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NONLINEAR PRICING WITH IMPERFECTLY INFORMED CONSUMERS

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A monopolist sells a single product to a population of consumers. The cost per unit of supplying this product is constant and equal to \( c \). Consumers have utility functions of the form \( u(q, \theta) - T \), where \( q \) is the quantity consumed, \( \theta \) is a parameter affecting demand, and \( T \) is the payment for consumption. The function \( u \) satisfies \( u(0, \theta) \equiv 0, u_\theta \geq 0 \) and \( u_{q\theta} \geq 0 \). Consumers gain information about their preferences in two stages: first they learn a parameter \( \alpha \), which does not enter directly into their utility function, then they learn \( \theta \). The distribution of \( \theta \) depends on \( \alpha \), and write the distribution function for \( \theta \) given \( \alpha \) as \( F(\theta, \alpha) \). We assume that higher values of \( \alpha \) make higher values of \( \theta \) more likely, i.e. that \( F_\alpha(\theta, \alpha) \leq 0 \). Crucially, we make the assumption that the support of \( \theta \) does not depend on \( \alpha \), and say this support is \([\theta_L, \theta_H] \). The distribution function for \( \alpha \) is \( G(\alpha) \) with support \([\alpha_L, \alpha_H] \).

The firm offers a family of tariffs from which a consumer must choose after \( \alpha \) is known but before \( \theta \) is known. Let the family of tariffs be indexed by \( \alpha \), and so a consumer is free to choose to buy from any tariff \( T(q, \alpha) \). Given a particular family of tariffs \( T(q, \alpha) \), define

\[
s(\theta, \alpha) \equiv \max_{q \geq 0} : u(q, \theta) - T(q, \alpha)
\]

and write \( q(\theta, \alpha) \) to be the quantity that solves the above problem. Then, in the usual way, if the type \( \alpha \) consumer chooses the tariff \( T(\cdot, \alpha) \) she obtains expected surplus

\[
v(\alpha, \hat{\alpha}) \equiv \int_{\theta_L}^{\theta_H} u_\theta(q(\theta, \hat{\alpha}), \theta)(1 - F(\theta, \alpha)) \, d\theta + s(\theta_L, \hat{\alpha})
\]

and the firm obtains expected profit of

\[
\int_{\theta_L}^{\theta_H} \{ [u(q(\theta, \hat{\alpha}), \theta) - c q(\theta, \hat{\alpha})] f(\theta, \alpha) - u_\theta(q(\theta, \hat{\alpha}), \theta)(1 - F(\theta, \alpha)) \} \, d\theta
- s(\theta_L, \hat{\alpha}) .
\]

(Here, \( f \equiv F_\theta \).) Thus in doing this we have eliminated the underlying tariff \( T(\cdot, \hat{\alpha}) \) and expressed consumer surplus and profit given \( \alpha \) and \( \hat{\alpha} \) in terms of the demand profile \( q(\theta, \hat{\alpha}) \) and the minimal surplus term \( s(\theta_L, \hat{\alpha}) \). Clearly, provided the function \( q(\theta, \hat{\alpha}) \) is (weakly) increasing in \( \theta \), a tariff \( T(\cdot, \hat{\alpha}) \) can be found that induces the
demand profile \( q(\theta, \hat{\alpha}) \). We can therefore think of the firm as choosing \( q(\theta, \hat{\alpha}) \) and \( s(\theta_L, \hat{\alpha}) \) rather a family of tariffs \( T(\cdot, \alpha) \).

What remains to do is to ensure that the scheme is incentive compatible and that the type \( \alpha \) consumer chooses \( \hat{\alpha} = \alpha \). Write

\[
V(\alpha) = \max_{\alpha_L \leq \hat{\alpha} \leq \alpha_H} : v(\alpha, \hat{\alpha})
\]

where \( v \) is given by (1). Clearly, if the type \( \alpha \) chooses \( \hat{\alpha} = \alpha \) then

\[
V'(\alpha) = -\int_{\theta_L}^{\theta_H} u_\theta(q(\theta, \alpha), \theta)F_\alpha(\theta, \alpha) \, d\theta \geq 0 .
\]

In particular, \( V(\alpha) \) is increasing in \( \alpha \) and so if the participation constraint is satisfied for the lowest type \( \alpha = \alpha_L \) it is satisfied for all types. Therefore, it must be optimal from the firm’s point of view to set \( V(\alpha_L) = 0 \). We deduce from (3) that under any incentive compatible scheme that satisfies the participation constraints, the rent of the type \( \alpha \) is given by

\[
V(\alpha) = -\int_{\theta_L}^{\theta_H} u_\theta(q(\theta, \alpha), \theta)F_\alpha(\theta, \alpha) \, d\theta \, d\hat{\alpha} .
\]

From (1), the term \( s(\theta_L, \alpha) \) must then be given by

\[
s(\theta_L, \alpha) = V(\alpha) - \int_{\theta_L}^{\theta_H} u_\theta(q(\theta, \alpha), \theta)(1 - F(\theta, \alpha)) \, d\theta
\]

where \( V(\alpha) \) is given by (4).

