Understanding the Impact of Weights Constraints in Portfolio Theory

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Abstract

In this article, we analyze the impact of weights constraints in portfolio theory using the seminal work of Jagannathan and Ma (2003). They show that solving the global minimum variance portfolio problem with some constraints on weights is equivalent to use a shrinkage estimate of the covariance matrix. These results may be easily extended to mean variance and tangency portfolios. From a financial point of view, the shrinkage estimate of the covariance matrix may be interpreted as an implied covariance matrix of the portfolio manager. Using the universe of the DJ Eurostoxx 50, we study the impact of weights constraints on the global minimum variance portfolio and the tangency portfolio. We illustrate how imposing lower and upper bounds on weights modify some properties of the empirical covariance matrix. Finally, we draw some conclusions in the light of recent developments in the asset management industry.

Keywords: Global minimum variance portfolio, Markowitz optimization, tangency portfolio, Lagrange coefficients, shrinkage methods, covariance matrix.

JEL classification: G11, C60.

1 Introduction

We consider a universe of n assets. We denote by μ the vector of their expected returns and by Σ the corresponding covariance matrix. Let us specify the Markowitz problem in the following way:

$$\begin{aligned}&\min \frac{1}{2} w^{\top} \Sigma w \\ &\text{u.c.} \left\{ \begin{array}{l} \mathbf{1}^{\top} w = 1 \\ w \in \Omega \bigcap \mathcal{C} \end{array} \right.\end{aligned} \tag{1}$$

where w is the vector of weights in the portfolio and Ω is the search space. For example, if $\Omega = \mathbb{R}^n$, the optimisation problem defines the global minimum variance portfolio. if

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$\Omega = \left\{w \in \mathbb{R}^n : \mu^\top w \geq \mu^\star\right\}$, we obtain the efficient portfolio where $\mu^\star$ is the desired expected return of the investor. The tangency portfolio is the efficient portfolio which maximizes the Sharpe ratio. $\mathcal{C}$ is the set of weights constraints. We consider two definitions of $\mathcal{C}$:

1. $\mathcal{C}$ is equal to $\mathbb{R}^n$. In this case, the solution is unconstrained and we note it $w^\star$ or $w^\star(\mu, \Sigma)$.

2. We may impose some bounds $w_i^- \leq w_i \leq w_i^+$ on the weight of the asset i. In this case, we note $\mathcal{C} = \mathcal{C}(w^-, w^+)$ and we define $\tilde{w}$ as the solution of the corresponding optimisation problem.

The idea of this paper is to analyse the impact of constraints $\mathcal{C}(w^-, w^+)$ on the discrepancy between $w^\star$ and $\tilde{w}$. Following Jagannathan and Ma (2003), we may show that the constrained solution may be obtained by solving the unconstrained problem with another specification of μ and Σ. We have also:

$$\tilde{w} = w^\star\left(\tilde{\mu}, \tilde{\Sigma}\right)$$

where $\tilde{\mu}$ and $\tilde{\Sigma}$ are perturbations of the original vector of expected returns μ and the original covariance matrix Σ. Traditionally, the impacts of weights constraints are analysed by studying the difference between $w^\star$ and $\tilde{w}$. In this paper, we analyse these impacts by studying the difference between the implied parameters $\tilde{\mu}$ and $\tilde{\Sigma}$ and the original parameters μ and Σ.

The paper is organized as follows. In section two, we review the main results of Jagannathan and Ma (2003) and we illustrate these results with an example. In section three, we consider an empirical application on the DJ Eurostoxx 50 universe. We focus on the global minimum variance portfolio and the tangency portfolio, because Demey *et al.* (2010) has shown that restrictive constraints on weights should be imposed for these methods in order to avoid extreme concentration in optimized portfolios. We illustrate how these constraints impact the covariance matrix, in terms of volatilities, correlations and risk factors. Finally, section 4 draws some conclusions in the light of recent developments in the asset management industry.

2 The effects of constraints in portfolio theory

In this section, we review the main results of Jagannathan and Ma (2003) for three portfolio optimisation problems (global minimum variance portfolio, mean variance portfolio and tangency portfolio). We illustrate each problem with the same generic example of 4 assets and the covariance matrix given in Table 1.

Table 1: Specification of the covariance matrix Σ (in %)

σ_i	$\rho_{i,j}$			
15.00	100.00			
20.00	10.00	100.00		
25.00	40.00	70.00	100.00	
30.00	50.00	40.00	80.00	100.00

2.1 The global minimum variance portfolio

2.1.1 Analytics of the solution

The global minimum variance portfolio corresponds to the solution of the optimisation problem (1) when $\Omega = \mathbb{R}^n$ and $\mathcal{C} = \mathbb{R}^n$. We define the Lagrange function as:

$$f(w;\lambda_0) = \frac{1}{2} w^\top \Sigma w - \lambda_0 \left(\mathbf{1}^\top w - 1\right)$$

with $\lambda_0 \geq 0$. The first order conditions are:

$$\begin{cases} \Sigma w - \lambda_0 \mathbf{1} = 0 \\ \mathbf{1}^\top w - 1 = 0 \end{cases}$$

We deduce that the optimal solution is:

$$w^\star = \frac{1}{\mathbf{1}^\top \Sigma \mathbf{1}} \Sigma^{-1} \mathbf{1}$$

This solution depends only on the covariance matrix Σ and we note $w^\star = w^\star(\Sigma)$.

