Supplement to "Martingale properties of self-enforcing debt"

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Abstract

We present some complementary results to Bidian and Bejan (2012). Part 1 provides necessary and sufficient transversality conditions for an agent's optimization problem. They are extensions to stochastic environments of the conditions given by Kocherlakota (1992), or alternatively, extensions to nonzero debt constraints of the corresponding conditions in Forno and Montrucchio (2003). Part 2 presents an elementary proof of the characterization of NTT debt limits (Theorem 3.5 in the main paper) for the case when debt constraints bind in bounded time, that requires no martingale techniques or boundedness assumptions on the discounted debt limits. Part 3 complements results in Section 5.1 (in the main paper), showing that all the equilibria that can sustain bubbles under an interdiction to trade can be achieved from fixed, zero initial wealth for the agents. Thus endogeneity of debt limits causes multiplicity of not only asset prices (through bubbles), but also of real equilibrium allocations.

1 Transversality conditions

We analyze the problem $P_t(\hat{a}_t, \phi, p)$ of a consumer that faces debt bounds ϕ, pricing kernel p and starts with wealth $\hat{a}_t$ ($\mathcal{F}_t$-measurable) at period t (see Section 3 in the

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main text). Let $(\bar{c},\bar{a}) \in C_t(\hat{a}_t,\phi,p)$ be the optimal consumption (assumed positive) and asset holdings for the agent. Familiar variational arguments show that $(\bar{c},\bar{a})$ satisfies the following Kuhn-Tucker necessary conditions, for all $s \geq t$:

$$u'_s(\bar{c}_s) - u'_{s+1}(\bar{c}_{s+1})\frac{p_s}{p_{s+1}} \geq 0, \tag{1.1}$$

$$\left(u'_s(\bar{c}_s) - u'_{s+1}(\bar{c}_{s+1})\frac{p_s}{p_{s+1}}\right)(\bar{a}_{s+1} - \phi_{s+1}) = 0. \tag{1.2}$$

Let $\bar{e}_s := e_s + \phi_s - E_s\frac{p_{s+1}}{p_s}\phi_{s+1}$, for all $s \geq t$. Adapting the arguments of Forno and Montrucchio (2003), we obtain the following necessary transversality condition:[1]

Lemma 1.1 (Necessary transversality condition)**.** *The optimal path* $(\bar{c},\bar{a})$ *satisfies*

$$\lim_{t\to\infty} E_t u'_s(\bar{c}_s)(\bar{a}_s - \phi_s) = 0. \tag{1.3}$$

Proof. Fix an $\bar{\varepsilon} > 0$ a period $s > t$. Concavity implies that for any $0 < \varepsilon < \bar{\varepsilon}$ and $n \geq t$,

$$u_n(\bar{c}_n) - u_n(\bar{c}_n + \varepsilon(\bar{e}_n - \bar{c}_n)) \leq \frac{\varepsilon}{\bar{\varepsilon}}\left(u_n(\bar{c}_n) - u_n(\bar{c}_n + \bar{\varepsilon}(\bar{e}_n - \bar{c}_n))\right).$$

We construct the alternative asset holdings process $(a_n(\varepsilon))_{n=t}^{\infty}$ where $a_n(\varepsilon) = \bar{a}_n$ if $t \leq n \leq s$, and $a_n(\varepsilon) = (1-\varepsilon)\bar{a}_n + \varepsilon\phi_n$ if $n \geq s+1$. It sustains the feasible consumption process $(c_n(\varepsilon))_{n=s}^{\infty}$ defined by $c_n(\varepsilon) = \bar{c}_n$ if $t \leq n < s$, $c_s(\varepsilon) = \bar{c}_s + E_s\frac{p_{s+1}}{p_s}(\bar{a}_{s+1} - \phi_{s+1})$, and $c_n(\varepsilon) = \bar{c}_n + \varepsilon(\bar{e}_n - \bar{c}_n)$ for $n > s$. Optimality of $\bar{c}$ implies that

$$0 \leq E_t\left(u_s(\bar{c}_s) - u_s(c_s(\varepsilon))\right) + \limsup_{T\to\infty} E_t \sum_{n=s+1}^{T}\left(u_n(\bar{c}_n) - u_n(c_n(\varepsilon))\right). \tag{1.4}$$

