An elementary characterization of the Gini index

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Abstract

The Gini coefficient or index is perhaps one of the most used indicators of social and economic conditions. In this paper we characterize the Gini index as the unique function that satisfies the properties of scale invariance, symmetry, proportionality and convexity in similar rankings. Furthermore, we discuss a simpler way to compute it.

Keywords: Gini index, income inequality, axiomatization.
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1 Introduction

In this article we offer an original and novel characterization, based on the characterization of the contour lines generated by the Gini index on simplex sets, which represent all possible income distributions, when we impose an axiom of scale invariance on any numeric function that aspires to be an inequality index. In the case of three individuals, these contour lines are hexagons whose edges are line segments on distribution sets that maintain a similar order (see Foster and Sen (1997)); there are six orders among individuals, this leads six flat parts on any contour line of the Gini index. The basic idea is to define appropriate axioms that reflect this fact in the general case. The treatment of extreme points of subsets of distributions that maintain the same income ordering is essential. If we ask that

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the index has a certain value in these extreme points, we can use it as a basis to extend the index to the interior by assuming a principle of restricted linearity on this kind of distributions.

This characterization is substantially different from others and in the next section we will discuss the main differences. The main idea of the characterization is the following. An inequality index is a real function with domain vectors of nonnegative real numbers. We impose four properties or axioms directly on functions that represent potential inequality indexes. Our first two axioms are well known, simple and natural. The first one, *scale invariance*, allows us to restrict the domain of the index to the corresponding simplex. A second property, *symmetry* or anonymity, allows us to restrict our domain even more; it would be sufficient to define the index on income distribution sets that remain invariant the order among individuals and incomes. In other words, an income distribution and any ordering of such distribution will maintain the same inequality value. Each possible increasing order is associated to a specific type of permutation of individuals; all distributions with the same order form a very special set in which we impose the remaining two axioms on it. The third axiom, *proportionality*, serves to define the index at the extreme points of these special sets. In these extreme points, there are only two possible values: zero or identical values with the characteristic that the total sum is one. For the case of three individuals and a distribution with order $y_1 \leq y_2 \leq y_3$, the extreme points are $(1/3, 1/3, 1/3)$, $(0, 1/2, 1/2)$ and $(0, 0, 1)$. The proportionality axiom allows us to define the inequality index in extreme points as the ratio formed by the number of individuals with no income and the total number of individuals. Finally, the fourth axiom, *convexity in similar rankings*[1], serves to extend linearly the inequality index, from the values generated by the proportionality axiom into the corresponding special set, which represents ordered distributions with the same permutation.

Among the advantages of the characterization we note the following: it works very well for the discrete case and is simple, the axioms are imposed directly on the indexes and not on welfare orderings induced by distributions, there is no need to appeal to axioms of rationality (completeness and transitivity) and continuity to obtain numeric representations of welfare orderings. Moreover, a simpler way to compute it is derived.

The Gini index has been used primarily as a tool for comparison of income distributions among countries or geographical regions. At the same time, it is used as one of the most important indicators to take into account for the allocation of public resources. For example, Sen (1973) suggests that the comparison of welfare among different countries is not limited to the valuation of GDP per capita, he proposes to weigh the GDP by the degree of equality of the income distribution. Thus, between two countries with the same per capita income, it is considered a better country the one with less inequality in the distribution of resources. The Gini index is also used as one of the factors that explain poverty (for instance, see Sen (1973) and Foster et al (1984)).

The coefficient attributed to Gini (1912) has been presented and analyzed by Dalton (1920), Atkinson (1970) and Sen (1973), among others.

As is well known, the Gini index is constructed traditionally in two ways, both with roots from statistical measures of dispersion: (1) the discrete version as an standardized average of all income differences between individuals or groups and (2) the continuous version through the Lorenz curve. In this last version, the Gini coefficient is associated with the area between the Lorenz curve and the 45 degree diagonal line of the unit square. The Lorenz curve, $L(p)$ is a non-decreasing continuous function with domain the unit interval, which measures the proportion of accumulated wealth from individuals with the lowest income to the income accumulated in the p percent of the population, ordered from lower to higher incomes and standardized in the unit interval. Two times the area enclosed between $L(p)$ and the identity line is precisely the Gini coefficient. When it is zero, there is no inequality because the income would be distributed equally.

