# Validity and precision of estimates in the classical newsvendor model with exponential and rayleigh demand

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## ABSTRACT

In this paper we consider the classical newsvendor model with profit maximization. When demand is fully observed in each period and follows either the Rayleigh or the exponential distribution, appropriate estimators for the optimal order quantity and the maximum expected profit are established and their distributions are derived. Measuring validity and precision of the corresponding generated confidence intervals by respectively the actual confidence level and the expected half-length divided by the true quantity (optimal order quantity or maximum expected profit), we prove that the intervals are characterized by a very important and useful property. Either referring to confidence intervals for the optimal order quantity or the maximum expected profit, measurements for validity and precision take on exactly the same values. Furthermore, validity and precision do not depend upon the values assigned to the revenue and cost parameters of the model. To offer, therefore, a-priori knowledge for levels of precision and validity, values for the two statistical criteria, that is, the actual confidence level and the relative expected half-length are provided for different combinations of sample size and nominal confidence levels 90%, 95% and 99%. The values for the two criteria have been estimated by developing appropriate Monte-Carlo simulations. For the relative-expected half-length, values are computed also analytically.

**Keywords:** Inventory Control; Classical newsvendor model; Exponential and Rayleigh Distributions; Confidence Intervals; Monte-Carlo Simulations.

**JEL Codes:** C13: Estimation; C15: Statistical simulation methods; C44: Operations Research; D24: Production & cost; M11: Production Management.

## 1. Introduction

Inventory management is a crucial task in the operation of firms and enterprises. For products whose life-cycle of demand lasts a relatively short period (daily and weekly newspapers and magazines, seasonal goods etc), newsvendor (or alternatively newsboy) models offer quantitative tools to form effective inventory policies. The short period where demand refers to and inventory decisions should be made represents an inventory cycle. In general, applications of such models are found in fashion industry, airline seats pricing, and management of perishable food supplies in supermarkets.

Among the alternative forms of newsvendor models, the classical version refers to the purchasing inventory problem where newsvendors decide on a one-time basis and their decisions are followed by a stochastic sales outcome (Silver et al., 1998). In such cases, newsvendors have to predict order quantities in the beginning of each inventory cycle (or period) and products cannot be sold in the next time period if the actual demand is greater than the order quantity, as any excess inventory is disposed of by buyback arrangements. At the same time, there is an opportunity cost of lost profit in the opposite situation (Chen and Chen, 2009), where at the end of the inventory cycle an excess demand is observed.

During the last decades, a number of researchers have explored the issue of the optimal order quantity for cases of uncertainty in demand. Alternative extensions of newsvendor models have been published in the literature, and Khouja (1999) provides an extensive search of these works till 1999. Since then, various papers have explored the newsboy-type inventory problem like Schweitzer and Cachon (2000), Casimir (2002), Dutta et al. (2005), Salazar-Ibarra (2005), Matsuyama (2006), Benzion et al. (2008), Wang and Webster (2009), Chen and Chen (2010), Huang et al. (2011), Lee and Hsu (2011), and Jiang et al. (2012). However, the crucial condition of applying these models in practice is that parameters of demand distributions should be known, something that does not hold. And, unfortunately, the extent of applicability of newsvendor models in inventory management to determine the level of customer service depends upon the estimation of demand parameters. The problem of uncertainty becomes even more severe for certain types of product, like seasonal clothing, for which data on demand are available only for few inventory cycles (ensuring that market conditions do not change), and this makes estimation procedures to be under question.

Research on studying the effects of demand estimation on optimal inventory policies is limited (Conrad 1976; Nahmias, 1994; Agrawal and Smith, 1996; Hill, 1997; Bell, 2000). Besides, none of these works has addressed the problem of how sampling variability of estimated values of demand parameters influences the quality of estimation concerning optimal ordering policies. Recently, it has been recognized that effective applications of newsvendor models to form reliable inventory policies depend upon the variability of estimates for the parameters of probabilistic laws which generate demand in successive inventory cycles. Assuming that demand follows the normal distribution, for the classical newsvendor model, Kevork (2010) developed appropriate estimators to explore the variability of estimates for the optimal order quantity and the maximum expected profit. His analysis showed that the weak point of applying this model to real life situations is the significant reductions in precision and stability of confidence intervals for the true maximum expected profit when high shortage costs occur.

Su and Pearn (2011) developed a statistical hypothesis testing methodology to compare two newsboy-type products and to select the one that has a higher probability of achieving a target profit under the optimal ordering policy. The authors provided tables with critical values of the test and the sample sizes which are required to attain designated type I and II errors. Prior to these two works, Olivares et al. (2008) presented a structural estimation framework to disentangle whether specific factors affect the observed order quantity either through the distribution of demand or through the overage/underage cost ratio.

In the current paper, for the classical newsvendor model with the demand in each period to be fully observed and to follow either the exponential or the Rayleigh distribution, we study the variability of estimates for the optimal order quantity and the maximum expected profit which is caused by the sampling distribution of known estimators of the parameter(s) of each distribution. For the exponential distribution, to estimate its parameter we use the sample mean, while to estimate the variance of the distribution we adopt an estimator, which belongs to the form of estimators which are produced from the product of a constant times the squared of the sample mean (Pandey and Singh, 1977; Singh and Chander, 2008). For the Rayleigh distribution its parameter is estimated using a maximum likelihood estimator (Johnson et al., 1994).

Putting these estimators into the forms that determine the optimal order quantity and the maximum expected profit, for the latter two quantities we establish analogous estimators, whose distributions are derived (a) for sufficiently large samples in the case of an exponential demand, and (b) for both small and large samples when demand follows the Rayleigh distribution. Using the derived distributions of the proposed estimators, to study the validity and precision of estimates for the optimal order quantity and the maximum expected profit, we consider two statistical criteria: (a) the actual confidence level which the generated confidence intervals attain and (b) their expected half-length divided by the true quantity (optimal order quantity or maximum expected profit).

A first reason for considering the Exponential and Rayleigh distributions was that their coefficients of variation, skewness, and excess kurtosis remain unaltered in changes of their parameter. But the most substantial reason of the choice of the two distributions under consideration is a momentous property that the generated confidence intervals possess. We prove that for each distribution under consideration and at any sample size, the actual confidence level and the relative expected half-length take on the same values either when the confidence interval refers to the optimal order quantity or when the interval has to do with the maximum expected profit. Additionally to this important remark, the values of the two statistical criteria do not depend upon the revenue and cost parameters of the newsvendor model, that is, price, salvage value, purchasing (or production) cost and shortage cost. This property indicates that being at the start of the period and estimating the optimal order quantity and the maximum expected profit, validity and precision depends upon only the available sample on demand data.

To provide, therefore, a framework where practitioners could find a-priori knowledge for the validity and precision attained under an exponential or Rayleigh demand, we have estimated the actual confidence level and the relative half- length by organizing appropriate Monte-Carlo simulations and we present their values for different combinations of alternative sample sizes and nominal confidence level. Especially, for the relative expected half-length, analytic forms have been obtained from which exact values have been also computed. The fact that discrepancies between estimated and true values for the relative expected half-length are negligible assures the reliability of simulation results.

The Exponential and Weibull (a special case of which is the Rayleigh) distributions have been extensively used in various papers related to inventory management, but with different aims than those which are considered in the current work. For the Newsboy problem, Khouja (1996) used an exponential distributed demand to illustrate the effect of an emergency supply, and Lau (1997) offered closed-form formulas for computing the expected cost and the optimal expected cost when demand follows the exponential distribution. Hill (1997) used the exponential distribution to perform analytical and numerical comparisons between the Frequentist and Bayesian approach for demand estimation. Exponential demand was also adopted in the works of Geng et al. (2010) who considered a single-period inventory system with a general S-shaped utility function and Grubbstrom (2010) who provided a compound variation of the newsboy problem where customers are arriving at different points in time and requiring amounts of product of varying size.