If we impose now the weights constraints $\mathcal{C}(w^-, w^+)$, the Lagrange function becomes:

$$\begin{aligned} f\left(w;\lambda_0,\lambda^-,\lambda^+\right) &= \frac{1}{2} w^\top \Sigma w - \lambda_0 \left(\mathbf{1}^\top w - 1\right) - \\ & \quad {\lambda^-}^\top \left(w - w^-\right) - {\lambda^+}^\top \left(w^+ - w\right) \end{aligned}$$

with $\lambda_0 \geq 0$, $\lambda_i^- \geq 0$ and $\lambda_i^+ \geq 0$. In this case, the Kuhn-Tucker conditions are:

$$\begin{cases} \Sigma w - \lambda_0 \mathbf{1} - \lambda^- + \lambda^+ = 0 \\ \mathbf{1}^\top w - 1 = 0 \\ \min\left(\lambda_i^-, w_i - w_i^-\right) = 0 \\ \min\left(\lambda_i^+, w_i^+ - w_i\right) = 0 \end{cases}$$

It is not possible to obtain an analytic solution but we may numerically solve the optimisation problem using a quadratic programming algorithm.

2.1.2 An implied covariance matrix

Given a constrained portfolio $\tilde{w}$, it is possible to find a covariance matrix $\tilde{\Sigma}$ such that $\tilde{w}$ is the solution of the global minimum variance portfolio. Let $\mathcal{E} = \left\{\tilde{\Sigma} > 0 : \tilde{w} = w^\star\left(\tilde{\Sigma}\right)\right\}$ denotes the corresponding set. We have:

$$\mathcal{E} = \left\{\tilde{\Sigma} > 0 : \left(\mathbf{1}^\top \tilde{\Sigma} \mathbf{1}\right) \cdot \tilde{\Sigma} \tilde{w} = \mathbf{1}\right\}$$

Of course, the set $\mathcal{E}$ contains several solutions. From a financial point of view, we are interested to covariance matrices $\tilde{\Sigma}$ which are closed to Σ. Jagannathan and Ma (2003) remark that the matrix $\tilde{\Sigma}$ defined by:

$$\tilde{\Sigma} = \Sigma + \left(\lambda^+ - \lambda^-\right) \mathbf{1}^\top + \mathbf{1} \left(\lambda^+ - \lambda^-\right)^\top \tag{2}$$

is a solution of $\mathcal{E}$. It is easy to show that $\tilde{\Sigma}$ is a positive definite matrix and we have:

$$\begin{aligned}
\tilde{\Sigma}\tilde{w} &= \Sigma\tilde{w} + \left(\lambda^{+} - \lambda^{-}\right)\mathbf{1}^{\top}\tilde{w} + \mathbf{1}\left(\lambda^{+} - \lambda^{-}\right)^{\top}\tilde{w} \\
&= \Sigma\tilde{w} + \left(\lambda^{+} - \lambda^{-}\right) + \mathbf{1}\left(\lambda^{+} - \lambda^{-}\right)^{\top}\tilde{w} \\
&= \lambda_0\mathbf{1} + \mathbf{1}\left(\lambda_0\mathbf{1} - \Sigma\tilde{w}\right)^{\top}\tilde{w} \\
&= \left(2\lambda_0 - \tilde{w}^{\top}\Sigma\tilde{w}\right)\mathbf{1}
\end{aligned}$$

Because $\tilde{\Sigma}\tilde{w}$ is a constant vector, it proves that $\tilde{w}$ is the solution of the unconstrained optimisation problem[1].

The implied covariance matrix defined by the equation (2) is very interesting for two points:

- This implied covariance matrix is easy to compute when we has solved the constrained optimisation problem, because it only requires the computation of the Lagrange coefficients.

- This implied covariance matrix has a natural interpretation. Indeed, we have:

$$\tilde{\Sigma}_{i,j} = \Sigma_{i,j} + \Delta_{i,j}$$

where the elements of the perturbation matrix are:

$(\Delta)_{i,j}$	w_i^-	$\left]w_i^-, w_i^+\right[$	w_i^+
w_j^-	$-\left(\lambda_i^- + \lambda_j^-\right)$	$-\lambda_j^-$	$\lambda_i^+ - \lambda_j^-$
$\left]w_j^-, w_j^+\right[$	$-\lambda_i^-$	0	λ_i^+
w_j^+	$\lambda_j^+ - \lambda_i^-$	λ_j^+	$\lambda_i^+ + \lambda_j^+$

