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Abstract: The paper introduces for the most frequently used three-dimensional panel data sets several random effects model specifications. It derives appropriate estimation methods for the balanced and unbalanced cases. An application is also presented where the bilateral trade of 20 EU countries is analysed for the period 2001-2006. The differences between the fixed and random effects specifications are highlighted through this empirical exercise.

Key words: panel data, multidimensional panel data, random effects, error components model, trade model, gravity model.

JEL classification: C1, C2, C4, F17, F47.

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1. Introduction

The use of multidimensional panel data sets has received momentum the last few years. Especially, three dimensional data bases are becoming readily available and frequently used to analyze different types of economic flows, like capital flows (FDI) for example, or most predominantly trade relationships (for a recent reviews of the subject see *Anderson* [2010] or *van Bergeijk and Brakman* [2010]). Several model specifications have been proposed in the literature to deal with the heterogeneity of these types of data sets, but all of them considered these heterogeneity factors as fixed effects, i.e., fixed unknown parameters. As it is pretty well understood from the use of "usual" two dimensional panel data sets, the fixed effects formulations are more suited to deal with cases when the panel, at least in one dimension, is short. On the other hand, for large data sets, the random effects specifications seems to be more suited, where the specific effects are considered as random variables, rather than parameters.

In this paper we present different types of random effects model specifications which mirror the fixed effects models used so far in the literature (some earlier versions were introduced in *Davis* [2002]), derive proper estimation methods for each of them and analyze their properties under some data problems. Finally, we present an interesting application.

2. Different Heterogeneity Formulations

The most widely used fixed effects model specifications have been proposed by *Baltagi et al.* [2003], *Egger and Pfanffermayr* [2003], *Baldwin and Taglioni* [2006], and *Baier and Bergstrand* [2007]. The straightforward direct generalization of the standard fixed effects panel data model (where the usual individuals are in fact the (ij) country pairs) takes into account bilateral interaction. The model specification is

$$y_{ijt} = \beta' x_{ijt} + \gamma_{ij} + \varepsilon_{ijt} \qquad i = 1, \ldots, N \quad j = 1, \ldots, N, \quad t = 1, \ldots, T$$

where the γ_{ij} are the bilateral specific fixed effects. If the specification is used in a macro trade model, for example, with say 150 countries involved, this explicitly or implicitly, means the estimation of $150 \times 150 = 22,500$ parameters. This looks very much like a textbook over-specification case. Instead we propose, like in a standard panel data context, the use of the much more parsimonious random effects specification

$$y_{ijt} = \beta' x_{ijt} + \mu_{ij} + \varepsilon_{ijt} \qquad i = 1, \ldots, N, \quad j = 1, \ldots, N, \quad t = 1, \ldots, T \qquad (1)$$

where $E(\mu_{ij}) = 0$, the random effects are pairwise uncorrelated, and

$$E(\mu_{ij}\mu_{i'j'}) = \begin{cases} \sigma_\mu^2 & i = i' \text{ and } j = j' \\ 0 & \text{otherwise} \end{cases}$$

A natural extension of this model is to include time effects as well

$$y_{ijt} = \beta' x_{ijt} + \mu_{ij} + \lambda_t + \varepsilon_{ijt} \qquad i = 1, \ldots, N \quad j = 1, \ldots, N, \quad t = 1, \ldots, T \tag{2}$$

where $E(\lambda_t) = 0$ and

$$E(\lambda_t\lambda_t') = \begin{cases} \sigma_\lambda^2 & t = t' \\ 0 & \text{otherwise} \end{cases}$$

Another form of heterogeneity is to use individual-time-varying effects

$$y_{ijt} = \beta' x_{ijt} + \alpha_{jt} + \varepsilon_{ijt}$$

The corresponding random effects specification now is

$$y_{ijt} = \beta' x_{ijt} + u_{jt} + \varepsilon_{ijt} \tag{3}$$

where $E(u_{jt}) = 0$, the random effects are pairwise uncorrelated, and

$$E(u_{ij}u_{j't'}) = \begin{cases} \sigma_u^2 & j = j' \text{ and } t = t' \\ 0 & \text{otherwise} \end{cases}$$

Or alternatively we can also have the following random effects specification

$$y_{ijt} = \beta' x_{ijt} + v_{it} + \varepsilon_{ijt} \tag{4}$$

where $E(v_{it}) = 0$, the random effects are pairwise uncorrelated, and

$$E(v_{it}v_{i't'}) = \begin{cases} \sigma_v^2 & i = i' \text{ and } t = t' \\ 0 & \text{otherwise} \end{cases}$$

The random effects specification containing both the above forms of heterogeneity now is

$$y_{ijt} = \beta' x_{ijt} + v_{it} + u_{jt} + \varepsilon_{ijt} \tag{5}$$

The model specification which encompasses all above effects is

$$y_{ijt} = \beta' x_{ijt} + \gamma_{ij} + \alpha_{it} + \alpha_{jt} + \varepsilon_{ijt}$$

The corresponding random effects specification now is

$$y_{ijt} = \beta' x_{ijt} + \mu_{ij} + v_{it} + u_{jt} + \varepsilon_{ijt} \tag{6}$$

where $E(\mu_{ij}) = 0$, $E(u_{jt}) = 0$, $E(v_{it}) = 0$, all random effects are pairwise uncorrelated, and

$$E(\mu_{ij}\mu_{i'j'}) = \begin{cases} \sigma_\mu^2 & i = i' \text{ and } j = j' \\ 0 & \text{otherwise} \end{cases}$$

$$E(u_{jt}u_{j't'}) = \begin{cases} \sigma_u^2 & j = j' \text{ and } t = t' \\ 0 & \text{otherwise} \end{cases}$$

$$E(v_{it}v_{i't'}) = \begin{cases} \sigma_v^2 & i = i' \text{ and } t = t' \\ 0 & \text{otherwise} \end{cases}$$

In order to estimate efficiently these random effects models their corresponding covariance matrices need to be derived

3. Covariance Matrices of the Different Random Effects Specifications

The standard way to estimate these models is with the Feasible GLS (FGLS) estimator. First, we need to derive the covariance matrix of each of the models introduced in Section 2, then the unknown variance components of these matrices need to be estimated.

For model (1) let us denote

$$u_{ijt}^\star = \mu_{ij} + \epsilon_{ijt} \tag{7}$$

So for all t observations

$$\begin{aligned} u_{ij}^\star &= \mu_{ij} \otimes l_T + \epsilon_{ij} \\ E\left[u_{ij}^\star {u^\star}'_{ij}\right] &= E\left[(\mu_{ij} \otimes l_T)(\mu_{ij} \otimes l'_T)\right] + E\left[\epsilon_{ij}\epsilon'_{ij}\right] \\ &= \sigma_\mu^2 J_T + \sigma_\epsilon^2 I_T \end{aligned}$$

where l_T is the $(T \times 1)$ vector of ones, J_T is the $(T \times T)$ matrix of ones and I_T is the $(T \times T)$ identity matrix. In all the paper matrix J will denote the matrix of ones, with the size in the index, and I the identity matrix, also with the size in the index. Now for individual i

$$\begin{aligned} u_i^\star &= \mu_i \otimes l_T + \epsilon_i \\ E\left[u_i^\star {u^\star}'_i\right] &= E\left[(\mu_i \otimes l_T)(\mu'_i \otimes l'_T)\right] + E\left[\epsilon_i\epsilon'_i\right] \\ &= \sigma_\mu^2 I_N \otimes J_T + \sigma_\epsilon^2 I_{NT} \end{aligned}$$

