



Munich Personal RePEc Archive

## **Population-ageing, structural change and productivity growth**

Stijepic, Denis and Wagner, Helmut

FernUniversität in Hagen

15 February 2009

Online at <https://mpra.ub.uni-muenchen.de/37005/>

MPRA Paper No. 37005, posted 29 Feb 2012 14:23 UTC

# Population-Ageing, Structural Change and Productivity Growth

by

**Denis Stijepic** (Corresponding Author)

University of Hagen, Department of Economics

Universitätsstr. 41, D-58084 Hagen, Germany

Tel.: +49(0)2331/987-2640

Fax: +49(0)2331/987-391

E-mail: [denis.stijepic@fernuni-hagen.de](mailto:denis.stijepic@fernuni-hagen.de)

and

**Helmut Wagner**

University of Hagen, Chair of Macroeconomics

Universitätsstr. 41, D-58084 Hagen, Germany

Tel.: +49(0)2331/987-2640

Fax: +49(0)2331/987-391

E-mail: [helmut.wagner@fernuni-hagen.de](mailto:helmut.wagner@fernuni-hagen.de)

Web: <http://www.fernuni-hagen.de/HWagner>

**This version: February 2012**

## *Abstract*

Population-ageing is one of the traditional topics of development and growth theory and a key challenge to most modern societies. We focus on the following aspect: Population-ageing is associated with changes in demand-structure, since demand-patterns change with increasing age. This process induces structural changes (factor-reallocations across technologically heterogeneous sectors) and, thus, has impacts on average productivity growth. We provide a neoclassical multi-sector growth-model for analyzing these aspects and elaborate potential policy-impact channels. We show that ageing has permanent and complex/multifaceted impacts on the growth rate of the economy and could, therefore, be an important determinant of long-run GDP-growth.

*Keywords:* Population-ageing, demand shifts, reallocation of factors, cross-sector technology-disparity, GDP-growth, multi-sector growth models, neoclassical growth models, structural change.

**JEL Classification Numbers:** J11, O14, O41

# 1. Introduction

Population-ageing – a term which in general refers to an increasing life span of an average member of a society – is one of the key stylized facts of the development process. It has had and will have some major impacts on economic and social structures in developing and industrialized economies. This fact is reflected by a large body of literature dealing with it. Well known examples of this literature are development and growth theories related to population growth, e.g. classical theories (like Malthusian development traps) and neoclassical growth theory (ranging from Solow-model to endogenous-growth theories), and most obviously theories of social security and pension systems. As well, these aspects are associated with actual policy making, including development policy (World Bank, UN, etc), population policy (e.g. in China), and major changes in pension and health systems in some industrialized economies. A very general discussion of population-ageing is provided by IMF (2004). The focus of our paper is on economic growth. (To some extent our paper has implications for pension systems as well). For an extensive discussion of models dealing with ageing and economic growth see, e.g., Gruescu (2007); for a short, but still very comprehensive, discussion see, e.g., Mc Morrow and Röger (2003). An overview of empirical studies is provided by, e.g., Groezen et al. (2005).

## *1.1 Focus of our paper*

In this paper we focus on an impact channel of ageing which seems to be rarely studied in this literature (at least there seems to be a shortage of theoretical models which analyze it): the impacts of ageing-induced demand-shifts on factor-allocation across technologically distinct sectors and their consequences for GDP-growth. Our results have also implications for old-age-pension-funding, since GDP is the basis for funding the pension systems. The working hypothesis is the following: An increase in the relative share of the “old” in an economy changes the structure of aggregate demand, since the “old” have a different structure of demand in comparison to the “young”. If there are some differences in technologies between sectors which produce the goods for the old and sectors which produce the goods for the young some effects on aggregate productivity growth and, thus, on GDP-growth and pension-to-output-ratios may arise. (We name this whole line of arguments “*factor-allocation-effects of ageing*”). In

other words, ageing may induce “structural change” (i.e. cross-technology factor-reallocation), hence causing changes in aggregate (or: average) productivity growth. Thus, the increasing old-age pension payments (due to the increasing number of recipients) are confronted with changes in the growth rate of the tax-base, which may require changes in the old-age-pension system.

This line of arguments seems to be quite obvious, especially when thinking of services, like health care services and geriatric nursing services: in general, the “old” demand more of such services in comparison to the “young”; furthermore, the “production process” of these services is regarded to be technologically distinct (i.e. relatively labour-intensive) in comparison to e.g. manufacturing goods (see also IMF (2004), chapter 3, and especially p. 159). However, there are also some other differences in demand between the old and the young, e.g. the young have a relatively larger demand for commodities and investment goods (e.g. housing, car and furniture, i.e. things which the old may already have). Furthermore, in general, the old seem to spend a larger share of budget on services (see Groezen et al. (2005)).

### ***1.2 Related empirical evidence***

Empirical evidence on such differences in demand patterns between the old and the young and their growing importance (not only for factor reallocation across sectors) has been presented by, e.g., Börsch-Supan (1993, 2003), Fuchs (1998) and Fougère et al. (2007). Furthermore, empirical evidence implies that there are strong differences in technology across products/sectors (e.g. when comparing some services and manufactured products or health care services and commodities production): Evidence on differences in TFP-growth across sectors/products is provided by, e.g., Baumol et al. (1985) and Bernard and Jones (1996). Evidence on differences in capital intensities across sectors is provided by, e.g., Close and Schulenburger (1971), Kongsamut et al. (1997), Gollin (2002), Acemoglu and Guerrieri (2008) and Valentinyi and Herrendorf (2008). Nordhaus (2008) presents some evidence on the relevance of cross-sector reallocations for aggregate growth.<sup>1</sup> Overall, this (partly indirect) evidence on factor-allocation-

---

<sup>1</sup> Further references on the relationship between structural change and growth are: Robinson (1971), Madisson (1987), pp.666ff, Dowrick and Gemmel (1991), Bernard and Jones (1996), Broadberry (1997,1998), Foster et al. (1998), Berthélmy and Söderling (1999), Poirson (2000), Caselli and Coleman (2001), Temple (2001), Disney et al. (2003), Penderer (2003), Broadberry and Irwin (2006), UN (2006), Restuccia et al. (2008) and Duarte and Restuccia (2010).

effects of ageing seems to provide sufficient incentive to take a look at their relevance from a theoretical perspective.

### ***1.3 Related theoretical literature***

Our model is related to the theoretical literature which postulates the importance of cross-sector technology-differences for GDP-growth, e.g. Baumol (1967) and Acemoglu and Guerrieri (2008). Baumol (1967) claims that cross-sector differences in (labour-)productivity-growth can cause (by themselves) a GDP-growth-slowdown via relative price changes (“Baumol’s cost disease”). However, Baumol (1967) does not analyze (ageing-induced) demand-shifts, and he makes as well some simplifying assumptions (e.g. he excludes capital accumulation), which may be not accurate for our goals as we will see later. Acemoglu and Guerrieri (2008) show that cross-sector differences in capital-intensities have an impact on aggregate growth. However, they as well do not include (ageing-induced) demand-shifts into their analysis. Furthermore, Rausch (2006) provides a two-sector Heckscher-Ohlin model with ageing, where ageing leads to an increase in the savings rate, since the old have relatively larger amounts of assets. He argues that ageing leads to changes in the relative sector-size (and, thus, in GDP-growth), provided that sectors differ by capital intensity (see Rausch (2006), pp. 20 ff.). He as well does not take account of the impacts of ageing-induced demand-shifts.

