



Munich Personal RePEc Archive

On the Smoothness of Value Functions

Strulovici, Bruno and Szydlowski, Martin

Northwestern University

7 January 2012

Online at <https://mpra.ub.uni-muenchen.de/36326/>

MPRA Paper No. 36326, posted 01 Feb 2012 07:26 UTC

On the Smoothness of Value Functions

Bruno Strulovici
Northwestern University

Martin Szydlowski
Northwestern University

January 31, 2012

Abstract

We prove that under standard Lipschitz and growth conditions, the value function of all optimal control problems for one-dimensional diffusions is twice continuously differentiable, as long as the control space is compact and the volatility is uniformly bounded below, away from zero. Under similar conditions, the value function of any optimal stopping problem is continuously differentiable. For the first problem, we provide sufficient conditions for the existence of an optimal control, which is also shown to be Markov. These conditions are based on the theory of monotone comparative statics.

Keywords: Super Contact, Smooth Pasting, HJB Equation, Optimal Control, Markov Control, Comparative Statics, Supermodularity, Single-Crossing, Interval Dominance Order.

1 Control Problem

We are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ that satisfies the usual conditions, and whose outcomes are identified with the paths of a standard Brownian motion, denoted B .¹

We consider a process $\{X_t\}_{t \in \mathbb{R}_+}$ controlled by another process $\{A_t\}_{t \in \mathbb{R}_+}$, taking values in a nonempty closed interval \mathcal{X} of \mathbb{R} with (possibly infinite) endpoints $\underline{x} < \bar{x}$, and following the dynamic equation

$$\begin{aligned} dX_t &= \mu(X_t, A_t) dt + \sigma(X_t, A_t) dB_t \\ x_0 &= x. \end{aligned} \tag{1}$$

¹We refer the reader to Karatzas and Shreve (1998) for the standard concepts used in this paper.

ASSUMPTION 1 *There exists a compact metric space \mathcal{K} such that $A_t \in \mathcal{K}$ for all t .*²

A control process satisfying Assumption 1 and such that (1) has a unique strong solution is said to be *admissible*. The set of admissible control processes is denoted by \mathcal{A} .

Given an admissible control A , the agent's expected payoff is given by

$$v(x, A) = E \left[\int_0^\kappa e^{-rt} f(X_t^A, A_t) dt + e^{-r\kappa} g(X_\kappa^A) \right]$$

where $f(x_t, a_t)$ is the flow payoff at time t , X_t^A is the process starting at x and controlled by A , and κ is the first exit time of the process X_t from \mathcal{X} . Assumptions 2 and 3, stated shortly, guarantee that the expected payoff is well-defined for any admissible control.³

The (optimal) *value function*⁴ of the problem starting at x , denoted $v(x)$, is defined by

$$v(x) = \sup_{A \in \mathcal{A}} v(x, A). \tag{2}$$

An admissible control is said to be *optimal* if $v(x, A) = v(x)$.

We make the following Lipschitz assumption on μ , σ , and f .

ASSUMPTION 2 *There exists $K > 0$ such that, for all $(x, a) \in \mathcal{X} \times \mathcal{K}$ and $x' \in \mathcal{X}$*

$$\begin{aligned} |\mu(x, a) - \mu(x', a)| &\leq K|x - x'| \\ |\sigma(x, a) - \sigma(x', a)| &\leq K|x - x'| \\ |f(x, a) - f(x', a)| &\leq K|x - x'|. \end{aligned}$$

*The functions $\mu(x, \cdot)$, $\sigma(x, \cdot)$, $f(x, \cdot)$ are continuous in a , for each x .*⁵

The last assumption contains several bounds on the primitives: standard linear growth conditions, a uniform lower bound on σ , and a condition guaranteeing that, for any control, the state grows at a rate slower than the discount rate.

²The analysis can be extended to the case the control set depends on x .

³Precisely, the growth conditions on μ and f guarantee that $f(X_t, A_t)$ grows at a strictly slower rate than r . This implies that the integral payoff is bounded. The terminal payoff is bounded above by $\max\{g(\underline{x})1_{\underline{x} > -\infty}, g(\bar{x})1_{\bar{x} < +\infty}\}$, which is also finite.

⁴To avoid confusion, we reserved the expression “value function” to the *optimal* expected payoff, and use the expression “expected payoff” when the control is arbitrary.

⁵When \mathcal{K} is a finite set, this latter assumption is vacuous.

ASSUMPTION 3 *There exist strictly positive constants K_1^μ , K_2^μ , K^σ , K^f and σ such that for all $(x, a) \in \mathcal{X} \times \mathcal{K}$,*⁶

$$\begin{aligned} |\mu(x, a)| &\leq K_1^\mu + K_2^\mu |x| \\ |\sigma(x, a)| &\leq K^\sigma (1 + |x|) \\ |f(x, a)| &\leq K^f (1 + |x|) \\ |\sigma(x, a)| &\geq \sigma \\ K_2^\mu &< r. \end{aligned}$$

2 Twice Differentiability of the Value Function

Our objective is to prove that the value function is twice differentiable in the interior of \mathcal{X} and solves the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \sup_{a \in \mathcal{K}} -rv(x) + f(x, a) + \mu(x, a)v'(x) + \frac{1}{2}\sigma^2(x, a)v''(x) \quad (3)$$

with given boundary conditions

$$v(\underline{x}) = g(\underline{x}) \text{ if } \underline{x} \text{ is finite, and } v(\bar{x}) = g(\bar{x}) \text{ if } \bar{x} \text{ is finite.} \quad (4)$$

We will prove the following result.

THEOREM 1 *Under Assumptions 1–3, the value function is twice continuously differentiable and is the unique solution to the HJB equation (3) at all $x \in \mathcal{X}$.*

The proof consists of the following steps: 1) Prove the existence of a solution, w , to the HJB equation; 2) Construct a control process based on this solution; 3) Prove that the solution is the value function of the problem, and that either the control constructed in 2) is optimal, or that it can be approximated by a sequence of admissible controls.

These steps imply that *any* solution to the HJB equation must coincide with the value function of the problem and, therefore, they will show the uniqueness claimed in Theorem 1.

⁶Several of the growth conditions of Assumption 3 are implied by the Lipschitz conditions of Assumption 2. For example, Assumption 2 implies that, fixing some $x_0 \in \mathcal{X}$, we have $|f(x, a)| \leq |f(x_0, a)| + K|x - x_0| \leq K_f(1 + |x|)$ for $K_f = \max\{K, \max_{a \in \mathcal{K}} |f(x_0, a)| + K|x_0|\}$. Due to their distinct role in the analysis, we follow the usual convention of separating these conditions.

We first show existence of a solution to the HJB equation. This follows from Proposition 1 below, which is proven in the Appendix. The proposition relies on the following conditions for an arbitrary function $\bar{H}(x, p, q)$ defined on $\mathcal{X} \times \mathbb{R}^2$.

CONDITION 1 *There exist constants M and K such that*

$$i) |\bar{H}(x, p, q)| \leq M(1 + |p| + |q|)$$

$$ii) |\bar{H}(x, p, q) - \bar{H}(x, \tilde{p}, \tilde{q})| \leq K(|p - \tilde{p}| + |q - \tilde{q}|).$$

iii) \bar{H} is continuous in x for each (p, q) .