And combining all these results we get for the covariance matrix of model (1)

$$\begin{aligned} u^\star &= \mu \otimes l_T + \epsilon \\ E\left[u^\star u^{\star\prime}\right] &= E\left[(\mu \otimes l_T)\left(\mu' \otimes l_T'\right)\right] + E\left[\epsilon\epsilon'\right] \\ &= \sigma_\mu^2 I_{N^2} \otimes J_T + \sigma_\epsilon^2 I_{N^2T} = \Omega \end{aligned}$$

Deriving likewise the covariance matrix for model (2)

$$\begin{aligned} u_{ij}^\star &= \mu_{ij} \otimes l_T + \lambda + \epsilon_{ij} \\ E\left[u_{ij}^\star u^{\star\prime}_{ij}\right] &= E\left[(\mu_{ij} \otimes l_T)(\mu_{ij} \otimes l_T)'\right] + E\left[\lambda\lambda'\right] + E\left[\epsilon_{ij}\epsilon_{ij}'\right] \\ &= \sigma_\mu^2 J_T + \sigma_\lambda^2 I_T + \sigma_\epsilon^2 I_T \end{aligned}$$

and

$$\begin{aligned} u_i^\star &= \mu_i \otimes l_T + l_N \otimes \lambda + \epsilon_i \\ E\left[u_i^\star u^{\star\prime}_i\right] &= E\left[(\mu_i \otimes l_T)(\mu_i \otimes l_T)'\right] + E\left[(l_N \otimes \lambda)(l_N \otimes \lambda)'\right] + E\left[\epsilon_i\epsilon_i'\right] \\ &= \sigma_\mu^2 I_N \otimes J_T + \sigma_\lambda^2 J_N \otimes I_T + \sigma_\epsilon^2 I_{NT} \end{aligned}$$

so we obtain

$$\begin{aligned} u^\star &= \mu \otimes l_T + l_{N^2} \otimes \lambda + \epsilon \\ E\left[u^\star u^{\star\prime}\right] &= E\left[(\mu \otimes l_T)(\mu \otimes l_T)'\right] + E\left[(l_{N^2} \otimes \lambda)(l_{N^2} \otimes \lambda)'\right] + E\left[\epsilon\epsilon'\right] \\ &= \sigma_\mu^2 I_{N^2} \otimes J_T + \sigma_\lambda^2 J_{N^2} \otimes I_T + \sigma_\epsilon^2 I_{N^2T} = \Omega \end{aligned}$$

Let us turn now to models (3) and (4) which can be dealt with in the same way as they are completely symmetric

$$u_{ijt}^\star = u_{jt} + \epsilon_{ijt} \tag{8}$$

$$\begin{aligned} u_{ij}^\star &= u_j + \epsilon_{ij} \\ E\left(u_{ij}^\star u^{\star\prime}_{ij}\right) &= E\left[u_j u_j'\right] + E\left[\epsilon_{ij}\epsilon_{ij}'\right] = \sigma_u^2 I_T + \sigma_\epsilon^2 I_T \end{aligned}$$

$$\begin{aligned} u_i^\star &= u + \epsilon_i \\ E\left(u_i^\star u^{\star\prime}_i\right) &= E\left[uu'\right] + E\left[\epsilon_i\epsilon_i'\right] = \sigma_u^2 I_{NT} + \sigma_\epsilon^2 I_{NT} \\ u^\star &= l_N \otimes u + \epsilon \\ E\left(u^\star u^{\star\prime}\right) &= E\left[(l_N \otimes u)(l_N' \otimes u')\right] + E\left[\epsilon\epsilon'\right] = \sigma_u^2 J_N \otimes I_{NT} + \sigma_\epsilon^2 I_{N^2T} = \Omega \end{aligned}$$

Using the same approach, the covariance matrix for model (5) is

$$u^{\star}_{ijt} = u_{jt} + v_{it} + \epsilon_{ijt}$$

$$u^{\star}_{ij} = u_j + v_i + \epsilon_{ij}$$

$$\begin{aligned} E\left(u^{\star}_{ij} {u^{\star}}'_{ij}\right) &= E\left[u_j u'_j\right] + E\left[v_i v'_i\right] + E\left[\epsilon_{ij} \epsilon'_{ij}\right] \\ &= \sigma^2_u I_T + \sigma^2_v I_T + \sigma^2_\epsilon I_T \end{aligned}$$

$$\begin{aligned} u^{\star}_{i} &= l_N \otimes v_i + u + \epsilon_i \\ E\left(u^{\star}_{i} {u^{\star}}'_{i}\right) &= E\left[\left(l_N \otimes v_i\right)\left(l'_N \otimes v'_i\right)\right] + E\left[uu'\right] + E\left[\epsilon_i \epsilon'_i\right] = \\ &= \sigma^2_v J_N \otimes I_T + \sigma^2_u I_{NT} + \sigma^2_\epsilon I_{NT} \end{aligned}$$

and so

$$E\left(u^{\star} {u^{\star}}'\right) = \sigma^2_v (I_N \otimes J_N \otimes I_T) + \sigma^2_u (J_N \otimes I_{NT}) + \sigma^2_\epsilon I_{N^2} = \Omega$$

And finally the covariance matrix of the all encompassing model (6) is

$$u^{\star}_{ijt} = \mu_{ij} + u_{jt} + v_{it} + \epsilon_{ijt} \tag{9}$$

$$\begin{aligned} u^{\star}_{ij} &= \mu_{ij} \otimes l_T + u_j + v_i + \epsilon_{ij} \\ E\left(u^{\star}_{ij} {u^{\star}}'_{ij}\right) &= E\left[\left(\mu_{ij} \otimes l_T\right)\left(\mu_{ij} \otimes l'_T\right)\right] + E\left[u_j u'_j\right] + E\left[v_i v'_i\right] + E\left[\epsilon_{ij} \epsilon'_{ij}\right] \\ &= \sigma^2_\mu J_T + \sigma^2_u I_T + \sigma^2_v I_T + \sigma^2_\epsilon I_T \end{aligned}$$

$$\begin{aligned} u^{\star}_{i} &= \mu_i \otimes l_T + l_N \otimes v_i + u + \epsilon_i \\ E\left(u^{\star}_{i} {u^{\star}}'_{i}\right) &= E\left[\left(\mu_i \otimes l_T\right)\left(\mu'_i \otimes l'_T\right)\right] + E\left[\left(l_N \otimes v_i\right)\left(l'_N \otimes v'_i\right)\right] + E\left[uu'\right] + E\left[\epsilon_i \epsilon'_i\right] = \\ &= \sigma^2_\mu I_N \otimes J_T + \sigma^2_u I_{NT} + \sigma^2_v J_N \otimes I_T + \sigma^2_\epsilon I_{NT} \end{aligned}$$

and so

$$E\left(u^{\star} {u^{\star}}'\right) = \sigma^2_\mu (I_{N^2} \otimes J_T) + \sigma^2_u (J_N \otimes I_{NT}) + \sigma^2_v (I_N \otimes J_N \otimes I_T) + \sigma^2_\epsilon I_{N^2 T} = \Omega$$