To our knowledge, the model by Groezen et al. (2005) is the only one which explicitly includes ageing-induced demand-shifts into analysis, where ageing is incorporated into a two-sector overlapping-generations model. The old consume the output of a “backward” services sector; this sector uses labour-input only and does not have any productivity growth. The young consume the output of a “progressive” commodities sector; this sector uses capital and labour as input factors and generates capital and endogenous technological progress which increases its productivity with time. Groezen et al. (2005) focus on the trade-off between the positive “savings-effect of longevity”<sup>2</sup> and the negative “factor-

---

<sup>2</sup> This savings-effect works as follows: An increase in longevity implies more saving for retirement. An increase in savings is associated with additional generation of capital and technological progress. Thus, factors are reallocated to the commodities sector (since this sector generates capital and technological progress).

allocation-effects of ageing”<sup>3</sup>. They show the importance of the elasticity of substitution between capital and labour in the progressive sector. If this elasticity is equal to unity, the two effects offset each other and ageing has no impacts on growth in their model. However, if this elasticity is greater (smaller) than unity, ageing has a negative (positive) impact on growth.<sup>4</sup>

#### ***1.4 Model setup***

In contrast to Groezen et al. (2005), we do not study the *trade-off* between “savings-effect” and “factor-allocation-effect”, but focus on a detailed and in some sense “more general” study of the “factor-allocation-effect”.<sup>5</sup> We are able to provide a detailed discussion of the factor-reallocations and the “factor-allocation-effect (without simulations), because our paper is rooted in the “new” structural change literature, which is pioneered by Kongsamut et al. (1997, 2001), Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008). This literature focuses on neoclassical structural change modelling (capital accumulation and intertemporal utility maximization) and the usage of (partially) balanced growth paths (PBGPs). Especially, PBGPs facilitate the dynamic analysis significantly; see also the discussion in section 5.5.

Our model is a sort of disaggregated Ramsey-model<sup>6</sup> where the representative household(s) consume(s) two groups of goods: “senior-goods” (i.e. goods which are primarily consumed by “older” people) and “junior-goods” (i.e. goods which are primarily consumed by younger people). Ageing (i.e. an increasing ratio of old-to-young) yields an increasing weight of senior-needs in the aggregate utility function, hence leading to a demand-shift in direction of senior-goods. We assume that the production of senior-goods and the production of junior-goods differ by TFP-growth and by capital-intensity (i.e. output-elasticity of inputs), according to the empirical evidence discussed above. Moreover, we include intermediates production into the model; this allows for linkages between senior- and junior-goods-production, which have been stated to be important by Fougère

---

<sup>3</sup> The factor-allocation-effect has been described in section 1.1. In the Groezen-et-al.- (2005)-model this effect works as follows: ageing shifts factors to the “backward” services sector, since the “old” consume services only; thus, aggregate labour-productivity is lowered.

<sup>4</sup> A paper, which is to some extent related to this topic, since it deals with ageing-related choice of technology, is provided by Irmen (2009).

<sup>5</sup> In fact, the sort of “savings effect”, which is modelled by Groezen et al. (2005), does not exist in our model.

<sup>6</sup> For discussion of the standard (one-sector) Ramsey-model, see e.g. Barro and Sala-i-Martin (2004).

et al. (2007) and by Kuhn (2004); see also the discussion at the end of Section 2.2.

### ***1.5 Model results***

Overall, our results imply that the factor-allocation-effects of ageing on GDP-growth are “complex” (or: “multifaceted”), i.e. they are dependent on many parameters, consisting of several channels and potentially non-monotonous over time.

Furthermore, they seem to be very significant, from the theoretical point of view, since even a *one-time increase* in the old-to-young ratio causes permanent (non-transitory) impacts on the GDP-growth-rate. Thus, ageing seems to be an important determinant of GDP-growth.

For a more detailed summary of our results and their implications see section 5; especially, see section 5.4 for a comparison of our results to previous literature.

### ***1.6 Setup of the paper***

The rest of the paper is set up as follows: In sections 2 and 3 we present the assumptions and the solution of our model. In section 4 we analyze the impacts of ageing: first, we describe the dynamics of the equilibrium without ageing (section 4.1); subsequently, we compare this equilibrium to the equilibrium with ageing, where we present a simpler version of the model in section 4.2 (where only cross-sector-differences in TFP-growth exist) and the more sophisticated version of the model in section 4.3. Finally, we make some concluding remarks in section 5.

## **2. Model assumptions**

### ***2.1 Utility***

We assume an economy where two groups of goods exist: “junior-goods” (goods  $i = 1, \dots, m$ ) and “senior-goods” (goods  $i = m + 1, \dots, n$ ). The representative household consumes a mix of these goods and maximizes the following life-time utility function (in the following we omit the time-indices):

$$(1) U = \int_0^{\infty} u e^{-\rho t} dt$$

where

$$(2) u = \frac{L}{N} u_J + \left(1 - \frac{L}{N}\right) u_S$$

$$(3) u_J = \ln \left[ \prod_{i=1}^m (C_i - \theta_i)^{\omega_i} \right]$$

$$(4) u_S = \ln \left[ \prod_{i=m+1}^n (C_i - \theta_i)^{\omega_i} \right]$$

$$(5a) \sum_{i=1}^m \theta_i = 0, \quad \sum_{i=m+1}^n \theta_i = 0$$

$$(5b) \sum_{i=1}^m \omega_i = 1, \quad \sum_{i=m+1}^n \omega_i = 1$$

$$(6) 0 < \omega_i < 1 \quad \forall i, \quad \frac{\dot{N}}{N} = g_N, \quad \frac{\dot{L}}{L} = g_L$$

where  $C_i$  stands for consumption of good  $i$  and  $\rho$  is the time-preference rate.  $N$  is an index of overall-population (including the young and the old) growing at constant exogenous rate;  $L$  is an index of the young (working) population growing at constant exogenous rate. Hence, the ratio  $L/N$  is an index of the share of the young as part of overall population, and a decreasing  $L/N$  can be interpreted as ageing.

The utility function is based on the Stone-Geary-preferences, where the  $\theta_i$ s can be respectively interpreted as the subsistence levels (if  $\theta_i > 0$ ) or as levels of home-production (if  $\theta_i < 0$ ). The income-elasticity of demand differs across goods; the price-elasticity of demand differs across goods as well and is *not* equal to unity. (See also Kongsamut et al (1997, 2001) for a discussion of a similar utility function.)

In fact, this utility function introduces ageing-induced demand shifts in the simplest way. The utility function implies that ageing (a decreasing  $L/N$ ) makes the consumption of senior-goods relatively more contributing to aggregate utility, and, as we will see later, this leads to a shift of demand towards senior-goods. In order to focus on the effects of ageing we introduce the restrictions (5a) and (5b).



In this way we ensure that there are no other shifts in demand between the junior and senior sector, beside of those induced by ageing (a decreasing  $L/N$ ): Provided that  $L/N$  is constant (no ageing), the demand for senior-goods and the demand for junior-goods grow at the same rate, yielding no factor reallocations between the senior- and the junior-sector. (Nevertheless, there are still demand shifts and reallocations within these sectors, due to the  $\theta_i$  s.)

Alternatively, the functions  $u_j$  and  $u_s$  could be assumed to be of type Cobb-Douglas or CES. We chose Stone-Geary-preferences, since in this way we can add additional sources of demand-shifts (others than ageing) by omitting the restriction (5a) and (5b). This will be of importance later.