The perturbation $\Delta_{i,j}$ may be negative, nul or positive. It is nul when the optimized weights do not reach the constraints $\tilde{w}_i \neq \left(w_i^-, w_i^+\right)$ and $\tilde{w}_j \neq \left(w_j^-, w_j^+\right)$. It is positive (resp. negative) when one asset reaches its upper (resp. lower) bound whereas the second asset does not reach its lower (resp. upper) bound. Introducing weights constraints is also equivalent to apply a shrinkage method to the covariance matrix (Ledoit and Wolf, 2003). Lower bounds have a negative impact on the volatility whereas upper bounds have a positive impact on the volatility:

$$\tilde{\sigma}_i = \sqrt{\sigma_i^2 + \Delta_{i,i}}$$

The impact on the correlation coefficient is more complex. In the general case, we have:

$$\tilde{\rho}_{i,j} = \frac{\rho_{i,j}\sigma_i\sigma_j + \Delta_{i,j}}{\sqrt{\left(\sigma_i^2 + \Delta_{i,i}\right)\left(\sigma_j^2 + \Delta_{j,j}\right)}}$$

The correlation may increase or decrease depending on the magnitude of the Lagrange coefficients with respect to the parameters $\rho_{i,j}$, σ_i and σ_j.

[1] The lagrange coefficient $\lambda_0^{\star}$ for the unconstrained problem is $2\tilde{\lambda}_0 - \tilde{w}^{\top}\Sigma\tilde{w}$ where $\tilde{\lambda}_0$ is the Lagrange coefficient for the constrained problem.

2.1.3 An illustrative example

Let us consider the universe of 4 assets with the covariance matrix specified in Table 1. Given these parameters, the global minimum variance portfolio is equal to:

$$w^{\star} = \left(\begin{array}{r} 72.742\% \\ 49.464\% \\ -20.454\% \\ -1.753\% \end{array} \right)$$

In this portfolio, we have two long positions on the first and second assets and two short positions on the third and four assets. Suppose now that we impose a no short-selling constraint. All the results (in %) are reported in Table 2. In the constrained optimized portfolio, the weights of the third and fourth assets are set to zero. Imposing the constraints $w_i \geq 0$ implies also to decrease the volatility of these two assets. Indeed, the implied volatility $\tilde{\sigma}_i$ of the third asset is equal to 22.4% whereas its volatility σ_i is equal to 25%. Concerning the correlations, we notice that they are lower than the original ones, but the difference is small.

Table 2: Global minimum variance portfolio when $w_i \geq 0$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
65.487	0.000	0.000	15.000	100.000			
34.513	0.000	0.000	20.000	10.000	100.000		
0.000	0.613	0.000	22.413	26.375	64.398	100.000	
0.000	0.725	0.000	27.478	37.005	30.483	75.697	100.000

In Table 3 and 4, we report the results when the lower bound is respectively 10% and 20%. It is interesting to notice that the ranking of volatilities is not preserved. Finally, we illustrate in Table 4 the case when both lower and upper bounds are imposed. In the previous results, we observe a decrease of volatilities and correlations. In Table 4, the effect is more complex and correlations may increase and decrease.

Table 3: Global minimum variance portfolio when $w_i \geq 10\%$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
56.195	0.000	0.000	15.000	100.000			
23.805	0.000	0.000	20.000	10.000	100.000		
10.000	1.190	0.000	19.671	10.496	58.709	100.000	
10.000	1.625	0.000	23.980	17.378	16.161	67.518	100.000

Table 4: Global minimum variance portfolio when $w_i \geq 20\%$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
40.000	0.000	0.000	15.000	100.000			
20.000	0.390	0.000	17.944	−3.344	100.000		
20.000	2.040	0.000	14.731	−24.438	40.479	100.000	
20.000	2.670	0.000	19.131	−14.636	−19.225	45.774	100.000

Table 5: Global minimum variance portfolio when $0\% \leq w_i \leq 50\%$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
50.000	0.000	1.050	20.857	100.000			
50.000	0.000	0.175	20.857	35.057	100.000		
0.000	0.175	0.000	24.290	46.881	69.087	100.000	
0.000	0.000	0.000	30.000	52.741	41.154	79.937	100.000

2.2 The mean variance portfolio

Let us now consider the problem when we impose to reach an expected return:

$$\mu^\top w = \mu^\star$$

Without constraints on bounds, the Lagrange function is:

$$f(w;\lambda_0,\lambda_1) = \frac{1}{2}w^\top \Sigma w - \lambda_0\left(\mathbf{1}^\top w - 1\right) - \lambda_1\left(\mu^\top w - \mu^\star\right)$$

with $\lambda_0 \geq 0$ and $\lambda_1 \geq 0$. The first order conditions are:

$$\left\{\begin{array}{l} \Sigma w - \lambda_0 \mathbf{1} - \lambda_1 \mu = 0 \\ \mathbf{1}^\top w - 1 = 0 \\ \mu^\top w - \mu^\star = 0 \end{array}\right.$$

