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Garcia-Martinez, Jose A.

Universidad Miguel Hernandez

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José A. García-Martínez

Departamento de Estudios Económicos y Financieros. Universidad Miguel Hernández

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E-mail address: Jose.Garciam@umh.es

Phone 1: +34 966658886

Address for manuscript correspondence:
José A. García-Martínez
Departamento de Estudios Económicos y Financieros
Universidad Miguel Hernández
Avenida de la Universidad s/n, Edificio Torreblanca
E-03202, Elche, Alicante (SPAIN)

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Abstract

A selection process and a hierarchical promotion system in a dynamic model are considered as in Harrington (1998) and Garcia-Martinez (2010), where agents are "climbing the pyramid" in a rank-order contest based on the "up or out" policy. The population at any level of the hierarchy is matched in groups of n agents, and each group faces a particular environment. Agents are ranked according to the quality of their performances in each particular environment. The top k performing agents from each group are promoted. The fraction $\frac{k}{n}$ characterizes the local selectivity of the process. The role of the degree of local selectivity in the dynamic process where agents' types differ in their expected performances is studied. For low selectivity, the selection process is not strong enough to overcome the inertia of the initial population. If selectivity increases, only the best-performing type of agent will survive. If the selectivity is increased far enough, the worst-performing type also survives, and the proportion for which they account at equilibrium increases as selectivity increases. Therefore, surprisingly, *no matter how low the expected success rate of a type is, if the selection process has a high enough level of selectivity, agents of that type survive in the long run: Too much selectivity is always harmful to the best-performing type.*

JEL classification: D00; D23; C73; D72

Keywords: Social hierarchy; Selection; Selectivity; Promotion

1 Introduction

Almost all of economic analysis focuses on how individual choice determines group outcomes but it is also the case that selection has a role to play. Selection processes are common in all kinds of society. Broadly speaking, agents interact with one another within organizations and institutions, and as a result of that interaction certain individuals are promoted over others and achieve higher status in the form of greater political or economic power, increased prestige, more responsibility and wider intellectual influence. In such selection processes the characteristics of the individual agents obviously play an important role, but institutional factors such as the selectivity of the promotion system and hierarchical structures are also highly influential.

The concept of "degree of selectivity" is determined as in Garcia-Martinez (2010) by the fraction of agents promoted. For example, consider a set of agents ranked by the outcomes of their activities or by any other "agent's characteristic". A fraction $\alpha \in (0,1)$ of the highest-ranked agents is selected to be promoted. The closer α is to zero, the greater the selectivity of the process. The degree of selectivity of the selection process can be defined as $1-\alpha$.

A family of selection systems in a hierarchical institution is analyzed. The selection systems considered are characterized according to the amount of local selectivity implemented, and some interesting, counterintuitive properties are found.

In a dynamic model, a selection process and a hierarchical structure is considered as in Harrington (1998) and Garcia-Martinez (2010), where agents are "climbing the pyramid" in a rank-order contest based on the "up or out" policy. We generalize the selection process used in Harrington (1998) and Garcia-Martinez (2010) as shown at the end of this introduction.

As in the aforementioned papers, strategic interaction is not considered and each agent is endowed, at $t=0$, with one of two actions or behavioral rules: A and B. Thus, only two types of agent are considered: type A and type B. The environment where type A outperforms B is more frequent than the environment where B outperforms A. In this sense, A is a better type and has a higher expected success rate.

At any level of the hierarchy, the population of a level is matched in groups of n agents, and each group faces a particular environment. Agents are ranked according to the quality of their performances in this particular environment. The top k performing agents from each group are promoted: this is called *local selection* process[1]. Local selectivity increases as the ratio $\frac{k}{n}$ decreases.

In the next period, the agents promoted compete with one another again on their new level for promotion to the following level, always under the "up or out" policy. Thus, the non-promoted agents are no longer considered for promotion.

A-agents perform well more often than B-agents. Therefore, increases in selectivity might be expected

[1] In Garcia-Martinez (2010), another kind of selection, referred to as "global selection" is considered and parameterized, and local selection is fixed with $n=2$ and $k=1$.

to punish B-agents, and the proportion of B-agents promoted to decrease as selectivity increases. However, selectivity plays a different role. If the level of selectivity is low enough, the selection process is not strong enough to overcome the inertia of the initial population. The dynamic depends on the initial conditions, and the population eventually becomes homogeneous, i.e. either type A or type B. If selectivity increases enough, the whole population will come to be type A for any initial mixed population. In that case, the selection process is strong enough to eventually select agents of type A, the best performers. Finally, if the selectivity is increased far enough, type B agents also survive, and the proportion for which they account at equilibrium increases as selectivity increases. Thus, surprisingly, *no matter how low the success rate of a type is, if the selection process has a high enough level of selectivity, agents of that type survive in the long run. Too much selectivity is always harmful to the best performers.* Therefore, for low and intermediate levels of selectivity, the selection process favors homogeneity. At the beginning, the increase in selectivity favors the best performers and makes the process independent of the initial conditions. However, higher increases make the population heterogeneous in the long run. It seems that selectivity favors diversity. The intuition behind our results is broadly explained in Section 3.

The structure of our model is similar to the way in which some sports competitions are organized. There are different levels, players compete in separate groups at each level and the best players in each group are promoted to the next level. The final goal of each player is to get the top of the pyramid.

A sales company that promotes people according to their success in selling would be a stylized example. The company employs men and women and men sell better to men and women sell better to women. If the potential market has more men than women, men could be the A-agents and women the B-agents. In another example, rules (A and B) can be thought of as different available technologies: one of them is the best more often than the other, and agents are proficient in either technology A or B. The promotion of B-agents may or may not be desirable depending on the nature of the situation and the preferences of the institutions concerned. It is not easy to find an application that fits all of the model's elements because the model seeks to represent a family of complex institutions in a very stylized manner to point out a very specific characteristic of a selection process. Obviously, in any real situation the selection process is influenced by many more factors. However, we believe that the properties identified in our model are robust enough to play a role in more complex situations.

In a dynamic model such as the one in the present paper, Garcia-Martinez (2010) analyzes a promotion system that works in two steps. The first step is like the system considered in the present paper but always with $k = 1$ and $n = 2$, local selectivity is fixed. Agents are matched in pairs and one of them is selected. In the second step, those agent selected in the first step are pooled together and the top fraction θ of best-performing agents is eventually promoted; this is referred to as "global selection". Selectivity increases as θ decreases. However, the *effective* selectivity is limited and neither low nor high levels of selectivity can be considered. By contrast, in the present paper, effective selectivity is not limited and can be increased and decreased as much as desired. In any case, the nature of selectivity is different: In Garcia-Martinez (2010) a global selection process is considered and parameterized, while in the present paper only a local selection process is parameterized and considered. Global selectivity can be also detrimental to the best performers but only if the gap between the expected success rates of the agent types is not too high. However, with local selectivity, no matter how high this gap is, a high enough degree of local selectivity is always detrimental to best-performing type.

Harrington (1998, 2000) uses a selection process in a hierarchical structure to compare the performance of rigid behavior with that of flexible behavior. Harrington considers only one particular selection process with a unique level of selectivity, while we consider a family of selection processes that we characterize according to the amount of selectivity implemented. Harrington (1999a, 2003) follows the same line of study but now introduces the concept of "social learning", i.e. young agents who observe the older ones at the top of the hierarchy and imitate them. Finally, Harrington (1999b) adopts a strategic approach. In these papers, the degree of selectivity is always $\frac{1}{2}$. In fact, the promotion system that he uses in his papers is $S_{[n=2,k=1]}$. Thus, the level of selectivity is fixed. By contrast, Vega-Redondo (2000) employs only global selection, although with a different approach and purpose.

Although Harrington considers more kinds of agent behavior, our result can be applied to his model. Regarding Harrington (1998), which was the main inspiration of the present paper, it could be said that his results would change if the degree of selectivity increases or decreases. If the level of selectivity is increased enough, the stable equilibrium where the entire population follows the most successful rigid rule will lose its stability and there will be a heterogeneous population in the long run following different rules. And if the degree of selectivity is decreased enough, all the homogenous equilibria become locally stable equilibria. This assertion can be made because the intuition explained in Section 3 can be applied to his model.

The present paper is also related to the literature on tournaments developed since the seminal paper by Lazear and Rosen (1981). There are also some papers which have also focused on the selection role of contests, e.g., Rosen ((1986), Section V), and Hvide and Kristiansen (2002). The role of promotion in providing incentives is also analyzed in Fairburn and Malcomson (2001)

The rest of this paper is presented as follows: Section 2 describes the model and the dynamic equation, Sections 3 analyzes the dynamics and provides some intuitions, and Section 4 gives the conclusion.

2 The Model

A hierarchical system with a lowest level and no upper bound on the highest level is considered as in Harrington (1998). The initial population resides at the lowest level of the system, and comprises two types, A and B, which compete for promotion. The objective is to analyze how the proportions of A-agents and B-agents in this initial cohort of agents change as they migrate up through the hierarchy. This analysis is conducted for any degree of local selectivity. If the hierarchy is to be kept "full" then at the end of each round a fresh cohort of agents must enter the lowest level to replace those who have moved on. Another structure that can be considered is a hierarchy with just T levels where new agents imitate the agents at the top, as described in Harrington (1999a). Note that the two cases are equivalent if the new agents in this second case reproduce the profile of agents at the top.

Therefore, it is considered that at level t there is a large enough population of agents (a continuum), where $a_t \in [0,1]$ denotes the proportion of A-agents at level t, and $1-a_t$ the proportion of B-agents. We seek to specify a dynamic function $a_{t+1} = f(a_t)$ that relates the proportion of A-agents at level t to the proportion of A-agents after going through a selection process, i.e. the proportion of A-agents in the set of agents promoted to level $t+1$.

It is considered that the agents at level t are **randomly matched** in groups of $n \geq 2$. We assume that the random matching process has the following properties: *First*, the probability with which a given agent is matched with agents of given types equals the product of the proportions of agents of the respective types in the population. *Second*, the proportion of a given class of grouping is equal to the probability (ex-ante) of such a grouping. The existence of a random matching process with these properties is proved in Alós-Ferrer[2] (1999).

Thus, the proportion of *groups* containing x A-agents (and $(n-x)$ B-agents) is equal to the probability of such a group [3], i.e. $\binom{n}{x} a_t^x (1-a_t)^{n-x}$. This is also the proportion of *agents* in groups with x A-agents with regard to the initial population (level t) because the groups are composed of equal numbers of agents.