The rest of the paper is organized as follows. After giving a short review of the related literature, especially other characterizations, in Section 2, we recall the main basic features of the Gini index in Section 3. Section 4 states the set of axioms which are required an inequality index to satisfy in this work. Finally, Section 5 presents the main characterization theorem of the Gini index, which

[1] The precise definitions will be provided in Section 4.

constitutes the main result of this work.

2 Related literature

The discrete formula of the Gini coefficient as an standardized average of income differences is presented in several equivalent versions, a good explanation is available in the classic paper of Sen (1973) or in the survey of Dutta (2002). Atkinson (1970) provides the basis to discuss the relation between inequality and social welfare orders. When an income distribution x, presents a higher inequality than distribution of income z, the social welfare of the society with distribution z will be better than the social welfare under the distribution x. Where there is less inequality there will be less welfare. Atkinson proposes to measure inequality from social welfare orders. To this end, it is enough to have an income distribution associated with the indifferent point of equal distribution, in the given order of social welfare, and consider some distance away from such point. This will obviously depend on the fixed social welfare order. The approach is quite ordinal and has constructed parameterized family of inequality indices, coming from parameterized families of social evaluation functions.

Works like Weymark (1981), Donaldson and Weymark (1980), and Bossert (1990); present families of social evaluation functions and the corresponding family of inequality indices. However, there is a problem, two ordinally equivalent inequality indices, such as Gini and its square, can have associated social welfare functions that are not ordinally convertible into each other. Works like Ebert (1987), Blackorby and Donaldson (1984), Dutta and Esteban (1992), and Weymark (1999); present interesting alternatives to link social welfare with inequality. In addition to the parameterized family of inequality indices, there have been other attempts of characterization, among which are Mehran (1976), Kakwani (1980), Pyatt (1976), Yitzhaki (1973), but have not been satisfactory enough.

A closer related work to the point of view presented in this paper is the one studied by Thon (1982). This author proposes an axiomatization of the Gini coefficient in a general case. He works in a framework where income distributions whose aggregate incomes and populations are no necessarily the same. Its axioms are imposed directly over the numerical function representing the inequality index, as in our case. But the axioms used by Thon are essentially in a framework very different from the used in this paper. His axioms show us another properties of the Gini index, concerning basically with asymptotic properties when the size of population with income distribution grows. In our framework the size of population is fixed, we explore algebraic distribution properties instead of asymptotic characteristics of the Gini coefficient.

An axiomatization for the continuous version of the Gini index is provided in Aaberge (2001), it is based on the imposition of axioms to characterize an order over Lorenz curves, viewed as lotteries of the theory of choice under uncertainty. The axioms are almost the same as those that characterize the VNM utilities. This axiomatization is mathematically elegant but requires some kind of complexity greater than that proposed here for the discrete version. Our version does not require going through a representation theorem. The axioms are imposed directly on inequality indices, defined in the real n-dimensional space.

The axiomatization of Weymark (1981), presenting the Gini coefficient as an element of a parameterized family of inequality indices, his axioms generate social evaluation functions, i.e., representations of social welfare orders, which are transformed into inequality indices through the Atkinson methodology. The author is forced to use the Pigou-Dalton principle to achieve the characterization of the obtained family of indices. The characterization of Aaberge (2001), based on continuous Lorenz curves, use axioms strongly that imply a numerical representation of an order. They impose axioms of transitivity, completeness, continuity and independence on the orders of preference distributions. Our axioms are imposed directly on inequality indicators defined in a real n-dimensional space.

3 Framework and notation

In this section we give some concepts and notation related to mathematical symbols and a brief part of preliminaries of the Gini index.

The Gini index can be used to measure the dispersion of a distribution of income, or consumption, or wealth, or a distribution of any other kinds. But the kind of distributions where the Gini index is used most is the distribution of income. For this reason, and for simplicity, this paper will focus on the Gini index in the context of income distributions although its applications should not be limited to income distributions. For simplicity, the discussion in this paper is based on income distributions of the individuals within the population.

There are generally two different approaches for analyzing theoretical results of the Gini index: one is based on discrete distributions; the other on continuous distributions. The latter demands certain conditions on continuity while the former does not require such condition. The discrete income distribution is easy to understand in some cases while the continuous income distribution can simplify some derivations in some situations.

Let $N = \{1, 2, ..., n\}$ be a set of individuals. When the income distribution function is discrete, such incomes take n values that can be denoted by a vector $x \in \mathbb{R}^n_+$, where x_i denotes the income for the agent i.