With constraints on bounds, the Lagrange function becomes:

$$\begin{aligned} f\left(w;\lambda_0,\lambda^-,\lambda^+\right) &= \frac{1}{2}w^\top \Sigma w - \lambda_0\left(\mathbf{1}^\top w - 1\right) - \lambda_1\left(\mu^\top w - \mu^\star\right) - \\ &\quad {\lambda^-}^\top\left(w - w^-\right) - {\lambda^+}^\top\left(w^+ - w\right) \end{aligned}$$

with $\lambda_0 \geq 0$, $\lambda_1 \geq 0$, $\lambda_i^- \geq 0$ and $\lambda_i^+ \geq 0$. In this case, the Kuhn-Tucker conditions become:

$$\left\{\begin{array}{l} \Sigma w - \lambda_0 \mathbf{1} - \lambda_1 \mu - \lambda^- + \lambda^+ = 0 \\ \mathbf{1}^\top w - 1 = 0 \\ \min\left(\lambda_i^-, w_i - w_i^-\right) = 0 \\ \min\left(\lambda_i^+, w_i^+ - w_i\right) = 0 \end{array}\right.$$

We may show that the constrained portfolio $\tilde{w}$ is the solution of the unbounded optimization problem:

$$\tilde{w} = w^\star\left(\tilde{\mu},\tilde{\Sigma}\right)$$

with the following implied expected returns and covariance matrix[2]:

$$\left\{\begin{array}{l} \tilde{\mu} = \mu \\ \tilde{\Sigma} = \Sigma + \left(\lambda^+ - \lambda^-\right)\mathbf{1}^\top + \mathbf{1}\left(\lambda^+ - \lambda^-\right)^\top \end{array}\right.$$

[2]Indeed, we have:

$$\begin{aligned} \tilde{\Sigma}\tilde{w} &= \Sigma\tilde{w} + \left(\lambda^+ - \lambda^-\right)\mathbf{1}^\top\tilde{w} + \mathbf{1}\left(\lambda^+ - \lambda^-\right)^\top\tilde{w} \\ &= \lambda_0\mathbf{1} + \lambda_1\mu + \mathbf{1}\left(\lambda_0\mathbf{1} - \Sigma\tilde{w}\right)^\top\tilde{w} \\ &= \left(2\lambda_0 - \tilde{w}^\top\Sigma\tilde{w} + \mu^\star\lambda_1\right)\mathbf{1} + \lambda_1\mu \end{aligned}$$

It proves that $\tilde{w}$ is the solution of the unbounded optimization problem with Lagrange coefficients $\lambda_0^\star = 2\lambda_0 - \tilde{w}^\top\Sigma\tilde{w} + \mu^\star\lambda_1$ and $\lambda_1^\star = \lambda_1$.

We consider the previous example and we assume that expected returns are respectively 5%, 3%, 7% and 7%. In this case, the optimal portfolio for $\mu^\star = 6\%$ is $w_1^\star = 77.120\%$, $w_2^\star = -13.560\%$, $w_3^\star = 56.022\%$ and $w_4^\star = -19.582\%$. If we impose that the weights are between 0% and 40%, we obtain results in Table 6. We remark that the correlation between the first and second assets increases by 30% whereas the other implied correlations are very close to the original correlations. The underlying idea is to reduce the diversification component between the first two assets in order to decrease the weight of these two assets in the portfolio.

Table 6: Mean variance portfolio when $0\% \leq w_i \leq 40\%$ and $\mu^\star = 6\%$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
40.000	0.000	1.573	23.227	100.000			
5.000	0.000	0.000	20.000	40.308	100.000		
40.000	0.000	0.595	27.276	57.888	75.065	100.000	
15.000	0.000	0.000	30.000	54.857	40.000	80.595	100.000

2.3 The tangency portfolio

The optimisation problem to solve the tangency portfolio is:

$$\begin{aligned} & \max \frac{(\mu - r)^\top w}{\sqrt{w^\top \Sigma w}} \\ & \text{u.c.} \left\{ \begin{array}{l} \mathbf{1}^\top w = 1 \\ w \in \Omega \bigcap \mathcal{C} \end{array} \right. \end{aligned} \tag{3}$$

With this specification, it is difficult to use the previous framework. Nevertheless, since the seminal work of Harry Markowitz and because the tangency portfolio belongs to the efficient frontier, we know that the tangency portfolio is the solution of a quadratic programming problem. More formally, we have (Roncalli, 2010):

$$\begin{aligned} & \min \frac{1}{2} w^\top \Sigma w - \phi w^\top \mu \\ & \text{u.c.} \left\{ \begin{array}{l} \mathbf{1}^\top w = 1 \\ w \in \Omega \bigcap \mathcal{C} \end{array} \right. \end{aligned} \tag{4}$$

Let $w^\star$ be the tangency portfolio for the unconstrained problem $\Omega = \mathbb{R}^n$ and $\tilde{w}$ be the tangency portfolio for the constrained problem with $\mathcal{C} = \mathcal{C}(w^-, w^+)$. We have:

$$\tilde{w} = w^\star \left(\tilde{\mu}, \tilde{\Sigma}, \tilde{\phi} \right)$$

with $\tilde{\phi}$ the optimal value of ϕ for the constrained optimisation program and:

$$\left\{ \begin{array}{l} \tilde{\mu} = \mu \\ \tilde{\Sigma} = \Sigma + (\lambda^+ - \lambda^-) \mathbf{1}^\top + \mathbf{1} (\lambda^+ - \lambda^-)^\top \end{array} \right.$$