Note that there is a difference between demand-shifts which are modelled in standard structural change theory (e.g. in the paper by Kongsamut et al. (2001)) and ageing-induced demand shifts which are modelled in our paper. In *standard structural change theory* demand shifts are caused by differences in income-elasticity of demand across goods. Hence, some repercussions arise: changes in income  $\rightarrow$  demand shifts  $\rightarrow$  productivity impacts  $\rightarrow$  changes in income and so on. This repercussion does not arise in our model. In our model the chain of impacts is rather only in one direction: (income-independent) exogenous change in old-to-young ratio  $\rightarrow$  demand shifts  $\rightarrow$  productivity impacts  $\rightarrow$  change in income. Of course, one could postulate that changes in income are associated with changes in old-to-young ratio to some extent (e.g. due to improvement in medicine or do to some change in socio-cultural parameters which are associated with increasing income). This would imply that changes in the old-to-young ratio are endogenous. Although we believe that this is an interesting topic in general, a model with endogenous old-to-young ratio would yield very similar results as the *standard structural change theory*. The only difference would be that there is a further link in the chain of impacts: income-change  $\rightarrow$  change in old-to-young ratio  $\rightarrow$  demand shifts  $\rightarrow$  productivity impacts  $\rightarrow$  income-change and so on.

Therefore, we can summarize this discussion as follows: Ageing seems to cause productivity-impacts via demand-shifts in two ways: On the one hand, it acts similar like income-elasticity-differences across goods. This sort of impact is modelled implicitly in standard structural change theory. On the other hand, ageing acts like an exogenous, income-independent shift in demand. This sort of impact is modelled in our paper. Hence, in our model we assume that ageing

arises due to some (from economist's point of view) exogenous changes. For example, some socio-cultural parameters change (e.g. change in religiosity, emancipation) and/or some progress in medicine occurs independently of income level. Of course both factors depend on the income of a country to some extent; however, they must have some income-independent timely component.

Note that we are not the only ones, who model ageing as exogenous shifts in demand. For example, Groezen et al. (2005) model it in this way too. Overall, there seems to be a research gap in this field, which may be interesting to fill.

## 2.2 Production

According to the evidence discussed above, the senior-goods are not produced by the same technology as junior-goods; the technologies differ by TFP-growth and by output-elasticities of inputs (i.e. capital intensities differ between the senior- and the junior-sector). Furthermore, we assume that only the young supply labour on the market; hence  $L$  (and not  $N$ ) is input in production:

$$(7) Y_i = A(l_i L)^\alpha (k_i K)^\beta (z_i Z)^\gamma, \quad i = 1, \dots, m$$

$$(8) Y_i = B(l_i L)^\chi (k_i K)^\nu (z_i Z)^\mu, \quad i = m + 1, \dots, n$$

$$(9) \frac{\dot{A}}{A} = g_A, \quad \frac{\dot{B}}{B} = g_B$$

$$(10) 0 < \alpha, \beta, \gamma, \chi, \nu, \mu < 1; \quad \alpha + \beta + \gamma = 1; \quad \chi + \nu + \mu = 1$$

where  $Y_i$  denotes the output of sector  $i$ ;  $K$  denotes the aggregate stock of capital;  $Z$  is an index of intermediate inputs;  $l_i$ ,  $k_i$  and  $z_i$  denote respectively the fraction of labour, capital and intermediates devoted to sector  $i$ ;  $A$  and  $B$  are exogenous technology parameters, where we assume that TFP-growth differs between the junior- and the senior-sector.

We assume that each sector's output is consumed and used as intermediate input ( $h_i$ ); only sector- $m$ -output is used as capital:

$$(11) Y_i = C_i + h_i, \quad \forall i \neq m$$

$$(12) Y_m = C_m + h_m + \dot{K} + \delta K$$

where  $\delta$  is the depreciation rate of capital. Provided that it is assumed that senior-goods are rather services, the assumption that only the junior-sector produces capital seems to be consistent with empirical evidence which states that nearly all capital goods are produced by the manufacturing sector (see e.g. Kongsamut et al. (1997, 2001)).

The intermediate-inputs-index ( $Z$ ) is a Cobb-Douglas function of sectoral intermediate outputs ( $h_i$ ):

$$(13) \quad Z = \prod_{i=1}^n (h_i)^{\varepsilon_i}, \quad 0 < \varepsilon_i < 1 \quad \forall i, \quad \sum_{i=1}^n \varepsilon_i = 1$$

This intermediate structure is the same as the one assumed by Ngai and Pissarides (2007). Note that it is important to assume intermediates production within this model. In general, we can assume that the old and the young consume many goods which are nearly the same. (However, the manner of consumption is quite different.) For example, the young and the old consume food. However, while probably many young cook the food by themselves, some very old consume the food by being served in retirement homes or hospitals. Hence, although the old and the young eat similar things, the share of services is larger in the consumption of the old. If we did not assume some intermediate linkages between the junior and senior consumption goods we would not take account for the fact that the old are the same human beings as the young (i.e. having the same basic needs). For example, the assumption that the old and the young consume different goods (which have no intermediate linkages) would e.g. imply that the old do not eat food. It would not be necessary to take account for these facts if intermediates production were irrelevant for the ageing-effects. However, as we will see, *the output-elasticities of intermediate inputs determine among others the strength of the ageing impacts via structural change*. Hence, we have to include intermediate linkages between senior-goods and junior-goods into our model.

All labour, capital and intermediate inputs are used in production, i.e.

$$(14) \quad \sum_{i=1}^n l_i = 1; \quad \sum_{i=1}^n k_i = 1; \quad \sum_{i=1}^n z_i = 1$$

## 2.3 Numéraire

Let  $p_i$  denote the price of good  $i$ . We choose the output of sector  $m$  as numéraire.

Hence,

$$(15a) \quad p_m = 1$$

It should be noted here that in reality real GDP is calculated by using an average price as GDP-deflator; i.e. in general, a basket of all goods which have been produced is used as numéraire. (See also Ngai and Pissarides (2007), p. 435, and Ngai and Pissarides (2004), p. 21.) We choose the manufacturing output as numéraire, since in this way we can analyze the equilibrium growth paths in the most convenient manner. Nevertheless, we will always calculate the GDP by using an average price deflator as well. We use the following compound deflator, which may be regarded as the theoretical mirror image of the deflators which are used to calculate real GDP in reality:

$$(15b) \quad \bar{p} \equiv \sum_{i=1}^n \frac{p_i Y_i^N}{Y_N} p_i$$

where  $Y_i^N$  and  $Y_N$  denote respectively the net-output of sector  $i$  and aggregate net-output. “Net-output” means here gross-output minus real value of intermediates inputs; thus, net-output is equal to “real-value added”. Hence,  $Y_i^N$  is given by the following relation:

$$(15c) \quad p_i Y_i^N = p_i Y_i - z_i H$$

where  $H$  is the aggregate value of all intermediates which have been produced (see later equation (18) as well). We use “net output”, since in reality GDP does not include intermediates production in order to avoid “double counting of intermediates production”. (See, e.g., Landefeld et al. (2008) on intermediate inputs and GDP.) Furthermore, the relationship between gross-output and net-

output in our model can be seen in equation (A.25) from APPENDIX A and equations (16).)

Overall, our GDP-deflator (equation (15b)) is simply a weighted-average of prices, where we used net-outputs as weights. If we divide our aggregate net-output (expressed in manufacturing terms) by this deflator we have a GDP-measure which is similar to the one which is used in reality. *However, all the issues regarding the choice of the numéraire are irrelevant when looking at shares or ratios (since the changes in the numéraire of the numerator offset the changes in the (same) numéraire of the denominator).* For example, the capital-to-output ratio ( $K/Y_N$ ) is the same irrespective of the numéraire. (See also Ngai and Pissarides (2007), p.435 and Ngai and Pissarides (2004), p.21.)