Notice that

$$\sum_{n=s+1}^{T}\frac{1}{\varepsilon}\left(u_n(\bar{c}_n) - u_n(c_n(\varepsilon))\right) \leq \sum_{n=s+1}^{\infty}\frac{1}{\bar{\varepsilon}}\left(u_n(\bar{c}_n) - u_n(c_n(\bar{\varepsilon}))\right)^+,$$

[1] The proof works for general period utilities $u_t(\cdot)$, not necessarily of the discounted and bounded variety assumed in the text, if one uses a weak optimality criterion (Forno and Montrucchio 2003) and if there exists $\bar{\varepsilon} > 0$ such that $E\sum_{s=t}^{\infty}\left(u_s(\bar{c}_s) - u_s(\bar{c}_s + \bar{\varepsilon}(\bar{e}_s - \bar{c}_s))\right)^+ < \infty$.

and the term $\sum_{n=s+1}^{\infty} \frac{1}{\bar{\varepsilon}} \left(u_n(\bar{c}_n) - u_n(c_n(\bar{\varepsilon}))\right)^+$ is integrable, by hypothesis. Fatou's lemma gives

$$\begin{aligned}\limsup_{T\to\infty} E_t \sum_{n=s+1}^{T} \frac{1}{\varepsilon}\left(u_n(\bar{c}_n) - u_n(c_n(\varepsilon))\right) &\le E_t \limsup_{T\to\infty} \sum_{n=s+1}^{T} \frac{1}{\varepsilon}\left(u_n(\bar{c}_n) - u_n(c_n(\varepsilon))\right) \\ &\le E_t \sum_{n=s+1}^{\infty} \frac{1}{\bar{\varepsilon}}\left(u_n(\bar{c}_n) - u_n(c_n(\bar{\varepsilon}))\right)^+ . \qquad (1.5)\end{aligned}$$

Dividing both sides of (1.4) by ε and using (1.5),

$$-E_t \frac{1}{\varepsilon}\left(u_s(\bar{c}_s) - u_s(c_s(\varepsilon))\right) \le E_t \sum_{n=s+1}^{\infty} \frac{1}{\bar{\varepsilon}}\left(u_n(\bar{c}_n) - u_n(c_n(\bar{\varepsilon}))\right)^+ < \infty.$$

By the monotone convergence theorem, when $\varepsilon \searrow 0$, the left hand side of the above equation converges to $E_t u_s'(\bar{c}_s)\frac{p_{s+1}}{p_s}(\bar{a}_{s+1} - \phi_{s+1})$, which equals $E_t u_{s+1}'(\bar{c}_{s+1})(\bar{a}_{s+1} - \phi_{s+1})$, due to the Kuhn-Tucker equations (1.1),(1.2). The conclusion follows by letting $s \to \infty$. $\square$

We include for completeness the standard proof of sufficiency of the Kuhn-Tucker and transversality conditions for the optimality of a path.

Lemma 1.2 (Sufficient transversality condition)**.** *If a feasible path* $(\bar{c}, \bar{a}) \in B_t(\hat{a}_t, \phi, p)$ *satisfies the Kuhn-Tucker conditions* (1.1) *and* (1.2)*, then for any other feasible path* $(c, a) \in B_t(\hat{a}_t, \phi, p)$ *and any bounded stopping time* $T \ge t$*,*

$$E_t \sum_{s=t}^{T} (u_s(c_s) - u_s(\bar{c}_s)) \le E_t u_{T+1}'(\bar{c}_{T+1})(\bar{a}_{T+1} - \phi_{T+1}). \qquad (1.6)$$

Thus a sufficient condition for $(\bar{c}, \bar{a})$ *to be optimal for problem* $P_t(\hat{a}_t, \phi, p)$ *is*

$$\liminf_{s\to\infty} E_t u_s'(\bar{c}_s)(\bar{a}_s - \phi_s) = 0, \qquad (1.7)$$