Let us consider the special case where all the assets have the same Sharpe ratio (Martinelli, 2008), that is when expected excess returns are proportional to volatilities. This tangency portfolio is known as the MSR portfolio[3]. Using the previous example, we obtain results in Table 7. This example is interesting because it illustrates how imposing weights constraints may modify the underlying assumptions of portfolio theory. In the case of the MSR portfolio, the central assumption is that all the assets have the same Sharpe ratio. However, this assumption is only true in the unconstrained problem. If we impose some weight constraints, it is obvious that this assumption does not hold. The question is how far is the optimized portfolio from the key assumption. If we consider the optimized portfolio given in Table 7 and if we assume that the Sharpe ratio is 0.5 for all the assets, the implied Sharpe ratio does not change for the third and four assets, but is is respectively equal to 0.381 and 0.444 to the first and second assets.

Table 7: MSR portfolio when $0\% \leq w_i \leq 40\%$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
40.000	0.000	0.810	19.672	100.000			
40.000	0.000	0.540	22.539	37.213	100.000		
0.000	0.000	0.000	25.000	46.970	71.698	100.000	
20.000	0.000	0.000	30.000	51.850	43.481	80.000	100.000

3 Empirical results with the Eurostoxx 50 universe

In this section, we consider an application of the previous framework on risk-based indexation, which is part (with fundamental indexation) of alternative-weighted indexation. Since some years, capitalization-weighted indexes have faced some criticisms because of their trend-following style and their lack of risk diversification. The study of Demey *et al.* (2010) focuses on four popular risk-based indexation methods: the equally-weighted portfolio (EW), the ERC portfolio[4], the global minimum variance portfolio (MIN) and the MDP/MSR portfolio. For these two last methods, the authors show that we need to impose weights constraints in order to limit the portfolio concentration and the turnover. In this section, we consider the universe of the DJ Euro Stoxx 50 Index from January 1992 to December 2009. The estimated covariance matrix corresponds to the empirical covariance matrix with a one-year lag window and the portfolio is rebalanced every end of the month. In Figure 1, we report the Lorenz curve of weights and risk contributions of the different allocation methods as well as the capitalization method (MCAP). The MIN and MSR portfolios appear to be more concentrated than the MCAP portfolio. We remark that the maximum weight may reach respectively 60% and 40%. In average, the MIN portfolio contains 14 stocks whereas the MSR portfolio contains 19 stocks.

This strong concentration implies that the turnover of the MIN and MSR portfolios may be high. That explains that these two strategies are implemented with some weights constraints in practice. For example, if we consider an upper bound of 5%, we obtain results in Figure 2. These constrained portfolios are more balanced and Demey *et al.* (2010) show that the turnover is reduced by a factor of two. In this section, we extend this study in order to analyze the impact of imposing weights constraints on the implied covariance matrix.

[3] It is also called the "Most Diversified Portfolio" by Choueifaty and Coignard (2008).

[4] The ERC portfolio corresponds to the portfolio in which every asset has the same risk contribution (Maillard *et al.*, 2010)