4. Estimation of the Variance Components and the Feasible GLS Estimator

Turning now to the estimation of the variance components of the different models, let us start with model (1)

$$E\left[u^{\star 2}_{ijt}\right] = E\left[(\mu_{ij} + \epsilon_{ijt})^2\right] = E\left[\mu_{ij}^2\right] + E\left[\epsilon_{ijt}^2\right] = \sigma_\mu^2 + \sigma_\epsilon^2 \tag{10}$$

and let us introduce the appropriate Within transformation

$$u^\star_{ijt,within} = u^\star_{ijt} - \bar{u^\star}_{ij} = \epsilon_{ijt} - \bar{\epsilon}_{ij} \tag{11}$$

where $\bar{\epsilon}_{ij} = 1/T \sum_t \epsilon_{ijt}$ and $\bar{u^*}_{ij} = 1/T \sum_t u^*_{ijt}$, so we get

$$E\left[\left(u^\star_{ijt} - \bar{u^\star}_{ij}\right)^2\right] = E\left[(\epsilon_{ijt} - \bar{\epsilon}_{ij})^2\right] = E\left[\epsilon_{ijt}^2 - 2\epsilon_{ijt}\frac{1}{T}\sum_{t=1}^T \epsilon_{ijt} + \left(\frac{1}{T}\sum_{t=1}^T \epsilon_{ijt}\right)^2\right]$$

$$= E\left[\epsilon_{ijt}^2\right] - 2E\left[\epsilon_{ijt}\frac{1}{T}\sum_{t=1}^T \epsilon_{ijt}\right] + E\left[\left(\frac{1}{T}\sum_{t=1}^T \epsilon_{ijt}\right)^2\right]$$

$$= \sigma_\epsilon^2 - \frac{2}{T}\sigma_\epsilon^2 + \frac{1}{T}\sigma_\epsilon^2 = \sigma_\epsilon^2 - \frac{1}{T}\sigma_\epsilon^2 = \sigma_\epsilon^2 \frac{T-1}{T}$$

Let $\hat{u}^*$ be the OLS residual of model (1) and $\hat{u}^*_{within}$ the Within transformation of this residual. Then we can estimate the variance components as

$$\hat{\sigma}_\epsilon^2 = \frac{T}{T-1}\hat{u}^{\star\prime}_{within}\hat{u}^\star_{within}$$

$$\hat{\sigma}_\mu^2 = \frac{1}{N^2 T}\sum_{i=1}^N \sum_{j=1}^N \sum_{t=1}^T \hat{u}^{\star 2}_{ijt} - \hat{\sigma}_\epsilon^2$$

These estimators naturally should be adjusted to the actual degrees of freedom.

Continuing with model (2)

$$E\left[u^{\star 2}_{ijt}\right] = E\left[(\mu_{ij} + \lambda_t + \epsilon_{ijt})^2\right] = E\left[\mu_{ij}^2\right] + E\left[\lambda_t^2\right] + E\left[\epsilon_{ijt}^2\right]$$
$$= \sigma_\mu^2 + \sigma_\lambda^2 + \sigma_\epsilon^2$$

$$E\left[\left(\frac{1}{T}\sum_{t=1}^T u^\star{}_{ijt}\right)^2\right] = E\left[\left(\frac{1}{T}\sum_{t=1}^T \mu_{ij} + \lambda_t + \epsilon_{ijt}\right)^2\right]$$

$$= E\left[\mu_{ij}^2\right] + \frac{1}{T^2}E\left[\sum_{t=1}^T \lambda_t^2\right] + \frac{1}{T^2}E\left[\sum_{t=1}^T \epsilon_{ijt}^2\right]$$

$$= \sigma_\mu^2 + \frac{1}{T}\sigma_\lambda^2 + \frac{1}{T}\sigma_\epsilon^2$$

and

$$
\begin{aligned}
&E\left[\left(u^{\star}_{ijt} - \bar{u^{\star}}_{ij} - \bar{u^{\star}}_{t} + \bar{u^{\star}}\right)^2\right] = E\left[\left(\epsilon_{ijt} - \bar{\epsilon}_{ij} - \bar{\epsilon}_t + \bar{\epsilon}\right)^2\right] \\
&\qquad = E\left[\epsilon_{ijt}^2\right] + E\left[\left(\frac{1}{T}\sum_{t=1}^{T}\epsilon_{ijt}\right)^2\right] + \\
&\qquad + E\left[\left(\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\epsilon_{ijt}\right)^2\right] + E\left[\left(\frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\epsilon_{ijt}\right)^2\right] - \\
&\qquad - 2E\left[\epsilon_{ijt}\frac{1}{T}\sum_{t=1}^{T}\epsilon_{ijt}\right] - 2E\left[\epsilon_{ijt}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\epsilon_{ijt}\right] + \\
&\qquad + 2E\left[\epsilon_{ijt}\frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\epsilon_{ijt}\right] + 2E\left[\frac{1}{T}\sum_{t=1}^{T}\epsilon_{ijt}\cdot\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\epsilon_{ijt}\right] - \\
&\qquad - 2E\left[\frac{1}{T}\sum_{t=1}^{T}\epsilon_{ijt}\cdot\frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\epsilon_{ijt}\right] - 2E\left[\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\epsilon_{ijt}\cdot\frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\epsilon_{ijt}\right] \\
&\quad = \sigma_\epsilon^2 + \frac{1}{T}\sigma_\epsilon^2 + \frac{1}{N^2}\sigma_\epsilon^2 + \frac{1}{N^2T}\sigma_\epsilon^2 - \frac{2}{T}\sigma_\epsilon^2 - \frac{2}{N^2}\sigma_\epsilon^2 + \\
&\qquad + \frac{2}{N^2T}\sigma_\epsilon^2 + \frac{2}{N^2T}\sigma_\epsilon^2 - \frac{2}{N^2T}\sigma_\epsilon^2 - \frac{2}{N^2T}\sigma_\epsilon^2 = \\
&\quad = \sigma_\epsilon^2\frac{(N-1)(N+1)(T-1)}{N^2T}
\end{aligned}
$$

This leads to the estimation of the variance components

$$
\begin{aligned}
\hat{\sigma}_\epsilon^2 &= \frac{N^2T}{(N-1)(N+1)(T-1)}\hat{u^{\star}}'_{within}\hat{u^{\star}}_{within} \\
\hat{\sigma}_\mu^2 &= \frac{1}{N^2T(T-1)}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\left(\sum_{t=1}^{T}\hat{u}^{\star}_{ijt}\right)^2 - \sum_{t=1}^{T}\left(\hat{u}^{\star}_{ijt}\right)^2\right)\right) \\
\hat{\sigma}_\lambda^2 &= \frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\left(\hat{u}^{\star}_{ijt}\right)^2 - \hat{\sigma}_\mu^2 - \hat{\sigma}_\epsilon^2
\end{aligned}
$$

Turning now to models (3) and (4)

$$
E\left[{u^{\star}}^2_{ijt}\right] = E\left[\left(u_{jt} + \epsilon_{ijt}\right)^2\right] = E\left[u_{jt}^2\right] + E\left[\epsilon_{ijt}^2\right] = \sigma_u^2 + \sigma_\epsilon^2 \qquad (12)
$$

and the appropriate Within transformation now is

$$u^{\star}_{ijt,within} = u^{\star}_{ijt} - \bar{u^{\star}}_{jt} = \epsilon_{ijt} - \bar{\epsilon}_{jt} \qquad (13)$$

where $\bar{u}^{*}_{jt} = 1/N \sum_i u^{*}_{ijt}$ and $\bar{\epsilon}_{jt} = 1/N \sum_i \epsilon_{ijt}$ and

$$\begin{aligned}
E\left[\left(u^{\star}_{ijt} - \bar{u^{\star}}_{jt}\right)^2\right] &= E\left[\left(\epsilon_{ijt} - \bar{\epsilon}_{jt}\right)^2\right] \\
&= E\left[\epsilon^2_{ijt} - 2\epsilon_{ijt}\frac{1}{N}\sum_{i=1}^{N}\epsilon_{ijt} + \left(\frac{1}{N}\sum_{i=1}^{N}\epsilon_{ijt}\right)^2\right] \\
&= E\left[\epsilon^2_{ijt}\right] - 2E\left[\epsilon_{ijt}\frac{1}{N}\sum_{i=1}^{N}\epsilon_{ijt}\right] + E\left[\left(\frac{1}{N}\sum_{i=1}^{N}\epsilon_{ijt}\right)^2\right] \\
&= \sigma^2_{\epsilon} - \frac{2}{N}\sigma^2_{\epsilon} + \frac{1}{N}\sigma^2_{\epsilon} = \sigma^2_{\epsilon} - \frac{1}{N}\sigma^2_{\epsilon} = \sigma^2_{\epsilon}\frac{N-1}{N}
\end{aligned}$$