These agents face a stochastic environment which is the same for all the members of a particular group. However, the environment of each group is stochastically independent of other groups. We categorize all the different possible environments into two types. In a type 1 environment, A-agents respond correctly to the environment (they are successful) and B-agents respond wrongly (they are unsuccessful). In a type 2 environment, B-agents respond correctly and A-agents wrongly[4]. The probability[5] of an environment of type $i = 1, 2$ is P_i, with $P_1 + P_2 = 1$. Therefore, each agent faces an uncertain future environment, but there is no aggregate uncertainty because of our assumptions. Therefore, at each level after the random matching, a proportion P_1 (P_2) of the groups has a type 1 (2) environment. This is assumed to be i.i.d. across levels, so that the probability of an agent facing a given environment is independent of the environment that he/she has faced in the past.

Therefore, the proportion of agents in groups with a number x of A-agents under a type 1 environment is $\binom{n}{x} a_t^x (1-a_t)^{n-x} P_1$. In such groups A-agents outperform B-agents. The system selects the k top-performing agents from each group, with $k \leq n$. This process is called **local selection**. The agents selected from each group are promoted. The proportion of promoted agents is $\frac{k}{n}$ with regard to the initial population (level t). This ratio measures the selectivity of local selection. Thus, if a group in a type 1 environment has more A-agents than

[2] Alós-Ferrer (1999) gives a constructive existence proof for the case $n = 2$. The generalization to groups of n agents is straightforward.

[3] The x is distributed as a binomial distribution, $x \backsim B(n, a_t)$.

[4] Two more environments can be considered in which both rules are right answers or both rules are wrong. These environments add no new insights to the analysis, so we do not consider them.

[5] This probability P_i can also be seen as the expected success rate of an agent of type i, with $i = 1, 2$.

vacancies available (i.e. $x \geq k$), then all the agents selected from this group are successful, and the proportion of A-agents who are successful is $\frac{k}{n}\binom{n}{x}a_t^x(1-a_t)^{n-x}P_1$. However, if $x < k$ then x successful A-agents are selected and some unsuccessful B-agents have to be randomly chosen to fill the $k-x$ vacancies, and the proportion of selected A-agents who are successful is $\frac{x}{n}\binom{n}{x}a_t^x(1-a_t)^{n-x}P_1$. Analogously, a fraction P_2 of groups will face a type 2 environment and similar reasoning applies. Consequently, the total proportion of selected A-agents who are successful will be: $S_t^a = \sum_{x=0}^{k-1}\frac{x}{n}\binom{n}{x}a_t^x(1-a_t)^{n-x}P_1 + \sum_{x=k}^{n}\frac{k}{n}\binom{n}{x}a_t^x(1-a_t)^{n-x}P_1 = \sum_{x=0}^{n} min[x,k]\frac{1}{n}\binom{n}{x}a_t^x(1-a_t)^{n-x}P_1$. On the other hand, the proportion of selected A-agents who are unsuccessful comprises the A-agents selected from the groups under the type 2 environment that do not have enough B-agents to fill all the k vacancies, i.e. $x-(n-k)$ A-agents: $U_t^a = \sum_{x=n-(k-1)}^{n}(x-(n-k))\frac{1}{n}\binom{n}{x}a_t^x(1-a_t)^{n-x}P_2$

The proportion of A-agents selected will be $S_t^a + U_t^a$ with regard to the initial population (level t). Finally, the proportion of A-agents is $\frac{1}{\frac{k}{n}}(S_t^a + U_t^a)$ with regard to the population of agents selected (level $t+1$). Therefore the dynamic equation has this form:

$$a_{t+1} = f(a_t) = \frac{1}{\frac{k}{n}}(S_t^a + U_t^a) = \sum_{x=0}^{n}\frac{min[x,k]}{k}\binom{n}{x}a_t^x(1-a_t)^{n-x}P_1 + \sum_{x=n-(k-1)}^{n}\frac{(x-(n-k))}{k}\binom{n}{x}a_t^x(1-a_t)^{n-x}P_2 \tag{1}$$

We use $S_{[n,k]}$ to denote the promotion system that selects k agents from groups of n agents. We consider the following concepts of equilibrium.

The point $a^* \in [0,1]$ is said to be a *steady state* of equation (1) if it is a fixed point of $f(.)$, i.e. $f(a^*) = a^*$.

It is obvious that $f(0) = 0$ and $f(1) = 1$. Consequently, $a = 0$ *and* $a = 1$ *are always steady states.*

The point $a^* \in [0,1]$ is a *globally stable equilibrium* of the dynamic system given by $a_{t+1} = f(a_t)$ if for all $a_0 \in (0,1)$, $\lim_{t\to\infty} a_t = a^*$. The point $a^* \in [0,1]$ is a *locally stable equilibrium* if only for $a_0 \in B(a^*,\varepsilon) \subset (0,1)$, $\lim_{t\to\infty} a_t = a^*$, where $B(a^*,\varepsilon) = \{a \in (0,1)/\, |a-a^*| < \varepsilon\}$ with $\varepsilon > 0$.

$a^*\left[S_{[n,k]}\right]$ denotes an inner steady state a^* for the promotion system $S_{[n,k]}$, i.e. a^* belongs to the open interval $(0,1)$.

The level of local selectivity is characterized by the quotient $\frac{k}{n}$: in fact it is $1-\frac{k}{n}$. The smaller $\frac{k}{n}$, the greater the selectivity.

In the following section, the dynamics is analyzed and the intuition behind the result is provided.

3 Analysis

Let a^* be an inner root of the equation $f(a_t) - a_t = 0$ that belongs to the open interval $(0,1)$. This root exists and is unique if either $\frac{k}{n} < P_2$ or $\frac{k}{n} > P_1$ (see the proof of the following result in the Appendix). By definition, this root is a steady state.

The following result characterizes the dynamic.

Proposition 1 *Let* $P_1 > P_2$, $\frac{k}{n} < 1$ *and consider the selection process* $S_{[n,k]}$ *specified by the equation (1):*

1) If $\frac{k}{n} < P_2$ *there is only one inner steady state* a^* *and it is globally stable.* $a = 0$ *and* $a = 1$ *are unstable.* B*-agents survive.*

2) If $\frac{k}{n} \in [P_2, P_1]$ *there are no inner steady states.* $a = 0$ *is unstable and* $a = 1$ *is globally stable.* A*-agents are eventually the only survivors.*

3) If $\frac{k}{n} > P_1$ *there is only one inner steady state* a^**, which is unstable and divides the interval* $a \in (0, 1)$ *into two subintervals. The subinterval* $(0, a^*)$ *is the basin of attraction of* $a = 0$ *and the subinterval* $(a^*, 0)$ *that of* $a = 1$ *and both are locally stable. Initial conditions determine whether either* A*-agent or* B*-agents are the only survivors.*

Therefore, only three types of behavior are possible. *Type I*: $a = 0$ and $a = 1$ are unstable and there is an inner steady state that is globally stable, and B-agents survive. *Type II*: $a = 0$ is unstable and $a = 1$ is stable and $a = 1$ is globally stable; A-agents would be the only survivors. *Type III*: $a = 0$ and $a = 1$ are locally stable. There is one inner unstable steady state that determines the basin of attraction.

Under low enough levels of selectivity ($\frac{k}{n} > P_1$) in the promotion system, the selection process is not strong enough to overcome the inertia of the initial population. The dynamic depends on the initial conditions, and the population eventually becomes homogeneous, i.e. either type A or type B: the dynamic is *type III*. If selectivity increases enough ($\frac{k}{n} \in [P_2, P_1]$), the whole population will be type A for any initial population. In that case the selection process is strong enough to eventually select the A-agents: the dynamic is *type II*. Finally, if selectivity is increased far enough ($\frac{k}{n} > P_1$), type B agents also survive: the dynamic is *type I*. Therefore, *no matter how low the success rate of a type is, if the selection process has a high enough selectivity, agents of that type survive in the long run.*

To understand why this happens, it must first be observed that the dynamic of the system depends on the probabilities of promotion of each type of agent[6]. For example, if the system is in a period t and the probability of an A-agent being promoted is greater than that of a B-agent, then the proportion of A-agents in period $t + 1$ is greater than in t, i.e. the proportion of A-agents increases and the proportion of B-agents decreases. Now focus on a particular type of agent who faces one of the two following extreme scenarios:

- If agents of this particular type are scarce (say close to extinction), an agent of this type will generally match with agents of the other type[7]. Thus, in general, there will only be one agent of this particular type in a group, who will only be promoted if he/she is successful (responds in the right way to the environment). In this case, the rest of the agents in his/her group will be unsuccessful. Thus, if an agent of this scarce type is successful, he/she is almost sure to get promoted. In such a context, the probability of promotion of this particular type of agent is not influenced by an increase in the degree of selectivity in the system.

[6] See Lemma 5 in the appendix.
[7] This happens with a probability close to one.

His/her probability of promotion depends almost entirely on his/her probability of success, i.e. it is P_1 if the agent is type A and P_2 otherwise.

- However, when agents of this particular type abound (say the other type is close to extinction), an agent of this particular type will generally match with agents of his/her own type[7]. Thus, if all the agents in a group are of the same type, they respond in the same way to the same environment. The competitors of a particular agent in his/her own group are as successful (or unsuccessful) as he/she is. Thus, for purposes of promotion it does not matter at all whether this particular type of agent is successful or not: the probability of promotion depends almost entirely on how many people are promoted. Therefore, the probability of promotion of this particular type of agent is strongly influenced by an increase in the degree of selectivity in the system.

Therefore, an increase in selectivity tends to punish the more common type of agents because it decreases their probability of promotion but does not affect the relatively scarce type. If selectivity is high enough, no one type can be abundant enough to be the only survivor. Thus, *diversity can be favored or punished by tuning the level of selectivity.*

To give a clearer picture of just what is happening consider the following particular case. Assume that the dynamic of the model is *type II*. In that case the only global equilibrium is the whole population being type A ($a^* = 1$). Consequently, for any state of the system a_t, the probability of promotion of A-agents is greater than that of B-agents. The rest of the paragraph focuses on states in which $a_t \simeq 1$. When $a_t \simeq 1$, the probability of promotion of A-agents is approximately equal to the proportion of agents promoted ($\frac{k}{n}$), and the probability of promotion of B-agents is approximately equal to their probability of success (P_2). Obviously, if the dynamic is *type II*, then $\frac{k}{n} > P_2$. However, if $\frac{k}{n}$ is decreased (selectivity increases), the probability of promotion of A-agents decreases, while the probability of promotion of B-agents remains practically unchanged. Therefore, if $\frac{k}{n}$ decreases beyond P_2, then the probability of promotion of B-agents is greater than that of A-agents, and the proportion of A-agents will decrease in the next period. When this happens the homogeneous equilibrium $a^* = 1$ becomes unstable, and the system converges to a stable globally mixed equilibrium in which there are agents of both types. The dynamic changes from *type II* to *type I*.