Lau and Lau (2002) used both the exponential and Weibull distributions in a manufacturer-retailer channel to study the effects of retail-market demand uncertainty on revenues, order quantities and expected profits. For the multi-product Newsboy model with constraints, Areeratchakul and Abdel-Malek (2006) modeled demand as Exponential, Log-Normal, and Weibull to provide a solution methodology, which was based on quadratic programming and a triangular presentation of the area under the cumulative distribution function of demand. Considering that demand follows the Weibull distribution and including risk preferences of the inventory manager to the classical newsvendor problem, Jammernegg and Kischka (2009) showed that robust ordering decisions can be derived from assumptions on stochastic dominance. For the multi-product competitive newsboy problem, Huang et al. (2011) used the exponential distribution to test the validity of a static service-rate approximation for the dynamic and stochastic availability of each product.

The aforementioned arguments and remarks lead the rest of the paper to be structured as follows. In the next section we develop the estimators for the optimal order quantity and the maximum expected profit. The distributions of the estimators are derived in section 3, where the corresponding confidence intervals are also constructed. In section 4, we evaluate the performance of the generated confidence intervals and we report the values of their actual confidence levels and relative expected half-lengths for different combinations of sample sizes and nominal confidence levels. Finally, the last section concludes the paper summarizing the most important findings of the current work.

## 2. Estimators for the optimal order quantity and the maximum expected profit

For the classical newsvendor model (Khouja, 1999), the aim is to determine at the start of the period (or inventory cycle) the order quantity, Q, which maximizes the expected value of

$$\xi = \begin{cases} (p-c)Q - (p-v)(Q-X) & \text{if} \quad X \leq Q \\ (p-c)Q + s(Q-X) & \text{if} \quad X > Q \end{cases},$$

where, $\xi$ stands for profits per period, X is a random variable representing size of demand at any inventory cycle or period, p is the selling price, c is the purchase (or production) cost, v is the salvage value, and s stands for the shortage cost per unit. A vital assumption of the model is that any excess inventory at the end of the previous inventory cycle has been disposed of by buyback arrangements with the salvage value to take care of such arrangements. Thus at the beginning of every period, the stock level is set up equal to the ordered quantity. On the other hand, if excess demand is observed, shortage cost per unit is used as an indirect cost element which incorporates current losses and present value of future payoffs expected to be lost from present unsatisfied customers.

Let $f(x)$ and $F(x)$ be respectively the probability density function and the cumulative distribution function of demand. Taking first and second derivatives of $E(\xi)$ by using Leibniz's rule, the optimal order quantity maximizing expected profit function satisfies the equation,

$$\Pr(X \leq Q^*) = F(Q^*) = \frac{p-c+s}{p-v+s} = R, \qquad (1)$$

where R is a critical fractile whose value identifies the product as low profit (R<0,5) or high profit (R>0,5) according to the principle stated by Schweitzer and Cachon (2000). Using $Q^*$, the next proposition expresses the maximum expected profit in terms of the unconditional expected demand and the expected demand given that this is less than the optimal order quantity.

***Proposition 1:*** *Given $Q^*$ satisfying (1), the maximum expected profit takes the form,*

$$E(\xi)^* = (p-c)E(X|X \le Q^*) - s\{E(X) - E(X|X \le Q^*)\} \qquad (2)$$

***Proof:*** *See in the Appendix.*

When demand follows the exponential distribution with parameter λ, we know that $F(Q^*) = 1 - e^{-Q^*/\lambda}$. Thus the optimal order quantity will be given from

$$Q^*_{EX} = \lambda \cdot |\ln(1-R)|. \qquad (3)$$

To derive the maximum expected profit, we obtain first the next result by using (1) and (3):

$$E(X|X \le Q^*) = \frac{1}{F(Q^*)}\int_0^{Q^*} \frac{x}{\lambda} e^{-\frac{x}{\lambda}} dx = \frac{1}{R}\left(-Q^* e^{-\frac{Q^*}{\lambda}} - \lambda e^{-\frac{Q^*}{\lambda}} + \lambda\right) = \lambda\left\{\frac{(1-R)\ln(1-R)}{R} + 1\right\}.$$

Replacing it into (2), given that $E(X) = \lambda$, we take

$$E(\xi)^*_{EX} = \lambda\{(p-c) + (c-v)\cdot \ln(1-R)\}. \qquad (4)$$

Regarding the Rayleigh distribution with parameter σ, this is a special case of Weibull and is defined in Johnson et al. (1994). Its distribution function evaluated at $Q^*$ is $F(Q^*) = 1 - e^{-\frac{1}{2}\left(\frac{Q^*}{\sigma}\right)^2}$, from which we obtain

$$Q^*_{RY} = \sigma \cdot \sqrt{2|\ln(1-R)|}. \qquad (5)$$

To have a corresponding analytic form for the maximum expected profit, the expected value of the Rayleigh distribution singly truncated to the right at $Q^*$ is prerequisite. To the extent of authors' knowledge such a direct result is not available in the literature. The next proposition derives the required expected value.

***Proposition 2:*** *Given $Q^*$ satisfying (5), it holds that*

$$E\left(X \middle| X \le Q^*\right) = \frac{\sigma}{R}\left\{(1-R)\cdot\sqrt{-2ln(1-R)} + \sqrt{2\pi}\cdot\Phi_{\sqrt{-2ln(1-R)}} - \frac{\sqrt{2\pi}}{2}\right\}, \quad (6)$$

*with $\Phi_{\sqrt{-2ln(1-R)}}$ the distribution function of the standard normal evaluated at $\sqrt{-2ln(1-R)}$.*

***Proof:*** *See in the Appendix.*

Replacing, (6) into (2), and given that $E(X) = \sigma\sqrt{\pi/2}$,

$$E(\xi)^*_{RY} = \sigma\left\{(p-v+s)\left[-(1-R)\cdot\sqrt{-2\ln(1-R)} + \sqrt{2\pi}\cdot\Phi_{\sqrt{-2\ln(1-R)}} - \sqrt{\frac{\pi}{2}} - \frac{s}{p-v+s}\sqrt{\frac{\pi}{2}}\right]\right\}. \quad (7)$$

Suppose now that $X_1, X_2, \ldots, X_n$ is a sequence of independent random variables following either the exponential or the Rayleigh distribution and representing demand for a sample of the most recent n consecutive periods. On the basis of expressions (3), (4), (5), and (7), the following estimators for the optimal order quantity and the maximum expected profit are defined for period n+1:

***Exponential (λ):***

$$\hat{Q}^*_{EX} = \hat{\lambda}\cdot\left|\ln(1-R)\right|, \quad (8a)$$

$$\hat{E}(\xi)^*_{EX} = \hat{\lambda}\left\{(p-c)+(c-v)\cdot\ln(1-R)\right\}, \quad (8b)$$

where $\hat{\lambda} = \sum_{t=1}^{n} X_t \Big/ n$.

***Rayleigh (σ):***

$$\hat{Q}^*_{RY} = \hat{\sigma} \cdot \sqrt{2|\ln(1-R)|}\,, \tag{9a}$$

$$\hat{E}(\xi)^*_{ry} = \hat{\sigma} \cdot g_R \tag{9b}$$

where

$$g_R = (p - v + s)\left[-(1-R)\cdot\sqrt{-2\ln(1-R)} + \sqrt{2\pi}\cdot\Phi_{\sqrt{-2\ln(1-R)}} - \sqrt{\frac{\pi}{2}} - \frac{s}{p-v+s}\sqrt{\frac{\pi}{2}}\right]$$

and $\hat{\sigma} = \sqrt{\sum_{t=1}^{n} X_t^2 \Big/ 2n}$ is the maximum likelihood estimator of $\sigma$ (Johnson et al., 1994).

Before we proceed to derive the distributions (small sample size or asymptotic) of the aforementioned four estimators, we shall assume that in every period in the sample the salvage value had been set up at a level which ensured that if any excess inventory remained at the end of period, this was disposed of through either consignment stocks or buyback arrangements. So, the stock level at the beginning of any period in the sample was being set up equal to the ordered quantity, without being necessary to know how the order quantity had been determined. Being however at the start of period n+1, and having decided to adopt the newsvendor model to determine the order quantity, we should have knowledge of how to access the precision of estimates for the optimal order quantity and the maximum expected profit. This issue is addressed in the next section.