Proof. Let $\mu_{s+1} := u_s'(\bar{c}_s) - u_{s+1}'(\bar{c}_{s+1})\frac{p_s}{p_{s+1}}$. Consider an arbitrary feasible path

$(c, a) \in B_t(\hat{a}_t, \phi, p)$. Using concavity of $u_s(\cdot)$ and the budget constraints,

$$E_t \sum_{s=t}^{T} (u_s(c_s) - u_s(\bar{c}_s)) \leq E_t \sum_{s=t}^{T} u_s'(\bar{c}_s)(c_s - \bar{c}_s) =$$
$$= E_t \sum_{s=t}^{T} u_s'(\bar{c}_s) \left(a_s - \phi_s - E_s \frac{p_{s+1}}{p_s}(a_{s+1} - \phi_{s+1}) \right) -$$
$$-E_t \sum_{s=t}^{T} u_s'(\bar{c}_s) \left(\bar{a}_s - \phi_s - E_s \frac{p_{s+1}}{p_s}(\bar{a}_{s+1} - \phi_{s+1}) \right).$$

We analyze separately the last two terms. Using the Kuhn-Tucker conditions (1.1) and (1.2), which show that $\mu_{s+1} \geq 0$ for all $s \geq t$, it follows that

$$E_t \sum_{s=t}^{T} u_s'(\bar{c}_s) \left((a_s - \phi_s) - E_s \frac{p_{s+1}}{p_s}(a_{s+1} - \phi_{s+1}) \right)$$
$$= E_t \sum_{s=t}^{T} \left(u_s'(\bar{c}_s)(a_s - \phi_s) - \left(u_{s+1}'(\bar{c}_{s+1}) + \frac{p_{s+1}}{p_s}\mu_{s+1} \right) (a_{s+1} - \phi_{s+1}) \right)$$
$$\leq u_t'(\bar{c}_t)(a_t - \phi_t) - E_t u_{T+1}'(\bar{c}_{T+1})(a_{T+1} - \phi_{T+1}) \leq E_t u_t'(\bar{c}_t)(a_t - \phi_t).$$

Similarly, using both (1.1) and (1.2),

$$E_t \sum_{s=t}^{T} u_s'(\bar{c}_s) \left((a_s - \phi_s) - E_s \frac{p_{s+1}}{p_s}(a_{s+1} - \phi_{s+1}) \right)$$
$$= u_t'(\bar{c}_t)(\bar{a}_t - \phi_t) - E_t u_{T+1}'(\bar{c}_{T+1})(\bar{a}_{T+1} - \phi_{T+1}).$$

Moreover $a_t = \bar{a}_t$ since they equal the initial period t wealth of the consumer, $\hat{a}_t$. Thus

$$\liminf_{T\to\infty} E_t \sum_{s=t}^{T} (u_s(c_s) - u_s(\bar{c}_s)) \leq \liminf_{T\to\infty} E_t u_{T+1}'(\bar{c}_{T+1})(\bar{a}_{T+1} - \phi_{T+1}) = 0,$$

and therefore $(\bar{c}, \bar{a})$ is optimal for $P_t(\hat{a}_t, \phi_t, p)$. $\square$

2 The case when debt limits bind in bounded time

We give an elementary proof of Theorem 3.5 (in the paper) that does not use results from the theory of martingales, for the case when $\alpha(t)$ is bounded, for any $t \in \mathbb{N}$. Assumption 3.1 and Proposition 3.4 are not needed in this case. As in the paper, continuation utilities after default are the same under the two debt limits $\phi, \bar{\phi}$. We assume that for any period t, there exists a natural number $n(t)$ such that the bounds ϕ bind before period $n(t)$ along the optimal path for the problem $P_t(\phi_t, \phi, p)$.