And the estimators for the variance components are

$$\begin{aligned}
\hat{\sigma}^2_{\epsilon} &= \frac{N}{N-1}\hat{u}^{\star'}_{within}\hat{u}^{\star}_{within} \\
\hat{\sigma}^2_{\mu} &= \frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\hat{u}^{\star 2}_{ijt} - \hat{\sigma}^2_{\epsilon}
\end{aligned}$$

Now for model (5) the Within transformation is

$$u^{*}_{ijt,within} = (u^{*}_{ijt} - 1/N\sum_{i} u^{*}_{ijt} - 1/N\sum_{j} u^{*}_{ijt} + 1/N^2\sum_{i}\sum_{j} u^{*}_{ijt}) \qquad (14)$$

so we get

$$\begin{aligned}
&E\left[\left(u^{\star}_{ijt} - \bar{u^{\star}}_{jt} - \bar{u^{\star}}_{it} + \bar{u^{\star}}_{t}\right)^2\right] = E\left[\left(\epsilon_{ijt} - \bar{\epsilon}_{jt} - \bar{\epsilon}_{it} + \bar{\epsilon}_{t}\right)^2\right] \\
&= E\left[\epsilon^2_{ijt}\right] + E\left[\frac{1}{N^2}\left(\sum_{i=1}^{N}\epsilon_{ijt}\right)^2\right] + E\left[\frac{1}{N^2}\left(\sum_{j=1}^{N}\epsilon_{ijt}\right)^2\right] + E\left[\frac{1}{N^4}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\epsilon_{ijt}\right)^2\right] - \\
&- 2E\left[\epsilon_{ijt}\frac{1}{N}\sum_{i=1}^{N}\epsilon_{ijt}\right] - 2E\left[\epsilon_{ijt}\frac{1}{N}\sum_{j=1}^{N}\epsilon_{ijt}\right] + 2E\left[\epsilon_{ijt}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\epsilon_{ijt}\right] + \\
&+ 2E\left[\frac{1}{N^2}\sum_{i=1}^{N}\epsilon_{ijt}\sum_{j=1}^{N}\epsilon_{ijt}\right] - 2E\left[\frac{1}{N^3}\sum_{i=1}^{N}\epsilon_{ijt}\sum_{i=1}^{N}\sum_{j=1}^{N}\epsilon_{ijt}\right] - 2E\left[\frac{1}{N^3}\sum_{j=1}^{N}\epsilon_{ijt}\sum_{i=1}^{N}\sum_{j=1}^{N}\epsilon_{ijt}\right] = \\
&= \sigma^2_{\epsilon} + \frac{1}{N}\sigma^2_{\epsilon} + \frac{1}{N}\sigma^2_{\epsilon} + \frac{1}{N^2}\sigma^2_{\epsilon} - \frac{2}{N}\sigma^2_{\epsilon} - \frac{2}{N}\sigma^2_{\epsilon} + \frac{2}{N^2}\sigma^2_{\epsilon} + \frac{2}{N^2}\sigma^2_{\epsilon} - \frac{2}{N^2}\sigma^2_{\epsilon} - \frac{2}{N^2}\sigma^2_{\epsilon} = \\
&= \sigma^2_{\epsilon}\left(1 - \frac{2}{N} + \frac{1}{N^2}\right) = \sigma^2_{\epsilon}\left(\frac{N^2 - 2N + 1}{N^2}\right) = \sigma^2_{\epsilon}\frac{(N-1)^2}{N^2}
\end{aligned} \qquad (15)$$

And, also,

$$E\left[{u^{\star}}^2_{ijt}\right] = E\left[(u_{jt} + v_{it} + \epsilon_{ijt})^2\right] = \sigma^2_u + \sigma^2_v + \sigma^2_\epsilon$$

$$\begin{aligned} E\left[\left(\frac{1}{N}\sum_{i=1}^{N} u^{\star}_{ijt}\right)^2\right] &= E\left[\left(\frac{1}{N}\sum_{i=1}^{N}(u_{jt} + v_{it} + \epsilon_{ijt})\right)^2\right] \\ &= E\left[u^2_{jt}\right] + \frac{1}{N^2}E\left[\sum_{i=1}^{N} v^2_{it}\right] + \frac{1}{N^2}E\left[\sum_{i=1}^{N}\epsilon^2_{ijt}\right] \\ &= \sigma^2_u + \frac{1}{N}\sigma^2_v + \frac{1}{N}\sigma^2_\epsilon \end{aligned} \tag{16}$$

The estimators of the variance components therefore are

$$\begin{aligned} \hat{\sigma}^2_\epsilon &= \frac{N^2}{(N-1)^2}\hat{u}^{\star\prime}_{within}\hat{u}^{\star}_{within} \\ \hat{\sigma}^2_u &= \frac{1}{N^2T(N-1)}\left(\sum_{j=1}^{N}\sum_{t=1}^{T}\left(\left(\sum_{i=1}^{N}\hat{u}_{ijt}{}^{\star}\right)^2 - \sum_{i=1}^{N}\hat{u}_{ijt}{}^{\star 2}\right)\right) \\ \hat{\sigma}^2_v &= \frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}{\hat{u}^{\star}}^2_{ijt} - \hat{\sigma}^2_\epsilon - \hat{\sigma}^2_u \end{aligned}$$

Finally, to derive the estimators of the variance components for model (6), we need first the appropriate Within transformation

$$\begin{aligned} u^*_{ijt,within} = (&u^*_{ijt} - 1/T\sum_t u^*_{ijt} - 1/N\sum_i u^*_{ijt} - 1/N\sum_j u^*_{ijt} + 1/N^2\sum_i\sum_j u^*_{ijt} \\ &+ 1/(NT)\sum_i\sum_t u^*_{ijt} + 1/(NT)\sum_j\sum_t u^*_{ijt} - 1/(N^2T)\sum_i\sum_j\sum_t u^*_{ijt}) \end{aligned}$$

Carrying out the derivation as earlier, we get to the following estimators

$$\begin{aligned} \hat{\sigma}^2_\epsilon &= \frac{N^2T}{N(N-1)(T-1)+1}\hat{u}^{\star\prime}_{within}\hat{u}^{\star}_{within} \\ \hat{\sigma}^2_v &= \frac{1}{N^2T(N-1)}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\left(\sum_{j=1}^{N}\hat{u}^{\star}\right)^2 - \sum_{j=1}^{N}\hat{u}^{\star 2}\right)\right) \\ \hat{\sigma}^2_u &= \frac{1}{N^2T(N-1)}\left(\sum_{j=1}^{N}\sum_{t=1}^{T}\left(\left(\sum_{i=1}^{N}\hat{u}^{\star}\right)^2 - \sum_{i=1}^{N}\hat{u}^{\star 2}\right)\right) \\ \hat{\sigma}^2_\mu &= \frac{1}{N^2T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}{\hat{u}^{\star}}^2_{ijt} - \hat{\sigma}^2_\epsilon - \hat{\sigma}^2_v - \hat{\sigma}^2_u \end{aligned}$$

Now we have all the tools to properly use the FGLS estimators.