## 3. Confidence Intervals for $\hat{Q}^*$ and $\hat{E}(\xi)^*$

When X follows the exponential distribution with parameter λ, it is known that $\sum_{t=1}^{n} X_t$ follows the Gamma distribution with parameters n and λ. The scaling property of the Gamma distribution states that if we multiply $\sum_{t=1}^{n} X_t$ by a constant $m > 0$, the product follows again the Gamma distribution with parameters n and $\lambda \cdot m$. Setting, therefore, $m = |\ln(1-R)|/n$ we obtain

$$\frac{\sum_{t=1}^{n} X_t}{n} |\ln(1-R)| = \hat{\lambda}|\ln(1-R)| = \hat{Q}^*_{EX} \sim \text{Gamma}\left(n, \hat{\lambda}\frac{|\ln(1-R)|}{n}\right),$$

while with $m = \{(p-c) + (c-v) \cdot \ln(1-R)\}/n$, we take

$$\hat{E}(\xi)^*_{EX} \sim \text{Gamma}\left(n, \hat{\lambda}\frac{\{(p-c) + (c-v) \cdot \ln(1-R)\}}{n}\right).$$

However, constructing confidence intervals for $Q^*_{EX}$ and $E(\xi)^*_{EX}$ using these two exact distributional results is not recommended for two reasons: (a) the required critical values of the gamma distribution depend upon an estimate of $\lambda$ which is not a fixed quantity, and (b) we do not have always positive values for the expression $\{(p-c) + (c-v) \cdot \ln(1-R)\}$. Alternatively, approximate confidence intervals for $Q^*_{EX}$ and $E(\xi)^*_{EX}$ can be obtained from the asymptotic distributions of estimators (8a) and (8b). The next proposition establishes the prerequisite asymptotic distributional results.

***Proposition 3:*** *When demand follows the exponential distribution, then we have*

$$\sqrt{n}\left(\hat{Q}^*_{EX} - Q^*_{EX}\right) \xrightarrow{D} N\left(0, \kappa^2 [ln(1-R)]^2\right),$$

$$\sqrt{n}\left(\hat{E}(\xi)^*_{EX} - E(\xi)^*_{EX}\right) \xrightarrow{D} N\left(0, \kappa^2 [(p-c) + (c-v) \cdot ln(1-R)]^2\right),$$

*where* $\kappa^2 = Var(X)$.

***Proof:*** *See in the Appendix.*

From proposition 3, the $(1-\alpha)100\%$ confidence intervals for $Q^*_{EX}$ and $E(\xi)^*_{EX}$ can be approximated in finite samples from

$$\hat{Q}^*_{EX} \pm z_{\alpha/2} \frac{\hat{\kappa}}{\sqrt{n}} \left| \ln(1-R) \right|, \tag{10a}$$

and

$$\hat{E}(\xi)^*_{EX} \pm z_{\alpha/2} \frac{\hat{\kappa}}{\sqrt{n}} \left| (p-c) + (c-v) \cdot \ln(1-R) \right|, \tag{10b}$$

where $\hat{\kappa}^2 = \frac{n^2}{(n+2)(n+3)} \hat{\lambda}^2$ estimates the variance of the exponential distribution and belongs to the class of estimators of the form $M \cdot \bar{x}^2$, where M is a suitably chosen constant and $\bar{x}$ is the sample mean (Pandey and Singh, 1977; Singh and Chander, 2008).

On the contrary, for the Rayleigh distribution, small sample size confidence intervals for $Q^*_{RY}$ and $E(\xi)^*_{RY}$ can be constructed.

***Proposition 4:*** *When demand follows the Rayleigh distribution, an $(1-\alpha)100\%$ confidence interval for $Q^*_{RY}$ is obtained from*

$$\hat{\sigma} \sqrt{\frac{2n}{\Gamma_{1-\frac{\alpha}{2}}\left(n, \frac{1}{\left| ln(1-R) \right|}\right)}} \leq Q^*_{RY} \leq \hat{\sigma} \sqrt{\frac{2n}{\Gamma_{\frac{\alpha}{2}}\left(n, \frac{1}{\left| ln(1-R) \right|}\right)}} \tag{11a}$$

*while, for $E(\xi)^*_{RY}$ the corresponding interval is given by*

$$\hat{\sigma} \sqrt{\frac{2n}{\Gamma_{1-\alpha/2}\left(n, \frac{2}{g_R^2}\right)}} \leq E(\xi)^*_{RY} \leq \hat{\sigma} \sqrt{\frac{2n}{\Gamma_{\alpha/2}\left(n, \frac{2}{g_R^2}\right)}} \tag{11b}$$

*where $\Gamma_{\alpha/2}(v_1, v_2)$ and $\Gamma_{1-\alpha/2}(v_1, v_2)$ are respectively the lower and the upper $\alpha/2$ percentage points of the Gamma distribution with parameters $v_1$ and $v_2$.*

***Proof:*** *See in the Appendix.*

Additionally, as in the case of exponential distribution, confidence intervals for $Q^*_{RY}$ and $E(\xi)^*_{RY}$ can be constructed from the asymptotic distributions of the corresponding estimators. The next proposition states the required asymptotic distributional results related to estimators (9a) and (9b).

**Proposition 5:** *When demand follows the Rayleigh distribution, then we obtain*

$$\sqrt{n}\left[\hat{Q}^*_{RY} - Q^*_{RY}\right] \xrightarrow{D} N\left( 0, \sigma^2 \frac{|ln(1-R)|}{2} \right),$$

$$\sqrt{n}\left[\hat{E}(\xi)^*_{EX} - E(\xi)^*_{EX}\right] \xrightarrow{D} N\left( 0, \sigma^2 \frac{g_R^2}{4} \right)$$

***Proof:*** *See in the Appendix.*

Convergence to normality, as this is stated in proposition 5, enables us to approximate in finite samples the $(1-\alpha)100\%$ confidence intervals for $Q^*_{RY}$ and $E(\xi)^*_{RY}$ by using respectively the following expressions

$$\hat{Q}^*_{RY} \pm z_{\alpha/2} \frac{\hat{\sigma}}{\sqrt{n}} \sqrt{\frac{|\ln(1-R)|}{2}}, \tag{12a}$$

and

$$\hat{E}(\xi)^*_{RY} \pm z_{\alpha/2} \frac{\hat{\sigma}}{\sqrt{n}} \frac{|g_R|}{2}. \tag{12b}$$

## 4. Precision and Validity of Confidence Intervals

In the current section we evaluate the performance of each pair of confidence intervals given in (10), (11), and (12) by using two statistical criteria and estimating them through Monte-Carlo simulations. These criteria are the actual confidence level (ACL), which the interval attains at a given sample size, and the relative expected half-length (REHL) of the interval defined as the expected half-length divided by the true quantity, that is, $Q^*$ and $E(\xi)^*$ respectively. The first criterion assesses the validity of confidence interval at finite sample sizes, while the second measures its precision. Besides, the REHL resolves the problem of comparability of precision between alternative combination of values for R, p, c, v, and s. This problem arises since for different R the optimal order quantity, $Q^*$, changes, and given R, assigning different sets of values for p, c, v, and s, we result in different sizes for the maximum expected profit, $E(\xi)^*$.

Propositions 6 and 7 establish that for each one of the two distributions under consideration and for each pair of confidence intervals considered in (10), (11), and (12) respectively, the values of the aforementioned two criteria are exactly the same, and their values do not depend upon R, p, c, v, and s, as well as, the parameter of the distribution.