Figure 1: Statistics of weights

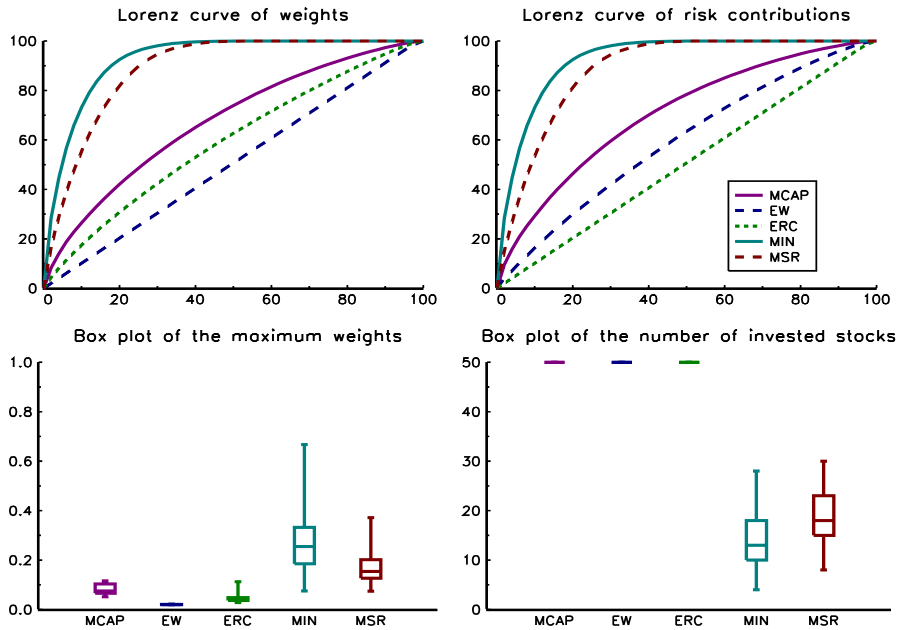
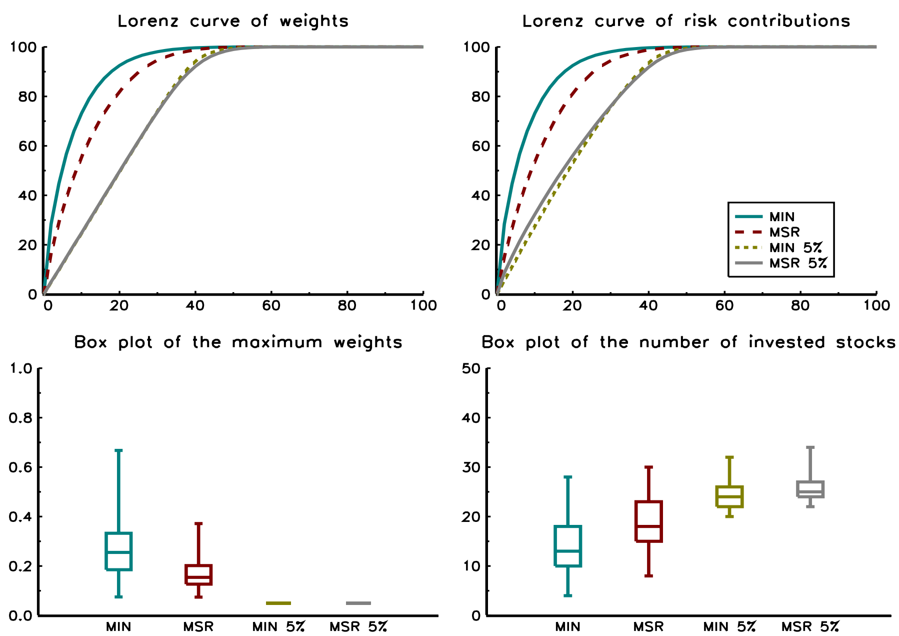


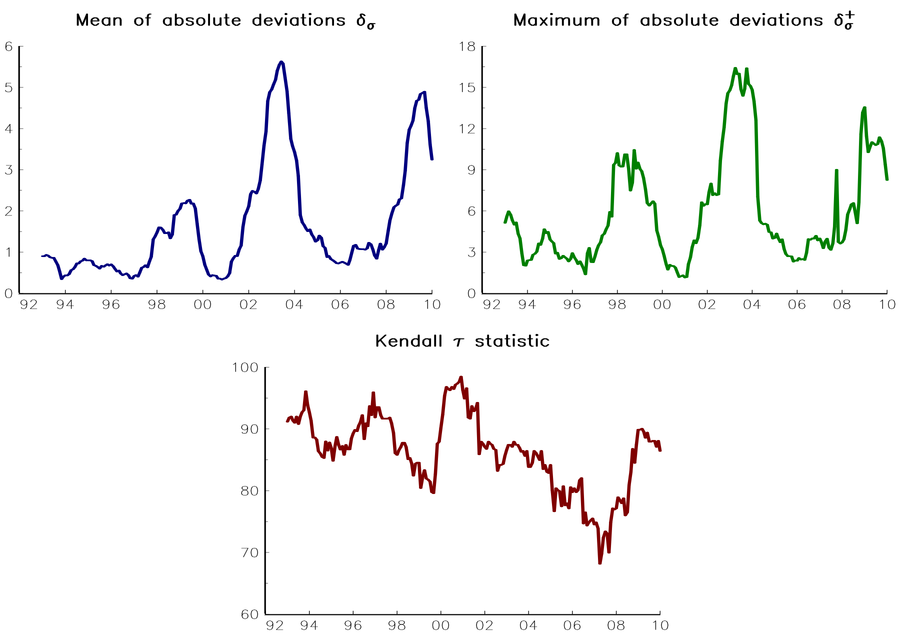
Figure 2: Statistics of weights with a 5% upper bound



3.1 Application to the global minimum variance portfolio

As Demey *et al.* (2010), we consider the weights constraints $\mathcal{C}\left(0\%, 5\%\right)$. At each rebalancing date t, we compute the one-year empirical covariance matrix Σ_t, we then estimate the constrained optimized portfolio $\tilde{w}_t$ and deduce the implied shrinkage covariance matrix $\tilde{\Sigma}_t$. In Figure 3, we report some results on the volatility. We define the mean and the maximum of absolute deviations as $\delta_\sigma = \frac{1}{n}\sum_{i=1}^{n}\left|\tilde{\sigma}_{i,t} - \sigma_{i,t}\right|$ and $\delta_\sigma^+ = \max_i \left|\tilde{\sigma}_{i,t} - \sigma_{i,t}\right|$. Generally, δ_σ takes a small value. Nevertheless, we observe two periods (Dec-02 to Feb-04 and May-08 to Dec-09) when the mean of absolute deviations is bigger than 2%. During these periods, we may observe deviations larger than 10% between implied and original volatilities. In Figure 3, we also report the Kendall τ statistic between the volatilities $\tilde{\sigma}_t$ and σ_t. This statistic measures the coherency of ranking. Generally, the rank correlation is very high, but there is one period when it falls below 80%.

