# Nonlinear pricing with imperfectly informed consumers

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**NONLINEAR PRICING WITH IMPERFECTLY INFORMED CONSUMERS**

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A monopolist sells a single product to a population of consumers. The cost per unit of supplying this product is constant and equal to $c$. Consumers have utility functions of the form $u(q,\theta)-T$, where $q$ is the quantity consumed, $\theta$ is a parameter affecting demand, and $T$ is the payment for consumption. The function $u$ satisfies $u(0,\theta)\equiv 0, u_\theta \geq 0$ and $u_{q\theta}\geq 0$. Consumers gain information about their preferences in two stages: first they learn a parameter $\alpha$, which does not enter directly into their utility function, then they learn $\theta$. The distribution of $\theta$ depends on $\alpha$, and write the distribution function for $\theta$ given $\alpha$ as $F(\theta,\alpha)$. We assume that higher values of $\alpha$ make higher values of $\theta$ more likely, i.e. that $F_\alpha(\theta,\alpha)\leq 0$. Crucially, we make the assumption that the support of $\theta$ does not depend on $\alpha$, and say this support is $[\theta_L,\theta_H]$. The distribution function for $\alpha$ is $G(\alpha)$ with support $[\alpha_L,\alpha_H]$.

The firm offers a family of tariffs from which a consumer must choose after $\alpha$ is known but before $\theta$ is known. Let the family of tariffs be indexed by $\alpha$, and so a consumer is free to choose to buy from any tariff $T(q,\alpha)$. Given a particular family of tariffs $T(q,\alpha)$, define

$$s(\theta,\alpha)\equiv \max_{q\,\geq\,0} : u(q,\theta)-T(q,\alpha)$$

and write $q(\theta,\alpha)$ to be the quantity that solves the above problem. Then, in the usual way, if the type $\alpha$ consumer chooses the tariff $T(\cdot,\hat{\alpha})$ she obtains expected surplus

$$v(\alpha,\hat{\alpha})\equiv \int_{\theta_L}^{\theta^H} u_\theta(q(\theta,\hat{\alpha}),\theta)(1-F(\theta,\alpha))\;d\theta + s(\theta_L,\hat{\alpha}) \tag{1}$$

and the firm obtains expected profit of

$$\int_{\theta_L}^{\theta_H} \left\{[u(q(\theta,\hat{\alpha}),\theta)-cq(\theta,\hat{\alpha})]\,f(\theta,\alpha)-u_\theta(q(\theta,\hat{\alpha}),\theta)(1-F(\theta,\alpha))\right\}\;d\theta \\ -s(\theta_L,\hat{\alpha})\ . \tag{2}$$

(Here, $f\equiv F_\theta$.) Thus in doing this we have eliminated the underlying tariff $T(\cdot,\hat{\alpha})$ and expressed consumer surplus and profit given $\alpha$ and $\hat{\alpha}$ in terms of the demand profile $q(\theta,\hat{\alpha})$ and the minimal surplus term $s(\theta_L,\hat{\alpha})$. Clearly, provided the function $q(\theta,\hat{\alpha})$ is (weakly) increasing in $\theta$, a tariff $T(\cdot,\hat{\alpha})$ can be found that induces the demand profile $q(\theta, \hat{\alpha})$. We can therefore think of the firm as choosing $q(\theta, \hat{\alpha})$ and $s(\theta_L, \hat{\alpha})$ rather a family of tariffs $T(\cdot, \alpha)$.

What remains to do is to ensure that the scheme is incentive compatible and that the type $\alpha$ consumer chooses $\hat{\alpha} = \alpha$. Write

$$V(\alpha) = \max_{\alpha_L \leq \hat{\alpha} \leq \alpha_H} : v(\alpha, \hat{\alpha})$$

where $v$ is given by (1). Clearly, if the type $\alpha$ chooses $\hat{\alpha} = \alpha$ then

$$V'(\alpha) = -\int_{\theta_L}^{\theta_H} u_\theta(q(\theta, \alpha), \theta) F_\alpha(\theta, \alpha)\, d\theta \ \geq 0 \ . \tag{3}$$

In particular, $V(\alpha)$ is increasing in $\alpha$ and so if the participation constraint is satisfied for the lowest type $\alpha = \alpha_L$ it is satisfied for all types. Therefore, it must be optimal from the firm's point of view to set $V(\alpha_L) = 0$. We deduce from (3) that under any incentive compatible scheme that satisfies the participation constraints, the rent of the type $\alpha$ is given by

$$V(\alpha) = -\int_{\alpha_L}^{\alpha} \int_{\theta_L}^{\theta_H} u_\theta(q(\theta, \hat{\alpha}), \theta) F_\alpha(\theta, \hat{\alpha})\, d\theta\, d\hat{\alpha} \ . \tag{4}$$

From (1), the term $s(\theta_L, \alpha)$ must then be given by

$$s(\theta_L, \alpha) = V(\alpha) - \int_{\theta_L}^{\theta^H} u_\theta(q(\theta, \hat{\alpha}), \theta)(1 - F(\theta, \alpha))\, d\theta \tag{5}$$

where $V(\alpha)$ is given by (4).

