The long run Dynamics of heterogeneous Product and Process Innovations for a Multi Product Monopolist

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Abstract

This paper introduces the dynamical framework which combines product and process innovations. The model contributes to the theoretical literature on innovations in two ways. First, it permits for the simultaneous dynamics of both types of innovations which is rarely considered in the literature. Second, the products being generated by the innovations are heterogeneous in their investment characteristics. This allows for the formation of the dynamic interdependency between both types of innovations. As a result the steady state levels of process innovations for each product are different and influence the dynamics of product innovations in turn.
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1 Introduction

The main goal of this paper is to introduce the new approach to modelling product and process innovations at the level of a single (possibly large) firm, which is a monopolist and produces multiple versions of the basic product in

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a given industry.

In this paper the multi-product monopolist firm is investing in product and process innovations simultaneously. The distinguishing feature of the model is the infinite number of possible new products as well as process innovations associated with these new products. The dynamics of the model does not contain any sort of resource constraints. However using the assumption of depreciation of products' qualities over time it is possible to define a natural ordering in the space of products according to the profitability of investments into all these products. This simplifies the problem and allows for explicit derivation of optimal investment policies as well as the evolution of the products space itself and process innovations for all new products. The paper uses a specific form of interdependence of product and process innovations. Namely, process innovations for every separate product depend from the introduction of new products only through the time of invention of new products and are independent from product innovations otherwise. Such form of dependence appears to be very simple. It is shown that in such a framework process innovations for separate products are waves generated by the introduction of new products. At the same time process of improvement of qualities of all the existing products simultaneously is a function of product innovations in the form of distribution of waves of technology whereas process innovations for each product are separate components of this process. In particular, the monopolist finds it optimal to develop qualities of all the products to different levels, while these levels are defined from the position of any given product in the ordered potential products space. Product innovations in turn depend on the profitability of process innovations into the next product to be invented and not on characteristics of all the products in the products space. This captures the idea of current value approach to investments value and established the dynamic and variable link between process and product innovations as every next product has different investment value then the preceding one and thus changes the incentives for further investments.

The model does not take into consideration the demand side of the industry and prices are not modelled as well as profit function. This is done to focus the analysis pn the interactions between different types of innovations which are driven by internal factors of the innovating agent. It is claimed that even without considering the market mechanisms the model captures key interdependencies in investment behaviour of such an agent.

Main findings of the paper may be summarized as following.

- There is the maximal level of quality, or, alternatively, production technology, up to which the monopolist refines the technology for each of the products.

- This level is different for all the products and depends on the efficiency of investments into the production of this product.
- The more products are introduced into the market, the slower is the pace of introduction of new products, as this becomes less attractive for the investor.
- Each next new product is developed to a lesser level of quality then earlier ones. The last product has zero level of production technology associated with it.
- Assumption on the form of investment efficiency function for process innovations plays crucial role: with increasing efficiency the dynamics is substantially different from the one being considered.

The rest of the paper is organized as follows: first we consider the current state of research in the area of theoretical modelling of product and process innovations; then the basic model together with the underlying assumptions is introduced; the next section presents main results of the model which are discussed at the end of the paper.

2 State of Research

Since late 1980-s it is widely acknowledged, that process and product innovations are not independent from each other. A lot of different empirical findings for different industries and economies support this conjecture (Faria and Lima 2009), (Salo, Karjaluoto, and Jussila 2007), (Chryssochoidis 2003), (Ram, Cui, and Wu 2010), (Kraft 1990).

The main purpose of theoretical modelling is then to capture in the simplest way possible the shape and form of this interdependency. This may be done through means of static models of equilibrium allocations of resources (intellectual, human, financial) between different types of innovative activity of a firm or through means of a dynamical model. The last one is more complicated but allows for the study of not only equilibrium allocations, but of the long-term dynamics of innovative processes and the short-run dynamics at the initial stage.

In this paper the dynamical perspective is adopted. The firm is free to choose between types of innovations at any given stage in its development and the evolution of both types of innovative activity may be analysed. This will give us more profound understanding of the key factors which affect innovations at the early and mature stages of the industry as well as to consider the differences in these factors across different types of innovations.

In the majority of models of product and process innovations the products which are being introduced are similar to each other, (Dixit and Stigliz 1977). That's why this type of innovations are called homogeneous ones. The other strand of research literature proposes that all the products which are considered as the results of innovations are different from each other in some basic characteristics (Lambertini and Orsini 2001), (Lin 2004). This last approach is relatively new to the literature and is considered as more fruitful for modelling large multi product monopolists which perform the major part of product innovations, as for example in the metal producing industry (Salo, Karjaluoto, and Jussila 2007) or pharmaceuticals industry. Hence the current paper follows the second approach. In the model proposed here products differ from each other by the difficulty of the optimization of their production processes which is reflected in their investment characteristics. These refer to the process innovations being associated with every new product. At the same time the monopolist is free to introduce new products to the market at a continuous basis. The rate of introduction defines the time when process innovations start. It is argued, that this kind of the dynamic dependence between different types of innovations reproduces the known stylized facts on innovations yet allowing for rather simple structural model, briefly outlined below.

Before presenting the model the quick overview of different types of models built to capture the dependence between product and process innovations is required.

First consider, what are the methods used to install the link between two types of innovations in the literature. Frequently this purpose is achieved through the construction of the 2-stage static game, where on the first stage the decision upon the introduction of new good is being made and on the second - how much investment to put into the development of quality of this newly introduced product (conditional upon the successful introduction of it on the first stage). One example of such papers is (Athey and Schmutzler 1995) which is mainly devoted not to the interaction between both types of innovations themselves, but to the interrelation between organizational structure of the firm and its innovative decisions. In this literature quality innovations are the same as process innovations in other papers.

It is shown, that the complementarity between process and product innovations is the direct consequence of the complementarity between firm's manufacturing capabilities and its research capabilities. Current work correlates with this kind of literature in the idea of simultaneous decision making upon innovations of both types. However, it differs form this kind of models in accounting for dynamic characteristics of these new products. Moreover, heterogeneity of investment characteristics of these products and their qualities plays an essential role in the suggested model.

One other paper which corresponds to some extent to the suggested analysis is of Boone, (Boone 2000). There the process of innovations is also formulated as the 2-stage game, but the author elaborates on the incentives to innovate and their relation to the particular characteristics of the profit function. Both these examples are static in nature and they do not handle multi-product situations.

Later on it has been noted, that real innovative companies are often multi-product monopolies. Papers by Lambertini (Lambertini and Orsini 2001), (Lambertini 2003) study the equilibrium characteristics of investments into innovations of such a monopoly. He allows for multi-product investments, and the number of existing products may also increase in the result of product innovations. However the whole model is static because it handles only the equilibrium points of innovative policy of a monopolist. Author does not study any dynamical characteristics of product and process innovations but only the equilibrium distribution of investments. In the second paper Lambertini claims that the equilibrium level of quality investments is higher for the monopolist then the social optimum. However, more recent paper by Lin (Lin 2004) suggests that this heavily depends on the level of economies of scope for the monopolist. In general to be able to answer this question one has to account for the dynamical perspective of multiple products development and the evolution of the product space. The recent paper by Lambertini (Lambertini 2009) employs the dynamic approach to analyse the multi-product innovations with of a monopolist. The suggested paper differs from this last by explicit consideration of multiple products with heterogeneous investment characteristics and not only the varying products range, but also qualities of all the products which already exist.