The process $(\hat{M}_s)_{s=t}^{\alpha(t)}$ from STEP 1 can be simply chosen to be $\hat{M}_s = E_s M_{\alpha(t)}$ (for any stopping time s such that $t \leq s \leq \alpha(t)$), and shown that $\hat{M} \leq M$ by backward induction. This proof is given in Proposition 2.1 below. Thus (3.12) (in the main text) follows therefore directly from the construction of $\hat{M}$. In STEP 2, $(\hat{M}_s)_{s=t}^{\infty}$ is obtained as before by letting $\hat{M}_s := E_s M_{\alpha^{k+1}(t)}$ for each $k \geq 1$ natural and each finite stopping time s such that $\alpha^k(t) + 1 \leq s \leq \alpha^{k+1}(t)$. The optimal solution $(\bar{c}, \bar{a})$ to the problem $P_t(\phi_t, \phi, p)$ is also an optimal solution for the "relaxed" problem $P_t(\phi_t, \hat{\phi}, p)$ (with $\hat{\phi} = \bar{\phi} + \hat{M}/p \leq \phi$), since for any feasible $(c, a) \in B_t(\phi_t, \hat{\phi}, p)$,

$$U_t(c) - U_t(\bar{c}) = \liminf_{k\to\infty} E_t \sum_{s=t}^{\alpha^k(t)-1} (u_s(c_s) - u_s(\bar{c}_s)) \leq \liminf_{k\to\infty} E_t u'_{\alpha^k(t)}(\bar{c}_{\alpha^k(t)})(\bar{a}_{\alpha^k(t)} - \hat{\phi}_{\alpha^k(t)}) = 0.$$

The inequality above follows from $\bar{a}_{\alpha^k(t)} - \hat{\phi}_{\alpha^k(t)} = \bar{a}_{\alpha^k(t)} - \phi_{\alpha^k(t)} = 0$. An identical argument shows that $M_t = \hat{M}_t$, hence $M_t = E_t M_{\alpha(t)}$. STEP 3 is unchanged.

Proposition 2.1. *Assume that ϕ are NTT. Let $T \in \mathbb{N}$ and $\omega \in \Omega$ such that there exists $n \in \mathbb{N}$ with the property that $\alpha(T) \leq T + n$ on $\mathcal{F}_T(\omega)$. Then*

$$M_T \geq E_T M_{\alpha(T)} \text{ on } \mathcal{F}_T(\omega). \tag{2.1}$$

Proof. We prove the Proposition by induction on n.

We show, first, that the claim in the Proposition is true for $n = 1$. When not explicit, all equalities and inequalities that follow are understood to hold on $\mathcal{F}(\omega)$. Since $T + 1 \leq \alpha(T) \leq T + n = T + 1$, it follows that $\alpha(T) = T + 1$. Assume by contradiction that $M_T < E_T M_{T+1}$. Let $(c, a) \in C_T(\phi_T, \phi, p)$. As ϕ are binding at

$T+1$,

$$c_T = e_T + \phi_T - E_T \frac{p_{T+1}}{p_T}\phi_{T+1} < e_T + \bar{\phi}_T - E_T \frac{p_{T+1}}{p_T}\bar{\phi}_{T+1} =: \bar{c}_T.$$

Let $\bar{a}_T := \bar{\phi}_T$ and $t > T$, choose $\bar{c}_t, \bar{a}_t$ such that $(\bar{c}, \bar{a}) \in B_{T+1}(\bar{\phi}_{T+1}, \bar{\phi}, p)$. It is immediate to check that $(\bar{c}, \bar{a}) \in B_T(\bar{\phi}_T, \bar{\phi}, p)$. We reached a contradiction, as the continuation utility after T of path $\bar{c}$ is less or equal to V_T^d and strictly dominates the continuation utility after T over c, which equals V_T^d:

$$V_T^d(p) = V_T(\phi_T, \phi, p) = U_T(c) = u_T(c_T) + E_T V_{T+1}^d < u_T(\bar{c}_T) + E_T V_{T+1}^d = U_T(\bar{c}) \leq V_T^d.$$

Suppose now that the claim in the proposition is true for arbitrary T and ω such that $\alpha(T) \leq T + n$ on $\mathcal{F}_T(\omega)$ for some $n = 1, \ldots, k$. Choose T and ω such that $\alpha(T) \leq T + k + 1$ on $\mathcal{F}_T(\omega)$. We need to show that $M_T \geq E_T M_{\alpha(T)}$ on $\mathcal{F}_T(\omega)$. Assume, by contradiction, that $M_T < E_T M_{\alpha(T)}$.