For a given $x \in \mathbb{R}^n_+$, let x^* denote the the vector obtained from x by arranging its coordinates in non-decreasing order ($x^*_1 \leq x^*_2 \leq \cdots \leq x^*_n$). For example, if $x = (9, 7, 0, 11, 3, 5, 9, 0)$ then $x^* = (0, 0, 3, 5, 7, 9, 9, 11)$.

Now, the group of permutations of N, $S_n = \{\theta : N \to N \mid \theta \text{ is bijective}\}$ acts on the space of income vectors $\mathbb{R}^n_+$ in a natural way; i.e., for $\theta \in S_n$ and $x \in \mathbb{R}^n_+$:

$$\theta \cdot (x_1, x_2, \ldots, x_n) = (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})^2$$

Notice that for an arbitrary $x \in \mathbb{R}^n_+$, there exists at least one permutation θ such that $\theta \cdot x = x^*$.

Let $x, y \in \mathbb{R}^n_+$ be two income distributions. We say that x and y provide the same *ranking* if there exists $\theta \in S_n$ such that $\theta \cdot x = x^*$ and $\theta \cdot y = y^*$. In other words, two income distributions give the same ranking if the income position for every agent is the same in both orderings x^* and y^*. For example, the income distributions $x = (6, 0, 3, 18, 1, 24)$ and $y = (11, 5, 10, 19, 7, 33)$ give the same ranking; since for both distributions, agent 2 is the poorest one, agent 5 is the second poorest, agent 6 is the richest one, and so on.

In general, an *inequality index* is a function $I : \mathbb{R}^n_+ \to \mathbb{R}$ that assigns to each income vector a real number, which represents the society's inequality level.

The attractiveness of the Gini index to many economists is that it has an intuitive geometric interpretation. That is, the Gini index can be defined geometrically as the ratio of the area that lies between the line of perfect equality and the Lorenz curve (which plots the proportion of the total income of the population that is cumulatively earned by the bottom of the population), over the total area under the line of equality.

Several alternative formulations in fact follow the same tradition. Sen (1973) defined the Gini index as a function $G : \mathbb{R}^n_+ \to \mathbb{R}$ such that

$$G(x) = \frac{1}{n}\left[n + 1 - 2\frac{\sum_{i=1}^{n}(n + 1 - i)x^*_i}{\sum_{i=1}^{n} x^*_i}\right] \tag{1}$$

This formulation illustrates that the income-rank-based weights are inversely associated with the sizes of incomes. That is, in the index the richer's incomes get lower weights while the poor's income get higher weights.

[2] This permutation implies a change of position of the individuals within the income distribution.

Example 1 *Let the hypothetical income distribution be* $x = (30, 20, 60, 10)$*, then* $x^* = (10, 20, 30, 60)$ *and*

$$\begin{aligned} G(x) &= \frac{1}{4}\left[5 - 2\frac{4(10) + 3(20) + 2(30) + 1(60)}{120}\right] \\ &= \frac{1}{3} \end{aligned}$$

This paper will show that the Gini index given by the above formulation is uniquely determined by a certain set of properties, described in the next section.

4 The axioms

Next, we define a set of axioms which are required an inequality index to satisfy in this work.

The first axiom deals with scales; that is, scaling the incomes x_i by a constant factor causes nothing to the inequality index. So, if the income is measured in other currencies, then the index remains the same.

Axiom 2 (Scale invariance) *The inequality index* I *is said to be scale-invariant if*

$$I(\lambda x) = I(x)$$

for all $x \in \mathbb{R}^n_+$ *and* $0 \neq \lambda \in \mathbb{R}$*.*

The next axiom requires that the inequality index be independent of any characteristic of the individuals other than income x. Formally,

Axiom 3 (Symmetry) *The inequality index* I *is symmetric if and only if*

$$I(\theta \cdot x) = I(x)$$

for every $\theta \in S_n$ *and* $x \in \mathbb{R}^n_+$*.*

Now, consider a society where there are only two types of individuals: rich people with 1 unit of income and the poor people with no income. The next property establishes that for this kind of societies, the inequality index is defined in direct proportion of the number of individuals with no income. Without loss of generality; for $k \in \mathbb{N}$, let $E^k = (\underbrace{0, \ldots 0}_{k}, \underbrace{1, \ldots, 1}_{n-k})$ be the income vector of such society.

Axiom 4 (Proportionality) *The inequality index* I *is said to be proportional if*

$$I(E^k) = \frac{k}{n}$$

According to the previous axiom, notice that a proportional index gives $I(1, \ldots, 1) = 0$ and $I(0, \ldots 0, 1) = \frac{n-1}{n}$.