CONDITION 2 *For all (x, q) , $\bar{H}(x, \cdot, q)$ is nonincreasing in p .*

CONDITION 3 *For all positive constants K_1, K_2 large enough and $\varepsilon \in \{-1, 1\}$,*

$$\bar{H}(x, K_1 + K_2|x|, \varepsilon K_2) < 0 \text{ and } \bar{H}(x, -K_1 - K_2|x|, \varepsilon K_2) > 0.$$

PROPOSITION 1 *Suppose \bar{H} satisfies Conditions 1 - 3. Then, for any \underline{v} and \bar{v} , there exists a twice continuously differentiable solution to the equation*

$$v'' + \bar{H}(x, v, v') = 0$$

which satisfies

$$v(\underline{x}) = \underline{v} \text{ if } \underline{x} \text{ is finite}$$

$$v(\bar{x}) = \bar{v} \text{ if } \bar{x} \text{ is finite.}$$

Moreover, there exists a positive constant K such that

$$|v(x)| \leq K(1 + |x|) \text{ for all } x \in \mathcal{X}.$$

We apply Proposition 1 to the HJB equation (3) by checking Conditions 1 - 3.

PROPOSITION 2 *Under Assumptions 1–3, the HJB equation (3) has a twice continuously differentiable solution on \mathcal{X} .*

Proof. Equation (3) can be rewritten as

$$v'' + H(x, v, v') = 0,$$

where⁷

$$H(x, p, q) = \max_{a \in \mathcal{K}} \frac{2}{\sigma^2(x, a)} (-rp + f(x, a) + \mu(x, a)q). \quad (5)$$

We show that H satisfies Conditions 1, 2, and 3.

Let

$$h(a, x, p, q) = \frac{2}{\sigma^2(x, a)} (-rp + f(x, a) + \mu(x, a)q),$$

so that $H(x, p, q) = \max_{a \in \mathcal{K}} h(a, x, p, q)$. Assumptions 1–3 guarantee that h is continuous in a and Lipschitz in x , uniformly in a . Because r and σ^2 are positive, h is decreasing in p and Condition 2 is satisfied.

To verify Condition 1, we use the following inequality⁸

$$|H(x, p, q) - H(x, \tilde{p}, \tilde{q})| \leq \max_{a \in \mathcal{K}} \frac{2}{\sigma^2(x, a)} |(-rp + f(x, a) + \mu(x, a)q) - (-r\tilde{p} + f(x, a) + \mu(x, a)\tilde{q})|.$$

This implies that

$$|H(x, p, q) - H(x, \tilde{p}, \tilde{q})| \leq \frac{2}{\sigma^2} (r|p - \tilde{p}| + (K_1^\mu + K_2^\mu|x|)|q - \tilde{q}|).$$

The growth condition follows because μ is bounded on any compact in support and $\sigma^2 \geq \underline{\sigma}^2$.

Continuity of H in x , the last part of Condition 1 is the key to guarantee that the value function is twice differentiable, even when the optimal control jumps, and is due to Berge's Maximum Theorem. Because the objective h is continuous in a and Lipschitz in x , uniformly in a , it is easy to show that h is jointly continuous in (x, a) .⁹ Since also the action set \mathcal{K} is compact, the Maximum Theorem applies, which proves that H is continuous in x . If, instead of a maximum selection $x \mapsto \hat{a}(x)$ of g , we had chosen an arbitrary selection $x \mapsto \tilde{a}(x)$, the resulting function $\tilde{H}(x, p, q) = h(x, \tilde{a}(x), p, q)$ would not be continuous in x . This hints at the reason why the expected payoff is twice differentiable for the optimal control, whereas it may fail to be so for a strictly suboptimal control.

There remains to verify that Condition 3 holds. To show that $H(x, K_1 + K_2|x|, \varepsilon K_2)$ is negative for large enough K_1, K_2 , it suffices to show that

$$-r(K_1 + K_2|x|) + K^f(1 + |x|) + (K_1^\mu + K_2^\mu|x|)K_2,$$

⁷Because all functions are continuous in a and σ is bounded below away from zero, the supremum is achieved as a maximum.

⁸More generally, if $H(\theta) = \max_{a \in \mathcal{K}} h(a, \theta)$, one can prove that $|H(\theta) - H(\tilde{\theta})| \leq \max_{a \in \mathcal{K}} |h(a, \theta) - h(a, \tilde{\theta})|$. For example, suppose that a, \tilde{a} maximize h at θ and $\tilde{\theta}$, respectively. Then, $H(\theta) - H(\tilde{\theta}) = h(a, \theta) - h(\tilde{a}, \tilde{\theta}) \leq h(a, \theta) - h(a, \tilde{\theta}) \leq \max_{a \in \mathcal{K}} |h(a, \theta) - h(a, \tilde{\theta})|$. The other inequality is proved similarly.

⁹This is shown as follows. We fix and omit from the notation some values for p and q , and suppose that (a_n, x_n) converges to (a, x) . We have $|h(x, a) - h(x_n, a_n)| \leq |h(x, a) - h(x, a_n)| + |h(x, a_n) - h(x_n, a_n)|$. The first term converges to zero by continuity of h with respect to a , while the second term converges to zero because h is Lipschitz in x , uniformly in a .

is negative. Since $K_2^\mu < r$, the result holds for all x , provided that K_1 and K_2 are large enough.

Thus the assumptions in Proposition 1 are satisfied, which shows existence of a solution to the HJB equation for arbitrary boundary conditions at \underline{x} and \bar{x} , whenever these points are finite.

To conclude the argument, we have to show that the solution w is indeed the value function. We split up the proof into two inequalities.

LEMMA 1 *Let w be a solution to the HJB equation (3) and let $v(x, A)$ be the expected value function given any admissible control A . Then $w(x) \geq v(x, A)$ for all $x \in \mathcal{X}$.*

Proof. The proof follows a standard verification argument. For any fixed, admissible control A and time T , Itô's formula implies, for the diffusion X^A controlled by A and starting at x , that

$$e^{-rT}w(X_T^A) = w(x) + \int_0^T e^{-rt} \left(-rw(X_t^A) + \mu(X_t^A, A_t)w'(X_t^A) + \frac{1}{2}\sigma^2(X_t^A, A_t)w''(X_t^A) \right) dt + \int_0^T e^{-rt} \sigma(X_t^A, A_t)dB_t. \quad (6)$$

The stochastic integral has zero mean as long as $|X_t^A|$ grows at a slower rate than r , which is guaranteed by our assumption that $K_2^\mu < r$. This, together with the linear growth restriction on v in Proposition 1, also guarantees that $\lim_{T \rightarrow \infty} e^{-rT}w(X_T^A) = 0$. Taking expectations and using (8), we get the inequality

$$E \left[\int_0^T e^{-rt} f(X_t^A, A_t) dt \right] \leq w(x) - E [e^{-rT}w(X_T^A)]. \quad (7)$$

Taking the limit as T goes to infinity yields

$$v(x, A) \leq w(x).$$

For the reverse inequality, we first obtain a candidate optimal control A_t^* from the solution to the HJB equation w . This candidate need not be admissible, because the stochastic differential equation (1) may fail to have a unique strong solution. We will use a result by Nakao (1972), who has shown that a one-dimensional SDE has a unique strong solution if its drift is measurable and its volatility has bounded variation and is bounded away from zero.¹⁰ We exploit this property to construct an approximation to the candidate control which is admissible and gets arbitrarily close to the desired inequality.