## 2.4 Aggregates and sectors

We define aggregate (gross-)output ( $Y$ ), aggregate net-output ( $Y_N$ ), real GDP ( $GDP$ ), aggregate consumption expenditures ( $E$ ) and aggregate value of intermediate inputs ( $H$ ) as follows:

$$(16a) \quad Y \equiv \sum_{i=1}^n p_i Y_i$$

$$(16b) \quad Y_N \equiv Y - H$$

$$(16c) \quad GDP \equiv \frac{Y_N}{\bar{p}}$$

$$(17) \quad E \equiv \sum_{i=1}^n p_i C_i$$

$$(18) \quad H \equiv \sum_{i=1}^n p_i h_i$$

Throughout the paper we use aggregate net-output instead of aggregate (gross-)output ( $Y$ ), since in general GDP does not include intermediates. (In our model  $Y$  is equal to the sum of investment, consumption and intermediates-value ( $H$ ); see equation (A.25) in APPENDIX A.)

The aggregate input-shares of the junior-sector ( $l_j, k_j, z_j$ ) and the aggregate input-shares of the senior-sector ( $l_s, k_s, z_s$ ) are given by:

$$(19) \quad l_J \equiv \sum_{i=1}^m l_i, \quad k_J \equiv \sum_{i=1}^m k_i, \quad z_J \equiv \sum_{i=1}^m z_i, \quad l_S \equiv \sum_{i=m+1}^n l_i, \quad k_S \equiv \sum_{i=m+1}^n k_i, \quad z_S \equiv \sum_{i=m+1}^n z_i$$

The aggregate consumption expenditures on junior-goods ( $E_J$ ) and senior-goods ( $E_S$ ) are given by:

$$(20) \quad E_J \equiv \sum_{i=1}^m p_i C_i \quad E_S \equiv \sum_{i=m+1}^n p_i C_i$$

$E_S$  could also be interpreted as the budget devoted to the old. Throughout the paper we assume that the aggregate budget is distributed across the old and the young according to the representative household utility function (social welfare function). That is, budgets are such to maximize social welfare.

### 3. Model equilibrium

#### 3.1 Optimality conditions

The model, as specified in the previous section, can be solved by maximizing life-time utility (equations (1)-(6)) subject to equations (7)-(15a), e.g. by using a Hamiltonian function. The intra- and intertemporal optimality conditions are (where we assume that there is free mobility of factors across sectors):

$$(21) \quad p_i = \frac{\partial Y_m / \partial (l_i L)}{\partial Y_i / \partial (l_i L)} = \frac{\partial Y_m / \partial (k_m K)}{\partial Y_i / \partial (k_i K)} = \frac{\partial Y_m / \partial (z_m Z)}{\partial Y_i / \partial (z_i Z)} = \frac{\partial Y_m}{\partial (z_m Z)} \frac{\partial Z}{\partial h_i}, \quad \forall i$$

$$(22) \quad p_i = \frac{\partial u(.) / \partial C_i}{\partial u(.) / \partial C_m}, \quad \forall i$$

$$(23) \quad -\frac{\dot{u}_m}{u_m} = \frac{\partial Y_m}{\partial (k_m K)} - \delta - \rho$$

where  $u_m \equiv \partial u(.) / \partial C_m$ . For a proof that these conditions are necessary and sufficient conditions for an optimum, see Stijepic (2011).

By using equations (1)-(20) these optimality conditions (equations (21)-(23)) can be transformed into the following equations (sections 3.2 and 3.3) describing the *aggregate and sectoral* behaviour of the economy (for a proof of these equations see APPENDIX A):

### 3.2 Aggregates

$$(24) \quad \dot{\hat{K}} = \lambda_m^{-c} (\alpha + \beta \lambda_m) \hat{K}^c - \hat{E} - (\delta + g_L + \frac{g_G}{1-c}) \hat{K}$$

$$(25) \quad \frac{\dot{\hat{E}}}{\hat{E}} = \beta \lambda_m^{1-c} \hat{K}^{c-1} - \delta - \rho - g_L - \frac{g_G}{1-c}$$

$$(26a) \quad \hat{Y} = \hat{K}^c \lambda_m^{-c} \frac{\alpha(\beta - v) + \beta(\chi - \alpha) \lambda_m}{\chi\beta - \alpha v}$$

$$(26b) \quad \hat{Y}_N = \hat{K}^c \lambda_m^{-c} (\alpha + \beta \lambda_m)$$

$$(26c) \quad \lambda_m = \frac{1 - \mu \varepsilon_S + \gamma \varepsilon_S \frac{v}{\beta} - \frac{\chi\beta - \alpha v}{\alpha\beta} \left(1 - \frac{L}{N}\right) \frac{\hat{E}}{\hat{K}^c \lambda_m^{-c}}}{1 - \mu \varepsilon_S + \gamma \varepsilon_S \frac{\chi}{\alpha}}$$

$$(26d) \quad \hat{H} = \hat{K}^c \lambda_m^{-c} \frac{\alpha(\beta\mu - v\gamma) + \beta(\chi\gamma - \alpha\mu) \lambda_m}{\chi\beta - \alpha v}$$

$$(27) \quad G\hat{D}P = \hat{K}^c \lambda_m^{-c} (\alpha + \beta \lambda_m)^2 \left[ \alpha + \beta \lambda_m - (1 - \lambda_m) \frac{\alpha\beta(1 - \mu)}{\chi\beta - \alpha v} (1 - p_S) \right]^{-1}$$

where  $0 < c \equiv \frac{\beta(1 - \varepsilon_S \mu) + \varepsilon_S \gamma v}{1 - \gamma(1 - \varepsilon_S) - \varepsilon_S \mu} < 1$ ,  $\lambda_m \equiv \frac{l_m}{k_m}$ ,  $\varepsilon_S \equiv \sum_{i=m+1}^n \varepsilon_i$ ,  $g_G \equiv \frac{\dot{G}}{G}$

$$G \equiv \left\{ A^{\frac{1 - \mu \varepsilon_S}{\gamma}} B^{\varepsilon_S} \left[ \left( \frac{\chi}{\alpha} \right)^\chi \left( \frac{v}{\beta} \right)^v \left( \frac{\mu}{\gamma} \right)^\mu \right]^{\varepsilon_S} \gamma \prod_{i=1}^n \varepsilon_i^{\varepsilon_i} \right\}^{\frac{\gamma}{1 - \gamma(1 - \varepsilon_S) - \mu \varepsilon_S}}, \text{ and}$$

$$p_S \equiv \left[ \left( \frac{\alpha}{\chi} \right)^{\alpha\chi} \left( \frac{\beta}{v} \right)^{\alpha v} \left( \frac{\gamma}{\mu} \right)^{\alpha\mu} \left( \gamma \prod_{i=1}^n \varepsilon_i^{\varepsilon_i} \right)^{\gamma\chi - \mu\alpha} \left( \frac{\hat{K}}{\lambda_m} \right)^{\frac{(\chi\beta - \alpha v)[\alpha + \varepsilon_S(\gamma\chi - \alpha\mu)]}{1 - \gamma + (\gamma - \mu)\varepsilon_S}} \frac{A^\chi}{B^\alpha} \right]^{\frac{1}{\alpha + \varepsilon_S(\gamma\chi - \alpha\mu)}}.$$

**Definition 1:**  $\hat{K}$ ,  $\hat{E}$ ,  $\hat{Y}$ ,  $\hat{GDP}$  and  $\hat{Y}_N$  stand for  $K, E, Y, GDP$  and  $Y_N$  expressed in

“labour-efficiency-units, i.e.  $\hat{Y} \equiv \frac{Y}{LG^{\frac{1}{1-c}}}$ ,  $\hat{K} \equiv \frac{K}{LG^{\frac{1}{1-c}}}$ ,  $\hat{E} \equiv \frac{E}{LG^{\frac{1}{1-c}}}$ ,

$$\hat{GDP} \equiv \frac{GDP}{LG^{\frac{1}{1-c}}} \text{ and } \hat{Y}_N \equiv \frac{Y_N}{LG^{\frac{1}{1-c}}}.$$

Note that this definition of variables in efficiency units makes our discussion about the equilibrium growth path easier later.