Methodologically the current model is closer to the recent literature on vintage capital models although it concentrates on another type of questions. It is this strand of literature where the distributed parameter optimal control models are extensively used to describe the investment policy of an agent which has capital with different dates of appearance at hand. Then his policy should depend on the distribution of the mass of his capital in the past and hence the dynamic problem the agent has to solve is of distributed parameter optimal control type. Examples of such models are (Bouccecine, Germain, Licandro, and Magnus 1998), (Fabbri and Gozzi 2008) and others. This strand of literature uses vintage capital idea to describe policy of investments on industrial level, like in (Fabbri and Gozzi 2008), (Fabbri and

Iacopetta 2002) and also to contribute to the growth theories with embodied technological progress of the neoclassical type.

One of the few dynamic approaches to modelling heterogeneous innovations is the work of Hopenhayn&Mitchell (Hopenhayn and Mitchell 2001), which handles the innovative process in a rather general way. However their work is mainly concentrated on the patent policy and handles innovative process in a sense of previous theories, namely of Shapiro, (Gilbert and Shapiro 1990).

The suggested approach combines ideas of Hopenhayn&Mitchell and of Lambertini and Lin in a way that innovations are assumed to differ in their characteristics from each other as in (Hopenhayn and Mitchell 2001) and at the same time the appearance of the new products on the market as in (Lambertini 2009), (Lin 2004) is allowed in the dynamic context.

On the conceptual level the existence of the interdependence of product and process innovations was already admitted in papers by Utterback (Albernathy and Utterback 1985), (Albernathy and Utterback 1978) and others, as well as the necessity for some integrated model of the firm which would include product and process innovations in a dynamic fashion. However in this line of literature both processes are handled as 1-dimensional. Equivalently, one may assume any number of coexisting homogeneous products as one simple product and process innovations as improvements in production technology.

In the suggested framework, however, new product innovations lead to the creation of new products with new associated quality dimension for each of them. Such new products are then positioned as different from the basic one in the market (for example, for market segmentation purposes). That's why one may assume these products as being sold at different segments of the market.

In this strand of literature it is assumed that process innovations depend on product innovations since they are the result of learning activities. In the multi-product framework of this paper stream of process innovations for every new product depends on the introduction of this product but not from other products. Thus every new product is separated from others in terms of refinement of its production techniques. However, the aggregate process of refinement of technologies is depending on product innovations at each point in time and is the result of learning activities. In such a way one may consider independent products from the one hand (and thus differentiated markets) with separate process innovations and the dependence of evolution of production technologies as a distribution of functions over the products space which is then the generator of process innovations.

The existing literature on theory of product and process innovations does not contain models similar to the suggested one. The model described further on in the paper has two important features, novel to the area of product and process innovations:

- It allows for simultaneous dynamical optimization of product and process innovations;
- It allows for different investment characteristics of all the new products;
- Product innovations depend on the investment characteristics of process innovations into the last invented product while process innovations as a whole depend on the introduction of new products;
- Every new product is different from all the other ones and hence is not sold on the same market as other introduced products, thus generating not only new output but the creation of new markets also.

These features allow to establish the dynamic and coherent link between both types of innovative activity. The formal exposition of the model follows.

3 Model

In this section the formal model is introduced together with the underlying economic intuition. The goal and structure of the model is also explained in this section.

3.1 Basic Structure

Assume there is a single firm (a monopolist) in a given industry. The industry is mature and no growth of the demand is expected for existing products variety. Hence this monopolist is maximizing its profit by developing new products, which are then introduced to the market. The natural objective of the monopolist is the maximization of its profits, $\pi(t) \rightarrow max$ for any given time period. This paper concentrates on just one part of activities of such a monopolist, namely on the process of its innovative activities. To put this in line with profit maximization behaviour we assumed that markets for all existing products are mature, yield some constant profit with stable prices and output. Production policy of the monopolist is assumed to follow standard rules of monopolistic behaviour under profit maximization: given (constant) demand, the monopolist is setting the price and production as to maximize its profit. In mature markets the process innovations reached their maximum and thus no further improvements to the production process may be made. Hence, the production costs are also constant in time. These considerations lead to the conclusion that in mature markets the monopolist's production and pricing (and hence profits) are constant.

Assumption 1 *For those products which are already in mature stage, the production, price and profit of the monopolist are constant.*

Because of this one may abstract from this part of monopolist's activities in the optimization problem. Now consider those products which are being introduced in the given time frame and which production is subject to process innovations. For these products the profit of the monopolist is proportional to the costs decrease which is the result of process innovations, denoted by $q_i(t)$. If we abstract from the pricing policy and assume constant demand for each of such products, this would result in the profit function per unit of production of a linear form: $\pi_i(t) = \delta * q_i(t)$. Then normalizing δ coefficient to one, we may have q_i as the only profit parameter for any product within the product range N.

Assumption 2 *The only source of new profit for the monopolist is the development of new products which leads to the increase in the existing range of products over time, $n(t) > 0$.*

Assume the process of development of products is continuous in time and yield new products (which are new versions of some basic for the industry product) with some rate. Let us call this rate the rate of variety expansion.

Assumption 3 *The product innovations, are continuous in time and new products appear at a continuous basis, $\dot{n}(t) \geq 0$.*

Assume that the range of these new products is limited from above. The product innovations are limited to upgrades of some basic product which defines the industry (e.g. cell phones industry produces different versions of cell phones but not computers). We do not model fundamental inventions, which introduce totally new products to the economy by this model and hence it is natural to require that there is limited capacity of the industry for the variety of products which are somewhat similar to each other.

Assumption 4 *Product innovations are limited by the maximal possible range of products, $n(t) \leq N$*

Assume these newly introduced products initially require very much resources for their production and hence the monopolist allocates part of its R&D capacity on process innovations related to these new products. Every new product is than intensively studied with respect to opportunities for its costs minimization. As there are numerous new products (continuum of) there are numerous streams of such cost-minimizing processes associated with every product.

Assumption 5 *Every product has its own dimension of process innovations or 'quality' which depends on time, $\forall i \in n(t) \exists q_i(t)$.*

Assume at each point in time, the monopolist has to choose optimally the level of investments being made into the development of new products (product innovations) and into the development of production of already existing products (process innovations). Both these investment streams cannot be negative.

Assumption 6 *Product innovations and process innovations require different types of investments, which vary over time, while process innovations for every product are also different $u(t) \geq 0, g_i(t) \geq 0$*

Assume also that the monopolist is the long-run player and does not restrict its planning to some certain length of time. Hence, the innovations of both types occur continuously up to infinite time.

Assumption 7 *There is no terminal time for both processes of innovations, $0 < t \leq \infty$*

The last point to mention is that we assume that all innovations are certain. This is rather strong limitation, but allows to concentrate on the key issues of this paper: heterogeneity and form of interdependence between different types of innovations.

Assumption 8 *All innovations do not have any uncertainty associated with them.*

3.2 Assumptions on Dynamics

In this section the form of dynamic laws, which govern the innovations of both types is explained.

Observe that under the assumptions stated above, one has the process of new products introduction, $n(t)$ which describes the range of products which are already available for production. Hence product innovations are defined by the rate of increase of this range over time, $\dot{n}(t)$. This last cannot be negative, as the product which is already introduced to the market cannot be forgotten. The increase in the range of products which are already developed is proportional to the investments being made by the monopolist in this direction, $u(t)$. We assume no other internal factors, which may affect this product innovations rate. Hence, the dynamics of product innovations is assumed to be described by the following differential equation:

$$\dot{n(t)} = \alpha \times u(t). \tag{1}$$

Here α is the efficiency of investments into the product innovations. It is constant and exogenously defined by the state of technology in the economy as a whole. According to this equation, the range of products which are already introduced cannot decrease over time. Eventually all the possible products are developed, as the range is restricted by N. This process is continuous and hence there is a continuous spectrum of products available on the market.