Figure 3: Impact (in %) on the volatilities



The impact on the correlations is more important than the impact on the volatilities. In Figure 4, we report the statistics[5] $\delta_\sigma = \frac{2}{n(n-1)}\sum_{i>j}\left|\tilde{\rho}_{i,j,t} - \rho_{i,j,t}\right|$, $\delta_\rho^+ = \max_{i,j}\left|\tilde{\rho}_{i,j,t} - \rho_{i,j,t}\right|$ and $\pi_\rho\left(x\right) = \frac{2}{n(n-1)}\sum_{i>j}\mathbf{1}\left\{\left|\tilde{\rho}_{i,j,t} - \rho_{i,j,t}\right| > x\right\}$. Moreover, we may observe periods for which an absolute deviation bigger than 10% may concern more than 30% of the correlations. As a consequence, we may think that weights constraints may have a significant impact on the risk factor decomposition of the covariance matrix. To verify this point, we consider a principal component analysis of the covariance matrices Σ_t and $\tilde{\Sigma}_t$. Let λ_j and $\tilde{\lambda}_j$ be the normalized eigenvalues. We report the differences $\lambda_j - \tilde{\lambda}_j$ in Figure 5. The first risk factor may be considered as the market risk factor. In average, this risk factor explains 70% of the variance of the stocks. We notice that weights constraints have a negative impact on this risk factor. It means that the representation quality of the market risk factor is lower for

[5] $\pi_\rho\left(x\right)$ indicates how many absolute deviations are larger than x in mean.

Figure 4: Impact (in %) on the correlations

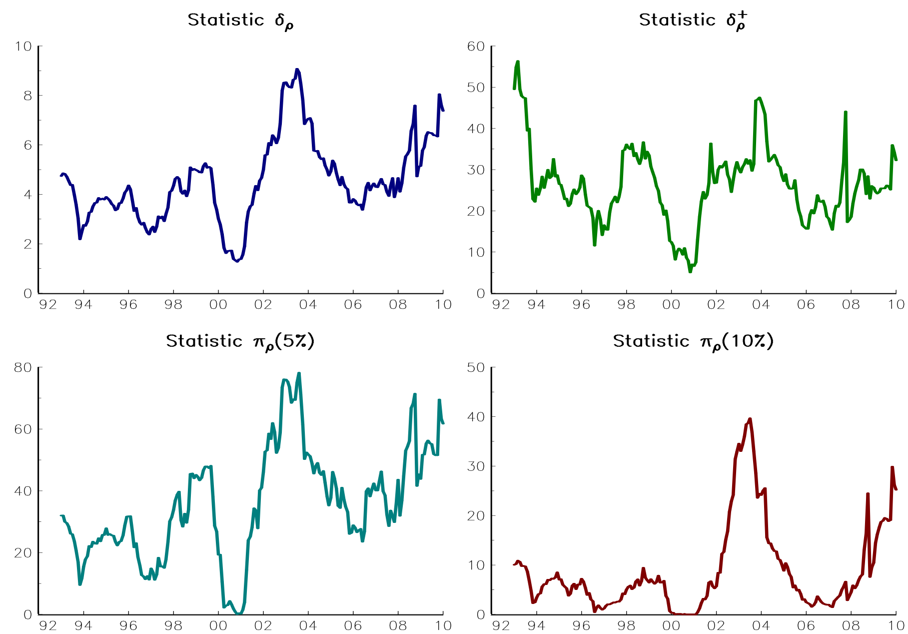
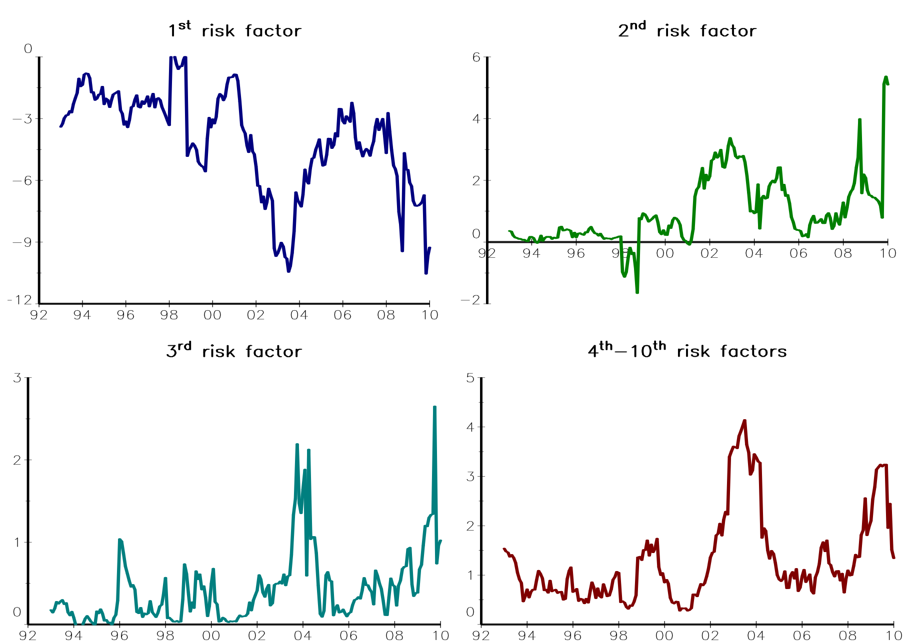