**Proposition 1:**  $p_S$  stands for the price of senior goods.  $p_S$  is given by:

$$p_S \equiv \left[ \left( \frac{\alpha}{\chi} \right)^{\alpha\chi} \left( \frac{\beta}{\nu} \right)^{\alpha\nu} \left( \frac{\gamma}{\mu} \right)^{\alpha\mu} \left( \gamma \prod_{i=1}^n \varepsilon_i^{\varepsilon_i} \right)^{\gamma\chi - \mu\alpha} \left( \frac{\hat{K}}{\lambda_m} \right)^{\frac{(\chi\beta - \alpha\nu)[\alpha + \varepsilon_S(\gamma\chi - \alpha\mu)]}{1 - \gamma + (\gamma - \mu)\varepsilon_S}} \frac{A^\chi}{B^\alpha} \right]^{\frac{1}{\alpha + \varepsilon_S(\gamma\chi - \alpha\mu)}}$$

**Proof:** Remember that senior goods are produced by the same production functions; hence each senior good has the same price,  $p_S$ . The rest of the proof is given in APPENDIX A. **Q.E.D.**

We can see that (beside of the GDP-measure) the optimum aggregate structure of this economy is quite similar to the optimum structure of the standard Ramsey-model (or sometimes also named “Ramsey-Cass-Koopmans model”).<sup>7</sup> For a given  $\lambda_m$ , equations (24)-(26b) determine the equilibrium savings rate of the model (optimal intertemporal allocation of factors), like in the normal Ramsey-model. (In fact, for  $\lambda_m = 1$  equations (24)-(26b) are the same as the corresponding equations of the standard Ramsey-model). Equation (26c) determines  $\lambda_m$  as function of cross-sectors demand patterns (see also later equations (30) and (31)).  $\lambda_m$  can be regarded as a productivity indicator of aggregate production: it captures the changes in aggregate productivity which are caused by factor-reallocation across technologically distinct sectors (junior and senior sector), since  $\lambda_m$  depends only on the allocation of labour across the junior and senior

<sup>7</sup> For discussion of the standard Ramsey-model, see e.g. Barro and Sala-i-Martin (2004).



sectors:  $\lambda_m = \left( l_J + \frac{\alpha v}{\beta \chi} l_S \right)$ .<sup>8</sup> Furthermore, equation (26c) determines the  $\lambda_m$

which is consistent with the efficient (intratemporal) allocation of factors across sectors, since equation (26c) can be derived from equations (14) and (21) (among others); see as well the derivations in APPENDIX A. (Equations (14) state that all factors must be used in production, i.e. no factors are wasted; equation (26c) requires that marginal rates of technical substitution are equal across sectors, i.e. factors are efficiently allocated across sectors.)

### 3.3 Sectors

$$(28) \quad l_J = \left( \frac{E_J}{Y} + \frac{H}{Y} \varepsilon_J + \frac{\dot{K} + \delta K}{Y} \right) \frac{\alpha(\beta - v) + \beta(\chi - \alpha)\lambda_m}{(\chi\beta - \alpha v)}$$

$$(29) \quad l_S = \left( \frac{E_S}{Y} + \frac{H}{Y} \varepsilon_S \right) \frac{\alpha(\beta - v) + \beta(\chi - \alpha)\lambda_m}{(\chi\beta - \alpha v)} \frac{\chi}{\alpha}$$

$$(30) \quad E_J = E \frac{L}{N}$$

$$(31) \quad E_S = E \left( 1 - \frac{L}{N} \right)$$

$$(32a) \quad \frac{k_i}{l_i} = \frac{\alpha v}{\chi\beta} \frac{k_m}{l_m} \quad \text{for } i = m+1, \dots, n$$

$$(32b) \quad k_i = \frac{k_m}{l_m} l_i \quad \text{for } i = 1, \dots, m$$

$$(33a) \quad \frac{z_i}{l_i} = \frac{\alpha\mu}{\chi\gamma} \frac{z_m}{l_m} \quad \text{for } i = m+1, \dots, n$$

$$(33b) \quad z_i = \frac{z_m}{l_m} l_i \quad \text{for } i = 1, \dots, m$$

where  $\varepsilon_J \equiv \sum_{i=1}^m \varepsilon_i$ .

For a proof of these equations see APPENDIX B.

We can see that ageing (i.e. changes in  $L/N$ ) induces demand shifts between the junior- and the senior-sector (equations (30) and (31)). These demand shifts lead

---

<sup>8</sup> This equation can be derived by using equation (A.23) from APPENDIX A and equations (14) and (19).

to changes in factor allocation between these two sectors (here shown by changes in employment shares; see equations (28) and (29)). Further factor-reallocation between the senior- and the junior-sector is caused by changes in aggregate capital demand (since only the junior-sector produces capital) and by changes in aggregate intermediates demand.

**Proposition 2:** Capital intensity in the senior sector ( $\frac{k_s K}{l_s L}$ ) is lower in comparison to capital-intensity in the junior sector ( $\frac{k_j K}{l_j L}$ ), provided that  $\alpha v < \chi \beta$  ( $\alpha \mu < \chi \gamma$ ). Intermediate-intensity in the senior sector ( $\frac{z_s Z}{l_s L}$ ) is lower in comparison to intermediate-intensity in the junior sector ( $\frac{z_j Z}{l_j L}$ ), provided that  $\alpha v < \chi \beta$  ( $\alpha \mu < \chi \gamma$ ).

**Proof:** Since capital intensity in a subsector  $i$  is given by  $\frac{k_i K}{l_i L}$ ,  $\forall i$ , and intermediates intensity in a subsector  $i$  is given by  $\frac{z_i Z}{l_i L}$ ,  $\forall i$ , equations (19), (32) and (33) imply this proposition. **Q.E.D.**

## 4. Effects of ageing

To study the effects of ageing we compare the economy without ageing ( $L/N = \text{constant}$ ) to the economy with ageing ( $L/N$  decreases), ceteris paribus. In all the following argumentation, ageing (i.e. a change in  $L/N$  or a change in  $\frac{N-L}{N}$ ) means that  $g_N$  changes and not  $g_L$ . That is, we assume that  $L$  is independent of ageing (i.e. it grows at constant rate  $g_L$  irrespective of whether ageing takes place or not). In this way we can clearly distinguish between growth-effects of ageing via factor-reallocation, which are in the focus of our paper, and growth effects of changes in labour supply (i.e. changes in the growth rate of  $L$ ). The latter are well known from standard (one-sector) models.

**(Working)Definition 2:** Ageing stands for an increase in  $N/L$ , where  $g_L$  is constant.

In the next section (4.1) we discuss the equilibrium without ageing. In section 4.2, we analyze the effects of ageing in a simpler version of our model, where only cross-sector-differences in TFP-growth are allowed for. In section 4.3 the effects of ageing are analyzed in the general version of the model, where it is allowed for cross-sector differences in input-elasticities as well.