On the other hand, it can be shown by a similar argument that if $\frac{k}{n}$ increases beyond P_1, the state $a = 0$ becomes locally stable. In that case, for states of the system close to $a = 0$, the probability of promotion for B-agents is greater than for A-agents. In addition, the state $a = 1$ changes from globally to locally stable, and the dynamic changes from *type II* to *type III*. The less selective a system is, the easier it is for it to be dominated by one type of agent and for it to achieve homogeneity.

Therefore, if selectivity increases two forces work together: On the one hand, the more selective a system is, the more important an agent's success or failure in the promotion becomes and, thus, the less important the effect of the initial proportions of the different types of agent is. On the other hand, an increase in selectivity tends to punish the more common type of agent because it decreases their probability of promotion but does not affect that of the relatively scarce type. Thus, selectivity can encourage diversity.

To obtain more insights about the inner steady state, two specific values of k are considered in the following sections. This allows us to obtain a closed form of equation (1).

3.1 Selection with $k = n - 1$, $S[n, k = n - 1]$

In this section $k = n - 1$ is assumed. A wide range of degrees of local selectivity can be considered. The level of local selectivity is characterized by the quotient $\frac{k}{n}$: In fact it is $1 - \frac{k}{n}$. By changing parameter n, levels of local selectivity[8] between 0 and $\frac{1}{2}$ can be considered. An increase in n decreases the level of selectivity because $\frac{k}{n} = \frac{n-1}{n}$ increases.

As $\frac{k}{n} = \frac{n-1}{n} \in (\frac{1}{2}, 1]$, necessarily $\frac{k}{n} > P_2 \Leftrightarrow \frac{n-1}{n} > P_2$. Thus, by Proposition 1 only behaviors of *type II* and *III* can be only found. Let $a^*[S_{[n,k=n-1]}]$ be the unique inner steady state if $\frac{n-1}{n} > P_1$.

The following result shows that $a^*[S_{[n,k=n-1]}]$ is increasing in n. The proof is in the appendix.

Proposition 2 $a^*[S_{[n,k=n-1]}] > a^*[S_{[n+1,k=n-1]}]$ *if* $\frac{n-1}{n} > P_1$. *If* $\frac{n-1}{n} < P_1$, $a = 1$ *is globally stable.*

If n increases enough (selectivity decreases), the basin of attraction of the locally stable steady state $a = 0$ increases and that of the locally stable steady state $a = 1$ decreases. Therefore, as selectivity increases the inner steady state decreases to zero (increasing the basin of attraction of $a = 1$), and $a = 1$ eventually becomes globally stable. The type shifts from III to II.

Proposition 3 *Let* $\frac{k}{n} = \frac{n-1}{n} > P_1$, *if the gap* $P_1 - P_2$ *increases, the inner steady state* $a^*[S_{[n,k=n-1]}]$ *decreases.*

The greater the gap between the success rates $(P_1 - P_2)$ is, the greater the advantage obtained by type A agents is. This increase produces a decrease in the basin of attraction of $a = 0$.

3.2 Selection with $k = 1$, $S[n, k = 1]$

In this section $k = 1$ is assumed. By changing parameter n, levels of local selectivity[9] between $\dfrac{1}{2}$ and 1 can be considered. An increase in n increases the level of selectivity.

As $\frac{k}{n} \in (0, \frac{1}{2}]$, necessarily $\frac{k}{n} < P_1 \Leftrightarrow \frac{1}{n} < P_1$. Thus, by Proposition 1 only behaviors of *Type I* and *II* can be found. Let $a^*[S_{[n,k=1]}]$ be the unique globally stable steady state.

The following result shows that $a^*[S_{[n,k=1]}]$ is decreasing in n. The proof is in the appendix.

Proposition 4 $a^*[S_{[n,k=1]}] \geq a^*[S_{[n+1,k=1]}]$, *and the inequality is strict if* $\frac{1}{n} > P_1$.

[8] Since we consider $k = n - 1$ in this section, the fraction $\frac{k}{n} = \frac{n-1}{n}$ belongs to $[\frac{1}{2}, 1)$, thus the level of local selectivity is in the interval $[0, \frac{1}{2})$.

[9] Since we consider $k = 1$ in this section, the fraction $\frac{k}{n}$ belongs to $(0, \frac{1}{2}]$, thus the level of local selectivity is in the interval $[\frac{1}{2}, 1)$.

If n increases enough (selectivity increases), type B agents not only survive but also increase in proportion. If selectivity is neither too high nor too low, $a = 1$ are globally stable and A-agents are the only survivors. However, if it is increased, B-agents also survive. The *type* shifts from *II* to *I*.

Proposition 5 *Let $\frac{k}{n} = \frac{1}{n} < P_2$, if the gap $P_1 - P_2$ increases, the globally stable inner steady state $a^*[S_{[n,k=1]}]$ increases.*

The greater the gap between the success rates ($P_1 - P_2$) is, the greater the advantage obtained by type A agents is. This increase produces an increase in the globally stable inner steady state $a^*[S_{[n,k=1]}]$.

4 Conclusion

The role of local selectivity is analyzed in a family of promotion systems within a hierarchy. The focus on local selectivity allows a complete range of degrees of selectivity to be considered[10]. Thus, three kinds of dynamic behavior are found depending on the level of local selectivity. The dynamic depends on the probability of promotion of each type of agent, which in turn depends on three factors: first, the composition of the population, i.e. the proportion of agents of each type; second, how strong the selection process is (which we measure with the level of local selectivity); and third, the probability of success in the activity undertaken by agents within the organization. As selectivity increases the initial conditions becomes less relevant, so for an intermediate level of selectivity the best performing agents are the only survivors. However, if selectivity is increased enough, the agents with the lowest expected success rate also survive, no matter how low their expected success rate is. Too much selectivity is always harmful to the best performers. An Increase in selectivity tends to punish the more common type of agent because it decreases their probability of promotion, while it does not affect the relatively scarce type. Consequently, care must be taken with the degree of selectivity in the promotion mechanism within hierarchical social systems. As we show, if there is a desire to increase the presence in the social system of certain agents with a high expected success rate then, in certain contexts, it may be necessary have to decrease the selectivity of the promotion mechanism rather than increasing it. By contrast, selectivity may have to be increased if the objective is to increase the presence of agents with low performances.

Our result depends largely on one particular critical assumption: In a group, all agents of the same type are either better or worse than other types simultaneously, thus their successes (or failures) correlate perfectly with one another. If two agents are under the same environment and are following the same rule and one of them is successful, then the other will also be successful, or at least more successful than other types, with a probability of one. If that is the case, selectivity will have this paradoxical effect in the dynamic of the process. In the real world it is not easy to find a situation where this correlation is so strong. We expect this paradoxical effect to become more noticeable as the correlation becomes stronger.

[10] Unlike Garcia-Martinez(2010), where another kind of selection is considered and effective selectivity is bounded above and below.

Appendix

Let $P_1 > P_2 > 0$, $P_1 + P_2 = 1$, $k \in \mathbb{N}$, $n \in \mathbb{N}$, $k > 0$, $n > k$, $a_t \in [0, 1]$, and the time subscript is omitted wherever it is not confusing to do so.

The binomial theorem

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x} \tag{2}$$

and the corollary

$$2^n = \sum_{x=0}^{n} \binom{n}{x} \tag{3}$$

are used in the following proofs.

The following basic identities of binomial coefficients are directly derived from the factorial formula $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ and are also used in the following proofs.

$$\binom{n}{x} = \binom{n}{n-x} \tag{4}$$

$$\binom{n}{x} = \binom{n-1}{x-1} + \binom{n-1}{x} \tag{5}$$

$$x\binom{n}{x} = n\binom{n-1}{x-1} \tag{6}$$

$$\binom{n}{x} = \binom{n}{x-1}\frac{n-x+1}{x} \tag{7}$$

$$\binom{n}{x+1} = \binom{n}{x+2}\frac{x+2}{n-x-1} \tag{8}$$

$$\binom{n}{n-x+1} = \binom{n}{(n-(x-1))} = \binom{n}{x-1} = \binom{n}{x+1}\frac{(x+1)x}{(n-x)(n-x+1)} \tag{9}$$

For example, expression (8) is proved: $\binom{n}{h+1} = \frac{n!}{(h+1)!(n-h-1)!} = \frac{n!}{(h+2)!(n-h-2)!}\frac{h+2}{n-h-1} = \binom{n}{h+2}\frac{h+2}{n-h-1}$.

The following Claim is also used in the following proofs.

Claim 1 $\sum_{x=0}^{k}(k-x)\binom{n}{x} = \sum_{x=n-(k-1)}^{n}(x-(n-k))\binom{n}{x}$

Proof

Because of the symmetry of binomial coefficient $\binom{n}{x}$=$\binom{n}{n-x}$,

$\sum_{x=0}^{k}\binom{n}{x} = \sum_{x=n-k+1}^{n}\binom{n}{x} \Leftrightarrow \sum_{x=0}^{k}(k-x)\binom{n}{x} = \sum_{x=n-k+1}^{n}(k-(n-x))\binom{n}{x}$

$\Leftrightarrow \sum_{x=0}^{k}(k-x)\binom{n}{x} = \sum_{x=n-(k-1)}^{n}(x-(n-k))\binom{n}{x}$ ■

The following lemmas are used in the proof of Proposition 1.

Lemma 1 *Let $h > 0$ be an integer, then,*

$\sum_{x=0}^{h}\left(\binom{n}{x}a^x(1-a)^{n-x}\left(a^2n(n-1) - 2a(n-1)x + x(x-1)\right)\right) = (1-a)^{n-h}a^{h+1}(a(n-1)-h)(h+1)\binom{n}{h+1}$

Proof

It is proved by induction on h.