LEMMA 2 *Let w be a solution to the HJB equation (3). Then $w(x) \leq v(x)$ for all $x \in \mathcal{X}$.*

¹⁰See also Revuz and Yor, 2001, p.392 for a related result.

Proof. We construct a candidate optimal control as follows. Take a solution to the HJB equation w , and define $\mathcal{M}(x) \subset \mathcal{K}$ as the set of maximizers in the equation

$$rw(x) = \max_{a \in \mathcal{K}} f(x, a) + \mu(x, a)w'(x) + \frac{1}{2}\sigma(x, a)^2w''(x) \quad (8)$$

Because f , μ , σ are continuous in a , the function h is continuous in a and, therefore, $\mathcal{M}(x)$ is nonempty, closed, and compact in \mathcal{K} . We can therefore apply the measurable maximum theorem,¹¹ which guarantees that there exists a measurable function $\hat{a}(x)$ such that $\hat{a}(x) \in \mathcal{M}(x)$ for all x . If the control $A_t^* = \hat{a}(X_t)$ is admissible, applying the previous verification argument, this time with an equality, shows that $w(x) = v(x, A^*)$ and therefore $w(x) \leq v(x)$. In general, the candidate control may not be admissible, because the volatility function $\hat{\sigma}(x) = \sigma(x, \hat{a}(x))$ can jump, violating the standard Lipschitz (or Hölder) continuity conditions that are usually assumed for existence results.¹² We circumvent this issue by the following approximation argument.

Fix any $\varepsilon > 0$ and let $\eta = \varepsilon/K$, where K is the Lipschitz constant included in Assumption 1. Consider a grid of \mathcal{X} with equally spaced intervals of length η , and define the Markovian control \tilde{a} by $\tilde{a}(x) = \hat{a}(\chi(x))$ where $\chi(x)$ is the element of the grid closest to x .¹³ By construction, \tilde{a} is piecewise constant. By Nakao (1972), the SDE

$$dX_t = \mu(X_t, \tilde{a}(X_t))dt + \sigma(X_t, \tilde{a}(X_t))dB_t \quad (9)$$

has a unique strong solution, because the function $x \mapsto \sigma(x, \tilde{a}(x))$ has bounded variation. Letting $\{\tilde{X}_t\}_{t \in \mathbb{R}_+}$ denote this solution, the control $\{\tilde{A}_t\}_{t \in \mathbb{R}_+}$ defined by $\tilde{A}_t = \tilde{a}(\tilde{X}_t)$ is admissible. Moreover, the Lipschitz property of μ , σ , and f guarantees that $\tilde{a}(x)$ satisfies for all x

$$f(x, \tilde{a}(x)) + \mu(x, \tilde{a}(x))w'(x) + \frac{1}{2}\sigma(x, \tilde{a}(x))^2w''(x) \geq \max_{a \in \mathcal{K}} f(x, a) + \mu(x, a)w'(x) + \frac{1}{2}\sigma(x, a)^2w''(x) - D\varepsilon$$

for some positive constant D . This guarantees that, in the verification argument, the inequality (7) is almost tight, with the gap being bounded above by $D\varepsilon/r$. This yields

$$v(x) \geq v(x, \tilde{A}) \geq w(x) - D\varepsilon/r.$$

Taking the limit as ε goes to zero yields the desired inequality $v(x) \geq w(x)$.

3 Existence of an Optimal Control

The previous section has established twice differentiability of the value function, but did not prove the existence of an optimal control.

¹¹See Aliprantis and Border (1999), p. 570.

¹²See, e.g., Karatzas and Shreve (1998), Chapter 5.2, Theorem 2.5.

¹³We can adopt any arbitrary convention when there are two points of the grid that are closest to x .

As observed earlier, the control constructed in Lemma 2 need generally not be admissible, because it may fail to generate a strong solution to the SDE (1). This section provides conditions under which there exists an admissible candidate optimal control. We start by stating the most general result:

THEOREM 2 *The control A_t^* constructed in Lemma 2 is admissible and optimal if the function $\hat{\sigma} : x \mapsto \sigma(x, \hat{a}(x))$ has locally bounded variation.*

Proof. Admissibility follows from Nakao (1972). Optimality follows from a standard verification argument. ■

The bounded variation condition is necessary for the result: Barlow (1982) provides a large class of stochastic differential equations for which the volatility does not have bounded variation and there does not exist a strong solution, even if the volatility is bounded below, away from zero.¹⁴

An easy case in which $\hat{\sigma}$ has bounded variation is if $\sigma(x, a)$ is independent of a , a case that arises in many economic applications. Apart from that case, one way of guaranteeing that $\hat{\sigma}$ has bounded variation is to check that the correspondence \mathcal{M} of maximizers has a Lipschitz continuous selection \hat{a} , and to assume that $\sigma(x, a)$ is Lipschitz in a , uniformly in x . This implies that $\hat{\sigma}$ is Lipschitz continuous and, therefore, has bounded variation.¹⁵ Unfortunately, this condition is very strong, and certainly does not hold if the control set \mathcal{K} is discrete, for in that case the selection \hat{a} is generally discontinuous in x .

Our next result follows a different route. In many economic problems, it is often the case that the optimal control is monotonic. For example, investment in a project is monotonic in the probability that this project succeeds. Consumption is increasing in wealth, etc. Sometimes, the optimal control is only piecewise monotonic. There are different ways of proving establishing this monotonicity, based on direct arguments, or by exploiting the HJB equation. If we can construct a selection \hat{a} that is piecewise monotonic, this will guarantee, under a mild and easily checked condition on σ , that $\hat{\sigma}$ has bounded variation.

We start by stating a general theorem, and then refine it by putting more structure on the primitive of the control problem.

PROPOSITION 3 *Suppose that σ is Lipschitz continuous in a ,¹⁶ uniformly on any compact subset of \mathcal{X} , and that there exists a nondecreasing sequence $\{x_n\}_{n \in \mathbb{Z}}$ of \mathcal{X} and a selection $\hat{a}(\cdot)$ of $\mathcal{M}(\cdot)$ such*

¹⁴We are grateful to Nicolai Krylov for pointing this reference out to us.

¹⁵Continuity of the control is not enough, as it does not imply bounded variation.

¹⁶The assumption that σ is Lipschitz in a cannot be easily relaxed: Josephy (1981, Theorem 4) has shown that for the composition $f \circ g$ to have bounded variations for all functions g with bounded variation, f must be Lipschitz.

that

i) $\lim_{n \rightarrow -\infty} x_n = \underline{x}$ and $\lim_{n \rightarrow +\infty} x_n = \bar{x}$,

ii) There exists $\alpha > 0$ such that for any $n \in \mathbb{Z}$, either $x_n = x_{n+1}$ or $x_{n+1} \geq x_n + \alpha$

iii) On any section $(x_n, x_{n+1}) \times \mathcal{A}$, \hat{a} is monotonic.

Then, there exists an optimal control, which is Markov and characterized by the selector \hat{a} , and such that $\hat{\sigma}$ has bounded variation.