### 4.1 Partially Balanced Growth Path (PBGP) without ageing

In this subsection we assume that there is no ageing, i.e.  $L/N = \text{constant}$ .

**Definition 3:** A “partially balanced growth path” (“PBGP”) is an equilibrium growth path where  $\hat{K}$ ,  $\hat{E}$ ,  $\hat{Y}$ ,  $\hat{Y}_N$  and  $\lambda_m$  are constant.

The name “partially balanced growth path” reflects the fact that along the PBGP some variables ( $Y, K, E$  and  $Y_N$ ) behave as if they were on a balanced growth path (steady state), while the other variables (e.g. GDP) do not behave in this manner, i.e. they grow at non-constant rates, as we will see soon. (This concept is similar to the concept of “aggregate balanced growth”, which is used by Ngai and Pissarides (2007).)

**Lemma 1:** There exists a unique PBGP of the dynamic equation system (24)-(26), provided that  $L/N$  is constant.

**Proof:** It can be seen at first sight that equations (24)-(26) imply that there is an equilibrium growth path where  $\hat{K}$ ,  $\hat{E}$ ,  $\hat{Y}$ ,  $\hat{Y}_N$  and  $\lambda_m$  are constant, provided that  $L/N$  is constant. **Q.E.D.**

**Lemma 2:** Along the PBGP, the growth rate of the variables  $Y, K, E$  and  $Y_N$  is given by

$$(34) \quad g^* = \frac{(1 - \mu\varepsilon_S)g_A + \gamma\varepsilon_S g_B}{(1 - \mu\varepsilon_S)\alpha + \gamma\varepsilon_S \chi} + g_L = \text{const.}$$

(where  $L/N$  is constant).

**Proof:** This lemma is implied by Definitions 1 and 3. *Q.E.D.*

**Lemma 3:** Along the PBGP, factors are not shifted between the senior- and the junior-sector, i.e.  $l_J$  and  $l_S$  are constant, (where  $L/N$  is constant).

**Proof:** This lemma is implied by equations (28)-(31), by Definitions 1 and 3 and by Lemma 2. *Q.E.D.*

**Definition 4:** An asterisk (\*) denotes the PBGP-value of the corresponding variable.

Now, we derive the PBGP-values of variables as functions of exogenous parameters:

**Lemma 4:** Along the PBGP, the variables  $\hat{K}$ ,  $\hat{E}$ ,  $\hat{Y}$ ,  $\hat{Y}_N$ ,  $G\hat{D}P$  and  $\lambda_m$  are given by the following functions of exogenous model parameters (where  $L/N$  is constant)

$$(35a) \quad \hat{K}^* = s^{\frac{1}{1-c}} \lambda_m^*$$

$$(35b) \quad \hat{E}^* = \alpha s^{\frac{c}{1-c}} + \rho s^{\frac{1}{1-c}} \lambda_m^*$$

$$(35c) \quad \hat{Y}^* = s^{\frac{c}{1-c}} \frac{\alpha(\beta - v) + \beta(\chi - \alpha)\lambda_m^*}{\chi\beta - \alpha v}$$

$$(35d) \quad \hat{Y}_N^* = s^{\frac{c}{1-c}} (\alpha + \beta\lambda_m^*)$$

$$(35e) \quad \lambda_m^* = \frac{\alpha}{\beta} \frac{\beta + (v\gamma - \mu\beta)\varepsilon_s - (\chi\beta - \alpha v) \frac{N-L}{N}}{\alpha + (\chi\gamma - \mu\alpha)\varepsilon_s + (\chi\beta - \alpha v) \frac{N-L}{N} \frac{\rho}{\beta} s}$$

$$(35f) \quad G\hat{D}P^* = s^{\frac{c}{1-c}} (\alpha + \beta\lambda_m^*)^2 \left[ \alpha + \beta\lambda_m^* - (1 - \lambda_m^*) \frac{\alpha\beta(1-\mu)}{\chi\beta - \alpha v} (1 - p_s^*) \right]^{-1}$$

where

(35g)

$$p_s^* = \left[ \left( \frac{\alpha}{\chi} \right)^{\alpha\chi} \left( \frac{\beta}{v} \right)^{\alpha v} \left( \frac{\gamma}{\mu} \right)^{\alpha\mu} \left( \gamma \prod_{i=1}^n \varepsilon_i^{\varepsilon_i} \right)^{\gamma\chi - \mu\alpha} \left( s^{\frac{1}{1-c}} \right)^{\frac{(\chi\beta - \alpha v)[\alpha + \varepsilon_s(\gamma\chi - \alpha\mu)]}{1 - \gamma + (\gamma - \mu)\varepsilon_s}} \frac{A^\chi}{B^\alpha} \right]^{\frac{1}{\alpha + \varepsilon_s(\gamma\chi - \alpha\mu)}}$$

$$(35h) \quad s \equiv \frac{\beta}{\delta + \rho + g_L + \frac{g_G}{1-c}}$$

**Proof:** To determine the PBGP-values of  $\hat{K}$ ,  $\hat{E}$ ,  $\hat{Y}$ ,  $\hat{Y}_N$  and  $\lambda_m$  we have to set  $\dot{\hat{K}} = 0$  and  $\dot{\hat{E}} = 0$  (because of Definition 3). Then equations (24)-(27) imply Lemma 4. Remember that in this section  $L/N$  is constant. **Q.E.D.**

**Lemma 5:** The young-to-old ratio  $\left(\frac{L}{N}\right)$  has an impact on the PBGP-levels of aggregate variables  $\hat{K}^*$ ,  $\hat{E}^*$ ,  $\hat{Y}^*$ ,  $\hat{Y}_N^*$ ,  $G\hat{D}P^*$  and  $\lambda_m^*$  (where  $L/N$  is constant).

**Proof:** This lemma is implied by equations (35). **Q.E.D.**

**Lemma 6:**  $G\hat{D}P^*$  does not grow at constant rate along the PBGP (even when  $N/L$  is constant).

**Proof:** This lemma is implied by (35f). Note that equation (35g) implies that  $p_s^*$  is not constant along the PBGP. **Q.E.D.**

Lemma 6 shows a quite convenient feature of our model: we can study the rich dynamics of the GDP (where the reallocation-effects of ageing cause unbalanced-growth of GDP) while the other variables are on a (partially) balanced growth path (partial steady state). This fact makes it possible to analyze the impacts of ageing without simulations.

**Lemma 7a:** A saddle-path, along which the economy converges to the PBGP, exists in the neighbourhood of the PBGP of the dynamic equation system (24)-(26).

**Lemma 7b:** If intermediates are omitted (i.e. if  $\gamma = \mu = 0$ ), the PBGP of the dynamic equation system (24)-(26) is locally stable.

**Proof:** see APPENDIX C.

**Corollary 1:** Even if the initial capital level is not given by equation (35a), the economy which is described by the aggregate equation system (24)-(26) converges to the PBGP, provided that  $L/N$  is constant.

**Proof:** This corollary follows from Lemmas 1, 4 and 7. **Q.E.D.**

## 4.2 Ageing and cross-sector differences in TFP-growth

In this subsection we provide a simpler version of our model, which is helpful to understand the general mechanism which leads to the reallocation effects of ageing. We assume now that input-elasticities are equal across sectors, i.e.  $\alpha = \chi$ ,  $\beta = \nu$  and, thus,  $\gamma = \mu$ . Furthermore, we assume that ageing takes place.