Each such product has the associated stream of process innovations. Denote by i the position of the product within the products range $n(t)$. Then to distinguish between process innovations for different products we will denote them by q_i. These innovations also grow only due to the investments being made into them. Since only one monopolist is modelled, there are no technology spillovers or acquisitions of competitors' innovations. At the same time, we assume that the improvements of every product are outdated as time flows. This process is that stronger the more refined the production technology already is. The given process innovation exhibits the decline of technology over time if no investments are being made into the production process innovation of this given product i. These considerations lead to the following form of dynamics of process innovations for every product i:

$$\dot{q_i(t)} = \gamma_i \times g_i(t) - \beta \times q_i(t). \tag{2}$$

Note, that this defines only the i-th component of the overall process of refinement of technology, which is the function of two variables, i, t. However, under the assumption of independent refinement technologies this total function is decomposed into functions of 1 variables for each product i.
Here γ_i denotes the efficiency of investments into the optimization of production of product i and β denotes the rate of decay of technology in the absence of investments. This one is assumed to be constant across all products, while the efficiency of investments is different. As a result the level of technology for every product might be different. The form of this difference depends on the form of the γ_i parameter dependence on i.

Observe that the equation (2) implies difference of dynamics of process innovations for all the products. Depending on the γ_i, the development of technology for every next product may be easier, harder or equally difficult then for preceding products. As in this paper we assumed that the product innovations is the process of appearance of new versions of the same basic product variety, it is natural to assume that next products are more complicated then preceding ones. To this end we specify the dependence of the efficiency of investments into production technology development as

(3)
$$\gamma(i) \stackrel{\text{def}}{=} \gamma \times \sqrt{N-i}.$$

This specification makes process innovations investments efficiency a convex from zero function of i:investments are less efficient for every next product and with the increasing speed of this increase in difficulty. For the last product in the available range, $i = N$ this means zero level of technology, as the efficiency will become zero. At the same time this is the increasing and convex from zero function of the products' range N: the wider is this range the higher is the efficiency of investments into the process innovations. This reflects the learning cycle as from (Albernathy and Utterback 1978): the more product versions might be invented, the faster is the development of new products qualities (production technologies). Hence the chosen specification accounts for two effects of different directions: the positive effect of the range of new products and the negative effect of the growing complexity of every next product. Figure 1 displays this function for $\gamma = 0.3$.

Observe, that γ term in (3) reflects some positive constant which measures

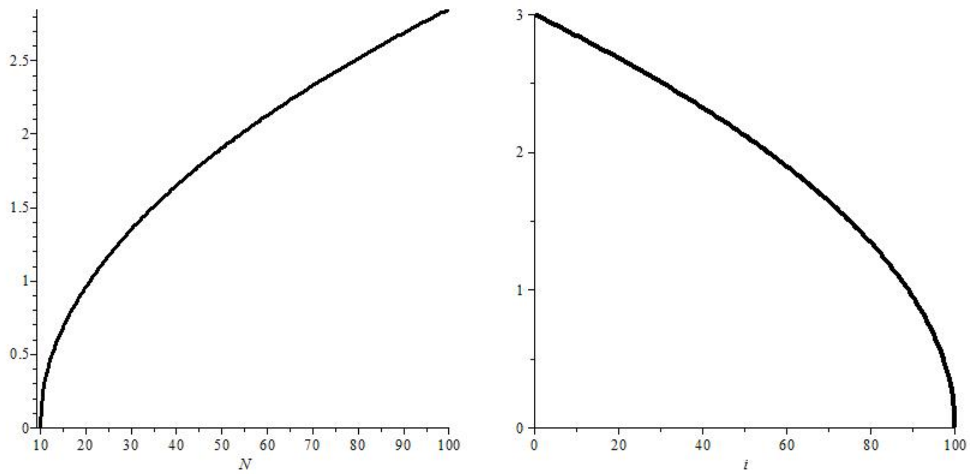

Figure 1: Investments efficiency $\gamma(i)$ as a function of N and i

average efficiency of investments across products. It positively affects the efficiency of investments for any product i. It has to be distinguished from the $\gamma(i)$ or γ_i, as the first of these two denotes the investments efficiency as a function of the products' space and the other the value of this function for a given i, while γ is the same for all products.

Now consider that product innovations actually introduce new products. These new products must have zero level of production technology at the moment of their appearance. This level might be increased through investments only. This requires a constraint on the initial level of production technology for any new product. At the same time, the position of this new product is dependent on the product innovations, as it may be seen from the formal form of constraint:

$$q_i(t)|_{i \geq n(t)} = 0. \tag{4}$$

This constraint introduces the notion of frontier or boundary product into the model. The boundary product is the current position of the product innovation process $n(t)$ in the available products range N. The quality of this boundary product is always zero. Observe, that every product among those which are to be developed becomes the boundary product exactly once, at the moment of its introduction. This moment, $t_i(0)$, triggers the process innovation associated with the product i and is in turn defined by the dynamics of product innovations, $n(t)$. Thus the last constraint is very important to the model: it establishes the dynamic dependence of process innovations for each product i from the product innovations as a whole, $n(t)$. Again, the total process, $q(i,t)$, depends on $n(t)$ at each point in time, since for any t there are some new emerging products and associated quality-improving processes. This dependence is dynamic and governs the intensity of process innovations. Indeed, the more new products are developed at each point in time, the more process innovations strats to increase at this point and the more aggregate investments into qualities, $\int\limits_0^{n(t)} g(i,t)di$ are being made then.

At the same time, the requirement of optimality of different types of investments establishes the dependence in the inverse direction: product innovations and rate of products variety expansion as the function of the process innovations. This form of dependence is the one being observed by empirical findings on the single firm's level (Kraft 1990). It is captured in the objective function described below.

3.3 Objective

The natural objective of the monopolist is the maximization of its profits, $\pi(t) \rightarrow max$ for any given time period. This paper concentrates on just one part of activities of such a monopolist, namely on the process of its innovative activities. The objective functional of the monopolist is defined as:

$$(5)\qquad J \stackrel{\text{def}}{=} \int_0^{\infty} e^{-rt}\left(\int_0^{n(t)}\Big[q(i,t)-\frac{1}{2}g(i,t)^2\Big]di-\frac{1}{2}u(t)^2\right)dt \to \max_{g(\bullet),u(\bullet)}$$

In what follows this form of the objective functional is discussed and explained.

Monopolist is maximizing integral sum of qualities (production technologies) of all products invented until each time t minus costs of investments being made to every invented product's quality and to the overall expansion process over the planning horizon. Both R&D cost functions are assumed to be quadratic in the amount of investments being made.

There is no sign of prices or profit in this formulation. The market clearing mechanism and all the mechanics behind the market structure are also omitted. Such a specification would give the independent from prices and profit dynamical system. It is equivalent to the linearity of profit function which is a standard assumption in innovation literature, (Lambertini and Orsini 2001), (Lin 2004).

One may treat the objective of a given form as an extension of the model in (Grossman and Helpman 1993), where the variety of intermediate capital goods is also modelled through integral sum over numerous products while every product has its own market price. Hence one may treat all the products in $n(t)$ as having separate (or differentiated) markets. Unlike the Grossman's model, however, one has the development of qualities (process innovations) for all of the invented products, while in the strand of literature on the varieties of products this is assumed constant.

The functional (5) includes also the assumption on the form of R&D costs. These are assumed to be quadratic in the amount of investments being made which is a standard way of modelling investment costs in the literature on capital accumulation which employ optimal control methods. The reason for this specific way of introduction of costs is the theoretical result which states the existence of optimal feedback strategies for linear-quadratic optimal control problems. To be linear-quadratic, the model has to include investments in a quadratic way, (Fattorini 1999).