**Lemma 1** If the function \( s(\theta_L, \alpha) \) in (1) is given by (5) above, then the type \( \alpha \) consumer will choose \( \hat{\alpha} = \alpha \) in (1) provided that \( q(\theta, \alpha) \) is (weakly) increasing in \( \alpha \).

**Proof.** Substituting for \( s(\theta_L, \hat{\alpha}) \) as defined in (5) into (1) and differentiating with respect to \( \hat{\alpha} \) yields

\[
v_\hat{\alpha}(\alpha, \hat{\alpha}) = \int_{\theta_L}^{\theta_H} u_{q_\theta}(q(\theta, \hat{\alpha}), \theta)q_\alpha(\theta, \hat{\alpha})[F(\theta, \hat{\alpha}) - F(\theta, \alpha)] \, d\theta .
\]

Therefore, since \( u_{q_\theta} \) is assumed to be non-negative and \( q_\alpha \) is assumed in the statement of the lemma to be non-negative, the function \( v(\alpha, \hat{\alpha}) \) is increasing in \( \hat{\alpha} \) for \( \hat{\alpha} \leq \alpha \) and increasing in \( \hat{\alpha} \) for \( \hat{\alpha} \geq \alpha \) and hence is maximized at \( \hat{\alpha} = \alpha \) as required. \( \Box \)

(Note that, although it is necessary for implementability that \( q \) be increasing in \( \theta \), we do not claim that it is necessary, only sufficient, that \( q \) be increasing in \( \alpha \).)
We can now write the firm’s total profit purely in terms of the demand profile \( q(\theta, \alpha) \). From (2), the firm’s profit from the type \( \alpha \) consumer is

\[
\int_{\theta_L}^{\theta_H} u(q(\theta, \alpha), \theta) f(\theta, \alpha) \, d\theta - V(\alpha)
\]

and so the firm’s total profit is just

\[
\pi = \int_{\alpha_L}^{\alpha_H} \left\{ \int_{\theta_L}^{\theta_H} [u(q(\theta, \alpha), \theta) - cq(\theta, \alpha)] f(\theta, \alpha) \, d\theta - V(\alpha) \right\} \, dG(\alpha) .
\]

But using integration by parts and the relationship (3) yields

\[
\int_{\alpha_L}^{\alpha_H} V(\alpha) \, dG(\alpha) = \int_{\alpha_L}^{\alpha_H} \int_{\theta_L}^{\theta_H} -u_\theta(q(\theta, \alpha), \theta) F_\alpha(\theta, \alpha)(1 - G(\alpha)) \, d\theta \, d\alpha,
\]

and hence total profits can be expressed as

\[
\pi = \int_{\alpha_L}^{\alpha_H} \int_{\theta_L}^{\theta_H} \{ [u(q(\theta, \alpha), \theta) - cq(\theta, \alpha)] f(\theta, \alpha) g(\alpha) + u_\theta(q(\theta, \alpha), \theta) F_\alpha(\theta, \alpha)(1 - G(\alpha)) \} \, d\theta \, d\alpha . \quad (6)
\]

Therefore, the candidate for the profit-maximizing quantity profile is

\[
q(\theta, \alpha) \text{ maximizes}_{q \geq 0} : u(q, \theta) - cq - u_\theta(q, \theta) \frac{F_\alpha(\theta, \alpha)(1 - G(\alpha))}{f(\theta, \alpha) g(\alpha)} . \quad (7)
\]

Provided this function is weakly increasing in both \( \theta \) and \( \alpha \), and this requires a joint condition on the functional forms of \( u, F \) and \( G \), then (7) certainly gives the profit-maximizing demand profile.

**EXAMPLE:** Let \( u(q, \theta) = \theta u(q) \) and \( F(\theta, \alpha) = 1 - e^{-\theta/\alpha} \).

In this case the utility function takes the multiplicative form often used in models of nonlinear pricing, and the parameter \( \theta \) is exponentially distributed with mean \( \alpha \). From (7), the candidate demand profile \( q(\theta, \alpha) \) maximizes

\[
\theta u(q) - cq - \theta u(q) \frac{1 - G(\alpha)}{\alpha g(\alpha)} = \left[ 1 - \frac{1 - G(\alpha)}{\alpha g(\alpha)} \right] \theta u(q) - cq .
\]

This function is increasing in both \( \theta \) and \( \alpha \) provided the standard hazard rate condition that \( (1 - G(\alpha))/(\alpha g(\alpha)) \) is decreasing holds. (Demand is zero when \( (1 - G(\alpha))/(\alpha g(\alpha)) \geq 1 \).) Notice that this example has the feature that each tariff \( T(q, \alpha) \) is just a two-part tariff with marginal price equal to

\[
cq \frac{1 - (1 - G(\alpha))/(\alpha g(\alpha))}{1 - (1 - G(\alpha))/(\alpha g(\alpha))}
\]

and so the profit-maximizing strategy is to offer consumers a menu of two-part tariffs.