condition (6) guarantees that $\mathbf{p}(k) > 0$ for all k. Now turn to $\mathbf{c}_1$. First, $\mathbf{p}(k) > 0$ yields $\mathbf{c}_1(k) > 0$ if $\sigma\left[f'(k) - \mu - \rho\right] - g > 0$. Observe that this latter condition is satisfied for all $k \in \left[0, \bar{k}\right)$ if $f'(0) - \mu > \rho + g/\sigma$. Now, observe using (6), that for $\sigma \in (0,1)$, one has $\bar{k} < \hat{k}$ and $\rho + g/\sigma > (n_I - \sigma\rho)/(1-\sigma)$, while for $\sigma > 1$, one has $(n_I - \sigma\rho)/(1-\sigma) < \rho + g/\sigma$. Hence, $\mathbf{c}_1(k) > 0$ and $\mathbf{p}(k) > 0$ for all $k \in \left[0, \bar{k}\right)$ provided that

$$f'(0) - \mu > \max\left\{(n_I - \sigma\rho)/(1-\sigma), \rho + g/\sigma\right\} \tag{48}$$

Then, the function $\mathbf{c}_1 : \left[0, \bar{k}\right) \to \mathbb{R}^+$ satisfies $\mathbf{c}_1(0) = 0$ and $\lim_{k\to\bar{k}} \mathbf{c}_1(k) = +\infty$. Conversely, the function $\mathbf{c}_2 : \mathbb{R}^+ \to \mathbb{R}$ satisfies $\mathbf{c}_2(0) \geq 0$. If $\mu + n_I + n_E + g > 0$, $\mathbf{c}_2$ is first increasing, until k satisfies $f'(k) = \mu + n_I + n_E + g$, and then strictly decreasing to $-\infty$. If $\mu + n_I + n_E + g \leq 0$, $\mathbf{c}_2$ is strictly increasing and satisfies $\mathbf{c}_2\left(\bar{k}\right) \ll +\infty$.

Consequently, the existence is trivial if $\mathbf{c}_2(0) > 0$ and requires $\mathbf{c}_1'(0) < \mathbf{c}_2'(0)$ if $\mathbf{c}_2(0) = 0$. This latter condition can be expressed:

$$\frac{n_E\left[(1-\sigma)\left[f'(0)-\mu\right]-n_I+\sigma\rho\right]}{\sigma\left[f'(0)-\mu-\rho\right]-g} < f'(0)-(\mu+n_I+n_E+g)\,. \tag{49}$$

Note that assumption $f'(0) > \mu+\rho+g/\sigma$ implies, using (6), that $f'(0) > \mu+n_I+g$. Hence, condition (49) writes:

$$f'(0) > \mu+\rho+(n_E+g)/\sigma \tag{50}$$

Conditions (48) and (50) ensure the existence of a non-trivial steady-state. Merging them yields (19).

Step 2. k^* *is unique.*

Let $\phi(k) \equiv \mathbf{c}_1(k) - \mathbf{c}_2(k)$. Hence,

$$\phi(k) = kn_E\frac{(1-\sigma)\left[f'(k)-\mu\right]-n_I+\sigma\rho}{\sigma\left[f'(k)-\mu-\rho\right]-g} - f(k) + (\mu+n_I+n_E+g)\,k \tag{51}$$

Its derivative is

$$\begin{aligned}\phi'(k) \quad = \quad & \frac{n_E\left[f'(k)-\mu-n_I-g\right]}{\sigma\left[f'(k)-\mu-\rho\right]-g} - \frac{\sigma kn_E f''(k)\left[\rho-n_I-\left(1-\frac{1}{\sigma}\right)g\right]}{\left[\sigma\left[f'(k)-\mu-\rho\right]-g\right]^2} \\ & -f'(k)+(\mu+n_I+g)\end{aligned} \tag{52}$$

Using $\mathbf{c}_1(k^*) = \mathbf{c}_2(k^*)$ one has:

$$\frac{n_E\left[f'(k^*)-\mu-n_I-g\right]}{\sigma\left[f'(k^*)-\mu-\rho\right]-g} = \frac{f(k^*)}{k^*} - (\mu+n_I+g) \tag{53}$$

Replacing (53) in (52) yields:

$$\phi'(k^*) = \frac{f(k^*)}{k^*} - f'(k^*) - \frac{\sigma kn_E f''(k^*)\left[\rho-n_I-\left(1-\frac{1}{\sigma}\right)g\right]}{\left\{\sigma\left[f'(k^*)-\mu-\rho\right]-g\right\}^2} \tag{54}$$

Since f is strictly concave and given condition (6), one has $\phi'(k^*) > 0$.