For any t satisfying $T+1 \leq t \leq \alpha(T)$, using the induction hypothesis and applying the law of iterated expectations a finite number of times, we get $M_t \geq E_t M_{\alpha(T)}$. Let $(c, a) \in C_T(\phi_T, \phi, p)$. Let $\bar{a}_t := \bar{\phi}_T$ and for t satisfying $T + 1 \leq t < \alpha(T)$, construct

$$\bar{a}_t := a_t - \frac{1}{p_t} E_t M_{\alpha(T)} \geq \phi_t - \frac{1}{p_t} E_t M_{\alpha(T)} = \bar{\phi}_t + \frac{1}{p_t}(M_t - E_t M_{\alpha(T)}) \geq \bar{\phi}_t.$$

For $t \geq \alpha(T)$, let $(\bar{c}, \bar{a}) \in B_{\alpha(T)}(\bar{\phi}_{\alpha(T)}, \bar{\phi}, p)$. For $t \in [T, \alpha(T))$, let

$$\bar{c}_t := e_t + \bar{a}_t - \frac{1}{p_t} E_t p_{t+1} \bar{a}_{t+1}.$$

Notice that $\bar{c}_t = c_t$ for $T < t < \alpha(T)$, and

$$\bar{c}_T = e_T + \bar{\phi}_T - \frac{1}{p_T} E_T p_{T+1} \bar{a}_{T+1} = c_T - \frac{1}{p_T}(M_T - E_T M_{\alpha(T)}) > c_T.$$

Moreover, $(\bar{c}, \bar{a}) \in B_T(\bar{\phi}_T, \bar{\phi}, p)$. It follows that the path $\bar{c}$ dominates c in terms of utility after T, and a contradiction is obtained in exactly the same manner as for the case $n = 1$ treated above. □

3 One period transition to a cyclical equilibrium

All cyclical AJ-equilibrium allocations described in Proposition 5.3 in the main text (where agents cannot borrow after default) can be achieved with zero initial wealth by the agents. In contrast, each non-autarchic cyclical equilibrium described in Proposition 5.1 in the paper (where agents cannot trade after default) requires specific non-zero initial wealth for the agents. However, we show here that all such cyclical equilibrium paths can be reached after a one period transition, when all agents start with zero wealth. With zero initial wealth, there exists an equilibrium in which the transfers from the high-type to low-type agents are constant after the first period and an infinite number of equilibria converging to autarchy. Therefore the endogeneity of debt limits causes multiplicity of not only asset prices (through bubbles), but also of real equilibrium allocations, for both types of punishment for default.

For the rest of this section we assume that the penalty for default is the interdiction to trade. We add an extra period and assume that time starts at -1 and that, in agreement with our convention, the even agent has low endowment at -1 while the odd agent has high endowment at -1. We investigate the equilibria where the high-type agent at period -1 (the odd agent) is a saver, transferring an amount x_{-1} to the the low-type at period -1, and the transfers (x_t) from high-type to low type agents for periods greater or equal to zero (and consumption, asset holdings, debt limits and bond prices) are described in Proposition 5.1.

Proposition 3.1. *Let* $\Big((p_t), (c_t^i), (a_t^i), (\phi_t^i), (V_t^{i,d})\Big)_{t\geq 0}$ *be a cyclical non-autarchic AJ-equilibrium associated to a sequence of transfers* $(x_t)_{t\geq 0}$*, as in Proposition 5.1 (in the main text). Let* $\eta := \min\{\bar{x}_{-1}, y^H - y^L - x_0\}$*, where* $\bar{x}_{-1}$ *is chosen such that* $f(\bar{x}_{-1}, x_0) = 0$ *(see (5.8) in the paper). Let* $x_{-1} \in [0, \eta]$*. Then* $(x_t)_{t\geq -1}$ *are transfers from high-type to low-type agents in an AJ-equilibrium* $\Big((p_t), (c_t^i), (a_t^i), (\phi_t^i), (V_t^{i,d})\Big)_{t\geq -1}$ *with initial wealth levels* a_{-1}^i *for the agents, where*[2]

$$p_{-1} := \frac{u'(y^H - x_{-1})}{\beta u'(y^L + x_0)},\ a_{-1}^o = a_0^o/p_{-1} - x_{-1},\ a_{-1}^e = a_{-1}^e := -a_{-1}^o.$$