If $h = 1$:

$\sum_{x=0}^{1}\left(\binom{n}{x}a^x(1-a)^{n-x}\left(a^2n(n-1) - 2a(n-1)x + x(x-1)\right)\right) =$

$= (1-a)^n a^2 n(n-1) + na(1-a)^{n-1}\left(a^2 n(n-1) - 2a(n-1)\right)$
$= (1-a)^{n-1} a^2 n(n-1)\left((1-a) + na - 2\right)$
$= (1-a)^{n-1} a^2 n(n-1)\left(a(n-1) - 1\right)$
$= (1-a)^{n-h} a^{h+1}(a(n-1)-h)(h+1)\binom{n}{h+1}\Big|_{h=1}$

It is proved that it holds for $h+1$ if it holds for h:

$\sum_{x=0}^{h+1}\left(\binom{n}{x} a^x (1-a)^{n-x}\left(a^2 n(n-1) - 2a(n-1)x + x(x-1)\right)\right)$
$= \sum_{x=0}^{h}\left(\binom{n}{x} a^x (1-a)^{n-x}\left(a^2 n(n-1) - 2a(n-1)x + x(x-1)\right)\right)$
$+ \left(\binom{n}{h+1} a^{h+1}(1-a)^{n-h-1}\left(a^2 n(n-1) - 2a(n-1)(h+1) + (h+1)(h)\right)\right)$
$= (1-a)^{n-h} a^{h+1}(a(n-1)-h)(h+1)\binom{n}{h+1}$
$+ \left(\binom{n}{h+1} a^{h+1}(1-a)^{n-h-1}\left(a^2 n(n-1) - 2a(n-1)(h+1) + (h+1)(h)\right)\right)$
$= (1-a)^{n-h-1} a^{h+2}\binom{n}{h+1}\left(\frac{(1-a)(a(n-1)-h)(h+1)}{a} + \frac{\left(a^2 n(n-1) - 2a(n-1)(h+1) + (h+1)(h)\right)}{a}\right)$

(using expression (8))

$= (1-a)^{n-h-1} a^{h+2}\left(\binom{n}{h+2}\frac{h+2}{n-h-1}\right)\left(\frac{(n(n-1)-(h+1)(n-1))a^2 + ((h+1)(h+n-1) - 2(h+1)(n-1))a}{a}\right)$
$= (1-a)^{n-h-1} a^{h+2}\left(\binom{n}{h+2}\frac{h+2}{n-h-1}\right)\left(\frac{(n-1)(n-h-1)a^2 - ((h+1)(n-h-1))a}{a}\right)$
$= (1-a)^{n-h-1} a^{h+2}\binom{n}{h+2}(h+2)\left(a(n-1) - (h+1)\right)$

The last expression is equal to $(1-a)^{n-h} a^{h+1}(a(n-1)-h)(h+1)\binom{n}{h+1}$ but with $h+1$ instead of h■

Lemma 2 *Let h be an integer and $h > 0$, then,*

$$\sum_{x=0}^{h}\left(x\binom{n}{x} a^x (1-a)^{n-x}\left(a^2 n(n-1) - 2a(n-1)x + x(x-1)\right)\right) = (1-a)^{n-h} a^{h+1}(an-h-1)(h+1)h\binom{n}{h+1}$$

Proof

The proof is analogous to the lemma above, and it is also proved by induction on h.

If $h = 1$:

$\sum_{x=0}^{1}\left(x\binom{n}{x} a^x (1-a)^{n-x}\left(a^2 n(n-1) - 2a(n-1)x + x(x-1)\right)\right)$
$= 0 + na(1-a)^{n-1}\left(a^2 n(n-1) - 2a(n-1)\right)$
$= n(1-a)^{n-1} a^2 (n-1)\left(an-2)\right)$
$= (1-a)^{n-h} a^{h+1}(an-h-1)(h+1)h\binom{n}{h+1}\Big|_{h=1}$

It is proved that it holds for $h+1$ if it holds for h:

$\sum_{x=0}^{h+1}\left(x\binom{n}{x} a^x (1-a)^{n-x}\left(a^2 n(n-1) - 2a(n-1)x + x(x-1)\right)\right)$
$= \sum_{x=0}^{h}\left(x\binom{n}{x} a^x (1-a)^{n-x}\left(a^2 n(n-1) - 2a(n-1)x + x(x-1)\right)\right)$
$+ \left((h+1)\binom{n}{h+1} a^{h+1}(1-a)^{n-h-1}\left(a^2 n(n-1) - 2a(n-1)(h+1) + (h+1)(h)\right)\right)$
$= (1-a)^{n-h} a^{h+1}(an-h-1)(h+1)h\binom{n}{h+1}$
$+ \left((h+1)\binom{n}{h+1} a^{h+1}(1-a)^{n-h-1}\left(a^2 n(n-1) - 2a(n-1)(h+1) + (h+1)h\right)\right)$

(using expression (8))

$= (1-a)^{n-h-1} a^{h+2}(h+1)\left(\binom{n}{h+2}\frac{h+2}{n-h-1}\right)\left(\frac{(1-a)h(an-h-1)}{a} + \frac{\left(a^2 n(n-1) - 2a(n-1)(h+1) + (h+1)h\right)}{a}\right)$
$= (1-a)^{n-h-1} a^{h+2}\left(an-h-2\right)(h+1)(h+2)\binom{n}{h+2}$

The last expression is equal to $(1-a)^{n-h} a^{h+1}(an-h-1)(h+1)h\binom{n}{h+1}$ but with $h+1$ instead of h■

PROOF OF PROPOSITION 1.

The dynamic of the selection process $S_{[n,k]}$ is given by the equation (1):

$$\begin{aligned} a_{t+1} &= f(a_t) = \sum_{x=0}^{n} \frac{Min[x,k]}{k} \binom{n}{x} a_t^x (1-a_t)^{n-x} P_1 + \sum_{x=n-(k-1)}^{n} \frac{(x-(n-k))}{k} \binom{n}{x} a_t^x (1-a_t)^{n-x} P_2 \\ &= \sum_{x=0}^{n} \frac{Min[x,k]P_1 + Max[x-(n-k),0]P_2}{k} \binom{n}{x} a_t^x (1-a_t)^{n-x} \end{aligned}$$

Assume $g(a_t) = f(a_t) - a_t$.

Thus, if $\hat{a} \in [0,1]$ is a root of $g(a_t)$, i.e. $g(\hat{a}) = 0$, then $\hat{a}$ is a *steady state* of $a_{t+1} = f(a_t)$, i.e. $\hat{a} = f(\hat{a})$. In addition, if $g(a_t) \gtreqless 0$ then $a_t \lesseqgtr a_{t+1}$.

Obviously, $a = 0$ (all the population is type B) and $a = 1$ (all the population is type A) are always steady states, and therefore roots of $g(a)$.

For example, in our model if $g(a) > 0$ for all $a \in (0,1)$, then $a = 1$ is a globally stable equilibrium, i.e. for any initial condition belonging to $(0,1)$ the system converges to $a = 1$. On the other hand, if there is only one inner root $\hat{a}$, i.e. $g(\hat{a}) = 0$, and $g(a) > 0$ for all $a \in (0,\hat{a})$, and $g(a) < 0$ for all $a \in (\hat{a},1)$, then $\hat{a}$ is a globally stable equilibrium[11]. However, if $g(a) < 0$ for all $a \in (0,\hat{a})$, and $g(a) > 0$ for all $a \in (\hat{a},1)$, then $\hat{a}$ is an unstable steady state, and $a = 0$ and $a = 1$ are locally stable with $(0,\hat{a})$ and $(\hat{a},1)$ as their basins of attraction respectively. An **inner root** is a root that belongs to the open interval $(0,1)$.

It is useful to start by presenting an outline of the proof. First it is shown that $g(a)$ is continuous in $a = [0,1]$ with $g(0) = g(1) = 0$. Second, the local stability in $a = 0$ and $a = 1$ is studied by means of the first derivative of $g(a)$. Then it is shown that if $\frac{k}{n} < P_2$, $a = 0$ and $a = 1$ are unstable because $g'(0) > 0$ and $g'(1) > 0$; if $\frac{k}{n} \in [P_2, P_1]$, $a = 0$ is unstable and $a = 1$ is locally stable because $g'(0) > 0$ and $g'(1) < 0$; if $\frac{k}{n} > P_1$, $a = 0$ and $a = 1$ are locally stable because $g'(0) < 0$ and $g'(1) < 0$. Third, it is proved that $g(a)$ has no more than one inner root in $a = (0,1)$. Fourth, there are no periodical points. Consequently, the proposition is proved.

As $f(a_t)$ is a polynomial, $g(a_t)$ is continuous. Obviously, $f(0) = 0$ and $f(1) = 1$, consequently, $g(0) = g(1) = 0$.

It is straightforward to show that the first derivative[12] of $g(a)$ is:

$g(a)' = f'(a) - 1 = \sum_{x=0}^{n} \frac{Min[x,k]P_1 + Max[x-(n-k),0]P_2}{k} \binom{n}{x} a^{x-1}(1-a)^{n-x-1}(x-na) - 1$

If $a = 0$, the terms of the series are equal to zero except for $x = 1$.

$g(0)' = \frac{P_1}{k} n - 1 \geq 0 \iff \frac{k}{n} \leq P_1$

If $a = 1$, the terms of the series are equal to zero except for $x = n-1$ and $x = n$.

$g(1)' = \frac{kP_1 + (k-1)P_2}{k} n(-1) + \frac{kP_1 + kP_2}{k} n - 1 = \frac{n}{k} P_2 \geq 0 \iff \frac{k}{n} \leq P_2$

To prove that $g(a) = 0$ has no more than one inner root in $a \in (0,1)$, it suffices to show that $g(a)$ has no more than one inflection point in $a \in (0,1)$ since $g(0) = g(1) = 0$ and $f'(a)$ is continuous in $(0,1)$.

It is straightforward to show that the second derivative[13] of $g(a)$ is:

$g(a)'' = f''(a) = \sum_{x=0}^{n} \left(\frac{Min[x,k]P_1 + Max[x-(n-k),0]P_2}{k} \binom{n}{x} a^{x-2}(1-a)^{n-x-2} \left(a^2 n(n-1) - 2a(n-1)x + x(x-1)\right)\right)$

Let $z(.) = \binom{n}{x} a^x (1-a)^{n-x} \left(a^2 n(n-1) - 2a(n-1)x + x(x-1)\right)$. The expression above is rearranged:

$g(a)'' = \sum_{x=0}^{n} \left(\frac{Min[x,k]P_1 + Max[x-(n-k),0]P_2}{k} \frac{z(.)}{a^2(1-a)^2} \right)$

$= \frac{1}{ka^2(1-a)^2} \left(\left(\sum_{x=0}^{k} (x\ z(.)) + k \sum_{x=k+1}^{n} z(.) \right) P_1 + \left(\sum_{x=n-(k-1)}^{n} ((x-(n-k)z(.)) \right) P_2 \right)$

[11] It is proved below that there are no periodical points.

[12] This is the sequence of the derivatives of each term. In $x = 0$ and $x = n$ it is necessary to simplify the expression to obtain the properly defined derivative function

[13] This is the sequence of the second derivatives of each term of $g(a)$. In $x = 0$, $x = 1$, $x = n-1$ and $x = n$, it is necessary to simplify the expression to obtain the properly defined derivative function.