Finally, the next property is a technical requirement referred to a couple of income distributions that provide the same ranking. Further suppose that these two distributions have the same total income. The fourth axiom requires that the index of the convex combination of such distributions is exactly the convex combination of the index of each one.

Axiom 5 (Convexity in similar rankings) [3] *The inequality index* I *is convex in similar rankings if*

$$I\left[\beta x + (1-\beta)z\right] = \beta I(x) + (1-\beta)I(z)$$

for every $\beta \in [0, 1]$, every $x, z \in \mathbb{R}^n_+$ that provides the same ranking and $\sum_{i=1}^n x_i = \sum_{i=1}^n z_i$.

[3] We will abbreviate it by 'Convexity'.

5 The main result

This section is devoted to a characterization of the Gini index by means of the previous axioms, which establishes the main result of this work. But first of all, we shall present some results that are used in the characterization.

The key idea is to show that the four axioms characterize the Gini index on the convex set

$$K = \left\{ y \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} y_i = 1 \text{ and } y_1 \leq y_2 \leq \cdots \leq y_n \right\}^{4}$$

and so, it will be characterized on the whole space of income vectors $\mathbb{R}^n_+$. As it is shown in:

Lemma 6 *Let $I : \mathbb{R}^n_+ \to \mathbb{R}$ be a symmetric and scale-invariant index. If I is defined on K, then it is defined on $\mathbb{R}^n_+$.*

Proof. If $x \in \mathbb{R}^n_+$, then there exists $\theta \in S_n$ such that $x_{\theta(1)} \leq x_{\theta(2)} \leq \cdots \leq x_{\theta(n)}$ and hence $\frac{1}{\sum_{i=1}^n x_i}(\theta \cdot x) \in K$. Therefore,

$$I\left[\frac{1}{\sum_{i=1}^n x_i}(\theta \cdot x)\right] = I(\theta \cdot x) = I(x)$$

■

As we will see, each element of K may be expressed as a convex combination of the vectors $\{v^j\}_{j=1}^n$; which are actually the extreme points of K[5] and they are given by

$$v_i^j = \begin{cases} \frac{1}{n-j+1} & \text{if } j \leq i \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Lemma 7 *For every $y \in K$, there exist unique non-negative scalars $\{\alpha_j\}_{j=1}^n$, such that $\sum_{j=1}^n \alpha_j = 1$ and*

$$y = \sum_{j=1}^{n} \alpha_j v^j \tag{3}$$

Moreover, the scalars are given by

$$\alpha_j = (n - j + 1)(y_j - y_{j-1})$$

for $j = 1, 2, \ldots n$, where $y_0 = 0$.

Proof. Let V be a $n \times n$ matrix with entries $V_{i,j} = v_i^j$, and let α be the $1 \times n$ matrix $\alpha^T = [\alpha_1\ \alpha_2 \cdots \alpha_n]^T$. Notice that $y = \sum_{j=1}^n \alpha_j v^j$ can be represented by the system of equations $y = V\alpha$. Since V is non-singular, then the scalars $\{\alpha_j\}_{j=1}^n$ are unique.

Now, for arbitrary $y \in K$:

$$\begin{aligned} \sum_{j=1}^{n} \alpha_j v_i^j &= \sum_{j=1}^{i} (n-j+1)(y_j - y_{j-1}) \frac{1}{n-j+1} \\ &= \sum_{j=1}^{i} y_j - \sum_{j=1}^{i} y_{j-1} = y_i \end{aligned}$$

[4] Note that if $y \in K$, then $y^* = y$.

[5] Since we can easily verify that each v^j is a vector in K and does not lie in any open line segment joining two vectors of K.

For the other direction:

$$\begin{aligned} y_i - y_{i-1} &= \sum_{j=1}^{n} \alpha_j v_i^j - \sum_{j=1}^{n} \alpha_j v_{i-1}^j = \sum_{j=1}^{i} \frac{\alpha_j}{n-j+1} - \sum_{j=1}^{i-1} \frac{\alpha_j}{n-j+1} \\ &= \frac{\alpha_i}{n-i+1} \end{aligned}$$

Hence,

$$\alpha_i = (n-i+1)(y_i - y_{i-1})$$

and they are not negative since $y \in K$.