***Proposition 6:*** *When demand follows the exponential distribution, for the pair of confidence intervals defined in (10a) and (10b), it holds that*

$$ACL = Pr\left( - z_{\alpha/2} \leq \frac{\sqrt{n}\left(\hat{\lambda} - \lambda\right)}{\hat{\kappa}} \leq z_{\alpha/2} \right),$$

$$REHL = z_{\alpha/2}\sqrt{\frac{n}{(n+2)(n+3)}}.$$

***Proof:*** *See in the Appendix.*

***Proposition 7:*** *When demand follows the rayleigh distribution, (a) for the pair of confidence intervals defined in (11a) and (11b), we obtain*

$$ACL = Pr\left( \sqrt{\frac{n}{\Gamma_{1-\alpha/2}(n,1)}} \leq \frac{\sigma}{\hat{\sigma}} \leq \sqrt{\frac{n}{\Gamma_{\alpha/2}(n,1)}} \right),$$

$$REHL = \frac{1}{2}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)}\left\{ \sqrt{\frac{1}{\Gamma_{\alpha/2}(n,1)}} - \sqrt{\frac{1}{\Gamma_{1-\alpha/2}(n,1)}} \right\},$$

*and (b) for the corresponding pair defined in (12a) and (12b) we take*

$$ACL = Pr\left( -\frac{z_{\alpha/2}}{2} \leq \frac{\sqrt{n}(\hat{\sigma}-\sigma)}{\hat{\sigma}} \leq \frac{z_{\alpha/2}}{2} \right),$$

$$REHL = \frac{z_{\alpha/2}}{2n}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)},$$

*where Γ(x) is the Gamma function evaluated at x.*

***Proof:*** *See the Appendix.*

The outcomes of propositions 6 and 7 are important regarding the extent of applicability of the classical newsvendor model from the estimation point of view when demand follows the exponential or the Rayleigh distribution. Being at the start of period n+1, and trying to predict $Q^*$ and $E(\xi)^*$ based on demand data available from a sample of the n most recent periods, management should have some knowledge about the sensitivity of the validity and the precision of estimates in changes of the revenue and cost parameters. And this happens because accordingly to the type of the newsvendor product the values of some parameters cannot be specified at the start of the period. For example, for seasonal clothing the salvage value cannot be fixed at the start of the period, but must be adjusted appropriately at the end of period in order to get rid of any remained stock. Also the shortage cost, being the present value of future profits expected to be lost from present unsatisfied customers (Lapin, 1994), cannot be accurately determined, especially with sales where personal communication with customers is missing.

Fortunately, such problems are resolved when demand follows the exponential or the Rayleigh distribution. By using confidence intervals (10), (11) and (12), a-priori knowledge for the revenue and cost parameters is not necessary. Whichever the values of R, p, c, v and s are, ACL and REHL depend upon only the sample size. This, however, is not true when demand follows for instance the normal distribution. Kevork (2010) concluded that, given the profit margin, the precision of confidence intervals that he generated is different for different combinations of permissible values for the shortage cost and the salvage value. Particularly, increasing the shortage cost, precision reduces considerably. In such a case, therefore, a-priori knowledge for the salvage value and the shortage cost are required in order management to know how close or how far away from the corresponding true quantities the estimates for the maximum expected profit lie.

To estimate ACL and REHL as they are stated in propositions 6 and 7, 10000 replications of maximum size 2000 observations each were generated from the exponential distribution with $\lambda$=300 and from the Rayleigh with $\sigma = 300\sqrt{2/\pi}$ . The specific choice of $\lambda$ and $\sigma$ ensures that both distributions have a common expected demand E(X) =300. In each replication, the series of 2000 observations was generated by applying traditional inverse-transform algorithms to a sequence of 2000 random numbers uniformly distributed on (0,1). The algorithms can be found in Law (2007), on pages 448 and 452 for the exponential and the Rayleigh respectively. Details for the random number generator can be found in Kevork (2010).

For each distribution and for each replication, estimates for λ and σ were taken at different sample sizes. Then for each sample size, setting R=0,8, estimates for $Q^*$ were computed using (8a) for the exponential distribution and (9a) for Rayleigh. Finally, for each combination of n and R, and using nominal confidence levels of 90%, 95%, and 99%, 10000 different confidence intervals were computed using (10a) for the exponential and (11a), (12a) for Rayleigh. Having available now 10000 different confidence intervals for each sample size, the ACL was estimated by the percentage of intervals containing the true quantity $Q^*$, and the REHL by dividing the average half-length over the 10000 confidence intervals by $Q^*$. Table 1 displays the estimated values for the ACL's while tables 2, 3 and 4 show both the estimated and the true values for REHL. The required critical values for the gamma distribution were obtained through the statistical package MINITAB. Also to compute precisely the ratio of the two gamma functions defined in proposition 7, we used the transformation $\exp(\ln\Gamma(n+0{,}5)-\ln(\Gamma(n)))$.

**Table 1:** Estimated values for the ACL

| Sample size | Exponential – asymptotic confidence intervals | | | Rayleigh – small sample size confidence intervals | | | Rayleigh – asymptotic confidence intervals | | |
|---|---|---|---|---|---|---|---|---|---|
| | Nominal Confidence Level | | | Nominal Confidence Level | | | Nominal Confidence Level | | |
| | 90% | 95% | 99% | 90% | 95% | 99% | 90% | 95% | 99% |
| 5 | 0,728 | 0,788 | 0,848 | 0,897 | 0,948 | 0,990 | 0,862 | 0,903 | 0,947 |
| 10 | 0,801 | 0,854 | 0,914 | 0,897 | 0,949 | 0,988 | 0,881 | 0,923 | 0,964 |
| 15 | 0,831 | 0,885 | 0,938 | 0,902 | 0,949 | 0,989 | 0,889 | 0,933 | 0,973 |
| 20 | 0,848 | 0,903 | 0,950 | 0,899 | 0,951 | 0,989 | 0,892 | 0,935 | 0,980 |
| 25 | 0,861 | 0,913 | 0,959 | 0,901 | 0,951 | 0,990 | 0,896 | 0,942 | 0,982 |
| 30 | 0,867 | 0,920 | 0,965 | 0,900 | 0,950 | 0,992 | 0,898 | 0,942 | 0,981 |
| 40 | 0,873 | 0,925 | 0,970 | 0,897 | 0,950 | 0,990 | 0,895 | 0,945 | 0,984 |
| 50 | 0,874 | 0,927 | 0,974 | 0,896 | 0,949 | 0,990 | 0,891 | 0,943 | 0,984 |
| 100 | 0,891 | 0,942 | 0,983 | 0,900 | 0,953 | 0,991 | 0,899 | 0,951 | 0,988 |
| 200 | 0,899 | 0,945 | 0,984 | 0,902 | 0,952 | 0,990 | 0,903 | 0,950 | 0,987 |
| 300 | 0,896 | 0,949 | 0,986 | 0,902 | 0,952 | 0,990 | 0,900 | 0,952 | 0,988 |
| 400 | 0,895 | 0,945 | 0,987 | 0,900 | 0,949 | 0,989 | 0,899 | 0,947 | 0,989 |
| 500 | 0,894 | 0,946 | 0,988 | 0,897 | 0,948 | 0,989 | 0,896 | 0,948 | 0,989 |
| 1000 | 0,898 | 0,949 | 0,989 | 0,900 | 0,952 | 0,989 | 0,901 | 0,950 | 0,990 |
| 2000 | 0,899 | 0,949 | 0,990 | 0,902 | 0,950 | 0,990 | 0,901 | 0,951 | 0,990 |

From table 1, small sample size confidence intervals held for the Rayleigh distribution attain at any sample estimated ACL's almost identical with the corresponding nominal confidence levels verifying the validity of the recommended intervals (11a) and (11b). Besides, ACL's for the asymptotic confidence intervals converge to the nominal confidence levels with different rates of convergence, which are faster for the Rayleigh distribution. Regarding tables 2, 3 and 4, discrepancies between estimated and true values for the REHL are negligible, verifying the validity of simulation results. Apart from very small samples, in any other case these discrepancies are starting to appear at least after the fourth decimal place. Finally, as it was expected, REHL's are greater for the exponential distribution which has larger coefficient of variation, skewness, and kurtosis, and for the Rayleigh, asymptotic confidence intervals provide smaller REHL's compared to those of small sample size intervals.

