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Abstract. In this paper, we present a simple axiomatization of the n-person egalitarian solution. The single axiom sufficient for characterization is a new condition which we call *symmetric decomposition*.

Keywords: Cooperative bargaining; egalitarian solution.

JEL Classification Numbers: C71, C78

1 Introduction

Since the pioneering work of Nash (1950) on cooperative bargaining theory, a voluminous number of bargaining solutions have been proposed and axiomatized in the literature. Among the solutions that have been most studied is the egalitarian solution which was recommended by Rawls (1971). Given a bargaining problem faced by individuals in the society, this solution implies maximization of the utility of the worst-off individual over the bargaining set. A characterization of the egalitarian solution when the number of individuals is fixed and the bargaining set is convex, compact, and comprehensive was proposed by Kalai (1977) using *symmetry*, *weak Pareto optimality*,

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and *strong monotonicity* conditions. *Symmetry* axiom says that if the bargaining set is invariant under all exchanges of agents, then the solution must be invariant too. Weak Pareto optimality requires that all gains from cooperation should be weakly exhausted, whereas *strong monotonicity* demands that all agents should benefit from any expansion of the bargaining set. Kalai (1977) also shows that strong monotonicity condition can be replaced by *step-by-step negotiation*, a decomposability condition which says that if the bargaining set expands from U to S, the solution on S can be calculated by first finding the solution on U (step 1) and then adding it to the solution on the set of individually rational options in S with respect to the solution in step 1.

In this paper, we propose a simple axiomatization of the egalitarian solution in the same bargaining domain as studied by Kalai (1977). We show that the n-person egalitarian solution is the only bargaining solution that satisfies a new condition which we call *symmetric decomposition*.

A number of studies in the literature have axiomatized the egalitarian solution in alternative domains of bargaining problems. As such, Thomson (1983a, 1984) consider bargaining problems where the number of bargaining individuals may vary, Conley and Wilkie (2000) relax the restriction that the bargaining set is convex, Rachmilevitch (2011) considers a restricted domain where the bargaining set is strictly comprehensive, and Conley and Wilkie (2012) study domains where the bargaining set is finite. There are also other studies that consider the characterization of related solutions, such as the lexicographic egalitarian solution and proportional solutions. See, for example, Myerson (1977), Roth (1979), Myerson and Thomson (1980), Myerson (1981), Thomson (1983b), and Chun and Thomson (1990).

The paper is organized as follows: In Section 2 we introduce the basic structures and in Section 3 we present our characterization result. Finally, Section 4 contains some concluding remarks.

2 Basic Structures

We consider a society with the set of individuals $N = \{1, 2, ..., n\}$. A bargaining problem for this society consists of a pair (S, d) where S is a non-empty subset of R^n_+ and $d \in S$. Here, S represents von Neumann-Morgenstern utilities attainable through the cooperative actions of n individuals. If the individuals fail to agree on an outcome in S, then the bargaining is settled at the point d, which is called the disagreement point. In this paper, we consider the domain Σ^n of bargaining problems where

(a) S is convex and compact, and there exists $x \in S$ such that $x > d$.[1]

(b) S is d-comprehensive; i.e., if $x \in S$, $y \in R^n_+$, $x \geq y \geq d$ and $x \neq y$ then $y \in S$ (the possibility of free disposal of utility).

We define the weak Pareto set of S as

$$WP(S) = \{x \in S \,|\, y > x \ \text{ implies } y \notin S\}$$

and the strong Pareto set of S as

$$P(S) = \{x \in S \,|\, y \geq x \text{ and } y \neq x \ \text{ implies } y \notin S\}.$$

A solution F is a mapping from Σ^n to R^n_+ such that for each $(S, d) \in \Sigma^n$, $F(S, d) \in S$. The egalitarian solution maps each bargaining problem $(S, d) \in \Sigma^n$ to the point $E(S, d)$ of $WP(S)$ such that $E_i(S, d) - d_i = E_j(S, d) - d_j$ for all $i, j \in N$.

Given a bargaining problem (S, d), we denote by $a_j(S, d)$ the maximal net utility attainable by agent $j \in N$; i.e., $a_j(S, d) = \max_{x \in S}(x_j - d_j)$. For any real $\beta \in (0, 1]$, we define the reference point $c(S, d, \beta) \in S$ such that $c_i(S, d, \beta) = d_i + \beta \min_{j \in N} a_j(S, d)/2$ for all $i \in N$. Clearly, $c(S, d, 1) - d$ is the symmetric point in the Pareto frontier of the convex hull of the set of vectors $v^1, v^2, \ldots, v^n$ where for each $k \in N$, $v^k_k = \min_{j \in N} a_j(S, d)$ and $v^k_l = 0$ for each $l \in N \backslash \{k\}$.