Point (ii). The first inequality $f'(k^*)-\mu \geq \rho+g/\sigma$ follows directly from $\lim_{k\to\bar{k}}\mathbf{c}_2(k) = +\infty$. To prove the second inequality, let $\varepsilon < 1$ be such that $f'(k^*)-\mu = \rho + \left[g+n_E(1-\varepsilon)\right]/\sigma$. We will show that $\varepsilon > 0$. Replacing this expression in (53) to obtain:

$$\frac{f(k^*)}{k^*} = \frac{\rho-n_I-g\left(1-\frac{1}{\sigma}\right)}{1-\varepsilon} + \frac{n_E}{\sigma} + (\mu+n_I+g)\,. \tag{55}$$

Now, since $f(k) > kf'(k)$, one has:

$$\left[\rho-n_I-g\left(1-\frac{1}{\sigma}\right)\right]\frac{\varepsilon}{1-\varepsilon} > -\frac{n_E}{\sigma}\varepsilon \tag{56}$$

Given condition (6) and $n_E > 0$, we conclude that $\varepsilon < 0$ is not possible. If $n_E = 0$, then $\varepsilon = 0$ is the only solution.

Point (iii). The Jacobian matrix computed at steady state is:

$$\mathbf{J} = \begin{bmatrix} f'(k^*) - (\mu + n_I + n_E + g) & -1 & 0 \\ \sigma f''(k^*) c^* - n_E p^* & \sigma [f'(k^*) - \mu - \rho] - g & -n_E k^* \\ -(1-\sigma) f''(k^*) p^* & 0 & p^* \end{bmatrix} \tag{57}$$

with $c^* = n_E p^* k^* \{\sigma [f'(k) - \mu - \rho] - g\}^{-1}$ and $p^* = (1-\sigma)[f'(k^*) - \mu] - n_I + \sigma\rho$. Matrix $\mathbf{J}$ admits three eigenvalues v_1, v_2, v_3, which solve:

$$-v^3 + T(\mathbf{J}) v^2 - B(\mathbf{J}) v + D(\mathbf{J}) = 0 \tag{58}$$

where $D(\mathbf{J})$ and $T(\mathbf{J})$ are respectively the determinant and the trace, where:

$$\begin{aligned} B(\mathbf{J}) &= \begin{vmatrix} f'(k^*) - (\mu + n_I + n_E + g) & -1 \\ \sigma f''(k^*) c^* - n_E p^* & \sigma [f'(k^*) - \mu - \rho] - g \end{vmatrix} \\ &+ \begin{vmatrix} \sigma [f'(k^*) - \mu - \rho] - g & -n_E k^* \\ 0 & p^* \end{vmatrix} + \begin{vmatrix} f'(k^*) - (\mu + n_I + n_E + g) & 0 \\ -(1-\sigma) f''(k^*) p^* & p^* \end{vmatrix} \end{aligned} \tag{59}$$

The determinant satisfies:

$$\begin{aligned} D(\mathbf{J}) = \;& \{[f'(k^*) - (\mu + n_I + g)][f'(k^*) - \mu - \rho - (g + n_E)/\sigma]\} \sigma p^* \\ & + \sigma c^* \left[\rho - n_I - g\left(1 - \tfrac{1}{\sigma}\right)\right] f''(k^*) \end{aligned} \tag{60}$$

Use the bounds for $f'(k^*)$ established in Point (ii) and condition (6) to conclude that $D(\mathbf{J}) < 0$. Consequently, either there is one eigenvalue with a negative real part and the two others with positive real parts, or all eigenvalues have a negative real part. We now prove that the latter case cannot be observed. We proceed by contradiction by showing the impossibility of having simultaneously $T(\mathbf{J}) < 0$ and $B(\mathbf{J}) > 0$. The trace is given by:

$$T(\mathbf{J}) = [f'(k^*) - (\mu + n_I + n_E + g)] + \sigma \left[f'(k^*) - \mu - \rho - \frac{g}{\sigma}\right] + p^* \tag{61}$$

which is negative only if $f'(k^*) < (\mu + n_I + n_E + g)$. Then, $B(\mathbf{J})$ satisfies:

$$\begin{aligned} B(\mathbf{J}) \;&= [f'(k^*) - (\mu + n_I + n_E + g)][f'(k^*) - \mu - g - n_I] \\ & + \sigma f''(k^*) c^* + \sigma [f'(k^*) - \mu - \rho - (g + n_E)/\sigma] p^* \end{aligned} \tag{62}$$

which is positive only if $f'(k^*) > (\mu + n_I + n_E + g)$. Consequently, two eigenvalues have positive real parts and the third one has a negative real part. The equilibrium is locally saddle-path stable. ■

proof of proposition 2 We need to show that γ strictly decreases with λ. Points (ii) and (iii) are immediate corollaries. Using equations (25) and (53) and taking the derivative of γ with respect to λ yields:

$$\frac{d\gamma}{d\lambda} = f''(k^*)\frac{dk^*}{d\lambda} \tag{63}$$

Applying the implicit function theorem to equation $\phi(k^*;\lambda) = 0$ with ϕ defined in (51) gives:

$$\frac{dk^*}{d\lambda} = -\frac{\partial\phi(k^*;\lambda)/\partial\lambda}{\partial\phi(k^*;\lambda)/\partial k} \tag{64}$$