We close this section by giving some recommendations for the common used in practice 95% nominal confidence level. Regarding the Rayleigh distribution and using the asymptotic confidence intervals, actual confidence levels more than 90% can be attained even with a sample of five observations. But to achieve a sampling error of at most 10% of the true value (optimal order quantity or maximum expected profit) a sample of over 100 observations must be available. On the contrary, to attain the same precision with the exponential distribution, the sample size should be considerably larger exceeding 400 observations.

**Table 2:** Estimated and true values for the REHL under an exponential demand

| | ESTIMATED VALUES | | | TRUE VALUES | | |
|---|---|---|---|---|---|---|
| | Nominal Confidence Level | | | | | |
| | 90% | 95% | 99% | 90% | 95% | 99% |
| 5 | 0,4954 | 0,5903 | 0,7758 | 0,4915 | 0,5857 | 0,7697 |
| 10 | 0,4170 | 0,4969 | 0,6530 | 0,4165 | 0,4962 | 0,6522 |
| 15 | 0,3646 | 0,4344 | 0,5709 | 0,3642 | 0,4339 | 0,5703 |
| 20 | 0,3275 | 0,3902 | 0,5129 | 0,3270 | 0,3897 | 0,5121 |
| 25 | 0,2997 | 0,3571 | 0,4693 | 0,2991 | 0,3564 | 0,4684 |
| 30 | 0,2778 | 0,3310 | 0,4350 | 0,2772 | 0,3304 | 0,4342 |
| 40 | 0,2449 | 0,2919 | 0,3836 | 0,2448 | 0,2917 | 0,3833 |
| 50 | 0,2216 | 0,2640 | 0,3470 | 0,2216 | 0,2640 | 0,3470 |
| 100 | 0,1604 | 0,1912 | 0,2512 | 0,1605 | 0,1912 | 0,2513 |
| 200 | 0,1148 | 0,1368 | 0,1798 | 0,1149 | 0,1369 | 0,1799 |
| 300 | 0,0941 | 0,1121 | 0,1474 | 0,0942 | 0,1122 | 0,1475 |
| 400 | 0,0816 | 0,0973 | 0,1278 | 0,0817 | 0,0974 | 0,1280 |
| 500 | 0,0731 | 0,0872 | 0,1145 | 0,0732 | 0,0872 | 0,1146 |
| 1000 | 0,0519 | 0,0618 | 0,0812 | 0,0519 | 0,0618 | 0,0813 |
| 2000 | 0,0367 | 0,0438 | 0,0575 | 0,0367 | 0,0438 | 0,0575 |

**Table 3:** Estimated and true values for the REHL under a Rayleigh demand, small sample size confidence intervals

| | ESTIMATED VALUES | | | TRUE VALUES | | |
|---|---|---|---|---|---|---|
| | Nominal Confidence Level | | | Nominal Confidence Level | | |
| | 90% | 95% | 99% | 90% | 95% | 99% |
| 5 | 0,4181 | 0,5171 | 0,7460 | 0,4165 | 0,5151 | 0,7430 |
| 10 | 0,2765 | 0,3355 | 0,4610 | 0,2764 | 0,3353 | 0,4608 |
| 15 | 0,2212 | 0,2667 | 0,3609 | 0,2211 | 0,2666 | 0,3607 |
| 20 | 0,1897 | 0,2280 | 0,3062 | 0,1895 | 0,2278 | 0,3060 |
| 25 | 0,1686 | 0,2024 | 0,2706 | 0,1685 | 0,2022 | 0,2704 |
| 30 | 0,1533 | 0,1838 | 0,2451 | 0,1532 | 0,1836 | 0,2448 |
| 40 | 0,1320 | 0,1580 | 0,2099 | 0,1320 | 0,1580 | 0,2099 |
| 50 | 0,1177 | 0,1408 | 0,1866 | 0,1177 | 0,1407 | 0,1866 |
| 100 | 0,0827 | 0,0987 | 0,1303 | 0,0827 | 0,0988 | 0,1303 |
| 200 | 0,0583 | 0,0695 | 0,0916 | 0,0583 | 0,0696 | 0,0916 |
| 300 | 0,0476 | 0,0567 | 0,0746 | 0,0476 | 0,0567 | 0,0747 |
| 400 | 0,0412 | 0,0491 | 0,0646 | 0,0412 | 0,0491 | 0,0646 |
| 500 | 0,0368 | 0,0439 | 0,0577 | 0,0368 | 0,0439 | 0,0577 |
| 1000 | 0,0260 | 0,0310 | 0,0408 | 0,0260 | 0,0310 | 0,0408 |
| 2000 | 0,0184 | 0,0219 | 0,0288 | 0,0184 | 0,0219 | 0,0288 |

**Table 4:** Estimated and true values for the REHL under a Rayleigh demand, asymptotic confidence intervals

| | ESTIMATED VALUES | | | TRUE VALUES | | |
|---|---|---|---|---|---|---|
| | Nominal Confidence Level | | | | | |
| | 90% | 95% | 99% | 90% | 95% | 99% |
| 5 | 0,3601 | 0,4291 | 0,5640 | 0,3587 | 0,4275 | 0,5618 |
| 10 | 0,2570 | 0,3062 | 0,4024 | 0,2568 | 0,3060 | 0,4022 |
| 15 | 0,2107 | 0,2511 | 0,3300 | 0,2106 | 0,2509 | 0,3298 |
| 20 | 0,1829 | 0,2179 | 0,2864 | 0,1828 | 0,2178 | 0,2862 |
| 25 | 0,1638 | 0,1952 | 0,2565 | 0,1637 | 0,1950 | 0,2563 |
| 30 | 0,1497 | 0,1784 | 0,2344 | 0,1495 | 0,1782 | 0,2342 |
| 40 | 0,1297 | 0,1545 | 0,2031 | 0,1296 | 0,1545 | 0,2030 |
| 50 | 0,1160 | 0,1383 | 0,1817 | 0,1160 | 0,1382 | 0,1817 |
| 100 | 0,0821 | 0,0979 | 0,1286 | 0,0821 | 0,0979 | 0,1286 |
| 200 | 0,0581 | 0,0692 | 0,0910 | 0,0581 | 0,0693 | 0,0910 |
| 300 | 0,0474 | 0,0565 | 0,0743 | 0,0475 | 0,0566 | 0,0743 |
| 400 | 0,0411 | 0,0490 | 0,0643 | 0,0411 | 0,0490 | 0,0644 |
| 500 | 0,0368 | 0,0438 | 0,0576 | 0,0368 | 0,0438 | 0,0576 |
| 1000 | 0,0260 | 0,0310 | 0,0407 | 0,0260 | 0,0310 | 0,0407 |
| 2000 | 0,0184 | 0,0219 | 0,0288 | 0,0184 | 0,0219 | 0,0288 |

## 5. Conclusions

In the current paper we enrich existent approaches of estimating optimal inventory policies. By considering the classical newsvendor model with profit maximization and assuming that demand per period follows either the exponential or the Rayleigh distribution, we explore the validity and precision of appropriate confidence intervals for the optimal order quantity and the maximum expected profit. The extraction of confidence intervals was based on small and/or large sample size distributions of estimators which we derived for the two quantities under consideration. Validity and precision were measured respectively from the actual confidence level attained by the intervals and from their expected half-length divided either by the optimal order quantity or the maximum expected profit. For the relative expected half-length, analytic forms have been also derived to compute it precisely.

When demand in each period is modeled either by the exponential or the Rayleigh distribution, validity and precision of estimates for the optimal order quantity and the maximum expected profit are characterized by an important and useful property which we prove in the current work. The actual confidence level and the relative expected half-length take exactly the same values either when the confidence intervals refers to the optimal order quantity or when the interval is constructed for the maximum expected profit. And the specific values do not depend upon the critical fractile and the revenue and cost parameters of the newsvendor model. Implications of this property are very important. Being at the start of the period and predicting the optimal order quantity and the maximum expected profit for the coming period, management should not worry about the sensitivity of predictions to different values of parameters of the newsvendor model, especially with those which cannot be fixed in the beginning of the period. The only factor that eventually determines validity and precision is the sample size.