Finally, those scalars add up 1:

$$\begin{aligned} \sum_{j=1}^{n} \alpha_j &= \sum_{j=1}^{n} (n-j+1)(y_j - y_{j-1}) \\ &= \sum_{j=1}^{n} (n-j+1) y_j - \sum_{j=0}^{n-1} (n-j) y_j \\ &= \sum_{j=1}^{n} y_j = 1 \end{aligned}$$

■

Lemma 8 *Let $w^j \in K$, $j = 1, \ldots, n$ and $\beta_j \geq 0$ be such that $\sum_{j=1}^{n} \beta_j = 1$. If $I : \mathbb{R}_+^n \to \mathbb{R}$ is a convex index, then*

$$I\left(\sum_{j=1}^{n} \beta_j w^j\right) = \sum_{j=1}^{n} \beta_j I(w^j)$$

Proof. First, it is clear that every pair of elements in K provides the same ranking, since by definition they all are ordered increasingly.

The proof is done by induction on n. Assume that the statement holds for $k \in \mathbb{N}$:

$$I\left(\sum_{j=1}^{k} \beta_j w^j\right) = \sum_{j=1}^{k} \beta_j I(w^j)$$

for $w^j \in K$, $j = 1, \ldots, k$ and $\beta_j \geq 0$ are such that $\sum_{j=1}^{k} \beta_j = 1$.

Now, let $w^j \in K$, $j = 1, \ldots, k+1$ and let $\gamma_j \geq 0$ be such that $\sum_{j=1}^{k+1} \gamma_j = 1$. Since I is a convex index:

$$\begin{aligned} I\left(\sum_{j=1}^{k+1} \beta_j w^j\right) &= I\left(\sum_{j=1}^{k} \beta_j w^j + \beta_{k+1} w^{k+1}\right) \\ &= (1-\beta_{k+1}) I\left(\sum_{j=1}^{k} \frac{\beta_j}{1-\beta_{k+1}} w^j\right) + \beta_{k+1} I(w^{k+1}) \end{aligned}$$

Notice that $\frac{\beta_j}{1-\beta_{k+1}} \geq 0$ and $\sum_{j=1}^{k} \frac{\beta_j}{1-\beta_{k+1}} = 1$. Thus, by the induction hypothesis:

$$\begin{aligned} I\left(\sum_{j=1}^{k+1} \beta_j w^j\right) &= (1-\beta_{k+1}) \sum_{j=1}^{k} \frac{\beta_j}{1-\beta_{k+1}} I(w^j) + \beta_{k+1} I(w^{k+1}) \\ &= \sum_{j=1}^{k+1} \beta_j I(w^j) \end{aligned}$$

■

Remark 9 *Note that the scale invariance and proportionality determines an inequality index on the extreme points of K, whereas the symmetry and convexity in similar rankings determines the index on the interior of K.*

We are now ready to state our main result.

Theorem 10 *Let $I : \mathbb{R}_+^n \to \mathbb{R}$. Then, I equals the Gini index, given by (1), if and only if it satisfies scale invariance, symmetry, proportionality and convexity axioms.*

Proof. It is straightforward to prove that the Gini index given by (1) satisfies the four properties.

For the converse, let $I : \mathbb{R}_+^n \to \mathbb{R}$ be any index satisfying the four axioms and let $z \in \mathbb{R}_+^n$ be an income vector. Define $x = \theta \cdot z$, where $\theta \in S_n$ is such that $\theta \cdot z = z^*$ and set $y_i = \frac{x_i}{\sum_{j=1}^n x_j}$. It is clear that $y \in K$ and so, it can be decomposed as $y = \sum_{j=1}^n \alpha_j v^j$ for unique real numbers $\left\{\alpha_j \geq 0 \mid \sum_{j=1}^n \alpha_j = 1\right\}_{j=1}^n$.

Thus, using the symmetry and scale invariance axioms, and Lemma 7:

$$I(z) = I(\theta \cdot z) = I(x) = I\left(\frac{1}{\sum_{j=1}^n x_j} x\right) = I(y) = I\left(\sum_{i=1}^n \alpha_i v^i\right)$$

Next, by Lemmas 7 and 8:

$$I(z) = \sum_{i=1}^n \alpha_i I(v^i) = \sum_{i=1}^n (n-i+1)(y_i - y_{i-1}) I(v^i)$$