Proof. Consider any interval (x_n, x_{n+1}) . We show that monotonicity of \hat{a} and the Lipschitz conditions on σ imply that $\hat{\sigma}$ has locally bounded variation on the interval. To see this, let $\{\chi_j\}_{j=1, \dots, m}$ denote a partition of some compact interval of (x_n, x_{n+1}) . We have

$$\sum_{j=1}^m |\bar{\sigma}(\chi_{j+1}) - \bar{\sigma}(\chi_j)| \leq \sum_{j=1}^m |\sigma(\chi_{j+1}, \hat{a}(\chi_{j+1})) - \sigma(\chi_j, \hat{a}(\chi_{j+1}))| + \sum_{j=1}^m |\sigma(\chi_j, \hat{a}(\chi_{j+1})) - \sigma(\chi_j, \hat{a}(\chi_j))|.$$

The first sum has bounded variation by Lipschitz continuity of σ in x , uniformly in a . The second term is bounded by $K_\chi \sum_j |\hat{a}(\chi_{j+1}) - \hat{a}(\chi_j)|$, where K_χ is the Lipschitz constant of σ with respect to a , which is uniform on $[\chi_0, \chi_m]$. Because $\hat{a}(\cdot)$ is monotonic on that domain, it has bounded variation, and so does the second sum. To conclude the proof, we define \hat{a} arbitrarily on the set $\{x_n : n \in \mathbb{Z}\}$, which is locally finite. The selector \hat{a} is clearly measurable. Consider now any bounded interval \mathcal{I} included in \mathcal{X} . By assumptions i) and ii), that interval is covered by finitely many intervals of the form $[x_n, x_{n+1}]$, which implies that $\hat{\sigma}$ has bounded variation on \mathcal{I} .¹⁷ ■

There are several ways of proving the existence of a piecewise-monotonic selection \hat{a} . Our approach is based on the theory of monotone comparative statics, which provide a general condition guaranteeing monotonicity of a maximizer with respect to a parameter. For the remainder of this section, we focus on one-dimensional controls, so that \mathcal{K} is a compact subset of \mathbb{R} .

A function $\rho(x, a)$ is *supermodular* on some domain $\mathcal{X}_0 \times \mathcal{K}_0$ if for all $\tilde{a} \geq a$ in \mathcal{K}_0 and $\tilde{x} \geq x$ in \mathcal{X}_0 , $\rho(a, x) + \rho(\tilde{a}, \tilde{x}) \geq \rho(\tilde{x}, a) + \rho(x, \tilde{a})$, and *submodular* if the reverse inequality holds on that domain (or, equivalently, if $-\rho$ is supermodular). When ρ is differentiable in a , supermodularity is equivalent to ρ_a being nondecreasing in x . When ρ is twice differentiable, supermodularity is equivalent to the cross partial being everywhere nonnegative. Supermodularity is a sufficient condition for the maximizer correspondence $x \mapsto \mathcal{M}(x)$ to be nondecreasing in the strong set order, which means that for all $x \leq \tilde{x}$, $a \in \mathcal{M}(x)$, and $\tilde{a} \in \mathcal{M}(\tilde{x})$, we have $\min\{a, \tilde{a}\} \in \mathcal{M}(x)$ and $\max\{a, \tilde{a}\} \in \mathcal{M}(\tilde{x})$.

¹⁷ $\bar{\sigma}$ can jump at each x_n , but these jumps are bounded, because \mathcal{K} is compact and σ is continuous in a (and x).

In particular, the selections constructed from the smallest and largest maximizers, respectively, are nondecreasing for any supermodular function, and nonincreasing for any submodular one.¹⁸

The following result exploits this property to guarantee piecewise monotonicity of the selection and, hence, bounded variation of $\hat{\sigma}$.

THEOREM 3 *Suppose that σ is Lipschitz continuous in a ,¹⁹ uniformly on any compact subset of \mathcal{X} , and let*

$$\rho(x, a) = \frac{1}{\sigma^2(x, a)} \left(-rv(x) + f(x, a) + \mu(x, a)v'(x) \right),$$

where v is the value function of the problem, and suppose that there exists a nondecreasing sequence $\{x_n\}_{n \in \mathbb{Z}}$ of \mathcal{X} such that

- i) $\lim_{n \rightarrow -\infty} x_n = \underline{x}$ and $\lim_{n \rightarrow +\infty} x_n = \bar{x}$,
- ii) There exists $\alpha > 0$ such that for any $n \in \mathbb{Z}$, either $x_n = x_{n+1}$ or $x_{n+1} \geq x_n + \alpha$
- iii) On any section $(x_n, x_{n+1}) \times \mathcal{A}$, ρ is supermodular or submodular.

Then, there exists an optimal control, which is Markov and characterized by a selector \hat{a} of \mathcal{M} such that $\hat{\sigma}$ has bounded variation.

Proof. On any interval (x_n, x_{n+1}) , let $\hat{a}(x) = \max \mathcal{M}(x)$. Whether ρ is supermodular or submodular, \hat{a} is monotonic on this interval. The result then follows from Proposition 3. ■

We now discuss several environments under which Theorem 3 can be applied. We can write ρ as

$$-rl(x, a)v(x) + \bar{f}(x, a) + \bar{\mu}(x, a)v'(x), \tag{10}$$

where $l(x, a) = 2/\sigma^2(x, a)$, $\bar{f} = lf$ and $\bar{\mu} = l\mu$ are all primitives of the optimal control problem.

In many economic problems (such as experimentation problems, or when the agent has an increasing concave utility flow), it is possible to show that the value function v has a constant sign, is monotonic, and is either convex or concave.²⁰ In such case, monotonicity of the selection can be guaranteed on the entire domain and one can find simple conditions on the primitives for which monotonicity of the selection obtains over the entire domain.

¹⁸This result is easy to check. See Topkis (1978) or Milgrom and Shannon (1994) for a proof. The strong set order is also called the Veinott set order (see Veinott, 1989).

¹⁹The assumption that σ is Lipschitz in a cannot be easily relaxed: Josephy (1981, Theorem 4) has shown that for the composition $f \circ g$ to have bounded variations for all functions g with bounded variation, f must be Lipschitz.

²⁰There are many techniques to establish this, either by analyzing the HJB equation, or by constructing various controls to directly show that the value function must be increasing and convex.

COROLLARY 1 *Suppose that the following conditions hold:*

- i) σ is Lipschitz in a , uniformly on any compact subset of \mathcal{X}*
- ii) The value function v is nonnegative (resp., nonpositive), increasing and convex*
- iii) $\bar{\mu}$ is nondecreasing in a and supermodular, \bar{f} is supermodular, l nonincreasing in a and submodular (resp., supermodular).*

Then, there exists an optimal control, and this control is Markov and nondecreasing in x .

The corollary is straightforward to prove: its conditions guarantee that each term in (10) is supermodular.

The supermodularity (or submodularity) assumed in Condition iii) of Theorem 3 can be weakened in several ways. Indeed, it suffices that ρ satisfies the single-crossing property in (a, x) , or obeys the *Interval Dominance Order* (IDO) property introduced by Quah and Strulovici (2009) for asserting the existence of a monotonic selection on any give interval.²¹ When ρ is differentiable with respect to a , the IDO property is guaranteed to hold on (x_n, x_{n+1}) if there exists a positive, nondecreasing function $\gamma_n(\cdot)$ of a such that, for each $x'' > x'$, $\rho_a(x'', a) \geq \gamma_n(a)\rho_a(x', a)$.²² If γ_n is constant over \mathcal{K} , the inequality implies that ρ satisfies the single crossing property in (a, x) . If γ_n is identically equal to 1, we recover the supermodularity condition.