$= \frac{\left(\sum_{x=0}^{k}(x\ z(.))P_1+k\left(\sum_{x=0}^{n}z(.)-\sum_{x=0}^{k}z(.)\right)P_1+\left(\sum_{x=0}^{n}((x-(n-k)z(.))-\sum_{x=0}^{n-k}((x-(n-k)z(.))\right)P_2\right)}{ka^2(1-a)^2}$

$= \frac{\left(\sum_{x=0}^{k}(x\ z(.))P_1+k\left(\sum_{x=0}^{n}z(.)-\sum_{x=0}^{k}z(.)\right)P_1+\left(\left(\sum_{x=0}^{n}((x\ z(.))-(n-k)\sum_{x=0}^{n}z(.)\right)-\left(\sum_{x=0}^{n-k}x\ z(.)-(n-k)\sum_{x=0}^{n-k}z(.)\right)\right)P_2\right)}{ka^2(1-a)^2}$

It is straightforward to show that $\sum_{x=0}^{n} z(.) = \sum_{x=0}^{n} xz(.) = 0$ from Lemma 1 and 2 with[14] $h = n$, thus:

$g(a)'' = \frac{\left(\sum_{x=0}^{k}(x\ z(.))P_1+k\left(0-\sum_{x=0}^{k}z(.)\right)P_1+\left((0-0)-\left(\sum_{x=0}^{n-k}x\ z(.)-(n-k)\sum_{x=0}^{n-k}z(.)\right)\right)P_2\right)}{ka^2(1-a)^2}$

$= \frac{\sum_{x=0}^{k}(x\ z(.))P_1-k\sum_{x=0}^{k}z(.)P_1-\left(\sum_{x=0}^{n-k}x\ z(.)-(n-k)\sum_{x=0}^{n-k}z(.)\right)P_2}{ka^2(1-a)^2}$

$= \frac{\sum_{x=0}^{k}(x\ z(.))P_1-k\sum_{x=0}^{k}z(.)P_1}{ka^2(1-a)^2} - \frac{\left(\sum_{x=0}^{n-k}x\ z(.)-(n-k)\sum_{x=0}^{n-k}z(.)\right)P_2}{ka^2(1-a)^2}$

By Lemma 1 and 2 with $h = k$, the first term of the expression above is

$\frac{\sum_{x=0}^{k}(x\ z(.))P_1-k\sum_{x=0}^{k}z(.)P_1}{ka^2(1-a)^2} = \frac{(1-a)^{n-k}a^{k+1}(an-k-1)(k+1)k\binom{n}{k+1}P_1-k\left((1-a)^{n-k}a^{k+1}(a(n-1)-k)(k+1)\binom{n}{k+1}\right)P_1}{ka^2(1-a)^2}$

$= \frac{(1-a)^{n-k}a^{k+1}(k+1)\binom{n}{k+1}(a-1)P_1}{a^2(1-a)^2}$

$= -(1-a)^{n-k-1}a^{k-1}(k+1)\binom{n}{k+1}P_1$

and by Lemma 1 and 2 with $h = n-k$ the second term is:

$\frac{\left(\sum_{x=0}^{n-k}x\ z(.)-(n-k)\sum_{x=0}^{n-k}z(.)\right)P_2}{ka^2(1-a)^2}$

$= \frac{\left((1-a)^{n-(n-k)}a^{n-k+1}(an-(n-k)-1)(n-k+1)(n-k)\binom{n}{n-k+1}-(n-k)(1-a)^{n-(n-k)}a^{n-k+1}(a(n-1)-(n-k))(n-k+1)\binom{n}{n-k+1}\right)P_2}{ka^2(1-a)^2}$

$= \frac{P_2(1-a)^{n-(n-k)}a^{(n-k)+1}(n-k)((n-k)+1)\binom{n}{n-k+1}(a-1)}{ka^2(1-a)^2}$

$= -\frac{P_2(1-a)^{k-1}a^{(n-k)-1}(n-k)((n-k)+1)\binom{n}{n-k+1}}{k}$

(using expression (9))

$= -\frac{P_2(1-a)^{k-1}a^{(n-k)-1}(n-k)((n-k)+1)\binom{n}{k+1}\frac{(k+1)k}{(n-k)(n-k+1)}}{k}$

$= -P_2(1-a)^{k-1}a^{(n-k)-1}\binom{n}{k+1}(k+1)$

Therefore,

$g(a)'' = -P_1(1-a)^{n-k-1}a^{k-1}(k+1)\binom{n}{k+1} - (-P_2(1-a)^{k-1}a^{(n-k)-1}\binom{n}{k+1}(k+1))$

$= -(k+1)\binom{n}{k+1}(1-a)^{-k-1}a^{-k-1}(P_1(1-a)^n a^{2k} - P_2(1-a)^{2k}a^n)$

If $\bar{a}$ is an inflection point then $g(\bar{a})'' = 0$. Below it is shown that there is no more than one $\bar{a}$ in $a \in (0,1)$.

$$g(\bar{a})'' = 0 \Leftrightarrow -(k+1)\binom{n}{k+1}(1-\bar{a})^{-k-1}\bar{a}^{-k-1}(P_1(1-\bar{a})^n\bar{a}^{2k} - P_2(1-\bar{a})^{2k}\bar{a}^n) = 0 \qquad (10)$$

$\Leftrightarrow P_1(1-\bar{a})^n\bar{a}^{2k} - P_2(1-\bar{a})^{2k}\bar{a}^n = 0$

$\Leftrightarrow \frac{P_1(1-\bar{a})^n\bar{a}^{2k}}{P_2(1-\bar{a})^{2k}\bar{a}^n} = 1$

$\Leftrightarrow \frac{(1-\bar{a})^{n-2k}}{\bar{a}^{n-2k}}\frac{P_1}{P_2} = 1$

$\Leftrightarrow \frac{1-\bar{a}}{\bar{a}}\left(\frac{P_1}{P_2}\right)^{\frac{1}{n-2k}} = 1$

$\Leftrightarrow \bar{a} = \frac{1}{1+\left(\frac{P_1}{P_2}\right)^{\frac{1}{2k-n}}}$

Therefore the equation (10) can have only one real root in the interval $a \in (0,1)$. Consequently, there is no more than one inflection point in $a \in (0,1)$. Note that $f'(a)$ is continuous in $a \in [0,1]$. As $g(0) = g(1) = 0$, the function $g(a)$ has no more than one root in $a \in (0,1)$.

Before concluding, it is proved that there are no **periodic points**[15].

[14] If $h = n$, then $\binom{n}{n+1} = 0$

[15] It is possible in difference equations for a solution not to be a steady point. Thus, point b is called a *periodic point* of $x_{t+1} = f(x_t)$ if $f^k(b) = b$ for a positive integer k, i.e. b is again reached after k iterations. See Elaydi (1996).

The function $f(a_t)$ has either only one inner steady state in $(0,1)$ or none at all. If it has none, there are obviously no periodic points because either $a_t > a_{t+1}$ for all $a_t \in (0,1)$ or $a_t < a_{t+1}$ for all $a_t \in (0,1)$. If there is one steady state in $(0,1)$, if the function $f(a_t)$ is increasing any possibility of there being periodic points completely disappears because for all a_t equal to or greater than the inner steady state $(\hat{a})$ either $a_t < a_{t+1}$ for all $a_t \in (\hat{a},1)$ or $a_t > a_{t+1}$ for all $a_t \in (\hat{a},1)$, and always $a_{t+1} \geq \hat{a}$. Thus, it suffices to prove that the function $f(a_t)$ is increasing.

The following claim is needed to prove that $f'(a_t) > 0$

Claim 2 $\sum_{x=0}^{n} \binom{n}{x} a^x (1-a)^{n-x}(x-an) = 0$

Proof.

Using basic identities of the binomial coefficient and the binomial theorem,

$\sum_{x=0}^{n} \binom{n}{x} a^x(1-a)^{n-x}(x-an) = \sum_{x=0}^{n} x\binom{n}{x} a^x(1-a)^{n-x} - an\sum_{x=0}^{n} \binom{n}{x} a^x(1-a)^{n-x}$

(using binomial theorem)

$= \sum_{x=0}^{n} x\binom{n}{x} a^x(1-a)^{n-x} - an$

(using expression (6))

$= \sum_{x=0}^{n} n\binom{n-1}{x-1} a^x(1-a)^{n-x} - an$

(using expression (5))

$= \sum_{x=0}^{n} n\left(\binom{n}{x} - \binom{n-1}{x}\right) a^x(1-a)^{n-x} - an$

$= \sum_{x=0}^{n} n\binom{n}{x} a^x(1-a)^{n-x} - \sum_{x=0}^{n} n\binom{n-1}{x} a^x(1-a)^{n-x} - an$

(using binomial theorem)

$= n - n\sum_{x=0}^{n} \binom{n-1}{x} a^x(1-a)^{n-1-x}(1-a) - an$

$= n - n\left(\sum_{x=0}^{n-1} \binom{n-1}{x} a^x(1-a)^{n-1-x} + \binom{n-1}{n} a^n(1-a)^{-1}\right)(1-a) - an$

(using binomial theorem)

$= n - n\left((a+(1-a))^{n-1} + 0\right)(1-a) - an = 0$■

Let $r(x) = \binom{n}{x} a^x(1-a)^{n-x}(x-an)$. Thus, $r(x)$ is negative[16] if $x \leq \lfloor an \rfloor$ and positive if $x > \lfloor an \rfloor$. Where $\lfloor an \rfloor$ gives the highest integer less than or equal to an.

Therefore, $\sum_{x=0}^{n} \binom{n}{x} a^x(1-a)^{n-x}(x-an) = \sum_{x=0}^{n} r(x) = 0$

$\Longleftrightarrow \sum_{x=0}^{\lfloor an \rfloor} r(x) + \sum_{x=\lfloor an \rfloor+1}^{n} r(x) = 0 \Longleftrightarrow -\sum_{x=0}^{\lfloor an \rfloor} r(x) = \sum_{x=\lfloor an \rfloor+1}^{n} r(x)$

Let $h(x) > 0$ be a function that is increasing in x.

Obviously, $-\sum_{x=0}^{\lfloor an \rfloor} h(x)r(x) < \sum_{x=\lfloor an \rfloor+1}^{n} h(x)r(x) \Leftrightarrow \sum_{x=0}^{n} h(x)r(x) > 0$

That $f'(a_t) > 0$ is now proved:

$f'(a_t) = \sum_{x=0}^{n} \frac{Min[x,k]P_1 + Max[x-(n-k),0]P_2}{k} \binom{n}{x} a^{x-1}(1-a)^{n-x-1}(x-na) > 0$

$\Longleftrightarrow \sum_{x=0}^{n} \left(Min[x,k]P_1 + Max[x-(n-k),0]P_2\right) r(x) > 0$

[16] It can be also zero if $x = \lfloor an \rfloor = an$. However the rationale is the same.