[1]Given two vectors x and y in R^n_+, $x \geq y$ means $x_i \geq y_i$ for all $i \in N$ and $x > y$ means $x_i > y_i$ for all $i \in N$.

Given a bargaining problem (S, d), we denote by $IR(S, d)$ the individually rational allocations; i.e., $IR(S, d) = \{x \in S \mid d \leq x\}$. For any bargaining set $S \subset R^n_+$ and any $z \in R^n_+$ we define $S - z = \{x \in R^n \mid \exists\, y \in S \text{ such that } x = y - z\}$. Apparently, $IR(S, c(S, d, \beta)) - c(S, d, \beta) = IR(S - c(S, d, \beta), 0) \in \Sigma^n$ for any $(S, d) \in \Sigma^n$ and for any $\beta \in (0, 1]$. In Fig. 1, we plot a 2-person bargaining problem with $\beta = 1$.

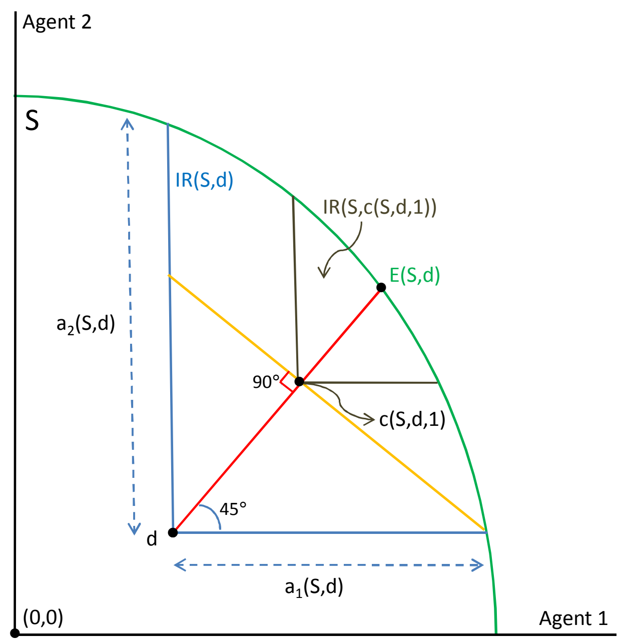

Fig. 1

We investigate the existence of solutions satisfying the following new axiom:

Symmetric decomposition. There exists $\beta \in (0, 1]$ such that $F(S, d) = c(S, d, \beta) + F(IR(S - c(S, d, \beta), 0), 0)$ for all $(S, d) \in \Sigma^n$.

The above axiom says that the solution $F(S, d)$ can be calculated in two steps, by first obtaining, for some $\beta \in (0, 1]$, the symmetric reference point $c(S, d, \beta)$ in S and then taking it to be the starting point for the distribution of the utilities in S.

3 Characterization Result

Theorem 1. *A solution satisfies symmetric decomposition if and only if it is the egalitarian solution.*