4 Optimal Stopping and Smooth Pasting

This section establishes that, under similar conditions to those of Section 1, the value function of any optimal stopping problem is everywhere differentiable. This implies, in particular, that it is differentiable at any threshold at which stopping becomes optimal, which is commonly known as the *smooth pasting* property. For presentation clarity, we separate optimal control and optimal stopping problems, which imply different smoothness levels for the value function. The problems can be combined, however, with an appropriate extension of Theorems 1 and 4.

Consider the optimal stopping problem

$$v(x) = \sup_{\tau \in \mathcal{T}} E \left[\int_0^\tau e^{-rt} f(X_t) dt + e^{-r\tau} g(X_\tau) \right],$$

²¹The IDO property generalizes the single-crossing property (Milgrom and Shannon, 1994), which is itself more general than supermodularity (Topkis, 1978). It also generalizes families of functions that are quasi-concave with increasing peaks, which were used, among many others, by Karlin and Rubin (1956) and Lehmann (1988).

²²See Quah and Strulovici (2009, Proposition 2)

where \mathcal{T} is the set of all stopping times adapted to the initial filtration \mathcal{F} , and $\{X_t\}_{t \in \mathbb{R}_+}$ solves the equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$$

subject to the initial condition $X_0 = x$. We maintain the same assumptions as before on μ , σ and f , which guarantee that the SDE has a unique strong solution, and, along with assumption 4, that the expected payoff is well defined for all stopping times.

We make the additional assumption that the termination function g is continuously differentiable.

ASSUMPTION 4 $g(\cdot)$ is C^1 and has bounded linear growth: $|g(x)| \leq K^g(1 + |x|)$ for some constant $K^g \geq 0$.

THEOREM 4 Under Assumptions 2–4²³ v is continuously differentiable on the interior of \mathcal{X} .

Proof. Let \mathcal{Y} denote the subset of \mathcal{X} for which $v(x) = g(x)$. \mathcal{Y} consists of all the states at which it is optimal to stop immediately. By continuity of v and g , $\mathcal{X} \setminus \mathcal{Y}$ consists of disjoint open intervals $\{\mathcal{Z}_i\}_{i \in I}$. Pick any two points $x_1 < x_2$ in such an interval. The result stated in Appendix A guarantees the existence of C^2 solution w to the ordinary differential equation

$$w''(x) + \frac{1}{\frac{1}{2}\sigma(x)^2}(-rw(x) + f(x) + \mu(x)w'(x)) = 0. \quad (11)$$

with boundary conditions $w(x_1) = v(x_1)$ and $w(x_2) = v(x_2)$. A standard verification argument then shows that v coincides with w on any such interval and, therefore, that v is C^2 on such interval and, hence, on $\mathcal{X} \setminus \mathcal{Y} = \cup_{i \in I} \mathcal{Z}_i$.

Consider now the boundary of some interval \mathcal{Z}_i , for example the upper boundary, and call it x^* . By construction, $v(x) > g(x)$ for x in a left-neighborhood of x^* , and $v(x^*) = g(x^*)$. In particular, v'_l , the left derivative of v at x^* must be less than or equal to $g'(x^*)$.

We now show by contradiction that this inequality must be tight. Thus suppose that $v'_l(x^*) < g'(x^*)$, and consider the domain $[x_1, x_2 = x^* + \varepsilon]$ for some x_1 in \mathcal{Z}_i and some small $\varepsilon > 0$. From Appendix A, there exists a solution w to Equation (11) on $[x_1, x_2]$, with initial value $w(x_1) = v(x_1)$ and initial slope $w'(x_1) = v'(x_1)$. Moreover, this solution satisfies $w(x^*) = v(x^*)$, and $w'(x^*) = v'_l(x^*)$, because v solves the same initial value problem (IVP) on the domain $[x_1, x^*]$, and the solution is unique (see Lemma 4). Therefore, $w(x) < g(x)$ for x in a right neighborhood of x^* , and without loss, on (x^*, x_2) . Taking a slightly higher slope $s' > s^*$, consider the solution \hat{w} to the IVP on the domain $[x_1, x_2]$ with initial slope s' and initial value $v(x_1)$. For s' close to s , this solution hits g at some $\hat{x} \in (x_1, x_2)$, because solutions to the IVP are continuous in s (see Lemma 5 in the Appendix).

²³The pure stopping problem that we consider is equivalent to reducing the control set \mathcal{K} of Section 1 to a singleton.

Moreover Lemma 3, which will be stated and proved shortly, implies that $\hat{w}(x) > w(x)$ for all $x \in (x_1, x^*]$ and, therefore, that $\hat{x} > x^*$. We redefine \hat{w} by setting $\hat{w}(x) = g(x)$ for all $x > \hat{x}$. By construction, $\hat{w}(x^*) > g(x^*) = v(x^*)$. Moreover, \hat{w} corresponds to the expected payoff obtained if the following stopping strategy is used: starting from x^* , continue until either x_1 or \hat{x} is reached. If \hat{x} is reached first, stop. If x_1 is reached first, follow the initial strategy leading to value $v(x_1)$. This strategy thus strictly improves on the value $v(x^*)$, starting from x^* , a contradiction.

We have proved so far that v is differentiable at x in the following cases: i) x lies in the interior of some interval \mathcal{Z}_i , ii) x connects two intervals \mathcal{Z}_i and \mathcal{Z}_j (i.e., it is the upper bound of one interval, and the lower bound of the other), and iii) x is a bound of some interval \mathcal{Z}_i , and $v(y) = g(y)$ in some neighborhood on the other side of x . Moreover, we showed that in cases ii) and iii) the derivative is given by $v'(x) = g'(x)$. In all these cases, it is straightforward to show that the derivative is continuous at x because the solution to (11) on any \mathcal{Z}_i is twice differentiable, and because g is C^1 .

To conclude the proof, we need to show the result when x is such that $v(x) = g(x)$, but x is an accumulation point of stopping and non stopping regions, on either its right side or its left side, or both. Without loss of generality, we set $x = 0$ and prove that $v_r(0) = g'(0)$, where v_r is the right derivative of v at 0. We wish to show that $\lim_{\eta \downarrow 0} (v(\eta) - v(0))/\eta$ converges to $g'(0)$. Consider any $\eta > 0$. The difference $v(\eta) - v(0)$ is either equal to $g(\eta) - g(0)$, if η belongs to \mathcal{Y} , or else η belongs to some interval \mathcal{Z}_i close to 0. Let y denote the lower bound of \mathcal{Z}_i . By twice differentiability of v on (y, η) , and because the right derivative of v at y is equal to $g'(y)$, we have $v(\eta) = v(y) + g'(y)(\eta - y) + v_r''(z_1)(\eta - y)^2$ for some $z_1 \in (y, \eta)$. Since $v(y) = g(y)$, we have $g(\eta) = v(y) + g'(z_2)(\eta - y)$ for some $z_2 \in (y, \eta)$. Therefore, $v(\eta) = g(\eta) + (g'(y) - g'(z_2))(\eta - y) + v_r''(z_1)(\eta - y)^2$. Thus, we have

$$\frac{v(\eta) - v(0)}{\eta} = \frac{g(\eta) - g(0)}{\eta} + 1_{\eta \notin \mathcal{Y}} \frac{\eta - y}{\eta} (g'(y) - g'(z_2) + v_r''(z_1)(\eta - y)).$$