Figure 5: Difference $\lambda_j - \tilde{\lambda}_j$ (in %)

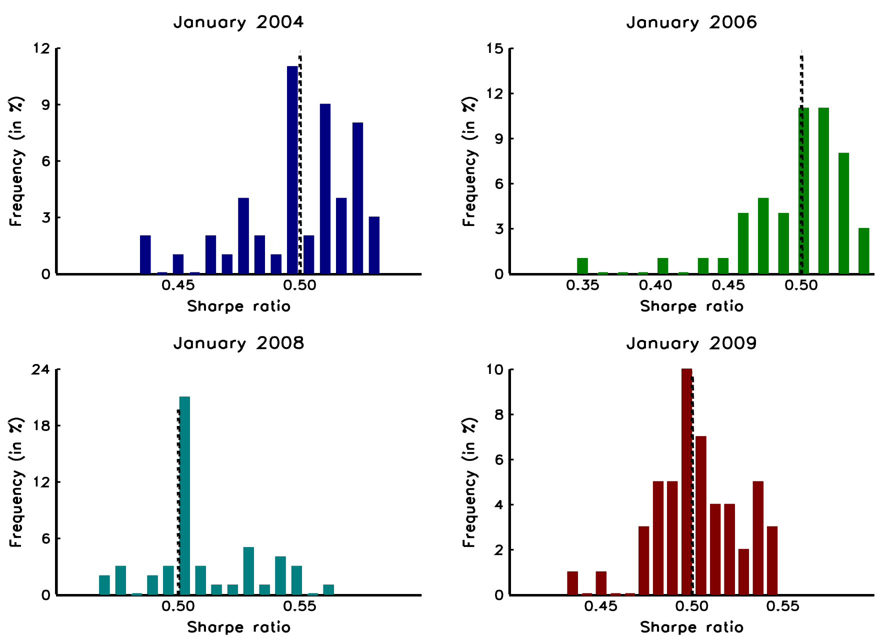


the implied covariance matrix than for the empirical covariance matrix. In the same time, the representation quality of the other factors increases as illustrated in Figure 5. This result is disturbing because the general principle of shrinkage methods is to decrease the representation quality of the last risk factors in order to reinforce the first risk factors.

3.2 Application to the MSR portfolio

If we consider the MSR portfolio, we obtain similar results but the difference between the covariance matrix Σ_t and $\tilde{\Sigma}_t$ are generally smaller than for the MIN portfolio[6]. More interesting is the impact on the Sharpe ratio. In the theory of the MSR portfolio, all the assets present the same constant Sharpe ratio. But this assumption is not valid in the case of weights constraints. We report the empirical probability density function of the implied Sharpe ratio for several dates in Figure 6. We notice that Sharpe ratio varies between 0.30 and 0.60 whereas the theoretical Sharpe ratio is 0.5, and the differences depend of the rebalancing date.

Figure 6: Density of the implied Sharpe ratio



Remark 1 *The differences between the values of the theoretical Sharpe and the implied Sharpe depends on the number of assets in the universe and the weights constraints. Of course, we observe more differences when the bounds are tighter. The relationship with the universe is more complex. Our experience shows that the differences generally increase and then decrease with the number of assets in average. Nevertheless, with very large universe, we may find some assets which present outlier implied Sharpe values.*

[6]It is more true for the correlations than for the volatilities.

4 Conclusion

In this article, we apply the framework of Jagannathan and Ma (2003) to analyse the implied covariance matrix of the constrained optimisation portfolio. Using an empirical application on the DJ Eurostoxx 50 universe, we show that weights constraints may modify substantially the covariance matrix.

Weights constraints are used by (almost) all portfolio managers. However, a few of them have a critical view on their constraints. Generally, they consider several sets of weights constraints and show their impact on the optimized portfolio weights. We think that it may be useful to complete this analysis by studying the impact on the covariance matrix. When the portfolio manager adds some constraints, he would like to obtain an optimized portfolio which satisfy his views. Using the previous framework, he may verify that its weights constraints are compatible with its views[7] on volatilities, correlations, risk factors, Sharpe ratios, etc.

This framework is also useful to analyse some alternative-weighted indexes. Since some years, we observe a large development of these investments products based on portfolio theory. Generally, index providers impose some bounds on the portfolio weights in order to obtain a more robust portfolio with lower turnovers, smaller concentrations, etc. The approach of Jagannathan and Ma (2003) is a very powerful tool to understand the impact of the bounds on these index portfolios, in particular when the bounds are sharp.

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[7] Indeed, the approach of Jagannathan and Ma (2003) may be relied to the Black-Litterman model (Yanou, 2010).