Organizing and performing appropriate Monte-Carlo simulations, for each proposed type of confidence interval, we have estimated the actual confidence level and the relative expected half-length for different combinations of sample sizes and nominal confidence level. Additionally, we have computed analytically the relative expected half-length from the available mathematical expressions which as we mentioned we have also derived. The results are displayed to appropriately structured tables so that in case where demand follows one of the two distributions under consideration to have knowledge about validity and precision which can be attained from the available sample of demand data. The general conclusion is that although acceptable actual confidence levels can be attained with relatively small samples, to achieve satisfactory levels of precision large samples are required. And unfortunately, for certain types of newsvendor products, due to their nature and to market conditions, past data for demand are limited. Seasonal clothing belongs to this category of products. In such cases, it is recommended the application of the classical newsvendor model to be made with great caution otherwise it is very likely the estimated values to lie far away from the optimal ones.

# APPENDIX

***Proof of proposition 1***

Taking expected values to both sides of (1),

$$E(\xi) = (p - c + s)\cdot Q - (p - v + s)\cdot Q \cdot F(Q) + (p - v)\cdot E(X|X \le Q)\cdot F(Q) -$$

$$- s \cdot E(X|X > Q)\cdot \{1 - F(Q)\}. \qquad \text{(A1)}$$

Using properties of truncated distributions,

$$E(X) = \int_{-\infty}^{Q} x \cdot f(x)\cdot dx + \int_{Q}^{+\infty} x \cdot f(x)\cdot dx =$$

$$= F(Q)\left\{\frac{1}{F(Q)}\int_{-\infty}^{Q} x \cdot f(x)\cdot dx\right\} + \{1 - F(Q)\}\left\{\frac{1}{1 - F(Q)}\int_{-\infty}^{Q} x \cdot f(x)\cdot dx\right\} =$$

$$= F(Q)\cdot E(X|X \le Q) + \{1 - F(Q)\}\cdot E(X|X \le Q). \qquad \text{(A2)}$$

Solving (A2) with respect to $E(X|X \le Q)$, and replacing to (A1),

$$E(\xi) = (p - c + s)\cdot Q - (p - v + s)\cdot Q \cdot F(Q) + (p - v + s)\cdot E(X|X \le Q)\cdot F(Q) - s \cdot E(X)). \qquad \text{(A3)}$$

Setting $Q = Q^*$, $F(Q^*) = R$, and $p - v + s = (p - c + s)/R$, the first two terms of the right hand side of (A3) vanish. Rearranging the remaining two terms of (A3) completes the proof of proposition 1.

***Proof of proposition 2***

Consider the integral,

$$\int_0^{Q*} x\cdot f(x)\cdot dx = \int_0^{Q*} \frac{x^2}{\sigma^2}\cdot e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx = \frac{\sqrt{2\pi}}{\sigma}\int_0^{Q*} x^2 \frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx \qquad \text{(A4)}$$

The $m_2$ truncated moment expression for the incomplete Normal Distribution is given in table 1 of Jawitz (2004) as

$$m_2 = \frac{\sigma}{\sqrt{2\pi}}\left[(\ell+\mu)e^{-\frac{1}{2}\left(\frac{\ell-\mu}{\sigma}\right)^2} - (u+\mu)e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2}\right] + \frac{\mu^2+\sigma^2}{2}\left[\operatorname{erf}\left(\frac{u-\mu}{\sigma\sqrt{2}}\right) - \operatorname{erf}\left(\frac{\ell-\mu}{\sigma\sqrt{2}}\right)\right] \qquad \text{(A5)}$$

Setting in (A5) $\ell = 0$, $u = Q^*$, and $\mu = 0$, and since $\operatorname{erf}\left(\frac{Q^*/\sigma}{\sqrt{2}}\right) = 2\Phi_{Q^*/\sigma} - 1$, and $\operatorname{erf}\left(\frac{0}{\sqrt{2}}\right) = 0$,

$$\int_0^{Q^*} x^2 \frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx = \frac{\sigma}{\sqrt{2\pi}}\left[-Q^*\cdot e^{-\frac{1}{2}\left(\frac{Q^*}{\sigma}\right)^2}\right] + \frac{\sigma^2}{2}\left[\operatorname{erf}\left(\frac{Q^*/\sigma}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{0}{\sqrt{2}}\right)\right] =$$

$$= -\frac{\sigma}{\sqrt{2\pi}} Q^*\cdot e^{-\frac{1}{2}\left(\frac{Q^*}{\sigma}\right)^2} + \sigma^2\Phi_{Q^*/\sigma} - \frac{\sigma^2}{2}. \qquad \text{(A6)}$$

As $Q^* = \sigma\sqrt{-2\ln(1-R)}$, and $e^{-\frac{1}{2}\left(\frac{Q^*}{\sigma}\right)^2} = 1-R$, replacing (A6) into (A4),.

$$E\left(X \middle| X \le Q^*\right) = \frac{1}{F(Q^*)}\int_0^{Q^*} \frac{x^2}{\sigma^2}\cdot e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx = \frac{1}{R}\left[-Q^*\cdot e^{-\frac{1}{2}\left(\frac{Q^*}{\sigma}\right)^2} + \sigma\sqrt{2\pi}\cdot\Phi_{Q^*/\sigma} - \frac{\sigma\sqrt{2\pi}}{2}\right] =$$

$$= \frac{1}{R}\left[-\sigma(1-R)\sqrt{-2\ln(1-R)} + \sigma\sqrt{2\pi}\cdot\Phi_{\sqrt{-2\ln(1-R)}} - \frac{\sigma\sqrt{2\pi}}{2}\right],$$

which completes the proof of proposition 2.

***Proof of proposition 3***

From the Central Limit Theorem $\sqrt{n}\left(\hat{\lambda}-\lambda\right) \xrightarrow{D} N\left(0\,,\kappa^2\right)$. Also, from the Weak Law of Large Numbers, $\text{p}\lim\hat{\lambda}=\lambda$, and

$$\text{p}\lim\hat{Q}^*_{EX}=\left|\ln\left(1-R\right)\right|\cdot\text{p}\lim\hat{\lambda}=Q^*_{EX},$$

$$\text{p}\lim\hat{E}\left(\xi\right)^*_{ex}=\left\{\left(p-c\right)+\left(c-v\right)\cdot\ln\left(1-R\right)\right\}\text{p}\lim\hat{\lambda}=E\left(\xi\right)^*_{EX}.$$

Setting, therefore,

$$\hat{Q}^*_{EX}=h_1\left(\hat{\lambda}\right)=\hat{\lambda}\left|\ln\left(1-R\right)\right|, \text{ and}$$

$$\hat{E}\left(\xi\right)^*_{ex}=h_2\left(\hat{\lambda}\right)=\hat{\lambda}\left\{\left(p-c\right)+\left(c-v\right)\cdot\ln\left(1-R\right)\right\},$$

the application of the univariate delta method (knight, 1999) gives

$$\sqrt{n}\left[h_1\left(\hat{\lambda}\right)-h_1\left(\lambda\right)\right]=\sqrt{n}\left[\hat{Q}^*_{EX}-Q^*_{EX}\right]\xrightarrow{D}N\left(0\,,\kappa^2\left\{\left.\frac{dh_1}{d\hat{\lambda}}\right|_{\hat{\lambda}=\lambda}\right\}^2\right),$$

and

$$\sqrt{n}\left[h_2\left(\hat{\lambda}\right)-h_2\left(\lambda\right)\right]=\sqrt{n}\left[\hat{E}\left(\xi\right)^*_{ex}-E\left(\xi\right)^*_{ex}\right]\xrightarrow{D}N\left(0\,,\kappa^2\left\{\left.\frac{dh_2}{d\hat{\lambda}}\right|_{\hat{\lambda}=\lambda}\right\}^2\right),$$

which complete the proof as $\left.\frac{dh_1}{d\hat{\lambda}}\right|_{\hat{\lambda}=\lambda}=\left|\ln\left(1-R\right)\right|$ and $\left.\frac{dh_2}{d\hat{\lambda}}\right|_{\hat{\lambda}=\lambda}=\left(p-c\right)+\left(c-v\right)\cdot\ln\left(1-R\right)$.