It is easy to verify that $I(v^i) = \frac{i-1}{n}$, by the scale-invariance and proportionality of I. Therefore, we obtain:

$$\begin{aligned}
I(z) &= \sum_{i=1}^n (n-i+1)(y_i - y_{i-1}) \frac{i-1}{n} \\
&= \sum_{i=1}^n \frac{(n-i+1)(i-1)}{n} y_i - \sum_{i=2}^n \frac{(n-i+1)(i-1)}{n} y_{i-1} \\
&= \sum_{i=1}^n \frac{(n-i+1)(i-1)}{n} y_j - \sum_{i=1}^{n-1} \frac{(n-i)i}{n} y_i \\
&= \frac{n-1}{n} y_n + \sum_{i=1}^{n-1} \left[\frac{(n-i+1)(i-1)}{n} - \frac{(n-i)i}{n}\right] y_i \\
&= \frac{n-1}{n} y_n + \sum_{i=1}^{n-1} \frac{2i-n-1}{n} y_i = \sum_{i=1}^n \frac{2i-n-1}{n} y_i \\
&= \frac{1}{n}\left[-n-1+2\sum_{i=1}^n i y_i\right] = \frac{1}{n}\left[-n-1+2\sum_{i=1}^n i\left(\frac{x_i}{\sum_{j=1}^n x_j}\right)\right] \\
&= \frac{1}{n}\left[n+1-2\frac{\sum_{i=1}^n (n+1-i)x_i}{\sum_{j=1}^n x_j}\right] = \frac{1}{n}\left[n+1-2\frac{\sum_{i=1}^n (n+1-i)z_i^*}{\sum_{j=1}^n z_j^*}\right] \\
&= G(z)
\end{aligned}$$

which coincides with (1). ■

Example 11 *As an illustration of the above derivation (and using the same notation), we compute the Gini index as Example 1.*

For the income distribution $z = (30, 20, 60, 10) \Rightarrow y = \left(\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}\right)$. *Furthermore,*

$$\begin{aligned} y &= \left(\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}\right) = \frac{1}{3}v^1 + \frac{1}{4}v^2 + \frac{1}{6}v^3 + \frac{1}{4}v^4 \\ &= \frac{1}{3}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + \frac{1}{4}\left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + \frac{1}{6}\left(0, 0, \frac{1}{2}, \frac{1}{2}\right) + \frac{1}{4}(0, 0, 0, 1) \end{aligned}$$

and hence,

$$\begin{aligned} G(z) &= G(y) = \frac{1}{3}G\left(v^1\right) + \frac{1}{4}G\left(v^2\right) + \frac{1}{6}G\left(v^3\right) + \frac{1}{4}G\left(v^4\right) \\ &= \frac{1}{3}(0) + \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{6}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{3}{4}\right) \\ &= \frac{1}{3} \end{aligned}$$

Remark 12 *From the proof of the previous Theorem, a simpler way to calculate the Gini index is derived. That is, for an arbitrary income vector* $z \in \mathbb{R}^n_+$, *it is shown that the Gini index is obtained from*

$$G(z) = \sum_{i=1}^{n} \gamma_i y_i \tag{4}$$

where $\gamma_i = \frac{2i-n-1}{n}$ *and* $y_i = \frac{z_i^*}{\sum_{j=1}^n z_j^*}$.

In order to compute the coefficients $\{\gamma_i\}_{i=1}^n$, *it is sufficient to calculate half of them, since the relation* $\gamma_i + \gamma_{n+1-i} = 0$ *holds for every* $i = 1, 2, \ldots n$. *When* n *is an odd number, it turns out that the central coefficient* $\gamma_{\frac{n+1}{2}}$ *vanishes.*

Example 13 *Let us give the precise values of coefficients* γ_i *for the case* $n = 7$:

i	1	2	3	4	5	6	7
γ_i	$-\frac{6}{7}$	$-\frac{4}{7}$	$-\frac{2}{7}$	0	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{6}{7}$

Example 14 *As a final example, we present the computation of the Gini index from expression (4) for the income distribution* $z = (75, 5, 10, 20, 6, 58, 6)$.

According to the previous discussion, $y = \left(\frac{1}{36}, \frac{1}{30}, \frac{1}{30}, \frac{1}{18}, \frac{1}{9}, \frac{29}{90}, \frac{5}{12}\right)$ *and we then get:*

$$\begin{aligned} G(z) &= -\frac{6}{7}\left(\frac{1}{36}\right) - \frac{4}{7}\left(\frac{1}{30}\right) - \frac{2}{7}\left(\frac{1}{30}\right) + \frac{2}{7}\left(\frac{1}{9}\right) + \frac{4}{7}\left(\frac{29}{90}\right) + \frac{6}{7}\left(\frac{5}{12}\right) \\ &= \frac{164}{315} \approx 0.5206 \end{aligned}$$

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