5. Unbalanced Data

Like in the case of the usual panel data models, just more frequently, one may be faced with a situation when the data at hand is unbalance. In our framework of analysis this means that for all models (1)-(6) in general $t = 1, \ldots, T_{ij}$, $\sum_i \sum_j T_{ij} = T$ and T_{ij} often is not equal to $T_{i'j'}$. For this unbalanced data case, as we did when the data was balanced, we need to derive the covariance matrices of the models and the appropriate estimators for the variance components.

For model (1), using decomposition (7) we get

$$\begin{aligned}
u^{\star}_{ij} &= \mu_{ij} \otimes l_{T_{ij}} + \epsilon_{ij} \\
E\left[u^{\star}_{ij} {u^{\star}}'_{ij}\right] &= E\left[\left(\mu_{ij} \otimes l_{T_{ij}}\right)\left(\mu_{ij} \otimes l_{T_{ij}}\right)'\right] + E\left[\epsilon_{ij}\epsilon'_{ij}\right] = \\
&= \sigma^2_{\mu} J_{T_{ij}} + \sigma^2_{\epsilon} I_{T_{ij}} \\
\text{and} \qquad u^{\star}_{i} &= \tilde{\mu}_i + \epsilon_i \\
E\left[u^{\star}_{i} {u^{\star}}'_{i}\right] &= E\left[\tilde{\mu}_i \tilde{\mu}_i'\right] + E\left[\epsilon_i \epsilon'_i\right] \\
&= \sigma^2_{\mu} A + \sigma^2_{\epsilon} I_{\sum_{j=1}^{N} T_{ij}}
\end{aligned}$$

$$\text{where} \qquad \tilde{\mu}_i = \begin{pmatrix} \mu_{i1} \\ \vdots \\ \mu_{i1} \\ \mu_{i2} \\ \vdots \\ \mu_{i2} \\ \vdots \\ \mu_{iN} \\ \vdots \\ \mu_{iN} \end{pmatrix}, \quad A = \begin{pmatrix} I_{T_{i1}} & 0 & \ldots & 0 \\ 0 & I_{T_{i2}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_{T_{iN}} \end{pmatrix} \quad \text{of size} \sum_{j=1}^{N} T_{ij} \times \sum_{j=1}^{N} T_{ij}$$

and finally for the complete model

$$\begin{aligned}
u^{\star} &= \tilde{\mu} + \epsilon \\
E\left[u^{\star} {u^{\star}}'\right] &= E\left[\tilde{\mu}\tilde{\mu}'\right] + E\left[\epsilon\epsilon'\right] \\
&= \sigma^2_{\mu} B + \sigma^2_{\epsilon} I_T
\end{aligned}$$

where $\tilde{\mu} = \begin{pmatrix} \mu_{11} \\ \vdots \\ \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{12} \\ \vdots \\ \mu_{ij} \\ \vdots \\ \mu_{ij} \\ \vdots \\ \mu_{NN} \\ \vdots \\ \mu_{NN} \end{pmatrix}$, $B = \begin{pmatrix} J_{T_{11}} & 0 & \dots & 0 \\ 0 & J_{T_{12}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{T_{NN}} \end{pmatrix}$ of size $(T \times T)$

Continuing with model (2)

$$\begin{aligned}
u^\star_{ij} &= \mu_{ij} \otimes l_{T_{ij}} + \lambda + \epsilon_{ij} \\
E\left[u^\star_{ij} {u^\star}'_{ij}\right] &= E\left[\left(\mu_{ij} \otimes l_{T_{ij}}\right)\left(\mu_{ij} \otimes l_{T_{ij}}\right)'\right] + E\left[\lambda\lambda'\right] + E\left[\epsilon_{ij}\epsilon'_{ij}\right] \\
&= \sigma^2_\mu J_{T_{ij}} + \sigma^2_\lambda I_{T_{ij}} + \sigma^2_\epsilon I_{T_{ij}} \\
u^\star_i &= \tilde{\mu}_i + \tilde{\lambda}_i + \epsilon_i
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\lambda}'_i &= (\lambda_1, \lambda_2, \dots, \lambda_{T_{i1}}, \dots, \lambda_1, \lambda_2, \dots, \lambda_{T_{iN}}) \\
E\left[u^\star_i {u^\star}'_i\right] &= E\left[\tilde{\mu}_i \tilde{\mu}_i'\right] + E\left[\tilde{\lambda}_i \tilde{\lambda}_i'\right] + E\left[\epsilon_i \epsilon'_i\right] \\
&= \sigma^2_\mu A + \sigma^2_\lambda D_i + \sigma^2_\epsilon I_{\sum_{j=1}^N T_{ij}} \\
u^\star &= \tilde{\mu} + \tilde{\lambda} + \epsilon \\
E\left[u^\star {u^\star}'\right] &= E\left[\tilde{\mu}\tilde{\mu}'\right] + E\left[\tilde{\lambda}\tilde{\lambda}'\right] + E\left[\epsilon\epsilon'\right] \\
&= \sigma^2_\mu B + \sigma^2_\lambda E + \sigma^2_\epsilon I_T
\end{aligned}$$

with

$$E(E_{11}, E_{12}, \dots, E_{1N}, \dots, E_{N1}, E_{N2}, \dots, E_{NN})$$

$$E_{ij} = \begin{pmatrix} M_{T_{11} \times T_{ij}} \\ M_{T_{12} \times T_{ij}} \\ \vdots \\ M_{T_{NN} \times T_{ij}} \end{pmatrix} \quad \text{and} \quad D_i = \begin{pmatrix} I_{T_{i1}} & M_{T_{i1} \times T_{i2}} & \dots & M_{T_{i1} \times T_{iN}} \\ M_{T_{i2} \times T_{i1}} & I_{T_{i1}} & \dots & M_{T_{i2} \times T_{iN}} \\ \vdots & \vdots & \ddots & \vdots \\ M_{T_{iN} \times T_{i1}} & M_{T_{iN} \times T_{i2}} & \dots & I_{T_{iN}} \end{pmatrix}$$

where

$$M_{T_{ij} \times T_{lj}} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \quad \text{if} \quad T_{lj} > T_{ij}$$

and

$$M_{T_{ij} \times T_{lj}} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{if} \quad T_{lj} < T_{ij}$$

Doing the same exercise for model (3) using decomposition (8) we end up with

$$\begin{aligned} u^\star_{ij} &= u_j + \epsilon_{ij} \\ E\left(u^\star_{ij} {u^\star}'_{ij}\right) &= E\left[u_j u'_j\right] + E\left[\epsilon_{ij}\epsilon'_{ij}\right] = \sigma^2_u I_{T_{ij}} + \sigma^2_\epsilon I_{T_{ij}} \\ u^\star_i &= u + \epsilon_i \\ E\left(u^\star_i {u^\star}'_i\right) &= E\left[u u'\right] + E\left[\epsilon_i \epsilon'_i\right] = \sigma^2_u I_{\sum_{j=1}^N T_{ij}} + \sigma^2_\epsilon I_{\sum_{j=1}^N T_{ij}} \\ u^\star &= \tilde{u} + \epsilon \end{aligned}$$

and so for the complete model we get

$$E\left(u^\star {u^\star}'\right) = E\left[\tilde{u}\tilde{u}'\right] + E\left[\epsilon\epsilon'\right] = \sigma^2_u C + \sigma^2_\epsilon I_T$$

where

$$\tilde{u}' = (u_{11}, \dots, u_{1T_{11}}, \dots, u_{N1}, \dots, u_{NT_{1N}}, \dots, u_{11}, \dots, u_{1T_{N1}}, \dots, u_{N1}, \dots, u_{NT_{NN}})$$