Taking the limit as η goes to zero yields the result, if we can show that $g'(y) - g'(z_2) + v_r''(z_1)(\eta - y)$ converges to zero as $\eta \rightarrow 0$. The first two terms cancel each other in the limit, as they both converge to $g'(0)$ (since g is C^1). The last term converges to 0 if we can show that $v''(\cdot)$ is uniformly bounded on all the intervals \mathcal{Z}_i in a neighborhood of 0. This uniform bound is guaranteed by a comparison lemma provided in the Appendix (see Lemma 8, which guarantees a uniform upper bound on $|v'|$ and on $|v''|$).²⁴ To prove continuity of v' for this case, the argument is similar. For any $\varepsilon > 0$, there

²⁴More precisely, the solution w_i to the BVP problem on \mathcal{Z}_i with endpoints $[x_i, \bar{x}_i]$ can be extended to a solution on the domain $\mathcal{Z}'_i[x_i, \max\{x_i + \tilde{x}, \bar{x}_i\}]$, for some $\tilde{x} > 0$ that is independent of i , by considering the IVP problem with domain \mathcal{Z}'_i and keeping the same initial slope. That solution has uniformly bounded first and second derivatives, by Lemma 8 and the discussion that follows it. The uniform bounds do not depend on i : they only depend on \tilde{x} ,

exists $\bar{\eta}(\varepsilon)$ be such that $|g'(\eta) - g'(0)| \leq \varepsilon/2$ for all $\eta \leq \bar{\eta}(\varepsilon)$. As was shown earlier, v'' is uniformly bounded on the interval $[x, x + \bar{\eta}(\varepsilon)]$, by some constant M . Let $\eta(\varepsilon) = \min\{\bar{\eta}(\varepsilon), \varepsilon/M\}$. Consider any $\eta < \eta(\varepsilon)$. If $\eta \in \mathcal{Y}$, then $v'(\eta) = g'(\eta)$ and $|v'(\eta) - v'(0)| < \varepsilon/2$. Otherwise we have, using the variable y introduced earlier in the proof,

$$v'(\eta) = v'(y) + v''(y)(\eta - y) = g'(y) + v''(y)(\eta - y),$$

which implies that

$$|v'(\eta) - v'(0)| < |g'(y) - g'(0)| + M\eta < \varepsilon.$$

Proceeding similarly on the left of x , we conclude that v' is continuous at x . ■

LEMMA 3 *Consider $v_{s'}$ and v_s , two solutions to the IVP with starting slopes $s' > s$ on an interval $[x_1, x_2]$ which both satisfy $v_{s'}(x_1) = v_s(x_1) = v_1$. Then, $v_{s'}(x) > v_s(x)$ for all $x \in (x_1, x_2]$.*

Proof. Let $\hat{x} = \inf \{x : v_{s'}(x) \leq v_s(x)\}$. Note that $\hat{x} > \underline{x}$ because $v_{s'}(\underline{x}) > v_s(\underline{x})$ and both $v_{s'}$ and v_s are C^2 . By construction, $v_{s'}(\hat{x}) > v_s(\hat{x})$. Since both solutions satisfy the equation

$$v''(x) + \frac{1}{\frac{1}{2}\sigma(x)^2} (-rv(x) + f(x) + \mu(x)v'(x)) = 0$$

we have

$$\begin{aligned} v_{s'}''(\hat{x}) &= \frac{1}{\frac{1}{2}\sigma(\hat{x})^2} (rv_{s'}(\hat{x}) + f(\hat{x}) + \mu(\hat{x})v_{s'}'(\hat{x})) \\ &> \frac{1}{\frac{1}{2}\sigma(\hat{x})^2} (rv_s(\hat{x}) + f(\hat{x}) + \mu(\hat{x})v_s'(\hat{x})) \\ &= v_s''(\hat{x}) \end{aligned}$$

Since $v_{s'}'(x)$ must hit $v_s'(x)$ from above as x reaches \hat{x} , we obtain a contradiction.

A Proof of Proposition 1

A.1 General Results on Initial Value Problems

We start with two results pertaining to the existence of solutions to initial value problems (IVP) and their continuity with respect to the initial conditions. We start with some function $\bar{H} : (x, y) \mapsto \bar{H}(x, y)$ defined on $\mathcal{X} \times \mathbb{R}^n$ and taking values in \mathcal{R}^n , which satisfies the following condition:

an upper bound on v , and the fact that $|w''(x)| = |\frac{1}{\frac{1}{2}\sigma(x)^2} (-rw(x) + f(x) + \mu(x)w'(x))| \leq K_1 + K_2|w'(x)|$ for some constants K_1, K_2 .

CONDITION 4 *On any compact interval \mathcal{I} of \mathcal{X} ,*

$$i) |H(x, y)| \leq M(1 + |y|)$$

$$ii) |H(x, y) - H(x, y')| \leq K|y - y'|.$$

iii) *H is continuous in x for each y.*

LEMMA 4 *If Condition 4 holds, the ordinary differential equation*

$$y'(x) = H(x, y(x)) \tag{12}$$

with initial condition $y(x_0) = y_0$ has a unique continuously differentiable solution on \mathcal{X} , for any $x_0 \in \mathcal{X}$ and $y_0 \in \mathbb{R}^n$.

Let $y(x, y_0)$ denote the solution to (12) on \mathcal{X} with initial condition $y(x_0) = y_0$.

LEMMA 5 *Given Condition 4 $y(\cdot, y_0)$ is uniformly continuous in y_0 .*

The proofs are standard and omitted. For Lemma 4 see, e.g., Hartman (2002, Chapter 2, Theorem 1.1).²⁵ For Lemma 5, see Hartman (2002, Chapter 5, Theorem 2.1).

A.2 Proof of Proposition 1: Bounded Case

We specialize the results of Section A.1 to our setting: suppose that $y = (p, q)$ and $H(x, p, q)$ satisfies Condition 1. In this case, the function $\bar{H}(x, (p, q)) = (q, H(x, p, q))$ satisfies Condition 4.

The proof of Proposition 1 is based on the “shooting method” (see, e.g., Bailey, 1962). The general intuition for the argument is as follows. We start from some initial conditions $(\underline{x}, \underline{v})$ and consider the solution to the IVP

$$v'' + H(x, v, v') = 0 \tag{13}$$

subject to the initial conditions $v(\underline{x}) = \underline{v}$ and $v'(\underline{x}) = s$. Given our assumption on H , Lemma 4 guarantees that this IVP will have a unique, twice continuously differentiable solution. Lemma 5 guarantees that the solution continuously depends on the starting slope s . We can establish existence to a solution of the boundary value problem (BVP) if we can show that it is always possible to pick the slope s in such a way that at \bar{x} , the solution to the IVP will hit \bar{v} .

²⁵That theorem establishes local existence. The growth condition, i), guarantees that the solution can be extended to the entire domain \mathcal{I} .

The proof relies on constructing a particular compact, convex subset of (x, v) -plane, ending with a vertical segment at $x = \bar{x}$ that contains \bar{v} . We then define a mapping between the initial slope s of the solution to the IVP problem with initial value $w(\underline{x}) = \underline{v}$ and initial slope s , and the “last” point at which it hits the boundary, and show that the mapping is onto. That property then proves the existence of an initial slope such that the solution hits the value \bar{v} at \bar{x} .