The reason for omitting prices and quantities for the products is primarily the focus of the model on the interrelation between different innovations types rather then on the influence of innovations on the production or productivity of a firm. As a result the functional (5) does not include quantifies being sold by the monopolist at all. To avoid confusion it should be noted that q_i variables denote the level of production technology for every new product i and not the quantities of this product being sold. In this respect the assumption of different and separated markets for all the new products does not play crucial role, since the structure of markets may influence the total profit of a firm, but does not influence directly the form of dynamics of innovations. Some more details are in two first assumptions upon the modelling framework above.

Dynamics of process innovations and variety expansion process are governed by subsequent dynamic equations in accordance to assumptions listed and explained in two previous subsections of the paper:

$$\begin{aligned}
&\dot{n(t)} = \alpha u(t);\\
&\dot{q(i,t)} = \gamma(i)g(i,t) - \beta(i)q(i,t);\\
&\gamma(i) = \gamma \times \sqrt{N-i};\\
&\beta(i) = \beta;\\
&\forall i \in [0,..,N] = \mathbf{I} \subset \mathbb{R}_+;\\
&\forall t \in [0,..,\infty) = \mathbf{T} \subseteq \mathbb{R}_+.
\end{aligned} \tag{6}$$

Note that $\mathbf{I}$ denotes the potential products' space and $\mathbf{T}$ is the time domain of the problem.
There are some static constraints formally restating the assumptions given above.:

$$\begin{aligned}
&u(t) \geq 0;\\
&g(i,t) \geq 0;\\
&n(t) \leq N;\\
&q(i,t) \mid_{i=n(t)} = 0.
\end{aligned} \tag{7}$$

Note that the last constraint is equivalent to the (4) constraint together with continuity requirement on $q(i,t)$.

Equations (5), (6), (7) constitute a parameter-distributed optimal control problem of the multi-product monopolist. Due to the special structure of this problem, it is possible to transform it into the sequence of optimal control problems. These problems are in turn solved through the implementation of Hamilton-Jacobi-Bellman approach. Details of the solution procedure are presented in the next section of the paper.

4 Problem Decomposition

The basic idea of the solution method being employed in the paper relies on the fact that the total process of quality development through process innovations, $q(i,t)$ may be decomposed into the continuum of quality improving innovations for every separate product i. This can be done due to the absence of horizontal connections between qualities of different products: the model accounts for dependence of the process of variety expansion on the process of quality improvement and vice versa, but not for interdependencies between qualities of different products. This last may be considered as an interesting future extension of the suggested framework.

The other special feature is the dependence of variety expansion (product innovations) only on the boundary product's quality, $q(t)_{n(t)}$. Thus the overall problems may be reduced to the solution of only two related 1-dimensional dynamic problems instead of the full infinite-dimensional one. First the problem of quality development for any given product i is solved and then the results are used for the derivation of the dynamics of variety expansion. Details follow.

4.1 Qualities of products (process innovations)

First observe, that the problem of payoff maximization from the process innovations may be solved independently of the problem for product innovations due to the infinite time horizon in the model (Dockner, Jorgensen, Long, and Sorger 2000). In this case stream of investments into the development of quality of every product i, starting from the time of its introduction, $t_i(0)$, is fully defined by the rate of the monopolist's investments into this process.

Since the problem under consideration is the dynamic one, for each product the profit maximization has to be defined over all the infinite time horizon. Hence, the monopolist has to maximize his instantaneous payoff, which is equal to $q_i(t)$ over $0 < t \leq \infty$. The only parameter under the control of the monopolist is the investments into the process innovations which result in the increase of $q_i(t)$, reduction of costs and profit increase. Then the problem of profit maximization for every product i may be defined as:

$$V(q_i) = \int_0^\infty e^{-rt} \times (q_i(t) - \frac{1}{2} \times g_i(t)^2)dt \to \max_{g_i}. \tag{8}$$

under the condition of dynamics of process innovations governed by (2). The investments into the process innovations for each product i, $g_i(t)$ are defined by the monopolist and control the rate of process innovations for every product separately from others; r denotes the discount rate, which is defined from time preferences of the monopolist and is exogenous to the problem.

Proposition 1 *Process innovations q_i for each product i are independent from product innovations $n(t)$ except for the time of introduction of the prod-*

uct i into the market, $t_i(0)$. At the same time the total quality-improving process, $q(i,t)$ is a function of variety expansion $n(t)$.

To see this, consider function $q(i,t)$ as function of $n(t)$. At each point in time, t_i, some positive mass of products emerge from variety expansion process, $n(t) > 0 \forall t \geq 0$. The solution to (1) defines time of emergence for each product i as a function of variety expansion, $t_i(0) = f(n(t))$. This also defines $i = f(t_i(0))$ and hence $q(i,t) = f(n(t),t)$. At the same time for $i = const$ the projection of $q(i,t)$ on the products' space is a function of t only, $q(i,t)|_i = q_i(t)$. Hence the proposition above.

One may construct the value function of the problem of maximization of (8) and rewrite it in the form of HJB equation (together with (2)).

$$(9) \qquad rV(q_i) = \max_{g_i(\bullet)}\{q_i(t) - \frac{1}{2}g_i(t)^2 + \frac{\partial V(i)}{\partial q_i}(\gamma\sqrt{N-i}g_i(t) - \beta_i q_i(t))\}.$$

This HJB equation brings together the dynamics of process innovations for the product i and the value generated by such innovations, which is the increase in the profit from this product. Observe, that the time derivative of the value function $V(q_i)$ is absent from the equation since the time horizon is infinite. Thus the HJB equation is time-autonomous.

For this type of HJB equations the linear specification of the value function is the only relevant one (Dockner, Jorgensen, Long, and Sorger 2000). This means that the value generated by the process innovations depends linearly on the level of the production technology, q_i at each point in time. This leads to the constant rate of investments into this technology for every product. The results of this solution method are presented in the next section.

4.2 Products variety expansion (product innovations)

Product innovations are taking place simultaneously with process innovations. They result in the continuous increase in the variety of products which appear in the market. For each such product the potential profit over all its life-cycle is defined by (8). However, the product innovations process also has an influence on the value generation for the monopolist. This influence is describe by the introduction of new products. The likeliness of the introduction of new product is defined by the stream of potential profits from its subsequent production. Hence, the decision of increasing or decreasing investments into the product innovations is governed by this potential profit stream, as the newly invented product has zero level of production technology and as such cannot be produced with costs lower than its price. This is where the constraint (4) comes in. As a result, one may define the profit generated by the introduction of new product as the evaluation of potential profit stream from the subsequent development of the production of this product. Hence the total profit generated by the product innovations is the integral over all such potential profit streams:

$$(10) \qquad V(n(t)) = \max_{u(\bullet)} \int_0^{\infty} e^{-rt} \Big(\alpha u(t) \times V(q_i)_{i=n(t)} - \frac{1}{2} u(t)^2 \Big) dt.$$

Here the term $V(q_i)_{i=n(t)}$ is the current value of the quality growth problem for the next product to be invented $i = n(t)$ estimated at zero quality level for this product.

Now consider the product innovations. It has been noted before, that the value of this process to the monopolist consists solely in the introduction of new product which has to be developed afterwards. Hence the value of this part of the innovations process depends on the value being generated by every potential product, (9). The monopolist decides upon the intensity of introduction of new products into the market, $\alpha u(t)$, at each point in time. This intensity is controlled through the stream of investments into the product innovations, $u(t)$. At the same time careful consideration of (10) shows that at each point in time the value of the introduction of the next product depends only on this product's production technology level, which may be eventually reached.