**Lemma 8:** If  $\alpha = \chi$ ,  $\beta = \nu$  and, thus,  $\gamma = \mu$ , equations (24)-(35) become:

$$(24)' \quad \dot{\hat{K}} = (\alpha + \beta)\hat{K}^c - \hat{E} - (\delta + g_L + \frac{g_G}{1-c})\hat{K}$$

$$(25)' \quad \frac{\dot{\hat{E}}}{\hat{E}} = \beta\hat{K}^{c-1} - \delta - \rho - g_L - \frac{g_G}{1-c}$$

$$(26a)' \quad \hat{Y} = \hat{K}^c$$

$$(26b)' \quad \hat{Y}_N = \hat{K}^c (\alpha + \beta)$$

$$(26c)' \quad \lambda_m = 1$$

$$(26d)' \quad \hat{H} = \hat{Y} - \hat{Y}_N = \gamma\hat{Y}$$

$$(27)' \quad G\hat{D}P = \hat{K}^c (\alpha + \beta) \left[ 1 + l_S \frac{A-B}{B} \right]^{-1}$$

$$(28)' \quad l_J = \left( \frac{E}{Y} \frac{L}{N} + \frac{H}{Y} \varepsilon_J + \frac{\dot{K} + \delta K}{Y} \right)$$

$$(29)' \quad l_S = \left( \frac{E}{Y} \left( 1 - \frac{L}{N} \right) + \frac{H}{Y} \varepsilon_S \right)$$

$$(34)' \quad g^* = \frac{(1 - \gamma\varepsilon_S)g_A + \gamma\varepsilon_S g_B}{\alpha} + g_L$$

$$(35a)' \quad \hat{K}^* = s^{\frac{1}{1-c}}$$

$$(35b)' \quad \hat{E}^* = \alpha s^{\frac{c}{1-c}} + \rho s^{\frac{1}{1-c}}$$

$$(35c)' \hat{Y}^* = s^{\frac{c}{1-c}}$$

$$(35d)' \hat{Y}_N^* = s^{\frac{c}{1-c}} (\alpha + \beta)$$

$$(35f)' G\hat{D}P^* = s^{\frac{c}{1-c}} (\alpha + \beta) \left\{ 1 + \frac{A-B}{B} \left[ (\alpha + \rho s) \left( 1 - \frac{L}{N} \right) + \gamma \varepsilon_s \right] \right\}^{-1}$$

$$\text{where } 0 < c = \frac{\beta}{\alpha + \beta} < 1, \quad G \equiv A^{\frac{1}{\alpha + \beta}} \left\{ \gamma \left[ \frac{B}{A} \right]^{\varepsilon_s} \prod_{i=1}^n \varepsilon_i^{\varepsilon_i} \right\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}.$$

**Proof:** The proof is quite straight-forward. Therefore, we omit it here. Note that following steps are necessary to obtain equation (27)': By inserting equation (26c) into equation (27) the following equation can be obtained:

$$G\hat{D}P = \hat{K}^c \lambda_m^{-c} (\alpha + \beta \lambda_m)^2 \left[ \alpha + \beta \lambda_m - \frac{\gamma \varepsilon_s - [(N-L)/N] \hat{E} / (\hat{K} / \lambda_m)^c (1-\mu)(1-p_s)}{1 - \mu \varepsilon_s + \gamma \varepsilon_s \chi / \alpha} \right]^{-1}$$

This term can be reformulated by using the other equations to obtain  $G\hat{D}P = \hat{K}^c (\alpha + \beta) \left[ 1 - \gamma \varepsilon_s + [(N-L)/N] \hat{E} / \hat{Y} (1 - A/B) \right]^{-1}$ . Then, by using equations (26d)' and (29)', equation (27)' can be derived. **Q.E.D.**

**Lemma 9:** *If input-elasticities are equal across sectors, there exists a unique PBGP, irrespective of whether ageing takes place or not, and irrespective of the rate of ageing.*

**Proof:** Lemma 8 implies that equations (24)'-(26)' apply here. The proof of Lemma 9 can be seen directly from equations (24)'-(26c)', which are nearly the same as in the standard one-sector Ramsey model. Since equations (24)'-(26)' are not dependent on  $L/N$ , the existence of the PBGP is not affected by changes in  $L/N$ . **Q.E.D.**

**Lemma 10:** *If input-elasticities are equal across sectors, the growth rate of the variables  $Y, K, E$ , and  $Y_N$  is given by equation (34)' along the PBGP.*

**Proof:** This lemma is implied by Lemma 8 and Definitions 1 and 3. **Q.E.D.**

**Lemma 11:** *If input elasticities are equal across sectors, the PBGP is globally saddle-path stable, irrespective of whether ageing takes place or not.*

**Proof:** Lemma 8 implies that equations (24)'-(26)' apply. Equations (24)' and (25)' are the same as in the standard Ramsey-model regarding all relevant features. Therefore, the aggregate system of our model behaves like the standard Ramsey-model, i.e. it is globally saddle-path stable. (See also Ngai and Pissarides (2007) on the stability of such frameworks.). Since equations (24)'-(26)' are independent of  $L/N$ , ageing has no impact on the stability of the PBGP. **Q.E.D.**

**Corollary 2:** *When input-elasticities are equal across sectors, ageing is irrelevant regarding the development of the variables  $Y, K, E$ , and  $Y_N$  in our model: Neither the PBGP-growth rate  $g^*$  nor the PBGP-levels  $\hat{K}^*$ ,  $\hat{E}^*$ ,  $\hat{Y}^*$  and  $\hat{Y}_N^*$  are affected by (the level or the growth rate of)  $L/N$ . A change in  $L/N$  does not induce a deviation from the (initial) PBGP with respect to  $Y, K, E$ , and  $Y_N$ .*

**Proof:** This corollary is implied by Lemmas 8-10 and equations (35). **Q.E.D.**

Now we take a look at the disaggregated variables of the economy.

**Theorem 1:** *If input-elasticities are equal across sectors, ageing shifts demand from the junior-sectors to the senior-sectors along the PBGP. That is, decreases in  $L/N$  lead to decreases in  $E_j/E$  and increases in  $E_s/E$ .*

**Proof:** This theorem is implied by equations (30) and (31). Remember that, as argued in section 2, the choice of the numéraire is irrelevant when looking at shares or ratios. **Q.E.D.**

**Theorem 2:** *If input-elasticities are equal across sectors, ageing reallocates factors from the junior-sectors to the senior-sectors along the PBGP; i.e. decreases in  $L/N$  lead to decreases in  $l_j$  and increases in  $l_s$ .*

**Proof:** This theorem is implied by Lemma 8 and equations (28)' and (29)'. **Q.E.D.**

**Theorem 3:** *If input elasticities are equal across sectors, ageing reduces the growth rate of GDP along the PBGP, provided that the TFP-growth rate (and the*



*TFP-level) is lower in the senior sector in comparison to the junior sector. That is, a decreasing  $L/N$  causes a reduction of the GDP-growth rate, provided that  $A > B$  and  $g_A > g_B$ .*

**Proof:** This theorem is implied by Lemma 8 and equation (35f)'. **Q.E.D.**

**Corollary 3:** *If input-elasticities are equal across sectors, ageing shifts demand from the junior-sectors to the senior-sector. These demand shift cause factor reallocation from the junior-sector to the senior-sector. This reallocation process reduces the growth rate of GDP provided that the senior-sector has a relatively low TFP(growth-rate) in comparison to the senior sector.*

**Proof:** This corollary is implied by Theorems 1-3. **Q.E.D.**

Hence, whether ageing increases or decreases the GDP-growth-rate depends only on the TFP-relation between the junior and senior sectors. The factors which determine the *strength* of the ageing-impact are analyzed in the next section.