LEMMA 6 *For any positive constant K_v large enough, the functions $b_1(x) = -K_v(1 + |x|)$ and $b_2(x) = K_v(1 + |x|)$ satisfy the inequalities*

$$\begin{aligned} b_1'' + H(x, b_1, b_1') &> 0 \\ b_2'' + H(x, b_2, b_2') &< 0 \end{aligned}$$

for all $x \neq 0$, and the boundary constraints $\underline{v} \in (b_1(\underline{x}), b_2(\underline{x}))$ and $\bar{v} \in (b_1(\bar{x}), b_2(\bar{x}))$.

Proof. We have for $x \neq 0$

$$b_2''(x) + H(x, b_2(x), b_2'(x)) = H(x, K_v + K_v|x|, K_v \operatorname{sgn}(x)),$$

which is strictly negative, by Condition 3. The argument for v_1 is analogous. ■

LEMMA 7 *There exist s_1 and s_2 such that the solution to IVP (13) hits b_2 for all initial slopes $s \geq s_2$ and b_1 for all initial slopes $s \leq s_1$.*

Proof. By suitably translating the problem, we can without loss assume that $\underline{x} = \underline{v} = 0$.²⁶ We wish to show that for high enough initial slopes s , the solution $v(s)$ hits b_2 . Consider the auxiliary IVP

$$u'' + Ku' + H(x, u(x), 0) + \varepsilon = 0$$

subject to $u(0) = 0$ and $u'(0) = s$, where K is the Lipschitz constant of H and ε is a positive constant. We will show that, for s high enough, u is strictly increasing on $[0, \bar{x}]$. For fixed s , let $\tilde{x} > 0$ denote the first time that $u'(x) = 0$. On $[0, \tilde{x}]$, we have $u(x) \geq 0$. By Condition 2, we have $H(x, u(x), 0) \leq H(x, 0, 0)$ on that domain, and

$$u''(x) + Ku'(x) + M \geq 0,$$

where $M = \max_{x \in [0, \bar{x}]} H(x, 0, 0) + \varepsilon$. Applying Grönwall’s inequality to the function $g(x) = -u'(x) - M/K$, which satisfies the inequality

$$g'(x) \leq -Kg(x)$$

²⁶The translation is obtained by letting $\bar{v}(x) = v(x - \underline{x}) - \underline{v}$ and $\bar{H}(x, v, v') = H(x - \underline{x}, v + \underline{v}, v')$. \bar{H} inherits the Lipschitz and monotonicity properties of H , as is easily checked.

on $[0, \tilde{x}]$, we conclude that

$$u'(x) \geq [s + M/K] \exp(-Kx) - M/K$$

on that domain, which implies that \tilde{x} is bounded below by²⁷

$$\frac{1}{K} \log \left(\frac{s + |M|/K}{|M|/K} \right),$$

which exceeds \bar{x} , for s high enough. Moreover, the lower bound on u' also implies that u hits b_2 for s large enough.

To conclude the proof, we will show that the IVP solution v is above u for any fixed s . The Lipschitz property of H in its last argument implies that, for all x, u, u' ,

$$-Ku' \leq H(x, u, 0) - H(x, u, u')$$

From the definition of u , this implies that

$$u''(x) + H(x, u(x), u'(x)) \leq -\varepsilon < 0$$

for all x . This implies that v , the solution to the IVP, lies above u , for the following reason. At $x = 0$, u and v have the same starting values and slopes, but u has a lower second derivative, by at least ε , which implies that $u' < v'$ in a right neighborhood of 0. We will show that $u' < v'$ for all x in $(0, \bar{x}]$ and, therefore, that $u < v$ on that domain. Suppose by contradiction that there exists an $x > 0$ such that $u'(x) = v'(x)$, and let \tilde{x} be the first such point. Necessarily, $u(\tilde{x}) < v(\tilde{x})$. Moreover, we have

$$u''(\tilde{x}) < -H(\tilde{x}, u(\tilde{x}), u'(\tilde{x})) \leq -H(\tilde{x}, v(\tilde{x}), v'(\tilde{x})) = v''(\tilde{x}),$$

where the second inequality is guaranteed by 2, which yields a contradiction. ■

We can finally prove Proposition 1. Let

$$B = \{(x, v) | b_1(x) = v \text{ or } b_2(x) = v\} \cup [(\bar{x}, b_1(\bar{x})), (\bar{x}, b_2(\bar{x}))] \subset \mathbb{R}^2.$$

B consists of the graph of the functions b_1 and b_2 on \mathcal{X} , along with the vertical segment joining the endpoints of these graphs at \bar{x} . We also define the function $\mathcal{H} : [s_1, s_2] \rightarrow \mathbb{R}^2$ as the *last* hitting point of B for the solution of the IVP with slope s . This function is clearly well defined: if a solution does not cross b_1 or b_2 before \bar{x} , it has to hit the vertical segment joining $b_1(\bar{x})$ and $b_2(\bar{x})$. From Lemma 7, $\mathcal{H}(s)$ is on the graph of b_2 for s large and on the graph of b_1 for s small (for s large, for example, remember that u had a slope greater than s and hence does not cross b_2 again after

²⁷If $M = 0$, the inequality implies that u' is strictly positive.

hitting it once). Moreover, \mathcal{H} cannot jump from the graph of b_2 to the graph of b_1 as s changes, because Lemma 6 implies, for example, that after v crosses b_2 , it stays above v_2 for all x beyond the crossing point,²⁸ and hence cannot hit b_1 . Therefore, \mathcal{H} must connect the upper and lower bounds of B as s goes down. Finally, Lemma 5 implies that \mathcal{H} is continuous at any point s for which $\mathcal{H}(s)$ lies on the vertical segment, this implies that $\mathcal{H}(s)$ must take all values on that segment as it connects the graphs of b_2 and b_1 . Since (\bar{x}, \bar{v}) belongs to that segment, this proves existence of a solution that solves the BVP.

A.3 Proof of Proposition 1: Unbounded Domain

We now prove Proposition 1 when \mathcal{X} is unbounded, so that $\underline{x} = -\infty$ and/or $\bar{x} = +\infty$. Precisely, we establish the existence of a function v which satisfies

$$v'' = H(x, v, v') \quad (14)$$

and

$$|v(x)| \leq K_v(1 + |x|) \quad (15)$$

on \mathcal{X} , where K_v is the constant used to construct the bounds b_1, b_2 in Lemma 6. The arguments of this section are based on Schrader (1969).

From Section A.2, we know that the BVP will have a unique C^2 solution for any finite interval $[\underline{\chi}, \bar{\chi}]$ and boundary conditions $v(\underline{\chi}) = \underline{v}$ and $v(\bar{\chi}) = \bar{v}$. Further, we know that the solution satisfies $-K_v(1 + |x|) \leq v(x) \leq K_v(1 + |x|)$ on $[\underline{\chi}, \bar{\chi}]$. The constant K_v depends only on the primitive functions of the problem, and not on the particular interval chosen.