It is straightforward to show that $(Min[x,k]P_1 + Max[x-(n-k),0]P_2) \in [0,k]$ is greater than zero and is increasing in x. Therefore $f'(a_t) > 0$, and $f(a_t)$ is increasing. Periodic points are therefore not possible.

It can be concluded that:

1) If $\frac{k}{n} < P_2$ then $g(0)' > 0$ and $g(1)' > 0$. Thus, on the one hand, $g(a)$ is continuous and $g(0) = g(1) = 0$. On the other hand $g(a)$ is positive around $a = 0$ and negative around $a = 1$. Consequently, by Bolzano's Theorem there is at least one inner root. It has been proved that there cannot be more than one inner root. Therefore, there is only one inner root a^* and it is globally stable.

2) If $\frac{k}{n} \in [P_2, P_1]$ then $g(0)' > 0$ and $g(1)' < 0$. In that case, as $g(0) = g(1) = 0$, $g(a)$ is positive around $a = 0$ and around $a = 1$. As there cannot be more than one inner root, the function has to be positive in the domain. The only possibility of having an inner point is for the inner point to be a minimum of the function, so that the function would be positive in the domain. However, this is not possible because there is no more than one inflection point. Therefore, necessarily there is no inner root. $a = 0$ is unstable and $a = 1$ is globally stable.

3) If $\frac{k}{n} > P_1$ then $g(0)' < 0$ and $g(1)' < 0$. In that case, as $g(0) = g(1) = 0$, $g(a)$ is negative around $a = 0$ and positive around $a = 1$. Necessarily, there is only one inner root a^* for the same reason as in point 1). This unique inner root a^* has to be unstable and divides the interval $a \in (0,1)$ into two subintervals. The subinterval $(0, a^*)$ is the basin of attraction of $a = 0$ and $(a^*, 1)$ of $a = 1$ which are locally stable.■

The following lemma is used in the proof of Proposition 2 and 4.

Lemma 3 $g(\frac{1}{2}) = f(\frac{1}{2}) - \frac{1}{2} > 0$

Proof

The equation (1) evaluated in $a = \frac{1}{2}$ and minus $\frac{1}{2}$ is:

$$g(\tfrac{1}{2}) = f(\tfrac{1}{2}) - \tfrac{1}{2} = \sum_{x=0}^{n} \frac{min[x,k]}{k}\binom{n}{x}\left(\tfrac{1}{2}\right)^x (1-\tfrac{1}{2})^{n-x} P_1 + \sum_{x=n-(k-1)}^{n} \frac{(x-(n-k))}{k}\binom{n}{x}\left(\tfrac{1}{2}\right)^x (1-\tfrac{1}{2})^{n-x} P_2 - \tfrac{1}{2}$$

$$= \left(\sum_{x=0}^{n} \frac{min[x,k]}{k}\binom{n}{x}P_1 + \sum_{x=n-(k-1)}^{n} \frac{(x-(n-k))}{k}\binom{n}{x}P_2\right)\left(\tfrac{1}{2}\right)^n - \tfrac{1}{2} > 0$$

$$\Leftrightarrow \left(\sum_{x=0}^{n} min[x,k]\binom{n}{x}P_1 + \sum_{x=n-(k-1)}^{n}(x-(n-k))\binom{n}{x}P_2\right) - \tfrac{1}{2}k2^n > 0$$

(using Claim 1)

$$\Leftrightarrow \left(\sum_{x=0}^{n} min[x,k]\binom{n}{x}P_1 + \sum_{x=0}^{k}(k-x)\binom{n}{x}P_2\right) - \tfrac{1}{2}k2^n > 0$$

$$\Leftrightarrow \left(\sum_{x=0}^{k} x\binom{n}{x}P_1 + k\sum_{x=k+1}^{n}\binom{n}{x}P_1 + \sum_{x=0}^{k}(k-x)\binom{n}{x}P_2\right) - \tfrac{1}{2}k2^n > 0$$

$$\Leftrightarrow \left(\sum_{x=0}^{k} x\binom{n}{x}P_1 + k\left(\sum_{x=0}^{n}\binom{n}{x} - \sum_{x=0}^{k}\binom{n}{x}\right)P_1 + \sum_{x=0}^{k}(k-x)\binom{n}{x}P_2\right) - \tfrac{1}{2}k2^n > 0$$

(using (3))

$$\Leftrightarrow \left(\sum_{x=0}^{k} x\binom{n}{x}P_1 + k\left(2^n - \sum_{x=0}^{k}\binom{n}{x}\right)P_1 + \sum_{x=0}^{k}(k-x)\binom{n}{x}P_2\right) - \tfrac{1}{2}k2^n > 0$$

$$\Leftrightarrow \left(\sum_{x=0}^{k} x\binom{n}{x}P_1 - k\sum_{x=0}^{k}\binom{n}{x}P_1 + \sum_{x=0}^{k}(k-x)\binom{n}{x}P_2 + kP_1 2^n\right) - \tfrac{1}{2}k2^n > 0$$

$$\Leftrightarrow -\sum_{x=0}^{k}(k-x)P_1\binom{n}{x} + \sum_{x=0}^{k}(k-x)\binom{n}{x}P_2 + kP_1 2^n - \tfrac{1}{2}k2^n(P_1+P_2) > 0$$

$$\Leftrightarrow -\sum_{x=0}^{k}(k-x)\binom{n}{x}P_1 + \sum_{x=0}^{k}(k-x)\binom{n}{x}P_2 + \tfrac{1}{2}k2^nP_1 - \tfrac{1}{2}k2^nP_2 > 0$$

$$\Leftrightarrow \left(\tfrac{1}{2}k2^n - \sum_{x=0}^{k}(k-x)\binom{n}{x}\right)(P_1 - P_2) > 0$$

$$\Leftrightarrow \left(\tfrac{1}{2}k2^n - \sum_{x=0}^{k}(k-x)\binom{n}{x}\right) > 0$$

(using (3))

$$\Leftrightarrow \left(\tfrac{1}{2}k\sum_{x=0}^{n}\binom{n}{x}\right) - \left(\sum_{x=0}^{k}(k-x)\binom{n}{x}\right) > 0$$
$$\Leftrightarrow \left(\tfrac{1}{2}k\sum_{x=0}^{k}\binom{n}{x} + \tfrac{1}{2}k\sum_{x=k+1}^{n}\binom{n}{x}\right) - \left(\sum_{x=0}^{k}(k-x)\binom{n}{x}\right) > 0$$
$$\Leftrightarrow \sum_{x=0}^{k}\left(\tfrac{1}{2}k - k + x\right)\binom{n}{x} + \tfrac{1}{2}k\sum_{x=k+1}^{n}\binom{n}{x} > 0$$
$$\Leftrightarrow \sum_{x=0}^{k}\left(x - \tfrac{1}{2}k\right)\binom{n}{x} + \tfrac{1}{2}k\sum_{x=k+1}^{n}\binom{n}{x} > 0$$

Obviously, $\frac{1}{2}k\sum_{x=k+1}^{n}\binom{n}{x} > 0$. The expression $\sum_{x=0}^{k}\left(x-\frac{1}{2}k\right)\binom{n}{x}$ is also positive: First note that $\left(x-\frac{1}{2}k\right)$ is negative[17] for $0 \le x \le \left\lfloor\frac{k}{2}\right\rfloor$ and positive for $\left\lfloor\frac{k}{2}\right\rfloor < x \le k$. In addition, the value of $\left(x-\frac{1}{2}k\right)$ for $x=i$ is equal to $x=k-i$ in absolute value; the values are symmetrical. Second, $\binom{n}{x}$ takes increasing values from $x=0$ to $x=\left\lfloor\frac{n}{2}\right\rfloor$, and then it decreases symmetrically. As $\left\lfloor\frac{k}{2}\right\rfloor < \left\lfloor\frac{n}{2}\right\rfloor$, the sum of the positive terms has to be greater than the sum of the negative terms in absolute value:

$$\sum_{x=0}^{n}\left(x-\tfrac{1}{2}k\right)\binom{n}{x} > 0 \Leftrightarrow \sum_{x=0}^{\left\lfloor\frac{k}{2}\right\rfloor}\left(x-\tfrac{1}{2}k\right)\binom{n}{x} + \sum_{x=\left\lfloor\frac{k}{2}\right\rfloor+1}^{n}\left(x-\tfrac{1}{2}k\right)\binom{n}{x} > 0$$
$$\Leftrightarrow \sum_{x=\left\lfloor\frac{k}{2}\right\rfloor+1}^{n}\left(x-\tfrac{1}{2}k\right)\binom{n}{x} > -\sum_{x=0}^{\left\lfloor\frac{k}{2}\right\rfloor}\left(x-\tfrac{1}{2}k\right)\binom{n}{x}\blacksquare$$

It is first shown the proofs of Proposition **4** and **5** and then that of Proposition **2** and **3.**

PROOF OF PROPOSITION 4.

A closed form of $g(a)$ is first obtained.

The equation (1) with $k=1$:

$$f(a) = \sum_{x=0}^{n} min[x,1]\binom{n}{x}a^x(1-a)^{n-x}P_1 + \sum_{x=n}^{n}(x-(n-1))\binom{n}{x}a^x(1-a)^{n-x}P_2$$
$$= \sum_{x=1}^{n}\binom{n}{x}a^x(1-a)^{n-x}P_1 + \binom{n}{n}a^nP_2$$
$$= \left(\sum_{x=0}^{n}\binom{n}{x}a^x(1-a)^{n-x}P_1 - \binom{n}{0}a^0(1-a)^{n-0}\right)P_1 + a^nP_2$$

(using the binomial theorem)

$$= \left(1-(1-a)^n\right)P_1 + a^nP_2$$

Thus, $g(a;n) = f(a) - a = \left(1-(1-a)^n\right)P_1 + a^nP_2 - a$

Let $\tilde{a}$ be the unique inner root, with $n=\bar{n}$, of $g(a;\bar{n})=0$ if $\frac{k}{\bar{n}} = \frac{1}{\bar{n}} < P_2$. See Proposition 1.

Let $\check{a}$ be the unique inner root, with $n=\bar{n}+1$, of $g(a;\bar{n}+1)=0$ if $\frac{k}{\bar{n}+1} = \frac{1}{\bar{n}+1} < P_2$. The proof of Proposition 1 shows that $g(0;\bar{n}+1) = g(1;\bar{n}+1) = 0$ and, if $\frac{k}{\bar{n}+1} < P_2$, $g'(0;\bar{n}+1) > 0$ and $g'(1;\bar{n}+1) > 0$. Therefore, $g(a;\bar{n}+1) > 0$ if $a \in (0,\check{a})$ and $g(a;\bar{n}+1) < 0$ if $a \in (\check{a},1)$. Consequently, if $g(\tilde{a};\bar{n}+1) < 0$, then $\tilde{a} > \hat{a}$.