Proposition 2 *The current value of the product innovations process, $V(n)$, depends only on the expected production technology benefits of the next boundary product, $V(q_i) = V(q_i)_{i=n(t)}$ and not on the level of production technologies of the products which are already in the market, $i < n(t)$.*

This result follows from the form of the general objective functional (5): the variety expansion (product innovations) are present only in the limit of integration of the instantaneous payoff function,

$$(11) \qquad \left(\int_0^{n(t)} \Big[q(i,t) - \frac{1}{2} g(i,t)^2 \Big] di - \frac{1}{2} u(t)^2 \right)$$

and thus the only positive result of it for the payoff is in the addition of the new dimension to the products' space. Hence the value of this process is defined by future development of qualities for yet not invented products at every t. At the same time all other future products except for the boundary one have zero quality levels and the expected payoff from them is undefined. Hence the value of variety expansion depends only on the boundary product.

It does not depend on already invented products because it does not influence their quality development since their invention also.

The value generation for the product innovations process may be described by the HJB equation, which combines (10), (1) and the resulting optimal current value of the process innovation for the boundary product, $V(q_i)_{i=n(t)}$:

$$rV(n(t)) = \max_{u(\bullet)}\{\alpha u(t) \times V(q_i)_{i=n(t)} - \frac{1}{2}u(t)^2 + \alpha u(t) \times \frac{\partial V_{n(t)}}{\partial n(t)}\}. \tag{12}$$

After obtaining the optimal investment strategies for product innovations, $u^{opt}(t)$ and the subsequent optimal product innovations dynamics, $n^{opt}(t)$ one would have the full solution for the model. The results are described below.

5 Solution and Results

5.1 Process Innovations

To obtain optimal investment strategies for quality development of each product i, make use of the equation (9) to derive first-order conditions of maximization with respect to the investments g_i. For this differentiate the right-hand side of (12), which is the Hamiltonian function:

$$\frac{\partial \mathcal{H}_i}{\partial g_i} = \frac{\partial V(q_i)_i}{\partial q_i} \times \gamma\sqrt{N-i} - g_i(t) = 0. \tag{13}$$

where $\mathcal{H}_i$ denotes the right-hand side of the HJB equation for process innovations (9).
With the assumption of linear value function for each product, $V(q_i)_i$, this yield the formulation of optimal investments:

$$\begin{aligned} V^{ass}(q_i)_i &= A_i q_i(t) + B_i; \\ g_i(t)^{opt} &= A_i \times \gamma\sqrt{N-i}. \end{aligned} \tag{14}$$

Where A_i is defined from the system of equations on coefficients of the value function:

$$\begin{cases} (r+\beta)A_i - 1 = 0; \\ rB_i - \frac{1}{2}\gamma^2(N-i)A_i^2 = 0. \end{cases} \tag{15}$$

The resulting form of the value function for quality development of each new product is a function of q_i and the position of the product, i:

$$V(q_i)_i = \frac{1}{r+\beta} \times q_i + \frac{1}{2}\frac{\gamma^2(N-i)}{r(r+\beta)^2};$$
$$\forall i = const \in N. \tag{16}$$

Observe, that under the assumption of linear value function the slope of this function is the same for all the products i, but the level is different, as B_i coefficient is different. Hence one has N different value functions for N new products. They constitute the value of process innovations as a whole as a function of q_i, i exactly of the same form as above but with $i = f(n(t))$.

Such a value function yields investment rule for the monopolist which differs between products (in accordance to the investment efficiency, γ_i) but is otherwise constant. This investment strategy is valid only starting from the time of actual introduction of the product i into the market:

$$g_i^{opt} = \frac{\gamma \times \sqrt{(N-i)}}{(r+\beta)};$$
$$\forall t \geq t_i(0). \tag{17}$$

Hence the optimal investment strategy for each product i is a piecewise defined function of time and variety expansion:

$$g_i^{opt} = \begin{cases} 0, n(t) < i; \\ \\ \frac{\gamma \times \sqrt{(N-i)}}{(r+\beta)}, n(t) \geq i. \end{cases}$$

This investment rule yields constant investments rate since the invention of the product for every product and is proportional to the efficiency of investments, γ while is negatively influenced by the decay rate of technology, β. At the same time for every product i the rate of investments is lower than for preceding one, $i - \delta i, \delta \to 0$ because of the increasing complexity of the production technology. This is described by the investment efficiency specification (3). The observations derived from the (17) are summarized in the following Proposition.

Proposition 3 *Optimal investments of the monopolist into process innovations, g_i^{opt} differ for all new products i but are constant in time since the introduction of this product. They are proportional to the efficiency of investments γ and are in inverse relation to the decay rate of technology β.*

With this investment rule, one may define the optimal path of the production technology for each new product, q_i^{opt}, starting from the time of the product's introduction, $t_i(0)$. Up to this time the technology is at zero level. Hence, the technology dynamics is piecewise defined: it is zero till time $t_i(0)$ and is the solution to (2) with optimal investments from (17) afterwards:

$$q_i(t)^{opt} = \begin{cases} 0, t < t_i(0); \\ \frac{\gamma^2(N-i)}{\beta(r+\beta)^2} \times (1 - e^{-\beta t}), t \geq t_i(0). \end{cases} \tag{18}$$

The study of (18) leads to the following observation:

Proposition 4 *Production technology q_i never declines but increases with decreasing speed up to its maximal level, $\bar{q}_i$ which is different for different products.*

To see this, just compute the time derivative of the function (18) at $t > t_i(0)$. This amount to

$$\frac{\gamma^2(N-i)}{\beta(r+\beta)^2} \times \beta e^{-\beta t} \geq 0. \tag{19}$$

This expression is nonnegative provided $i \leq N$ which is true by definition of N. At the same time the second time derivative is nonpositive.

$$\frac{\gamma^2(N-i)}{\beta(r+\beta)^2} \times (-(\beta^2)e^{-\beta t}) \leq 0. \tag{20}$$

Both derivatives are going to zero as $t \to \infty$ thus giving the constant long-run level of quality achievable for each product. At the same time this level is different for all the products since the decreasing investments efficiency.

The decreasing speed of improvements in production technologies mean decreasing marginal productivity of process innovations investments. The decrease in marginal efficiency of innovations is observed by empirical studies. It is usually explained by increased complexity of further refinement of technologies and the increased burden of maintaining the existing level of technology, q_i, at later stages of product development. This is rather stylized approach to process innovations, as in reality the investments into the production technology depend on the existing level of technology. However, the general pattern of process innovations in the model is in line with empirical findings (Faria and Lima 2009), (Salo, Karjaluoto, and Jussila 2007). There is a limit for improving the production process of any product. The more refined technology is used the more difficult it is to improve this technology further. Eventually the monopolist reaches the point where no new refinements would be profitable: new investments are totally spend to maintaining the existing level of technology. At this point the product enters the mature stage of development and no new profit increases may be derived from it. This is the reason for the monopolist to continue with introduction of new products as their technology are easier to improve that of those which are already mature.

Proposition 5 *Process innovations q_i have maximal level of development $\bar{q_i}$, which decreases in i and is different for all products.*

This maximal level of development is obtained by equating to zero the dynamics of quality development, (2) with optimal investments g_i^{opt} from (17) and finding the $\bar{q_i}$ from the resulting equation:

$$\dot{q_i(t)} = \gamma \times \sqrt{(N-i)} \times (\frac{\gamma \times \sqrt{(N-i)}}{(r+\beta)}) - \beta \times q_i(t) = 0,$$

$$\bar{q_i} = \frac{\gamma^2(N-i)}{\beta^2(r+\beta)}. \tag{21}$$

The last equation defines the steady-state quality level for each product i as a function of the position of this product in the product space.