[2]The outside option at period -1 is autarchy, that is $V_{-1}^{i,d} := \sum_{t\geq -1} \beta^t u(e_t^i)$, and ϕ_{-1}^i is chosen to be NTT, that is $V_{-1}^i(\phi_{-1}^i, (\phi_t^i)_{t\geq 0}, (p_t)_{t\geq -1}) = V_{-1}^{i,d}$.

The initial wealth a^e_{-1} of the even agent is strictly increasing in x_{-1}, and $a^e_{-1} < 0$ for $x_{-1} = 0$ and $a^e_{-1} > 0$ for $x_{-1} = \eta$.

Proof. The first order condition of the high-type (odd) agent at -1 are satisfied, by the construction of p_{-1}. By (5.5) (in the paper), the first order condition for the low-type (even) agent at -1 is satisfied since $x_{-1}+x_0 \leq y^H - y^L - x_0 + x_0 \leq y^H - y^L$. The participation constraint at -1 of the even agent is satisfied, since $x_{-1} \geq 0$, therefore he receives a positive transfer at -1 and at 0 his continuation utility is equal to that provided by autarchy. The participation constraint at -1 if the odd agent is also satisfied, since $f(x_{-1}, x_0) \leq f(\bar{x}_{-1}, x_0) = 0$. Agents' budget constraints at -1 hold by construction, and their transversality conditions hold, being the same as in the equilibrium starting at 0.

Finally, $a^e_{-1} = x_{-1} - a^o_0/p_{-1}$ and is therefore strictly increasing in x_{-1}, since p_{-1} is strictly decreasing in x_{-1} and $a^o_0 > 0$. Indeed, by (5.12) (in the paper), the sequence $(p_t x_t)_{t\geq 0}$ is strictly decreasing, and therefore

$$a^o_0 \geq L_1 := \lim_{t\to\infty} \sum_{s=0}^{2t-1} (-1)^s p_s x_s > p_0 x_0 - p_1 x_1 > 0.$$

If $x_{-1} = 0$, then clearly

$$a^e_{-1} = -a^o_0 \frac{\beta u'(y^L + x_0)}{u'(y^H)} < 0.$$

Assume now that $x_{-1} = \eta$. If $\bar{x}_{-1} \leq y^H - y^L - x_0$, then $x_{-1} = \bar{x}_{-1}$ and $f(x_{-1}, x_0) = 0$. It follows that

$$u'(y^H - x_{-1})x_{-1} > u(y^H) - u(y^H - x_{-1}) = \beta\left(u(y^L + x_0) - u(y^L)\right) > \beta u'(y^L + x_0)x_0,$$

and thus $p_{-1} > x_0/x_{-1}$. Therefore

$$a^e_{-1} = \frac{p_{-1}x_{-1} - a^o_0}{p_{-1}} > \frac{p_{-1}x_{-1} - x_0}{p_{-1}} > 0,$$

since

$$a^o_0 \leq L_2 := \lim_{t\to\infty} \sum_{s=0}^{2t} (-1)^s p_s x_s < p_0 x_0 = x_0.$$

If $\bar{x}_{-1} > y^H - y^L - x_0$, then $x_{-1} = y^H - y^L - x_0$. Hence $p_{-1} = 1/\beta$, and

$$a^e_{-1} = y^H - y^L - x_0 - \beta a^o_0 > y^H - y^L - x_0 - \beta x_0 > y^H - y^L - (1+\beta)\frac{y^H - y^L}{2} > 0,$$

where we used the inequality $x_0 \leq (y^H - y^L)/2$. $\square$

Proposition 3.1 implies that all the cyclical equilibrium paths described in Proposition 5.1 in the main text (including the autarchic one) can be achieved after a one period transition if agents start with zero wealth.

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