***Proof of proposition 4***

For a random variable Y which follows the chi-squared distribution with v degrees of freedom, it is also true that $Y \sim \text{Gamma}(v/2,2)$. Cohen and Whitten (1988) and Balakrishnan and Cohen (1991) stated that $Y = 2n\hat{\sigma}^2/\sigma^2 \sim \chi^2_{2n}$ or $Y = 2n\hat{\sigma}^2/\sigma^2 \sim \text{Gamma}(n,2)$. Using also the scaling property of the Gamma distribution we take, $g_R^{-2} \cdot Y \sim \text{Gamma}(n, 2g_R^{-2})$ and $(2|\ln(1-R)|)^{-1} Y \sim \text{Gamma}(n, (|\ln(1-R)|)^{-1})$, or

$$\frac{2n\hat{\sigma}^2}{\left[\sigma\sqrt{2|\ln(1-R)|}\right]^2} = \frac{2n\hat{\sigma}^2}{\left(Q^*_{RY}\right)^2} \sim \text{Gamma}\left(n, \frac{1}{|\ln(1-R)|}\right), \tag{A7}$$

$$\frac{2n\hat{\sigma}^2}{(\sigma \cdot g_R)^2} = \frac{2n\hat{\sigma}^2}{\left(E(\xi)^*_{RY}\right)^2} \sim \text{Gamma}\left(n, \frac{2}{g_R^2}\right). \tag{A8}$$

From (A7),

$$\Pr\left\{\Gamma_{\frac{\alpha}{2}}\left(n, \frac{1}{|\ln(1-R)|}\right) \le \frac{2n\hat{\sigma}^2}{\left(Q^*_{RY}\right)^2} \le \Gamma_{1-\frac{\alpha}{2}}\left(n, \frac{1}{|\ln(1-R)|}\right)\right\} =$$

$$\Pr\left\{\frac{2n\hat{\sigma}^2}{\Gamma_{1-\frac{\alpha}{2}}\left(n, \frac{1}{|\ln(1-R)|}\right)} \le \left(Q^*_{RY}\right)^2 \le \frac{2n\hat{\sigma}^2}{\Gamma_{\frac{\alpha}{2}}\left(n, \frac{1}{|\ln(1-R)|}\right)}\right\} = 1-\alpha,$$

and from (A8)

$$\Pr\left\{\Gamma_{\frac{\alpha}{2}}\left(n, \frac{2}{g_R^2}\right) \le \frac{2n\hat{\sigma}^2}{\left(E(\xi)^*_{RY}\right)^2} \le \Gamma_{1-\frac{\alpha}{2}}\left(n, \frac{2}{g_R^2}\right)\right\} =$$

$$\Pr\left\{\frac{2n\hat{\sigma}^2}{\Gamma_{1-\frac{\alpha}{2}}\left(n, \frac{2}{g_R^2}\right)} \le \left(E(\xi)^*_{RY}\right)^2 \le \frac{2n\hat{\sigma}^2}{\Gamma_{\frac{\alpha}{2}}\left(n, \frac{2}{g_R^2}\right)}\right\} = 1-\alpha.$$

Taking the square root of each term inside each probability statement completes the proof.

***Proof of proposition 5***

From the exact distributional result $2n\hat{\sigma}^2/\sigma^2 \sim \chi^2_{2n}$ we take $E(\hat{\sigma}^2) = \sigma^2$, and $Var(\hat{\sigma}^2) = \sigma^4/n \to 0$, as $n \to \infty$. Thus $p\lim \hat{\sigma}^2 = \sigma^2$, and

$$p\lim \hat{Q}^*_{RY} = \sqrt{2|\ln(1-R)|} \cdot (p\lim \hat{\sigma}^2)^{0,5} = Q^*_{RY},$$

$$p\lim \hat{E}(\xi)^*_{ry} = g_R \cdot (p\lim \hat{\sigma}^2)^{0,5} = E(\xi)^*_{ry}.$$

Further, for n sufficiently large, $\dfrac{\dfrac{2n\hat{\sigma}^2}{\sigma^2} - 2n}{\sqrt{4n}} \xrightarrow{D} N(0,1)$, and $\sqrt{n}\left[\hat{\sigma}^2 - \sigma^2\right] \xrightarrow{D} N(0, \sigma^4)$.

Setting, therefore, $\hat{Q}^*_{RY} = h_1(\hat{\sigma}^2) = (\hat{\sigma}^2)^{1/2}\sqrt{2|\ln(1-R)|}$ and $\hat{E}(\xi)^*_{ry} = h_2(\hat{\sigma}^2) = (\hat{\sigma}^2)^{1/2} \cdot g_R$, the application of the univariate delta method gives the following asymptotic distributional result

$$\sqrt{n}\left[h_1(\hat{\sigma}^2) - h_1(\sigma^2)\right] = \sqrt{n}\left[\hat{Q}^*_{RY} - Q^*_{RY}\right] \xrightarrow{D} N\left(0, \sigma^4\left\{\left.\frac{dh_1}{d\hat{\sigma}^2}\right|_{\hat{\sigma}^2=\sigma^2}\right\}^2\right),$$

and

$$\sqrt{n}\left[h_2(\hat{\sigma}^2) - h_2(\sigma^2)\right] = \sqrt{n}\left[\hat{E}(\xi)^*_{ry} - E(\xi)^*_{ry}\right] \xrightarrow{D} N\left(0, \sigma^4\left\{\left.\frac{dh_2}{d\hat{\sigma}^2}\right|_{\hat{\sigma}^2=\sigma^2}\right\}^2\right),$$

which completes the proof as $\left.\dfrac{dh_1}{d\hat{\sigma}^2}\right|_{\hat{\sigma}^2=\sigma^2} = \sigma^{-2}\sqrt{\dfrac{|\ln(1-R)|}{2}}$ and $\left.\dfrac{dh_2}{d\hat{\sigma}^2}\right|_{\hat{\sigma}^2=\sigma^2} = \left(\dfrac{g_R}{2\sigma}\right)^2$.

***Proof of proposition 6***

The actual confidence level of (10a) is

$$ACL_Q = \Pr\left\{-z_{\alpha/2} \leq \frac{\sqrt{n}\left(\hat{Q}^*_{EX} - Q^*_{EX}\right)}{\hat{\kappa}\sqrt{\left|\ln(1-R)\right|}} \leq z_{\alpha/2}\right\}$$

and of (10b)

$$ACL_\xi = \Pr\left\{-z_{\alpha/2} \leq \frac{\sqrt{n}\left(\hat{E}(\xi)^*_{ex} - E(\xi)^*_{ex}\right)}{\hat{\kappa}\left|(p-c)+(c-v)\cdot\ln(1-R)\right|} \leq z_{\alpha/2}\right\}$$

From (3) and (8a), $\hat{Q}^*_{EX} - Q^*_{EX} = \left|\ln(1-R)\right|\left(\hat{\lambda}-\lambda\right)$, and from (4) and (8b)

$$\hat{E}(\xi)^*_{ex} - E(\xi)^*_{ex} = \{(p-c)+(c-v)\cdot\ln(1-R)\}\left(\hat{\lambda}-\lambda\right).$$

Replacing each difference to the numerator of the corresponding probability statement

$$ACL_Q = ACL_\xi = \Pr\left(-z_{\alpha/2} \leq \frac{\sqrt{n}\left(\hat{\lambda}-\lambda\right)}{\hat{\kappa}} \leq z_{\alpha/2}\right).$$