To illustrate the form of the dynamics of process innovations, consider the evolution of production technologies for several products: Observe that

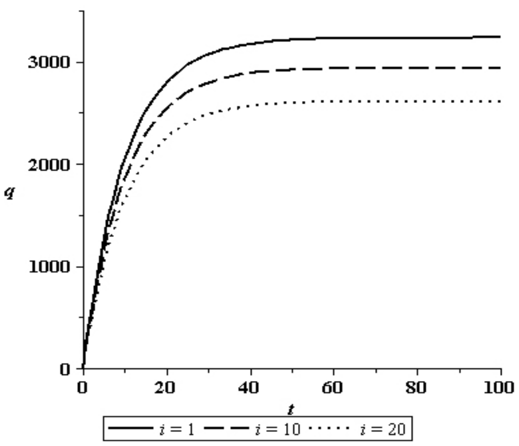

Figure 2: Difference in technologies for different products

all the technologies start to grow at different times since (4). On Figure 2 it is assumed that all the products are introduced at the same initial time $t = 0$ which is not the case. To define the time of introduction of every product, one has to solve for the product innovation process first, which is done in the next subsection.

The Figure 2 illustrates the dynamics of development of process innovations. For every next product the growth of technology is slower then for the preceding one. This is the result of increasing complexity of process innovations across products and decreasing in i rates of investments (17). Note, that these different dynamics of production technologies for different products are the direct consequence of the *heterogeneity* of investment characteristics of different new products i. The special form of this heterogeneity, assumed in the model establishes the specific form of such difference, namely that each next product has lower long-run production technology level then all the preceding ones. However this may be easily changed by assuming the form of the investment efficiency function γ_i different from (3).

5.2 Product Innovations

Now consider that together with process innovations q_i the monopolist is undertaking the continuous process of product innovations due to the fact of limited capacity for profit generation from improvements of technology for every product (limited maximal level of quality for each product). In terms of the model the monopolist decides upon the rate of introduction of new products which is governed by the investment strategy $u(t)$.

At each point in time the monopolist considers the potential profit from improving technology of the boundary product (the one he is going to introduce). This latter is described by the current value of the process innovations for the boundary product, $V(q_i)_i|_{i=n(t)}$. The exact form of this value is obtained from the value function (16) with $i = n(t)$. Observe, that *every* product in the products' space $\mathbf{I}$ becomes the boundary product at time $t_i(0)$ which is defined by the variety expansion (product innovations) process. The monopolist defines current investments into the introduction of new product $i = n(t)$ as a function of expected profit generation by this boundary product, $u(V(q_i)_i|_{i=n(t)})|_{t=t_{i=n(t)}(0)}$. For this product the production technology is at the zero level, $q_i = 0$ and the resulting value function (16) does not depend on the level of quality of the product, but only on the position of the product in the products' space $\mathbf{I}$. As a result the value $V(q_i)_i|_{i=n(t)}$ being used in the definition of product innovations intensity further on is:

$$V(q_i)_i|_{i=n(t),q_i=0} = \frac{\gamma^2 \times (N - n(t))}{2r(r+\beta)^2}. \tag{22}$$

This value function depends only on the level of product innovations at each point in time, $n(t)$ but not on the process innovations dynamics, $q_i(t)$. It depends on parameters of the efficiency of investments into the improvement of technology, γ and the decay rate of technology, β. The current value of a boundary product, (22) reflects the heterogeneous nature of the technologies for different products as it changes with changing $n(t)$ which is defined from product innovations dynamics.

Using the HJB equation for product innovations, (12) with (22) inserted into it, one may derive the first order conditions for optimal investment rule, $u^{opt}(t)$ and the resulting optimal investments strategy as functions of the value generated by the boundary product, $V(q_i)_i|_{i=n(t)}$ and the value function of product innovations, $V(n(t))$:

$$\frac{\partial \mathcal{H}_n}{\partial u} = \alpha \times \Big(\frac{\partial V(n(t)}{\partial n(t)} + \frac{\gamma^2 \times (N - n(t))}{2r(r+\beta)^2}\Big) - u(t) = 0. \tag{23}$$

where $\mathcal{H}_n$ denotes the right-hand side of the HJB equation for product innovations, (12).

Now assume the polynomial value function for product innovations. Unlike process innovations here the quadratic form is assumed, thus giving the linear-feedback form to the optimal investments:

$$V^{ass}(n) = A_n n(t)^2 + B_n n(t) + C_n;$$

$$\frac{\partial V^{ass}(n)}{\partial n} = 2A_n n(t) + B_n;$$

$$u^{opt}(t) = \alpha \times \Big(2A_n n(t) + B_n + \frac{\gamma^2 \times (N - n(t))}{2r(r+\beta)^2}\Big) \tag{24}$$

Inserting this together with the value function for process innovations into the next product, (22), into the HJB equation (12) yields a system of 3 algebraic equations on value function coefficients. Solving this system defines coefficients A_n, B_n, C_n as functions of exogenous parameters. Then their substitution into the optimal investments $u^{opt}(t)$ gives this last as a function of variety expansion process only:

$$u^{opt}(t) = \frac{\alpha r \gamma^2}{r(r+\beta)(\sqrt{r^4 + 2r^3\beta + r^2\beta^2 + 2\alpha^2\gamma^2} + r(r+\beta))}(N - n(t)). \tag{25}$$

The investment strategy for product innovation depends on the average efficiency of investments into process innovations, γ. Note that this is constant across products and differs from $\gamma(i)$ from (3). It also depends on the decay rate of production technology, β as well as from the efficiency of investments into the product innovations themselves, α. The study of the form of (25) leads to the conclusion, that the rate of investments is limited by the available potential for products introduction, N and negatively depends on the existing variety of products, $n(t)$. In fact it is the linear function of products' variety $n(t)$ for any fixed range of products N. For some plausible parameter settings this investment strategy is displayed on Figure 3.

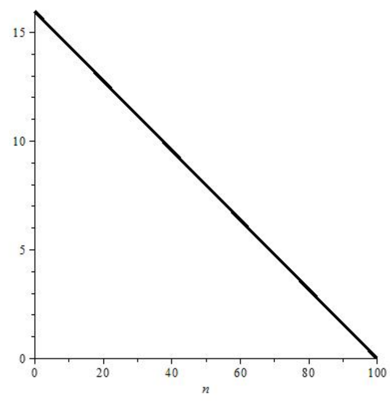

Figure 3: Investments into variety expansion

The higher is the existing variety, the wider is the range of products which are already developed. With the increase of this range the rate of product innovations is slowing down and reaches zero at the maximal level of variety, N. There is no explicit representation of the investment or research capacity of the monopolist in the model except for this maximal achievable level of products' variety N. However the dynamics of optimal investments into process and product innovations follows the pattern of limited research capacity: process innovations into the development of each product stop after reaching some mature stage, $\bar{q}_i$, while the development of new products provides new possibilities for improvements. At the same time new products development is slowing down with the range of already introduced products, that is,

$$\frac{\partial u}{\partial n} < 0.$$

Proposition 6 *Product innovations investments, $u^{opt}(t)$, are decreasing to zero while the process of variety expansion reaches its limit N, and all the possible versions of the basic product are already introduced, $n(t) = N$.*

This is the direct consequence of (25).