We define a sequence of boundary value problems which satisfy Equation (14) on $[\underline{x}_n, \bar{x}_n]$ with boundary conditions $v(\underline{x}_n) = \underline{v}_n$ and $v(\bar{x}_n) = \bar{v}_n$ for some values $\underline{v}_n, \bar{v}_n$ in $(b_1(\underline{x}_n), b_2(\underline{x}_n))$ and $(b_1(\bar{x}_n), b_2(\bar{x}_n))$, respectively, and let \underline{x}_n and/or \bar{x}_n be identically equal to \underline{x} or \bar{x} , or tend to $-\infty$ or $+\infty$, respectively according to whether \underline{x} and \bar{x} are finite or infinite. In the following, we use the Arzelà-Ascoli theorem and show that this procedure indeed yields a solution.

Let v_n denote the solution to the n^{th} BVP. We use the following comparison lemma.²⁹

LEMMA 8 *Let ϕ denote a nonnegative, continuous function on \mathbb{R}_+ , such that*

$$\int_0^\infty \frac{s}{\phi(s)} ds = \infty, \quad (16)$$

²⁸The proof of this result is similar to the proof that v stays above u in Lemma 7, showing that $v' \geq b'_2$ after the crossing point, and exploits the inequality $b''_2 + H(x, b_2, b'_2) < 0$.

²⁹See Hartman (2002), p. 428.

and let R, \tilde{x} denote two strictly positive constants. Then, there exists a number M such that if $v(x)$ is C^2 on $[0, \bar{x}]$ with $\bar{x} > \tilde{x}$ and satisfies $|v(x)| \leq R$ and $|v''(x)| \leq \phi(|v'(x)|)$, then $|v'(x)| \leq M$ on $[0, \bar{x}]$. The constant M depends only on R, ϕ and \tilde{x} .

We have

$$|v''(x)| = |H(x, v(x), v'(x))| \leq |H(x, v(x), 0)| + K|v'(x)| \leq K_1 + K|v'(x)| \quad (17)$$

for any bounded domain $\mathcal{X}_0 = [\underline{\chi}, \bar{\chi}]$, by the Lipschitz property of H (with constant K), where the constant K_1 is obtained by boundedness of v (which is contained between b_1 and b_2) and continuity of $H(\cdot, \cdot, 0)$ on the compact domain \mathcal{X}_0 .

Since $\phi(x) = K_1 + Kx$ satisfies (16) and v is bounded by the functions b_1 and b_2 , we can apply Lemma 8 to conclude that each v'_n is bounded on the compact domain \mathcal{X}_0 , and the bound is uniform for all n .³⁰ Moreover (17) implies that the second derivative of v is also uniformly bounded.

We now employ the following diagonalization procedure. Consider a finite domain $[\underline{x}_1, \bar{x}_1]$. By the previous argument, for each n , v_n, v'_n and v''_n are bounded on $[\underline{x}_1, \bar{x}_1]$, and the bounds are uniform in n . By Arzelà-Ascoli, there exists a subsequence such that v_n converges uniformly to a C^1 function \tilde{v}_1 on $[\underline{x}_1, \bar{x}_1]$.³¹ Moreover, the second derivatives $\{v''_n\}_{n \in \mathbb{N}}$ are also equicontinuous, because they satisfy $v''_n(x) = H(x, v_n(x), v'_n(x))$ with H continuous and v_n and v'_n equicontinuous. This implies that there is a subsequence of v_n that converges uniformly to a C^2 function \tilde{v}_1 on $[\underline{x}, \bar{x}]$. This also implies that the limit satisfies $\tilde{v}''_1(x) = -H(x, \tilde{v}_1(x), \tilde{v}'_1(x))$. By construction, $b_1(x) \leq v_n(x) \leq b_2(x)$ on $[\underline{x}_1, \bar{x}_1]$ and, therefore, \tilde{v}_1 is also contained between b_1 and b_2 .

To conclude, take the finite domain $[\underline{x}_2, \bar{x}_2] \supset [\underline{x}_1, \bar{x}_1]$. Applying Arzelà-Ascoli again,³² there exists a subsequence of the first subsequence such that v_n converges uniformly to a limit function \tilde{v}_2 on $[\underline{x}_2, \bar{x}_2]$. The functions \tilde{v}_1 and \tilde{v}_2 coincide on $[\underline{x}_1, \bar{x}_1]$. Proceeding iteratively, we can cover the entire domain \mathcal{X} . The function v defined by $v(x) = \tilde{v}_k(x)$ for $x \in [\underline{x}_k, \bar{x}_k] \setminus [\underline{x}_{k-1}, \bar{x}_{k-1}]$, solves the BVP and is bounded by b_1 and b_2 .

³⁰More precisely, we can apply the Lemma to the function $\bar{v}(x) = v(x - \underline{\chi})$, so as to have the 0 origin assumed in the Lemma.

³¹More precisely, we use the following version: if a sequence of C^1 functions have equicontinuous and uniformly bounded derivatives, and is bounded at one point, then it has a subsequence that converges uniformly to a C^1 function. Here equicontinuity of the derivatives is guaranteed by the uniform bound on the second derivative.

³²Note that the bounds for the domains $[\underline{x}_2, \bar{x}_2]$ and $[\underline{x}_1, \bar{x}_1]$ are different. However, since we are fixing the domain, we are still able to obtain a convergent subsequence.

References

- ALIPRANTIS, C., BORDER, K. (2006) *Infinite Dimensional Analysis*, Second Edition, Springer-Verlag.
- BAILEY, P. (1968) *Nonlinear Two Point Boundary Value Problems*, Academic Press.
- BARLOW, M (1982) “One Dimensional Stochastic Differential Equations with No Strong Solution,” *Journal of the London Mathematical Society*, Vol s2-26, pp. 335-347.
- HARTMAN, P. (2002) *Ordinary Differential Equations*, Second Edition, Classics in Applied Mathematics, SIAM.
- JOSEPHY, M. (1981) “Composing Functions of Bounded Variation,” *Proceedings of the American Mathematical Society*, Vol. 83, pp. 354–356.
- KARATZAS, I., SHREVE, S. (1998) *Brownian Motion and Stochastic Calculus*, Second Edition, Springer.
- KARLIN, S., RUBIN, H. (1956) “The Theory of Decision Procedures for Distributions With Monotone Likelihood Ratio,” *Annals of Mathematical Statistics*, Vol. 27, pp. 272-299.
- KRYLOV, N. (1980) *Controlled Diffusion Processes*, Springer Verlag, Springer.
- LEHMANN, E. (1988) “Comparing Location Experiments,” *Annals of Statistics*, Vol. 16, pp. 521-533.
- MILGROM, P., SHANNON, C. (1994) “Monotone Comparative Statics,” *Econometrica*, Vol. 62, pp. 157–180.
- NAKAO, S. (1972) “On the Pathwise Uniqueness of Solutions of One-Dimensional Stochastic Differential Equations,” *Osaka Journal of Mathematics*, Vol. 9, pp. 513-518.
- QUAH, J., STRULOVICI, B. (2009) “Comparative Statics, Informativeness, and the Interval Dominance Order,” *Econometrica*, Vol. 77, pp. 1949-1992.
- REVUZ, D., YOR, M. (2001) *Continuous Martingales and Brownian motion*, Third Edition (Corrected), Springer-Verlag.
- SCHRADER, K. (1969) “Existence theorems for second order boundary value problems,” *Journal of Differential Equations*, Vol. 5, pp. 572–584.
- TOPKIS, D. (1978) “Minimizing a Submodular Function on a Lattice,” *Operations Research*, Vol. 26, 305–321.

VEINOTT, A. (1989) "Lattice Programming," Unpublished Lecture Notes, Stanford University.