Proof. It is clear that the egalitarian solution satisfies *symmetric decomposition* since for all $\beta \in (0,1]$ we have $E(S,d) = c(S,d,\beta) + E(IR(S - c(S,d,\beta),0),0)$ for all $(S,d) \in \Sigma^n$. Conversely, let F be a solution on Σ^n satisfying *symmetric decomposition*. First, pick any $\beta \in (0,1]$ such that $F(S,d) = c(S,d,\beta) + F(IR(S - c(S,d,\beta),0),0)$ for all $(S,d) \in \Sigma^n$, and then pick any $(S,d) \in \Sigma^n$. Consider the sequence of bargaining problems $(S^t,d^t)_{t=0}^{\infty}$ where $S^0 = S$, $d^0 = d$, and $d^t = 0$ and $S^t = IR(S^{t-1} - c(S^{t-1},d^{t-1},\beta),0)$ for each integer $t \geq 1$. We say that agent k determines the reference point $c(\tilde{S},\tilde{d},\beta)$ in a given bargaining problem $(\tilde{S},\tilde{d}) \in \Sigma^n$ if $k = min\{i \in N | \, i = argmin_{j \in N}\, a_j(\tilde{S},\tilde{d})\}$. Then, for any integer $m \geq 0$, there must exist an agent, say $k(m)$, determining the reference point in at least $m+1$ of the first $n^m + 1$ problems in the sequence $(S^t,d^t)_{t=0}^{\infty}$. For each integer $m \geq 0$, we denote by $(S^t,d^t)_{t \in \{t_1,t_2,\ldots,t_{m+1}\}}$ the first $m+1$ problems in which agent $k(m)$ determines the reference point. Clearly, we have $a_{k(m)}(S^{t_{i+1}},d^{t_{i+1}}) \leq (1 - \frac{\beta}{2}) a_{k(m)}(S^{t_i},d^{t_i})$ for all $i \in \{1,2,\ldots,m\}$ for each integer $m \geq 0$. Using the fact that $a_{k(m)}(S^{t_1},d^{t_1}) \leq a_{k(m)}(S^0,d^0)$, we have $a_{k(m)}(S^{t_{m+1}},d^{t_{m+1}}) \leq (1 - \frac{\beta}{2})^m a_{k(m)}(S^0,d^0)$ for each integer $m \geq 0$. Now suppose that $F(S^0,d^0) \neq E(S^0,d^0)$. Pick any integer $\hat{m}$ such that $(1 - \frac{\beta}{2})^{\hat{m}} a_{k(\hat{m})}(S^0,d^0) < |F_{k(\hat{m})}(S^0,d^0) - \min_{j \in N} F_j(S^0,d^0)|$. Then $a_{k(\hat{m})}(S^{t_{\hat{m}+1}},d^{t_{\hat{m}+1}}) < |F_{k(\hat{m})}(S^0,d^0) - \min_{j \in N} F_j(S^0,d^0)|$ and therefore $F(S^0,d^0) - \sum_{\tau=1}^{t_{\hat{m}}} c(S^\tau,d^\tau,\beta) \notin S^{t_{\hat{m}+1}}$. However, we have $F(S^{t+1},d^{t+1}) = F(S^0,d^0) - \sum_{\tau=1}^{t} c(S^\tau,d^\tau,\beta)$ for all integer $t \geq 0$ by *symmetric decomposition*. This implies that $F(S^{t_{\hat{m}+1}},d^{t_{\hat{m}+1}}) \notin S^{t_{\hat{m}+1}}$, a contradiction. Therefore, $F(S^0,d^0) = E(S^0,d^0)$. Since $(S^0,d^0) = (S,d) \in \Sigma^n$ was arbitrarily picked, F must be equal to the the egalitarian solution on Σ^n. □

4 Concluding Remarks

In this paper, we have considered an alternative characterization of the egalitarian solution in a class of bargaining problems with convex, compact and comprehensive bargaining sets. We show that the n-person egalitarian solution is the only bargaining solution that satisfies a new condition which we call *symmetric decomposition*. This single condition replaces the three conditions in a characterization result of Kalai (1977), *step-by-step negotiation*, *symmetry* and *weak Pareto optimality*. Dropping *symmetry* in Kalai (1977), any n-person weighted egalitarian solution (n-person proportional solution) with weights α in the $n-1$ dimensional simplex, selecting the maximal point of the bargaining set in the direction of α also become admissible. Alternatively, dropping *weak Pareto optimality* in the characterization result of Kalai (1977), any n-person contracted egalitarian solution with the contraction factor in $[0,1]$ also become admissible.

The reason why the decomposition condition of ours successfully strengthens that of Kalai (1977) is that given any bargaining problem $(S,0)$, *step-by-step negotiation* requires any bargaining solution F to be decomposable with regard to the reference point $F(T,0)$ for each $T \subseteq S$, whereas symmetric decomposition requires the decomposability, for some $\beta \in (0,1]$, only with regard to the symmetric point $c(T,0,\beta)$ of the greatest symmetric and strictly comprehensive set T contained by S such that T has a linear Pareto frontier. Thus, the reference point in our case is proportional to the egalitarian solution on T; i.e., $c(T,0,\beta) = \beta E(T,0)$, where $\beta \in (0,1]$. Since it is true that $\beta E(T,0) = E(\beta T,0)$ for any $\beta \in (0,1]$, the reference point we use in our characterization is the symmetric and weakly Pareto optimal point in βT for our particular choice of T. Indeed, what eliminate the admissibility of any n-person contracted, non-egalitarian proportional solution in the sole presence of decomposability are entirely the notions of *symmetry* and *weak Pareto optimality* embedded in our decomposition condition.

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