Proposition 1 shows that $\partial\phi(k^*;\lambda)/\partial k > 0$. Moreover,

$$\frac{\partial\phi(k^*;\lambda)}{\partial\lambda} = \frac{mk^*}{\sigma(r^*-\rho)-g}\{\sigma(r^*-\rho)-g-n_E-(r^*-n_I-g)\} \tag{65}$$

with $r^* = f'(k^*) - \mu$. Using the upper bound of the interest rate, we have $\sigma(r^*-\rho) < n_E + g$. Using the lower bound, we have $r^* > \rho + g/\sigma$. In addition, $\rho > \rho_{\lim}$ implies that $r^* > n_I + g$. Consequently, $dk^*/d\lambda > 0$. ■

proof of proposition 3 To establish (i), (ii) and (iii), we need to show that γ strictly increases with δ. Using equations (25) and (53) and taking the derivative of γ with respect to δ yields:

$$\frac{d\gamma}{d\delta} = f''(k^*)\frac{dk^*}{d\delta} + 1 \tag{66}$$

Applying the implicit function theorem to equation $\phi(k^*;\delta) = 0$ with ϕ defined in (51) gives:

$$\frac{dk^*}{d\delta} = -\frac{\partial\phi(k^*;\delta)/\partial\delta}{\partial\phi(k^*;\delta)/\partial k} \tag{67}$$

which, as in the previous proof, has the sign of $-\partial\phi(k^*;\delta)/\partial\delta$. Then,

$$\frac{\partial\phi(k^*;\delta)}{\partial\delta} = \frac{k^*}{\sigma(r^*-\rho)-g}\{n_E-\sigma(r^*-\rho)+g\} \tag{68}$$

As $g < \sigma(r^*-\rho) < n_E + g$, one has $dk^*/d\delta < 0$ and $d\gamma/d\delta > 0$.

To establish (iv), note that $\lim_{\delta\to\infty}\gamma = \infty$. The result follows from the fact that γ is strictly increasing in δ. ■

proof of proposition 4 To establish (i), use equations (25) and (53) and take the derivative of γ with respect to σ to obtain:

$$\frac{d\gamma}{d\sigma} = f''(k^*)\frac{dk^*}{d\sigma} \tag{69}$$

Applying the implicit function theorem to equation $\phi(k^*;\sigma)=0$ with ϕ defined in (51) gives:

$$\frac{dk^*}{d\sigma}=-\frac{\partial\phi(k^*;\sigma)/\partial\sigma}{\partial\phi(k^*;\sigma)/\partial k} \tag{70}$$

where,

$$\frac{\partial\phi(k^*;\sigma)}{\partial\sigma}=\frac{-k^*n_E(r^*-\rho)(r^*-n_I-g)}{[\sigma(r^*-\rho)-g]^2} \tag{71}$$

which is negative according to the proof of proposition 2. Consequently, $d\gamma/d\sigma<0$.

To establish (ii) take the derivative of γ with respect to g to obtain:

$$\frac{d\gamma}{dg}=f''(k^*)\frac{dk^*}{dg}-1 \tag{72}$$

Applying the implicit function theorem to equation $\phi(k^*;g)=0$ with ϕ defined in (51) gives:

$$\frac{dk^*}{dg}=-\frac{\partial\phi(k^*;g)/\partial g}{\partial\phi(k^*;g)/\partial k} \tag{73}$$

where

$$\frac{\partial\phi(k^*;g)}{\partial g}=k^*n_E\frac{(1-\sigma)r^*-n_I+\sigma\rho}{[\sigma(r^*-\rho)-g]^2}+k^* \tag{74}$$

Recall with the proof of proposition 1 that $(1-\sigma)r^*-n_I+\sigma\rho$ is the aggregate propensity to consume; since it is necessarily positive, conclude that $dk^*/dg<0$. However, the sign of $d\gamma/dg$ is indeterminate. One may nonetheless observe that,

$$sign\left\{\lim_{g\to+\infty}\gamma\right\}=\begin{cases}-\infty \text{ if } \sigma<1\\ <0 \text{ if } \sigma=1\\ +\infty \text{ if } \sigma>1\end{cases} \tag{75}$$

(iii) Observe from (51) that $k^*\to 0$ when $n\to+\infty$. Hence,

$$sign\left\{\lim_{n\to+\infty}\gamma\right\}=sign\left\{\lim_{k\to 0}(\alpha(k)-\sigma)\right\} \tag{76}$$

The result follows from the continuity of γ with respect to n. ■

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