$C = (C_1, C_2, C_3)$

$$C_1 = \begin{pmatrix} I_{T_{11}} & 0 & \dots & 0 \\ 0 & I_{T_{12}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{T_{1N}} \\ M_{T_{21} \times T_{11}} & 0 & \dots & 0 \\ 0 & M_{T_{22} \times T_{12}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_{T_{2N} \times T_{1N}} \\ \vdots & \vdots & \ddots & \vdots \\ M_{T_{N1} \times T_{11}} & 0 & \dots & 0 \\ 0 & M_{T_{N2} \times T_{12}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_{T_{NN} \times T_{1N}} \end{pmatrix}$$

$$C_2 = \begin{pmatrix} M_{T_{11} \times T_{21}} & 0 & \dots & 0 & \dots \\ 0 & M_{T_{12} \times T_{22}} & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & \dots & M_{T_{1N} \times T_{2N}} & \dots \\ I_{T_{21}} & 0 & \dots & 0 & \dots \\ 0 & I_{T_{22}} & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & \dots & I_{T_{2N}} & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ M_{T_{N1} \times T_{21}} & 0 & \dots & 0 & \dots \\ 0 & M_{T_{N2} \times T_{22}} & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ 0 & 0 & \dots & M_{T_{NN} \times T_{1N}} & \dots \end{pmatrix}$$

$$C_3 = \begin{pmatrix} M_{T_{11}\times T_{N1}} & 0 & \dots & 0 \\ 0 & M_{T_{12}\times T_{N2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_{T_{1N}\times T_{NN}} \\ M_{T_{21}\times T_{N1}} & 0 & \dots & 0 \\ 0 & M_{T_{22}\times T_{N2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_{T_{2N}\times T_{NN}} \\ \vdots & \vdots & \ddots & \vdots \\ I_{T_{N1}} & 0 & \dots & 0 \\ 0 & I_{T_{N2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{T_{NN}} \end{pmatrix}$$

Let us now turn to model (4). Following the same steps as above, we get for the covariance matrix $(\sigma_v^2 D + \sigma_\epsilon^2 I_T)$ where

$$D = \begin{pmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ 0 & 0 & \dots & D_N \end{pmatrix}$$

Models (5) and (6) can be dealt with together using decomposition (9)

$$\begin{aligned}
u^\star_{ij} &= \mu_{ij} \otimes l_T + u_j + v_i + \epsilon_{ij} \\
E\left(u^\star_{ij} {u^\star}'_{ij}\right) &= E\left[\left(\mu_{ij} \otimes l_{T_{ij}}\right)\left(\mu_{ij} \otimes l'_{T_{ij}}\right)\right] + E\left[u_j u'_j\right] + E\left[v_i v'_i\right] + E\left[\epsilon_{ij} \epsilon'_{ij}\right] \\
&= \sigma_\mu^2 J_{T_{ij}} + \sigma_u^2 I_{T_{ij}} + \sigma_v^2 I_{T_{ij}} + \sigma_\epsilon^2 I_{T_{ij}} \\
u^\star_i &= \tilde{\mu_i} + \tilde{v_i} + u + \epsilon_i \\
E\left(u^\star_i {u^\star}'_i\right) &= E\left[\tilde{\mu_i}\tilde{\mu_i}'\right] + E\left[\tilde{v_i}\tilde{v_i}'\right] + E\left[u u'\right] + E\left[\epsilon_i \epsilon'_i\right] \\
&= \sigma_\mu^2 A + \sigma_u^2 I_{\sum_{j=1}^N T_{ij}} + \sigma_v^2 D_i + \sigma_\epsilon^2 I_{\sum_{j=1}^N T_{ij}} \\
u^\star &= \tilde{\mu} + \tilde{v} + \tilde{u} + \epsilon \\
\text{where} \quad \tilde{v_i}' &= (v_{i1}, v_{i2}, \dots, v_{iT_{i1}}, v_{i1}, v_{i2}, \dots, v_{iT_{i2}}, \dots, v_{i1}, v_{i2}, \dots, v_{iT_{iN}}) \\
\tilde{v}' &= (\tilde{v_1}, \tilde{v_2}, \dots, \tilde{v_N},) \\
E\left(u^\star {u^\star}'\right) &= E\left[\tilde{\mu}\tilde{\mu}'\right] + E\left[\tilde{v}\tilde{v}'\right] + E\left[\tilde{u}\tilde{u}'\right] + E\left[\epsilon\epsilon'\right] = \\
&= \sigma_\mu^2 B + \sigma_u^2 C + \sigma_v^2 D + \sigma_\epsilon^2 I_T
\end{aligned}$$

For model (5) the appropriate covariance matrix is the same with $B = 0$.

Now that we derived the covariance matrices for unbalanced data it is time to turn to the estimation of the variance components. Using (10) and (11)

$$E\left[\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(u^{\star}_{ijt} - \bar{u^{\star}}_{ij}\right)^2\right] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left[\left(\epsilon_{ijt} - \bar{\epsilon}_{ij}\right)^2\right]$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left[\epsilon_{ijt}^2 - 2\epsilon_{ijt}\frac{1}{T_{ij}}\sum_{t=1}^{T_{ij}}\epsilon_{ijt} + \left(\frac{1}{T_{ij}}\sum_{t=1}^{T_{ij}}\epsilon_{ijt}\right)^2\right]$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(E\left[\epsilon_{ijt}^2\right] - 2E\left[\epsilon_{ijt}\frac{1}{T_{ij}}\sum_{t=1}^{T_{ij}}\epsilon_{ijt}\right] + E\left[\left(\frac{1}{T_{ij}}\sum_{t=1}^{T_{ij}}\epsilon_{ijt}\right)^2\right]\right)$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\sigma_\epsilon^2 - \frac{2}{T_{ij}}\sigma_\epsilon^2 + \frac{1}{T_{ij}}\sigma_\epsilon^2\right) = \sigma_\epsilon^2\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{T_{ij}-1}{T_{ij}}$$

so for the variance components we get the following estimators

$$\hat{\sigma}_\epsilon^2 = \frac{N^2}{\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{T_{ij}-1}{T_{ij}}}\hat{u}^{\star\prime}_{within}\hat{u}^{\star}_{within}$$

$$\hat{\sigma}_\mu^2 = \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T_{ij}}\hat{u^{\star}}^2_{ijt} - \hat{\sigma}_\epsilon^2$$

For model (3) (and similarly for model (4)), using (12) and (13) and using the same derivations as there we get

$$\hat{\sigma}_\epsilon^2 = \frac{N}{N-1}\hat{u}^{\star\prime}_{within}\hat{u}^{\star}_{within}$$

$$\hat{\sigma}_u^2 = \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T_{ij}}\hat{u}^{\star 2}_{ijt} - \hat{\sigma}_\epsilon^2$$

Turning now to model (5), as (14) and (15) are the same in the unbalanced case we get

$$\hat{\sigma}_\epsilon^2 = \frac{N^2}{(N-1)^2}\hat{u}^{\star\prime}_{within}\hat{u}^{\star}_{within}$$

$$\hat{\sigma}_u^2 = \frac{1}{N-1}\left(\sum_{i=1}^{N}\frac{1}{\sum_{j=1}^{N}T_{ij}}\sum_{j=1}^{N}\sum_{t=1}^{T_{ij}}\left(\frac{1}{N}\sum_{i=1}^{N}\hat{u^{\star}}_{ijt}\right)^2 - \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T_{ij}}\hat{u}^{\star 2}_{ijt}\right)$$

$$\hat{\sigma_v^2} = \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T_{ij}}\hat{u}^{\star 2}_{ijt} - \hat{\sigma}_\epsilon^2 - \hat{\sigma}_u^2$$