Now with the help of optimal investment strategy (25) one may derive the evolution path of the product innovation process $n(t)$ through its dynamics, (1). This is an ordinary differential equation of the first order which has the explicit solution.

$$\dot{n(t)} = \frac{\alpha^2 r\gamma^2}{r(r+\beta)(\sqrt{r^4 + 2r^3\beta + r^2\beta^2 + 2\alpha^2\gamma^2} + r(r+\beta))}(N - n(t));$$

$$n^{opt}(t) = N - e^{-\frac{\alpha^2\gamma^2 rt}{\sqrt{(r+\beta)^2 r^2 (r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+(r+\beta)^2 r^2}}(N - n_0). \tag{26}$$

Here n_0 is the range of products already introduced to the market at the initial time.

The form of evolution of products' range demonstrates the positive rate of product innovations at each point in time. However the decreasing rate of investments over time, $u^{opt}(t)$ yields decreasing intensity of new products' introduction as the process approaches the limit N. To see this, consider optimal investments as a function of time. For this substitute the solution from (26) into the (25). The comparison of intensity of product innovations for $\alpha = 0.3, \alpha = 0.5$ is displayed on Figure 4

Product innovations depend on the process innovations in a dynamical way. This may be seen directly from the form of (26), but also from the fact that $n^{opt}(t)$ is the function of the value generated by the boundary product technology, (22). The last observation comes from the form of the HJB equation for product innovations, (12). At every point in time the product innovations and their intensity $\alpha \times u(t)$ are governed by the expected current value of development of the technology associated with the boundary product, $q_{i=n(t)}$ which is different and decreasing across products as (18) demonstrates. This establishes the dynamic and time-varying dependence of products variety expansion (product innovations) from process innovations (quality-improving technology).

Proposition 7 *Product innovations $n(t)$ depend in a dynamic way from process innovations $q(i,t)$ and the form of this dependence is defined by the investment efficiency function, $\gamma(i)$*

To see this, consider that product innovations investments, $u(t)$ are the function of efficiency of investments into the quality development of the boundary

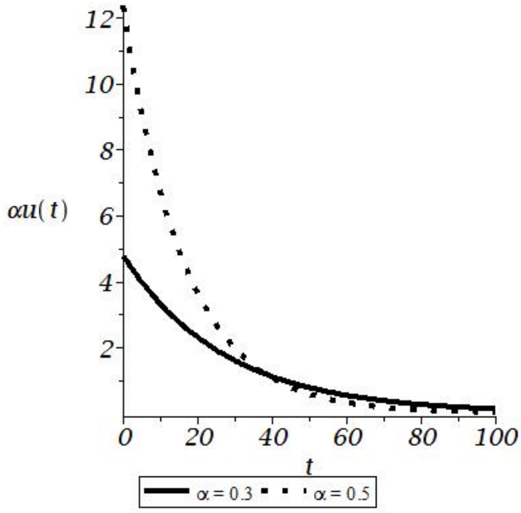

Figure 4: Intensity of product innovations

product, $\gamma(n(t)) = \gamma \times \sqrt{N - n(t)}$, while variety expansion $n(t)$ is the function of investments.

As the direct consequence of this observation evolution paths of product innovations being started with different initial ranges of already introduced products, n_0 are convergent. The higher is the range of products which are already introduced, the less opportunities the monopolist has to develop new products, as the total range which might be developed in the given industry with given basic product is limited by N. At the same time the wider range of introduced products broaden the opportunities for process innovations, $q(i, t)$. Recall that the form of process innovations investment efficiency, (3) implies that it is more profitable to develop simpler products from the starting range, than new ones. These latter are developed at a slower pace and to a lesser extent. However this does not mean that process innovations as a whole depend negatively on product innovations, since the aggregate process innovations $\int\limits_0^{n(t)} q(i, t)di$ are growing with $n(t)$.

The intensity of product innovations are governed by the expected profit from the development of next boundary product. With higher initial range n_0 current value of the profit from the development of next product is lower then for the case of low initially developed range. Thus the product innovations' pace will be slower in the first case. So the convergence of product innovations evolution paths is the direct consequence of the proposition above.

Corollary 1 *With decreasing efficiency of process innovations, γi, products' variety expansion paths are convergent for any initially available range of products, n_0.*

As a result the influence of the initially available range of products eventually wears down. This is illustrated by the Figure 5.

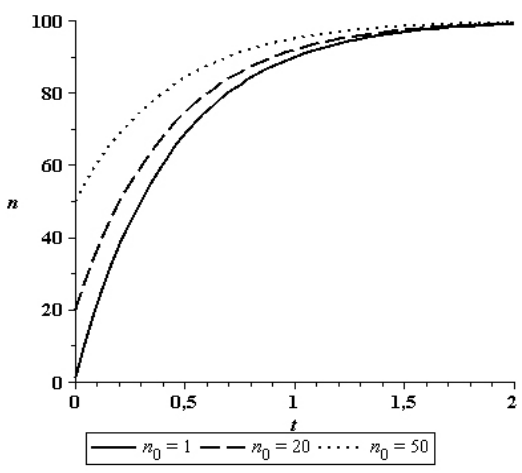

Figure 5: Convergence of different product innovations paths

5.3 Interdependence between both types of innovations

Now consider the general pattern of innovative activities of the monopolist. Putting together the results derived from (18) and (26) one may observe the overall process. At each point in time the rate of product innovations is defined from the current value of future profit from the development of the next boundary product, $V(q_i)_{i=n(t)}$ and the efficiency of investments themselves, α. Process innovations q_i for each such a new product start only after the introduction of the product i, $t(0)_i$ which is defined by the product innovations process $n(t)$ as its inverse function.

$$t(0)_i = f^{-1}(n^{opt}(t))|_{n(t)=i};$$

$$t(0)_i = -\frac{\sqrt{r^2(r+\beta)^2(r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+r^2(r+\beta)^2}{\alpha^2\gamma^2 r}\times$$

$$\times \ln(\frac{N-i}{N-n_0}) > 0. \tag{27}$$

This moment is different for all the potential products. The rate of investments into the improvement of production technology for each such product depends on its efficiency of investment, γ_i and decay rate of technology, identical for all products, β. The efficiency of investments is defined as a function of the product's position relative to other products, i, in the products' range N. In this paper it is assumed to reflect the increased complexity of investments in every next product and is defined as (3). Observe that the function (27) specifies explicitly what is the time of the start of development of process innovations for every product i. From the function $t(0)_i$ one may conclude that the rate of introduction of new products slows down as the position of the new product approaches the limit range, N. Then at every such moment in time the investment opportunities for the monopolist are different. These opportunities are lesser for every new product introduced and developed, since the space of products **I** has the maximal range of different versions of the same basic product which may be introduced into the market, N. As a result every next product differs from all others by the level of its production technology which may be achieved, $\bar{q}_i$.

The Figure 6 puts together all the information about the monopolist's behaviour: the solid black line represents the product innovations while red lines are process innovations for every of the introduced products i. It may be seen that every such a process eventually reaches its boundary. Starting from that point the production technology for this product cannot be further refined and the profit from the product i cannot further increase. The profit-seeking behaviour of the monopolist pushes him to the product innovations (i. e. introduction of further new versions of the basic product) and the improvement of technology for other yet underdeveloped products. In infinite time this process reaches its limits when the monopolist introduces all possible versions of the product, $i = N$. At this point the industry as a whole enters the mature stage of its technological cycle and new, fundamental inventions are necessary to boost its growth.