It is to be proven that $g(\tilde{a};\bar{n}+1) < 0$.

On the one hand, $g(\tilde{a};\bar{n}) = \left(1-(1-\tilde{a})^{\bar{n}}\right)P_1 + \tilde{a}^{\bar{n}}P_2 - \tilde{a} = 0$

$\Longleftrightarrow \left(1-(1-\tilde{a})^{\bar{n}}\right)P_1 + \tilde{a}^{\bar{n}}P_2 - \tilde{a}(P_1+P_2) = 0$

$\Longleftrightarrow P_1 - (1-\tilde{a})^{\bar{n}}P_1 - \tilde{a}(P_1+P_2) = -\tilde{a}^{\bar{n}}P_2$

[17] It can be also zero if $x = \left\lfloor\frac{k}{2}\right\rfloor = \frac{k}{2}$. However the rationale is the same.

$\Longleftrightarrow ((1-\tilde{a})-(1-\tilde{a})^{\bar{n}}) P_1 = (\tilde{a}-\tilde{a}^{\bar{n}}) P_2$

$$\Leftrightarrow \frac{P_1}{P_2} = \frac{\tilde{a}-\tilde{a}^{\bar{n}}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}}} \tag{11}$$

On the other hand,

$g(\tilde{a};\bar{n}+1) = \left(1-(1-\tilde{a})^{\bar{n}+1}\right) P_1 + \tilde{a}^{\bar{n}+1} P_2 - \tilde{a} < 0$

$\Leftrightarrow P_1 - (1-\tilde{a})^{\bar{n}+1} P_1 + \tilde{a}^{\bar{n}+1} P_2 - \tilde{a}(P_1+P_2) < 0$

$\Leftrightarrow P_1 - (1-\tilde{a})^{\bar{n}+1} P_1 - \tilde{a}(P_1+P_2) < -\tilde{a}^{\bar{n}+1} P_2$

$\Leftrightarrow \left((1-\tilde{a})-(1-\tilde{a})^{\bar{n}+1}\right) P_1 < \left(\tilde{a}-\tilde{a}^{\bar{n}+1}\right) P_2$

$\Leftrightarrow \frac{\tilde{a}-\tilde{a}^{\bar{n}+1}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}+1}} > \frac{P_1}{P_2}$

(using (11))

$\Leftrightarrow \frac{\tilde{a}-\tilde{a}^{\bar{n}+1}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}+1}} > \frac{P_1}{P_2} = \frac{\tilde{a}-\tilde{a}^{\bar{n}}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}}}$

$\Leftrightarrow \frac{\tilde{a}-\tilde{a}^{\bar{n}+1}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}+1}} > \frac{\tilde{a}-\tilde{a}^{\bar{n}}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}}}$

$\Leftrightarrow \frac{\tilde{a}\left(1-\tilde{a}^{\bar{n}}\right)}{(1-\tilde{a})(1-(1-\tilde{a})^{\bar{n}})} > \frac{\tilde{a}\left(1-\tilde{a}^{\bar{n}-1}\right)}{(1-\tilde{a})(1-(1-\tilde{a})^{\bar{n}-1})}$

$\Leftrightarrow \frac{1-\tilde{a}^{\bar{n}}}{1-(1-\tilde{a})^{\bar{n}}} > \frac{1-\tilde{a}^{\bar{n}-1}}{1-(1-\tilde{a})^{\bar{n}-1}}$

$\Leftrightarrow \frac{1-\tilde{a}^{\bar{n}}}{1-\tilde{a}^{\bar{n}-1}} > \frac{1-(1-\tilde{a})^{\bar{n}}}{1-(1-\tilde{a})^{\bar{n}-1}}$

It is shown below that $\frac{1-a^{\bar{n}}}{1-a^{\bar{n}-1}}$ is increasing in a, and $\frac{1-(1-a)^{\bar{n}}}{1-(1-a)^{\bar{n}-1}}$is decreasing in a. In addition, the two terms are equal if $a=\dfrac{1}{2}$. Therefore, if $a>\dfrac{1}{2}$ then $\frac{1-a^{\bar{n}}}{1-a^{\bar{n}-1}} > \frac{1-(1-a)^{\bar{n}}}{1-(1-a)^{\bar{n}-1}}$,and finally, it is shown that $\tilde{a} > \frac{1}{2}$.

First, $\frac{d}{da}\left(\frac{1-a^{\bar{n}}}{1-a^{\bar{n}-1}}\right) = \frac{a^{\bar{n}}}{(a-a^{\bar{n}})^2}\left(\bar{n}-a\bar{n}+a^{\bar{n}}-1\right)$, and $\bar{n}-a\bar{n}+a^{\bar{n}}-1$ is decreasing in a, it takes the minimum value in $a=1$ and $\left.(\bar{n}-a\bar{n}+a^{\bar{n}}-1)\right|_{a=1}=0$, therefore $\frac{d}{da}\left(\frac{1-a^{\bar{n}}}{1-a^{\bar{n}-1}}\right)>0$.

Second, $\frac{d}{da}\left(\frac{1-(1-a)^{\bar{n}}}{1-(1-a)^{\bar{n}-1}}\right) = -\frac{(1-a)^{\bar{n}}}{\left(a+(1-a)^{\bar{n}}-1\right)^2}\left((1-a)^{\bar{n}}+a\bar{n}-1\right)$, and $-\left((1-a)^{\bar{n}}+a\bar{n}-1\right)$ is decreasing in a, it takes the maximum value in $a=0$ and $-\left.\left((1-a)^{\bar{n}}+a\bar{n}-1\right)\right|_{a=0}=0$, therefore $\frac{d}{da}\left(\frac{1-(1-a)^{\bar{n}}}{1-(1-a)^{\bar{n}-1}}\right)<0$

To conclude, it is shown that $\tilde{a} > \frac{1}{2}$. The following claim proves it.

Claim 3 $\tilde{a} = a^*[S_{[n,k=1]}] > \frac{1}{2}$

Proof

As $g(a;n)>0$ if $a \in (0, a^*[S_{[n,k=1]}])$ and $g(a;n)<0$ if $a \in (a^*[S_{[n,k=1]}], 1)$ (see proof of Proposition 1). If $g(\frac{1}{2})>0$, then $a^*[S_{[n,k=1]}] > \frac{1}{2}$. In Lemma 3, $g(\frac{1}{2})>0$ is proved.■

■

PROOF OF PROPOSITION 5.

Let $\frac{k}{n}=\frac{1}{n}<P_2$, if the gap P_1-P_2 increases the globally stable inner steady state $a^*[S_{[n,k=1]}]$ increases.

By Proposition **1**, if $\frac{k}{n}=\frac{1}{n}<P_2$ there exists only one inner steady state $a^*[S_{[n,k=1]}]$, which is globally stable. Let $a^*[S_{[n,k=1]}]=\tilde{a}$. Thus $g(\tilde{a})=0$

As $P_2=1-P_1$, the expression P_1-P_2 increases if and only if P_1 increases.

In the proof of Proposition **4**, it is shown that $g(\tilde{a})=0$ is equivalent to equation (11):

$g(\tilde{a})=0 \Longleftrightarrow \frac{P_1}{P_2} = \frac{\tilde{a}-\tilde{a}^n}{(1-\tilde{a})-(1-\tilde{a})^n}$

$\Longleftrightarrow \frac{P_1}{1-P_1} = \frac{\tilde{a}-\tilde{a}^n}{(1-\tilde{a})-(1-\tilde{a})^n}$

On the one hand, $\frac{P_1}{1-P_1}$ is increasing in P_1.

On the other hand,

$\frac{d}{d\tilde{a}}\left(\frac{\tilde{a}-\tilde{a}^n}{(1-\tilde{a})-(1-\tilde{a})^n}\right) = \frac{1}{\tilde{a}}\frac{\tilde{a}-\tilde{a}^n}{(1-\tilde{a})-(1-\tilde{a})^n} > 0$, it is increasing in $\tilde{a}$.

Therefore, if P_1-P_2 increases, P_1 increases, $\frac{P_1}{1-P_1}$ increases, and $\tilde{a}$ has to increase.■

The following Lemma is used in the proof of Proposition **2.**

Lemma 4 $\sum_{x=0}^{n} x\binom{n}{x}a^x(1-a)^{n-x} = an$

Proof

$\sum_{x=0}^{n} x\binom{n}{x}a^x(1-a)^{n-x}$

(using 6)

$= \sum_{x=0}^{n} n\binom{n-1}{x-1}a^x(1-a)^{n-x}$

(using 5)

$= \sum_{x=0}^{n} n\left(\binom{n}{x} - \binom{n-1}{x}\right)a^x(1-a)^{n-x}$

$= n\sum_{x=0}^{n} \binom{n}{x}a^x(1-a)^{n-x} - n\sum_{x=0}^{n} \binom{n-1}{x}a^x(1-a)^{n-x}$

(using binomial theorem)

$= n - n\sum_{x=0}^{n} \binom{n-1}{x}a^x(1-a)^{n-1-x}(1-a)$

(since $\binom{n-1}{n} = 0$)

$= n - n\sum_{x=0}^{n-1} \binom{n-1}{x}a^x(1-a)^{n-1-x}(1-a)$

$= n - n(1-a) = an$■

PROOF OF PROPOSITION 2.

The proof is analogous to the proof of Proposition **4** and some of the identical steps are omitted.

A close form of $g(a)$ is first obtained.