And finally for model (6) we get

$$E\left[{u^\star}_{ijt}^2\right] = E\left[(\mu_{ij} + u_{jt} + v_{it} + \epsilon_{ijt})^2\right] = \sigma_\mu^2 + \sigma_u^2 + \sigma_v^2 + \sigma_\epsilon^2$$

$$\begin{aligned} E\left[\left(\frac{1}{N}\sum_{i=1}^N u^\star_{ijt}\right)^2\right] &= E\left[\left(\frac{1}{N}\sum_{i=1}^N (\mu_{ij} + u_{jt} + v_{it} + \epsilon_{ijt})\right)^2\right] \\ &= \frac{1}{N}\sigma_\mu^2 + \sigma_u^2 + \frac{1}{N}\sigma_v^2 + \frac{1}{N}\sigma_\epsilon^2 \\ E\left[\left(\frac{1}{N}\sum_{j=1}^N u^\star_{ijt}\right)^2\right] &= E\left[\left(\frac{1}{N}\sum_{j=1}^N (\mu_{ij} + u_{jt} + v_{it} + \epsilon_{ijt})\right)^2\right] \\ &= \frac{1}{N}\sigma_\mu^2 + \frac{1}{N}\sigma_u^2 + \sigma_v^2 + \frac{1}{N}\sigma_\epsilon^2 \end{aligned}$$

$$E\left[\frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N\left(\frac{1}{T_{ij}}\sum_{j=1}^{T_{ij}} u^\star_{ijt}\right)^2\right] = \frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N E\left[\left(\frac{1}{T_{ij}}\sum_{j=1}^{T_{ij}} u^\star_{ijt}\right)^2\right]$$

$$= \frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N \frac{1}{T_{ij}^2} E\left[\left(\sum_{t=1}^{T_{ij}}(\mu_{ij} + u_{jt} + v_{it} + \epsilon_{ijt})\right)^2\right]$$

$$= \frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N \frac{1}{T_{ij}^2}\left(E\left[\sum_{t=1}^{T_{ij}}\mu_{ij}^2\right] + E\left[\sum_{t=1}^{T_{ij}}u_{jt}^2\right] + E\left[\sum_{t=1}^{T_{ij}}v_{it}^2\right] + E\left[\sum_{t=1}^{T_{ij}}\epsilon_{ijt}^2\right]\right)$$

$$= \left(\sigma_\mu^2 + \sigma_u^2 + \sigma_v^2 + \sigma_\epsilon^2\right)\frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N\frac{1}{T_{ij}}$$

Putting all these together, the estimators of the variance components are

$$\hat\sigma_\mu^2 = \frac{1}{N^2 - \sum_{i=1}^N\sum_{j=1}^N\frac{1}{T_{ij}}}\left(\sum_{i=1}^N\sum_{j=1}^N\left(\frac{1}{T_{ij}}\sum_{t=1}^{T_{ij}}{\hat u^\star}_{ijt}\right)^2 - \left(\sum_{i=1}^N\sum_{j=1}^N\frac{1}{T_{ij}}\right)\frac{1}{T}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^{T_{ij}}{\hat u^\star}_{ijt}^2\right)$$

$$\hat\sigma_u^2 = \frac{1}{N-1}\left(\sum_{i=1}^N\frac{1}{\sum_{j=1}^N T_{ij}}\sum_{j=1}^N\sum_{t=1}^{T_{ij}}\left(\frac{1}{N}\sum_{i=1}^N{\hat u^\star}_{ijt}\right)^2 - \frac{1}{T}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^{T_{ij}}{\hat u^\star}_{ijt}^2\right)$$

$$\hat\sigma_v^2 = \frac{1}{N-1}\left(\sum_{j=1}^N\frac{1}{\sum_{i=1}^N T_{ij}}\sum_{i=1}^N\sum_{t=1}^{T_{ij}}\left(\frac{1}{N}\sum_{j=1}^N{\hat u^\star}_{ijt}\right)^2 - \frac{1}{T}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^{T_{ij}}{\hat u^\star}_{ijt}^2\right)$$

$$\hat\sigma_\epsilon^2 = \frac{1}{T}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^{T_{ij}}{\hat u^\star}_{ijt}^2 - \hat\sigma_\mu^2 - \hat\sigma_u^2 - \hat\sigma_v^2$$

5. An Application: Modelling Within EU Trade

In order to highlight the differences between the usual fixed effects (FE) and the proposed random effects (RE) approach, let us use a typical empirical trade problem. In a gravity-like panel estimation exercise we explore the effects of geographical distance and membership in the European Union (EU) on bilateral trade flows. We compare the RE estimates to FE and Pooled OLS estimates.

We take a balanced panel data set of bilateral trade flows for all pairs formed by 20 EU member countries for years 2001-2006. Hence, the total number of country pairs is $N^2 = 400$ and $T = 6$. Twelve of the countries were members of the EU in the whole sample period (group A)[2], the remaining eight entered the EU in 2004 (group B)[3]. Apart from foreign trade, the database also includes self-trade, i.e. trade of a country within its own borders.[4] Self-trade for a given year is generated as gross output minus total exports of a country in that year. All data is in current euros.

A first look at the data suggests that countries in group A trade more with each other than countries in group B, and trade of group B countries increased much faster after 2004 than trade of group A countries (*Table 1*). The first fact can simply reflect that larger, more advanced and more strongly integrated economies trade more. The second may be evidence for the trade creating effect of entering the EU.

We fit a simple gravity-type model that explains bilateral trade with country incomes (GDP), bilateral geographical distance and a dummy for EU membership. For better tractability we restrict the elasticities of trade to income to unity and use income adjusted trade, denoted as y, as dependent variable. We take all variables (except the EU dummy) in logarithms: $y_{ijt} = \ln \text{trade}_{ijt} - \ln \text{income}_{it} - \ln \text{income}_{jt}$, and the explanatory variables are simply $[\ln \text{dist}_{ij}, \text{EU}_{ijt}]$. The dummy for EU membership is 1, if both the exporter and the importer countries are EU members, and 0 otherwise. Formally,

$$\text{EU}_{ijt} = \begin{cases} 1 & (i \in \text{A and } \ j \in \text{A}) \text{ or } \ t \geq 2004 \\ 0 & \text{otherwise} \end{cases}$$

---

[2] Group A: Austria, Germany, Denmark, Spain, Finland, France, Greece, Ireland, Italy, Portugal, Sweden, United Kingdom

[3] Group B: Czech Republic, Estonia, Hungary, Lithuania, Latvia, Poland, Slovenia, Slovakia

[4] We include self-trade to avoid bias of the Fixed Effects estimates. As it is stressed in *Hornok* [2011] and *Mátyás and Balázsi* [2011], the Fixed Effects within transformation formulas for some of the error structures considered here give biased estimates if self-trade is not included in the database.

Notice that, while distance is time-invariant, the EU dummy is time-varying, changing from 0 to 1 from 2003 to 2004 for country pairs with at least one type-B country.

The (composite) error term, can take the form of any of the error structures (1) to (6) discussed in the previous sections. Assuming an error structure, we estimate the model using FGLS. Estimated coefficients for each error structure (models (1) to (6)) are reported in *Table 2*. We report Pooled OLS, FE and RE estimates, as well as the estimated variances for the error components. Pooled OLS estimates are identical for each model. The parameter estimates for the distance coefficient are very stable across all models and methods as this coefficient is always identified from variation in the country pair dimension. As a consequence, in the FE models with country pair fixed effects (models (1), (2) and (6)) the distance coefficient is not identified. This highlights one important drawback of the FE approach: due to the large number of fixed effects related dummies, other important dummy-like variable often are not identified.