The relative expected half-length of (10a) and (10b) is defined respectively as

$$REHL_Q = \frac{z_{\alpha/2}\left|\ln(1-R)\right|}{Q^*_{EX}\sqrt{n}}E(\hat{\kappa}), \text{ and,}$$

$$REHL_\xi = \frac{z_{\alpha/2}\left|(p-c)+(c-v)\cdot\ln(1-R)\right|}{E(\xi)^*_{ex}\sqrt{n}}E(\hat{\kappa}),$$

But $E(\hat{\kappa}) = \frac{n}{\sqrt{(n+2)(n+3)}}E\left(\hat{\lambda}\right) = \frac{n\cdot\lambda}{\sqrt{(n+2)(n+3)}}$, and using again (3) and (4)

$$REHL_Q = REHL_\xi = z_{\alpha/2}\sqrt{\frac{n}{(n+2)(n+3)}}.$$

***Proof of proposition 7***

Using (4), and the scaling property of the Gamma distribution, the ACL for intervals (11a) and (11b) are given below:

$$ACL_Q = \Pr\left\{\hat{\sigma}\sqrt{\frac{2n}{\Gamma_{1-\frac{\alpha}{2}}\left(n,\frac{1}{|\ln(1-R)|}\right)}} \le Q^*_{RY} \le \hat{\sigma}\sqrt{\frac{2n}{\Gamma_{\frac{\alpha}{2}}\left(n,\frac{1}{|\ln(1-R)|}\right)}}\right\} =$$

$$= \Pr\left\{\sqrt{\frac{n}{|\ln(1-R)|\Gamma_{1-\frac{\alpha}{2}}\left(n,\frac{1}{|\ln(1-R)|}\right)}} \le \frac{\sigma}{\hat{\sigma}} \le \sigma\sqrt{\frac{n}{|\ln(1-R)|\cdot\Gamma_{\frac{\alpha}{2}}\left(n,\frac{1}{|\ln(1-R)|}\right)}}\right\} =$$

$$= \Pr\left\{\sqrt{\frac{n}{\Gamma_{1-\frac{\alpha}{2}}(n,1)}} \le \frac{\sigma}{\hat{\sigma}} \le \hat{\sigma}\sqrt{\frac{n}{\Gamma_{\frac{\alpha}{2}}(n,1)}}\right\}.$$

Similarly, using (5)

$$ACL_\xi = \Pr\left\{\hat{\sigma}\sqrt{\frac{2n}{\Gamma_{1-\alpha/2}\left(n,\frac{2}{g_R^2}\right)}} \le E(\xi)^*_{RY} \le \hat{\sigma}\sqrt{\frac{2n}{\Gamma_{\alpha/2}\left(n,\frac{2}{g_R^2}\right)}}\right\} =$$

$$= \Pr\left\{\sqrt{\frac{n}{\frac{g_R^2}{2}\Gamma_{1-\alpha/2}\left(n,\frac{2}{g_R^2}\right)}} \le \frac{\sigma}{\hat{\sigma}} \le \sqrt{\frac{n}{\frac{g_R^2}{2}\Gamma_{\alpha/2}\left(n,\frac{2}{g_R^2}\right)}}\right\} =$$

$$= \Pr\left\{\sqrt{\frac{n}{\Gamma_{1-\frac{\alpha}{2}}(n,1)}} \le \frac{\sigma}{\hat{\sigma}} \le \hat{\sigma}\sqrt{\frac{n}{\Gamma_{\frac{\alpha}{2}}(n,1)}}\right\} = ACL_Q$$

The expressions for the REHL of (11a) and (11b) are respectively

$$\mathrm{REHL}_Q = \frac{1}{2}\left\{\sqrt{\frac{2n}{\Gamma_{\alpha/2}\left(n, \frac{1}{|\ln(1-R)|}\right)}} - \sqrt{\frac{2n}{\Gamma_{1-\frac{\alpha}{2}}\left(n, \frac{1}{|\ln(1-R)|}\right)}}\right\}\frac{E(\hat{\sigma})}{Q^*_{RY}} =,$$

$$= \frac{1}{2}\left\{\sqrt{\frac{n}{\Gamma_{\alpha/2}(n,1)}} - \sqrt{\frac{n}{\Gamma_{1-\frac{\alpha}{2}}(n,1)}}\right\}\frac{E(\hat{\sigma})}{\sigma}, \qquad \text{(A9)}$$

and

$$\mathrm{REHL}_\xi = \frac{1}{2}\left\{\sqrt{\frac{2n}{\Gamma_{\alpha/2}\left(n, \frac{2}{g_R^2}\right)}} - \sqrt{\frac{2n}{\Gamma_{1-\alpha/2}\left(n, \frac{2}{g_R^2}\right)}}\right\}\frac{E(\hat{\sigma})}{E(\xi)^*_{RY}} =$$

$$= \frac{1}{2}\left\{\sqrt{\frac{n}{\Gamma_{\alpha/2}(n,1)}} - \sqrt{\frac{n}{\Gamma_{1-\alpha/2}(n,1)}}\right\}\frac{E(\hat{\sigma})}{\sigma}. \qquad \text{(A10)}$$

When $Y \sim \chi^2_v$, then $\sqrt{Y}$ follows the chi distribution with $E(\sqrt{Y}) = \sqrt{2}\,\Gamma((v+1)/2)/\Gamma(v/2)$.

From proposition (4), $Y = 2n\hat{\sigma}^2/\sigma^2 \sim \chi^2_{2n}$, and $E\left(\frac{\hat{\sigma}\sqrt{2n}}{\sigma}\right) = \sqrt{2}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)}$ or

$$E(\hat{\sigma}) = \frac{\sigma}{\sqrt{n}}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)}. \qquad \text{(A11)}$$

Replacing (A11) into (A9) and (A10)

$$\mathrm{REHL}_Q = \mathrm{REHL}_\xi = \frac{1}{2}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)}\left\{\sqrt{\frac{1}{\Gamma_{\alpha/2}(n,1)}} - \sqrt{\frac{1}{\Gamma_{1-\alpha/2}(n,1)}}\right\}.$$

Regarding the asymptotic confidence intervals (12a) and (12b), the corresponding ACL's are

$$ACL_Q = \Pr\left\{-z_{\alpha/2} \le \frac{\sqrt{2n}\left(\hat{Q}^*_{RY} - Q^*_{RY}\right)}{\hat{\sigma}\sqrt{\left|\ln(1-R)\right|}} \le z_{\alpha/2}\right\},$$

and

$$ACL_\xi = \Pr\left\{-z_{\alpha/2} \le \frac{2\sqrt{n}\left(\hat{E}(\xi)^*_{RY} - E(\xi)^*_{RY}\right)}{\hat{\sigma}\left|g_R\right|} \le z_{\alpha/2}\right\}.$$

Replacing the differences $\hat{Q}^*_{RY} - Q^*_{RY} = \sqrt{2\left|\ln(1-R)\right|}(\hat{\sigma}-\sigma)$ and

$\left(\hat{E}(\xi)^*_{RY} - E(\xi)^*_{RY}\right) = g_R(\hat{\sigma}-\sigma)$ ,

$$ACL_Q = ACL_\xi = \Pr\left(-\frac{z_{\alpha/2}}{2} \le \frac{\sqrt{n}(\hat{\sigma}-\sigma)}{\hat{\sigma}} \le \frac{z_{\alpha/2}}{2}\right).$$

Finally, the REHL of (12a) and (12b) are

$$REHL_Q = \frac{z_{\alpha/2}\sqrt{\left|\ln(1-R)\right|}}{Q^*_{RY}\sqrt{2n}} E(\hat{\sigma}) = \frac{z_{\alpha/2}}{2\sqrt{n}}\frac{E(\hat{\sigma})}{\sigma},$$

$$REHL_\xi = \frac{z_{\alpha/2}\left|g_R\right|}{2E(\xi)^*_{RY}\sqrt{n}} E(\hat{\sigma}) = \frac{z_{\alpha/2}}{2\sqrt{n}}\frac{E(\hat{\sigma})}{\sigma},$$

and using (A11)

$$REHL_Q = REHL_\xi = \frac{z_{\alpha/2}}{2}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)}.$$

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