Now consider the aggregate process of quality-improving innovations, which is given by $q(n(t), t)$. This function describes for each value of $n(t)$ the

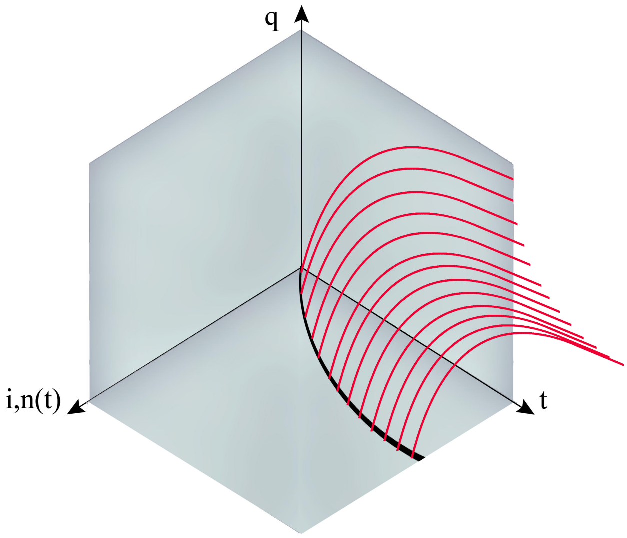

Figure 6: Reconstruction of product and process innovations

evolution of quality of the product $i = n(t)$ and also the process of emergence of new products, $n(t)$. This function depends positively on product innovations, as these latter increase the range of available process innovations. Thus with every new introduced product the intensity of process innovations as a whole increases. At the same time the process of refinement of production technology for each of the products separately does not depend on the product innovations after the introduction of the product. Formally this means:

$$q(n(t), t) = \int_0^{n(t)} q(i, t) di;$$

$$\frac{\partial q(n(t), t)}{\partial n(t)} = n(t) \times q(i, t)|_{i=n(t)} \geq 0;$$

$$\frac{\partial q(i, t)}{\partial n(t)} = \begin{cases} \frac{\partial q(i,t)|_{i \neq n(t)}}{\partial n(t)} = 0, \\ \\ \frac{\partial q(i,t)|_{i=n(t)}}{\partial n(t)} > 0. \end{cases}$$

Since the variety expansion process $n(t)$ is slowing down in time, this dependence is weakening in time also. Hence the interdependence of different innovations' types can be observed in both directions and is not monotonic on the aggregate level. This result is in line with general lines of (Albernathy and Utterback 1978) and follow-up papers.

6 Discussion

In this paper the simple and rather stylized model of the monopolist firm engaged in the product and process innovation activities is introduced. Main distinguishing features of the model are:

- The model is fully dynamic and both types of innovations are directly controlled through investments by the monopolist;
- Process innovations directly influence the profit generation and thus product innovations intensity while the latter define only the starting point of process innovations for each new product;
- However the aggregate process innovations are the increasing function of product innovations also;
- The product innovations is the generator of heterogeneous new products and process innovations for each of these products has different dynamics and long-run technology levels;
- With the increase of the range of invented products the intensity of introduction of other new products is decreasing, while process innovations are increasing. In the limit volume of product innovations investments is zero while the volume of process innovations investments reaches its maximum at $n(t) = N, t \to \infty$.

The first feature is rarely realized in the literature, as usually one or the other type of innovations is assumed to be fixed at some certain level while the other is directly controlled. It is argued that the simultaneous control over both types of innovative activities is essential for the optimal management of the multi product monopoly which is a typical situation in many industries. If such a monopolist would only introduce new products but will not develop production technology from its starting level, then this monopolist will not be able to derive new profit from these new products since they all have a zero level of process development from the start. Of course the profit is still derived from the product through sales but it is not optimal not to improve the production technology with the goal to cut production costs as this yields an immediate increase in profits without changes in price or production policies.

At the same time process innovations may not be stimulated without the product innovations and their dynamics. If one would assume the range of existing products constant and not changing eventually the monopolist will improve the technology of all the products which are already produced without the development of new ones. Then all the improvements to production technologies of already introduced products, n_0 will start at the same moment of time and the monopolist will not have the criteria for the time management of these innovations. The product innovations process gradually introduces new products and thus the monopolist has time to allocate his efforts between different products. One may conclude that the product innovations process is the generator of new process innovations for every new product although it does not directly influence the subsequent process of technology improvements for every product separately. This one-way dependence of product innovations from the process innovations is in line with the empirical literature (Kraft 1990). However, as it has been shown, the model contains the reverse dependence also in the form of aggregate distribution of process innovations and their total volume as a function of product innovations, $q(n(t), t)$. This last and its intensity has the same shape as predicted by (Albernathy and Utterback 1978): volume of process innovations increases with invention of new products while volume of product innovations intensity decreases. It is possible to define analytically and exactly the point in time when one stage of innovative activity is changed to another, since dynamic laws for both types of innovations are known. Hence one may exactly define the length of all the stages and the relative measure of all the innovative activities.

The next important feature of the model is the explicit heterogeneity of the products which are being introduced into the market by the monopolist. This heterogeneity is covered by the form of the investment efficiency parameter, γ_i. With $\gamma_i = \gamma$ all the new products would be identical in their investment characteristics, while any other form of this function makes them different. In the current setting it is assumed that every next product has lesser efficiency of investments then all the preceding ones. As it can be seen from the previous section, this leads to different levels and intensities of technology development for all the new products. Every next introduced product has the maximal level of process innovations lower then the preceding one. Due to the increasing weight of maintaining the existing level of technology it is less profitable to develop the production technology of the existing versions of the basic product only and refuse from the introduction and development of new ones even if they are more complicated. It is the increasing burden of maintaining the refined production technology which drives the introduction of new, more complicated and otherwise less profitable products into the market. As long as these new potential products have positive expected payoff, the intensity of introduction of new products is positive. However the increasing complexity of these new products lowers the expected benefits from their development (more investments are required for the development of this products) gradually decreases the intensity of this process. As more and more products are introduced, more investments are required for simply maintaining the existing level of technology for all these products, which is given by the quantity $\int\limits_0^{n(t)} g(i,t)di$ while the process of new products introduction $\alpha u(t)$ slows down. Observe that without the heterogeneity this will not be the case since every next product will essentially be the same as the basic one in terms of its production technology and no additional investments will be then needed for its development.

In such a situation there will be no limits for the introduction of new versions of the basic product while this is obviously not the case. For any given industry there is a limited research capacity for the refinements of the basic product being sold by the industry. In the other case no new industries will be never formed, while the cliometric analysis (Crafts 2010) clearly indicates that the process of the creation of new industries and the decay of the old ones is one of the important components of the technological growth in the modern economy (Schumpeter 1942).

At last observe that the dynamics of innovations, generated by the suggested stylized model is in line with empirical findings in the field (Faria and Lima 2009), (Salo, Karjaluoto, and Jussila 2007). While the industry is young, there are a lot of opportunities for the introduction of new products and for the development of process innovations associated with them. At this stage product innovation investments outweigh the process ones since there are not very much products which require process innovations. As the industry matures, these opportunities shrink and it is harder to bring something new into the industry. Hence the product innovations decrease in volume while process innovations grow because there are more products which require refinement. At the same time for each of the product process innovations has the decreasing intensity in time due to the burden of maintaining technology. Unlike the space of ideas which marks more fundamental level of difference between invention and innovation this work concentrates solely on two types of innovations which are always limited in their nature without the underlying process of creation of new knowledge in academia.

The suggested approach allows for a number of immediate and fruitful extensions for analysis of more applied questions. One may model the strategic interactions between several firms instead of a single monopolist in this framework. Such strategic interactions would possibly lead to the endogenous specialization of innovative activities of agents. Another extension is to allow for limited life cycles of new introduced products. This would allow for the analysis of patenting policy efficiency in the framework of heterogeneous innovative products.

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