**Lemma 1** *If the function $s(\theta_L, \alpha)$ in (1) is given by (5) above, then the type $\alpha$ consumer will choose $\hat{\alpha} = \alpha$ in (1) provided that $q(\theta, \alpha)$ is (weakly) increasing in $\alpha$.*

**Proof.** Substituting for $s(\theta_L, \hat{\alpha})$ as defined in (5) into (1) and differentiating with respect to $\hat{\alpha}$ yields

$$v_{\hat{\alpha}}(\alpha, \hat{\alpha}) = \int_{\theta_L}^{\theta_H} u_{q\theta}(q(\theta, \hat{\alpha}), \theta) q_\alpha(\theta, \hat{\alpha})[F(\theta, \hat{\alpha}) - F(\theta, \alpha)]\, d\theta \ .$$

Therefore, since $u_{q\theta}$ is assumed to be non-negative and $q_\alpha$ is assumed in the statement of the lemma to be non-negative, the function $v(\alpha, \hat{\alpha})$ is increasing in $\hat{\alpha}$ for $\hat{\alpha} \leq \alpha$ and increasing in $\hat{\alpha}$ for $\hat{\alpha} \geq \alpha$ and hence is maximized at $\hat{\alpha} = \alpha$ as required. $\square$

(Note that, although it is necessary for implementability that $q$ be increasing in $\theta$, we do not claim that it is necessary, only sufficient, that $q$ be increasing in $\alpha$.)

We can now write the firm's total profit purely in terms of the demand profile $q(\theta,\alpha)$. From (2), the firm's profit from the type $\alpha$ consumer is

$$\int_{\theta_L}^{\theta_H} u(q(\theta,\alpha),\theta) f(\theta,\alpha)\, d\theta - V(\alpha)$$

and so the firm's total profit is just

$$\pi = \int_{\alpha_L}^{\alpha_H} \left\{ \int_{\theta_L}^{\theta_H} \left[ u(q(\theta,\alpha),\theta) - cq(\theta,\alpha) \right] f(\theta,\alpha)\, d\theta - V(\alpha) \right\} dG(\alpha)\ .$$

But using integration by parts and the relationship (3) yields

$$\int_{\alpha_L}^{\alpha_H} V(\alpha)\, dG(\alpha) = \int_{\alpha_L}^{\alpha_H} \int_{\theta_L}^{\theta_H} -u_\theta(q(\theta,\alpha),\theta) F_\alpha(\theta,\alpha)(1-G(\alpha))\, d\theta d\alpha$$

and hence total profits can be expressed as

$$\begin{aligned} \pi \quad &= \quad \int_{\alpha_L}^{\alpha_H} \int_{\theta_L}^{\theta_H} \left\{ \left[ u(q(\theta,\alpha),\theta) - cq(\theta,\alpha) \right] f(\theta,\alpha) g(\alpha) \right. \\ &+ \qquad \left. u_\theta(q(\theta,\alpha),\theta) F_\alpha(\theta,\alpha)(1-G(\alpha)) \right\}\, d\theta\, d\alpha\ . \end{aligned} \tag{6}$$

Therefore, the candidate for the profit-maximizing quantity profile is

$$q(\theta,\alpha) \text{ maximizes}_{q\,\geq\,0} : u(q,\theta) - cq - u_\theta(q,\theta) \frac{-F_\alpha(\theta,\alpha)(1-G(\alpha))}{f(\theta,\alpha)g(\alpha)}\ . \tag{7}$$

Provided this function is weakly increasing in both $\theta$ and $\alpha$, and this requires a joint condition on the functional forms of $u, F$ and $G$, then (7) certainly gives the profit-maximizing demand profile.

**EXAMPLE:** Let $u(q,\theta) = \theta u(q)$ and $F(\theta,\alpha) = 1 - e^{-\theta/\alpha}$.

In this case the utility function takes the multiplicative form often used in models of nonlinear pricing, and the parameter $\theta$ is exponentially distributed with mean $\alpha$. From (7), the candidate demand profile $q(\theta,\alpha)$ maximizes

$$\theta u(q) - cq - \theta u(q) \frac{1-G(\alpha)}{\alpha g(\alpha)} = \left[ 1 - \frac{1-G(\alpha)}{\alpha g(\alpha)} \right] \theta u(q) - cq\ .$$

This function is increasing in both $\theta$ and $\alpha$ provided the standard hazard rate condition that $(1-G(\alpha))/(\alpha g(\alpha))$ is decreasing holds. (Demand is zero when $(1-G(\alpha))/(\alpha g(\alpha)) \geq 1$.) Notice that this example has the feature that each tariff $T(q,\alpha)$ is just a two-part tariff with marginal price equal to

$$\frac{c}{1-(1-G(\alpha))/(\alpha g(\alpha))}$$

and so the profit-maximizing strategy is to offer consumers a menu of two-part tariffs.