The equation (1) with $k = n-1$:

$f(a) = \sum_{x=0}^{n} \frac{min[x,n-1]}{n-1}\binom{n}{x}a^x(1-a)^{n-x}P_1 + \sum_{x=2}^{n} \frac{(x-1)}{n-1}\binom{n}{x}a^x(1-a)^{n-x}P_2$

$= \sum_{x=0}^{n-1} \frac{x\binom{n}{x}a^x(1-a)^{n-x}P_1 + (n-1)\binom{n}{n}a^nP_1}{n-1} + \sum_{x=2}^{n} \frac{(x-1)\binom{n}{x}a^x(1-a)^{n-x}P_2}{n-1}$

$= \frac{\sum_{x=0}^{n} x\binom{n}{x}a^x(1-a)^{n-x}P_1 - a^nP_1}{n-1} + \frac{\sum_{x=2}^{n} x\binom{n}{x}a^x(1-a)^{n-x}P_2 - \sum_{x=2}^{n}\binom{n}{x}a^x(1-a)^{n-x}P_2}{n-1}$

(using Lemma 4)

$= \frac{anP_1 - a^nP_1}{n-1} + \frac{\left(\sum_{x=0}^{n} x\binom{n}{x}a^x(1-a)^{n-x}P_2 - \binom{n}{1}a(1-a)^{n-1}P_2\right) - \left(\sum_{x=0}^{n}\binom{n}{x}a^x(1-a)^{n-x}P_2 - \binom{n}{0}(1-a)^nP_2 - \binom{n}{1}a(1-a)^{n-1}P_2\right)}{n-1}$

$= \frac{anP_1 - a^nP_1}{n-1} + \frac{\left(\sum_{x=0}^{n} x\binom{n}{x}a^x(1-a)^{n-x}P_2\right) - \left(\sum_{x=0}^{n}\binom{n}{x}a^x(1-a)^{n-x}P_2 - (1-a)^nP_2\right)}{n-1}$

(using Lemma 4 and the binomial theorem)

$= \frac{anP_1 - a^nP_1}{n-1} + \frac{anP_2 - P_2 + (1-a)^nP_2}{n-1}$

$= \frac{1}{n-1}\left(an - a^nP_1 - P_2 + (1-a)^nP_2\right)$

Thus $g(a;n) = f(a) - a = \frac{1}{n-1}\left(an - a^nP_1 - P_2 + (1-a)^nP_2\right) - a$

$= \frac{1}{n-1}\left(an - a^nP_1 - P_2 + (1-a)^nP_2\right) - a$

Let $\tilde{a}$ be the unique inner root, with $n = \bar{n}$, of $g(a;\bar{n}) = 0$ if $\frac{k}{n} = \frac{n-1}{n} > P_1$. See Proposition 1.

Let $\check{a}$ be the unique inner root, with $n = \bar{n}+1$, of $g(a;\bar{n}+1) = 0$ if $\frac{k}{n} = \frac{n-1}{n} > P_1$. In the proof of Proposition 1, it is shown that $g(0) = g(1) = 0$ and, if $\frac{k}{\bar{n}+1} > P_2$, $g'(0;\bar{n}+1) < 0$ and $g'(1;\bar{n}+1) < 0$. Therefore, $g(a;\bar{n}+1) < 0$ if $a \in (0,\check{a})$ and $g(a;\bar{n}+1) > 0$ if $a \in (\check{a},1)$. Consequently, if $g(\tilde{a};\bar{n}+1) < 0$, then $\tilde{a} < \hat{a}$.

It will be proven that $g(\tilde{a}; n+1) < 0$.

On the one hand, $g(\tilde{a}; \bar{n}) = \frac{1}{\bar{n}-1}\left(\tilde{a}\bar{n} - \tilde{a}^{\bar{n}}P_1 - P_2 + (1-\tilde{a})^{\bar{n}}P_2\right) - \tilde{a} = 0$

$\Leftrightarrow (\tilde{a}\bar{n} - \tilde{a}^{\bar{n}}P_1 - P_2 + (1-\tilde{a})^{\bar{n}}P_2) - (\bar{n}-1)\tilde{a} = 0$

$\Leftrightarrow -\tilde{a}^{\bar{n}}P_1 - P_2 + (1-\tilde{a})^{\bar{n}}P_2 + \tilde{a} = 0$

$\Leftrightarrow (1-(1-\tilde{a})^{\bar{n}})P_2 + \tilde{a}^{\bar{n}}P_1 - \tilde{a} = 0$

$\Leftrightarrow (1-(1-\tilde{a})^{\bar{n}})P_2 + \tilde{a}^{\bar{n}}P_1 - \tilde{a}(P_1+P_2) = 0$

$$\Leftrightarrow \frac{P_2}{P_1} = \frac{\tilde{a} - \tilde{a}^{\bar{n}}}{(1-\tilde{a}) - (1-\tilde{a})^{\bar{n}}} \tag{12}$$

On the other hand,

$g(\tilde{a}; \bar{n}+1) = \frac{1}{\bar{n}}\left(\tilde{a}(\bar{n}+1) - \tilde{a}^{\bar{n}+1}P_1 - P_2 + (1-\tilde{a})^{\bar{n}+1}P_2\right) - \tilde{a} = 0 < 0$

$\Leftrightarrow -(1-(1-\tilde{a})^{\bar{n}+1})P_2 - \tilde{a}^{\bar{n}+1}P_1 + \tilde{a} < 0$

$\Leftrightarrow (1-(1-\tilde{a})^{\bar{n}+1})P_2 + \tilde{a}^{\bar{n}+1}P_1 - \tilde{a} > 0$

$\Leftrightarrow \left(1-(1-\tilde{a})^{\bar{n}+1}\right)P_2 + \tilde{a}^{\bar{n}+1}P_1 - \tilde{a}(P_1+P_2) > 0$

$\Leftrightarrow \frac{\tilde{a}-\tilde{a}^{\bar{n}+1}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}+1}} < \frac{P_2}{P_1}$

(using equation (12))

$\Leftrightarrow \frac{\tilde{a}-\tilde{a}^{\bar{n}+1}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}+1}} < \frac{P_2}{P_1} = \frac{\tilde{a}-\tilde{a}^{\bar{n}}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}}}$

$\Leftrightarrow \frac{\tilde{a}-\tilde{a}^{\bar{n}+1}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}+1}} < \frac{\tilde{a}-\tilde{a}^{\bar{n}}}{(1-\tilde{a})-(1-\tilde{a})^{\bar{n}}}$

$\Leftrightarrow \frac{\tilde{a}\left(1-\tilde{a}^{\bar{n}}\right)}{(1-\tilde{a})(1-(1-\tilde{a})^{\bar{n}})} < \frac{\tilde{a}\left(1-\tilde{a}^{\bar{n}-1}\right)}{(1-\tilde{a})(1-(1-\tilde{a})^{\bar{n}-1})}$

$\Leftrightarrow \frac{1-\tilde{a}^{\bar{n}}}{1-(1-\tilde{a})^{\bar{n}}} < \frac{1-\tilde{a}^{\bar{n}-1}}{1-(1-\tilde{a})^{\bar{n}-1}}$

The proof of Proposition 4 showed that $\frac{1-a^{\bar{n}}}{1-a^{\bar{n}-1}}$ is increasing in a, and $\frac{1-(1-a)^{\bar{n}}}{1-(1-a)^{\bar{n}-1}}$is decreasing in a. In addition, the two terms are equal if $a = \dfrac{1}{2}$. Therefore, if $a < \dfrac{1}{2}$ then $\frac{1-a^{\bar{n}}}{1-a^{\bar{n}-1}} < \frac{1-(1-a)^{\bar{n}}}{1-(1-a)^{\bar{n}-1}}$. The following claim proves that $\tilde{a} < \frac{1}{2}$.

Claim 4 $\tilde{a} = a^*[S_{[n,k=n-1]}] < \frac{1}{2}$

Proof

As $g(a;n) < 0$ if $a \in (0, a^*[S_{[n,k=n-1]}])$ and $g(a;n) > 0$ if $a \in (a^*[S_{[n,k=n-1]}], 1)$. If $g(\frac{1}{2}) > 0$, then $a^*[S_{[n,k=n-1]}] < \frac{1}{2}$. In Lemma 3, $g(\frac{1}{2}) > 0$ is proved.■

■

PROOF OF PROPOSITION 3

The proof is analogous to the proof of Proposition 5

By Proposition **1**, if $\frac{k}{n} = \frac{n-1}{n} > P1$, there exists only one inner steady state $a^*[S_{[n,k=n-1]}]$. Let $a^*[S_{[n,k=n-1]}] = \tilde{a}$. Thus $g(\tilde{a}) = 0$

As $P_2 = 1 - P_1$, the expression $P_1 - P_2$ increases if and only if P_1 increases.

In the proof of Proposition **2**, it is shown that $g(\tilde{a}) = 0$ is equivalent to equation (12):

$g(\tilde{a}) = 0 \Longleftrightarrow \frac{P_2}{P_1} = \frac{\tilde{a}-\tilde{a}^n}{(1-\tilde{a})-(1-\tilde{a})^n}$

$\Longleftrightarrow \frac{1-P_1}{P_1} = \frac{\tilde{a}-\tilde{a}^n}{(1-\tilde{a})-(1-\tilde{a})^n}$

On the one hand, $\frac{1-P_1}{P_1}$ is decreasing in P_1.

On the other hand,

$\frac{d}{d\tilde{a}}\left(\frac{\tilde{a}-\tilde{a}^n}{(1-\tilde{a})-(1-\tilde{a})^n}\right) = \frac{1}{\tilde{a}}\frac{\tilde{a}-\tilde{a}^n}{(1-\tilde{a})-(1-\tilde{a})^n} > 0$, it is increasing in $\tilde{a}$.

Therefore, if $P_1 - P_2$ increases, P_1 increases, $\frac{1-P_1}{P_1}$ decreases, and $\tilde{a}$ has to decreases.■

Lemma 5 *Let $P_{A,t}$ be the probability of an A-agent being promoted in period t and $P_{B,t}$ that of a B-agent. If $P_{A,t} > P_{B,t}$, then $a_{t+1} > a_t$.*

Proof.

Let a_t be the proportion of A-agents at level t, and b_t be the proportion of B-agents.

$a_{t+1} = \frac{\text{A-agents promoted}}{\text{agents promoted}} = \frac{a_t \mathcal{P}_{A,t}}{a_t \mathcal{P}_{A,t} + b_t \mathcal{P}_{B,t}} = \frac{a_t \mathcal{P}_{A,t}}{\frac{k}{n}\theta} \Leftrightarrow P_{A,t} = \frac{a_{t+1}}{a_t}\frac{k}{n}\theta$

$b_{t+1} = \frac{b_t \mathcal{P}_{B,t}(prom)}{\frac{k}{n}\theta} \Leftrightarrow P_{B,t} = \frac{b_{t+1}}{b_t}\frac{k}{n}\theta$

Therefore,

$P_{A,t} > P_{B,t} \Leftrightarrow \frac{a_{t+1}}{a_t}\frac{k}{n}\theta > \frac{b_{t+1}}{b_t}\frac{k}{n}\theta \Leftrightarrow \frac{a_{t+1}}{a_t} > \frac{b_{t+1}}{b_t} \Leftrightarrow \frac{a_{t+1}}{a_t} > \frac{(1-a_{t+1})}{(1-a_t)} > 0 \Leftrightarrow \frac{a_{t+1}-a_t}{a_t(1-a_t)} > 0 \Leftrightarrow a_{t+1} > a_t \blacksquare$

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