In contrast, the coefficient for the EU dummy is always identified, but its estimates vary considerably across models. In models (1), (2) and (6) it is identified mostly (in the case of FE, only) from the time dimension, i.e., from the change in trade of type-B countries. In the other three models identification is based more on the cross-sectional dimension, i.e., comparing EU pairs to non-EU pairs before 2004. Apart from that, in this case, the RE parameter estimates happen to be quite close to the FE estimates (except for model (5)).

As a next step we change the estimating equation so that the EU dummy is broken up into three separate dummies. One for pairs of two type-A countries, one for pairs of two type-B countries and one for pairs with one type-A and one type-B country:

$$\mathrm{EU}_{AA} = \begin{cases} 1 & \mathrm{EU} = 1 \ \text{ and } \ i \in A \ \text{ and } \ j \in A \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{EU}_{BB} = \begin{cases} 1 & \mathrm{EU} = 1 \ \text{ and } \ i \in B \ \text{ and } \ j \in B \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{EU}_{AB} = \begin{cases} 1 & \mathrm{EU} = 1 \ \text{ and } \ \mathrm{EU}_{AA} \neq 1 \text{ and } \mathrm{EU}_{BB} \neq 1 \\ 0 & \text{otherwise} \end{cases}$$

This modification enables us to identify separate effects of the EU on the different groups of country pairs. Besides, it helps to bring to light again the already mentioned important disadvantage of the FE approach. When the within transformation nets out fixed effects in it and jt (and ij) dimensions, identification of other regressors (especially dummy variables) may not be possible, even if these regressors vary in the ijt dimension. This is due to perfect collinearity among the fixed effects and the regressors.[5] As reported in *Table 3*, the FE estimator is not able to identify the coefficients of EU_{BB} and EU_{AB} separately under models (5) and (6). In contrast, the RE method identifies all coefficients and reveals that there are indeed large and significant differences among the effects of EU on different groups of country pairs. These differences are in line with the raw data evidence in *Table 1*. The coefficient estimates for EU_{BB} are significantly larger than those for EU_{AB}, which are larger than the coefficient estimates for EU_{AA}. The estimates are in some cases negative, depending on whether the model identifies them mostly from the time series or from the cross section dimension. In an empirical research, this has of course important implications on which model to choose and how to interpret the estimates, an issue we do not deal with here.

6. Conclusion

In this paper we presented an alternative random effects approach to the usual fixed effects gravity models of trade, in a three-dimensional panel data setup. We showed that the random effects and fixed effects specifications, just like in the usual panel data cases, may lead to substantially different parameter estimates and inference, although in both cases the corresponding estimators are in fact consistent.

At the end of the day, the main question for an applied researcher, as in any panel data setup, is whether to use a fixed effects or random effects specification. In three (or multi-) dimensional models the fixed effects specification (due to the very large number of dummies to estimate) will result in a massive over-specification, which implies that much less data information will be available for the estimation of the main/focus variables. Also, again due to the fixed effects dummy variables, frequently other (say, for example policy, type, potentially important) dummy variables cannot be identified. On the other hand, in a random effects specification the data is not "burdened" by the massive estimation of the fixed effects parameters. In addition any reasonable covariance structure can be imposed on the disturbance terms, still the model can be estimated without to much trouble. The down side is, of course, that one has to keep an eye on the endogeneity problem. The choice unfortunately not obvious.

[5] This identification problem is also addressed in *Hornok* [2011].

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Table 1: Trade of EU Countries Before and After 2004

	2001-2003	2004-2006	% change
Foreign trade[1]			
A with A	7,659	8,561	11.8
A with B	1,064	1,471	38.3
B with B	379	712	87.9
Self-trade[2]			
A	268,348	285,006	6.2
B	29,102	32,803	12.7

Notes: Source of trade data (in millions of euros) is Eurostat. Self-trade is authors' calculation based on Eurostat and OECD data. [1] Average of annual pair-specific flows within group. [2] Average of annual country self-trade within group.

Table 2: Comparison of Estimators

	model 1	model 2	model 3	model 4	model 5	model 6
Pooled OLS						
EU			0.190631			
			(0.036527)			
ln dist			-1.645467			
			(0.019266)			
Fixed Effects						
EU	-0.030791	0.126275	-0.064553	-0.257496	0.593089	0.006564
	(0.011727)	(0.018922)	(0.054318)	(0.057025)	(0.072609)	(0.033426)
ln dist	–	–	-1.681144	-1.609525	-1.608665	–
	–	–	(0.017238)	(0.018097)	(0.015311)	–
Random Effects						
EU	-0.034560	0.110319	-0.114367	-0.250177	0.262012	-0.002745
	(0.011698)	(0.018636)	(0.048620)	(0.048929)	(0.059702)	(0.031547)
ln dist	-1.647862	-1.650085	-1.677159	-1.616094	-1.627702	-1.645157
	(0.045509)	(0.045181)	(0.017131)	(0.017937)	(0.014976)	(0.035335)
Variance Components						
σ_ε^2	0.052808	0.049494	0.508946	0.560954	0.345769	0.038615
σ_μ^2	0.639168	0.630406				0.344412
σ_λ^2		0.012077				
σ_u^2			0.183030		0.179277	0.179277
σ_v^2				0.131022	0.166930	0.129673

Notes: Dependent variable is log of income-adjusted bilateral trade. Standard errors in parenthesis. Estimation on a balanced panel of pairs of 20 EU countries for years 2001-2006.

Table 3: Comparison of Estimators with Three EU Dummies

	model 1	model 2	model 3	model 4	model 5	model 6
Pooled OLS						
EU_{AA}			-0.425864			
			(0.039215)			
EU_{BB}			0.628679			
			(0.065105)			
EU_{AB}			-0.250615			
			(0.043757)			
ln dist			-1.572309			
			(0.018913)			
Fixed Effects						
EU_{AA}	–	–	-0.058001	-0.250686	0.626499	–
	–	–	(0.052119)	(0.054984)	(0.071516)	–
EU_{BB}	0.041035	0.198101	1.078016	0.843686	–	–
	(0.023386)	(0.027204)	(0.098732)	(0.104160)	–	–
EU_{AB}	-0.054733	0.102333	0.228817	-0.024152	–	–
	(0.013502)	(0.019963)	(0.073639)	(0.077688)	–	–
ln dist	–	–	-1.631179	-1.557583	-1.577510	–
	–	–	(0.017071)	(0.018009)	(0.015482)	–
Random Effects						
EU_{AA}	-0.420115	-0.342579	-0.176215	-0.328905	0.216663	-0.091471
	(0.079668)	(0.078380)	(0.046842)	(0.047293)	(0.058454)	(0.086561)
EU_{BB}	0.057348	0.212035	0.870278	0.716491	1.002332	0.426399
	(0.023273)	(0.027062)	(0.083976)	(0.083104)	(0.135971)	(0.133636)
EU_{AB}	-0.060171	0.095017	0.041347	-0.141667	0.295330	0.202103
	(0.013480)	(0.019876)	(0.061452)	(0.060241)	(0.082291)	(0.067982)
ln dist	-1.627487	-1.627523	-1.625481	-1.562311	-1.596464	-1.637897
	(0.043166)	(0.042274)	(0.016964)	(0.017839)	(0.015113)	(0.035505)
Variance Components						
σ^2_ε	0.052504	0.049187	0.468526	0.521461	0.334464	0.038635
σ^2_μ	0.571179	0.548059				0.345164
σ^2_λ		0.026436				
σ^2_u			0.155157		0.140710	0.140710
σ^2_v				0.102222	0.148509	0.099174

Notes: Dependent variable is log of income-adjusted bilateral trade. Standard errors in parenthesis. Estimation on a balanced panel of pairs of 20 EU countries for years 2001-2006.