As argued in section 2, the choice of the numéraire is irrelevant when looking at shares or ratios. Hence, we can analyze the senior-goods-consumption-to-output ratio ( $E_S / Y_N$ ) without worrying about numéraire choice. The share of senior-budget in aggregate output ( $E_S / Y_N$ ) increases at the same rate as the old-to-young ratio (see equation (31) and remember that along the PBGP  $E$  and  $Y_N$  grow at the same rate).

All the results from this section are valid for the case that the budget devoted to seniors (e.g. old age pensions) develops according to the social welfare function (representative household utility function). If however political issues led to a reduction of old age pensions, the ageing-impacts would be weaker. We will discuss this case later.

### **4.3 Ageing and cross-sector differences in input-elasticities**

Now let us assume that input-elasticities differ across sectors, i.e.  $\alpha \neq \chi$ ,  $\beta \neq \nu$  and  $\gamma \neq \mu$ . (The TFP-growth rates differ across sectors as well.) Furthermore, in

this paper we analyze only the case where the capital intensity in the senior sector is lower in comparison to the junior sector (i.e.  $\alpha\nu < \chi\beta$ ), since this case is in general assumed in the literature (see also Proposition 2). We assume that initially the economy is in the equilibrium described in section 4.1 with  $L/N = \text{constant}$ . In sections 4.3.1 and 4.3.2 we analyze what happens if there is a *one time* decrease in  $L/N$  (according to Definition 2). (After this decrease  $L/N$  is constant again.) In section 4.3.3 we generalize our results to the case where  $L/N$  increases consecutively. Furthermore, in section 4.3.1 we analyze the effects of ageing on net-output and on the pension-to-output ratio and we derive the impact channels, whereas in section 4.3.2 we look at the differences in this analysis when our GDP-measure is taken into account.

### 4.3.1 Productivity effect: Impacts and channels

In this subsection the term “aggregates” refers only to  $Y, Y_N, E$  and  $K$  but *not* to GDP.

**Lemma 12:** *A one time decrease in  $L/N$  leads to a change of the PBGP. That is, the economy leaves the old PBGP and there is a transition period where the economy converges to the new PBGP. The growth rate of aggregates ( $g^*$ ) is the same along the old and the new PBGP.*

**Proof:** Remember that we assume here again that the input-elasticities differ across sectors; hence equations (24)-(35) apply here. Equations (35) imply that there *must* be a transition period, since the old and the new PBGP require different equilibrium capital levels; i.e.  $K^*$  depends on  $L/N$ . (That is, the capital level which exists when the decrease in  $L/N$  occurs is not the same as the capital level which brings the economy directly on the new PBGP; we know from the discussion of the standard one-sector Ramsey-model that this induces a transition period, where the economy is converging to the new PBGP.) Furthermore, Lemma 7 and Corollary 1 imply that the economy will converge to the new PBGP (provided that the decrease in  $L/N$  is not too strong). Equation (34) implies that the growth rate of aggregates ( $g^*$ ) is the same along the old and the new PBGP, since  $g^*$  does not depend on  $L/N$ . ***Q.E.D.***

**Lemma 13:** *A one-time decrease in  $L/N$  reduces the growth rate of aggregates during the transition period between the old and the new PBGP. That is, the growth-rate of aggregates ( $K, E$ , and  $Y_N$ ) during the transition period is lower in comparison to the growth rate of aggregates along the (old and new) PBGP ( $g^*$ ).*

**Proof:** To prove this lemma we need the following derivatives of equations (35). (Note that our key results would not change, if we calculated here elasticities instead of derivatives.)

$$(36a) \quad \frac{\partial \hat{K}^*}{\partial \lambda_m^*} = s^{\frac{1}{1-c}} > 0$$

$$(36b) \quad \frac{\partial \hat{E}^*}{\partial \lambda_m^*} = \rho s^{\frac{1}{1-c}} > 0$$

$$(36c) \quad \frac{\partial \hat{Y}^*}{\partial \lambda_m^*} = s^{\frac{c}{1-c}} \frac{\beta(\chi - \alpha)}{\chi\beta - \alpha\nu}$$

$$(36d) \quad \frac{\partial \hat{Y}_N^*}{\partial \lambda_m^*} = \beta s^{\frac{c}{1-c}} > 0$$

$$(36e) \quad \frac{\partial \lambda_m^*}{\partial \left( \frac{N-L}{N} \right)} = -(\chi\beta - \alpha\nu) \frac{\alpha}{\beta} \frac{\alpha + (\chi\gamma - \mu\alpha)\varepsilon_s + \rho s + (\gamma\nu - \mu\beta)\varepsilon_s \frac{\rho}{\beta} s}{\left[ \alpha + (\chi\gamma - \mu\alpha)\varepsilon_s + (\chi\beta - \alpha\nu) \frac{N-L}{N} \frac{\rho}{\beta} s \right]^2} < 0$$

From these equations we can see that a one-time decrease in  $L/N$  leads to a decrease in  $\lambda_m^*$ . The decrease in  $\lambda_m^*$  leads to a decrease in  $\hat{K}^*$ ,  $\hat{E}^*$ , and  $\hat{Y}_N^*$ .<sup>9</sup> Hence, the values of  $\hat{K}^*$ ,  $\hat{E}^*$ , and  $\hat{Y}_N^*$  along the new PBGP are lower in comparison to those of the old PBGP. Therefore, we can conclude that  $\hat{K}^*$ ,  $\hat{E}^*$ , and  $\hat{Y}_N^*$  decrease during the transition period. Hence, the growth rate of aggregates ( $K, E$  and  $Y_N$ ) during the transition period is lower than  $g^*$ . (Remember that Definitions 1 and 3 and Lemma 2 imply the following: if  $\hat{K}^*$ ,  $\hat{E}^*$  and  $\hat{Y}_N^*$  are constant, aggregates ( $K, E$  and  $Y_N$ ) grow at the constant rate

---

<sup>9</sup> Note that the effects of ageing on aggregate gross-output  $Y$  may be positive or negative depending on the sign of the term  $\chi - \alpha$ . This reflects the fact that depending on the input-elasticities ageing can lead to an increase in intermediates production that is stronger than the decrease in net-output and vice versa.

$g^*$ ; hence, if  $\hat{K}^*$ ,  $\hat{E}^*$  and  $\hat{Y}_N^*$  decrease, the growth rate of  $K, E$  and  $Y_N$  is lower than  $g^*$ . Note that this argumentation works, since “efficiency units” are the same along the old and the new PBGP: We express the variables in efficiency units (see Definition 1) as follows: e.g.  $\hat{Y} \equiv \frac{Y}{LG^{\frac{1}{1-c}}}$ ; since  $LG^{\frac{1}{1-c}}$  does not change due to ageing, efficiency units are the same along the old and the new PBGP.) **Q.E.D.**

We will discuss the intuition behind this lemma soon; at first we postulate two lemmas, which are helpful to understand Lemma 13.

**Lemma 14:** *A one-time decrease in  $L/N$  shifts demand from the junior-sector to the senior-sector. That is, along the new PBGP  $\frac{E_S}{E}$  is higher ( $\frac{E_J}{E}$  is lower) in comparison to the  $\frac{E_S}{E}$  ( $\frac{E_J}{E}$ ) of the old PBGP.*

**Proof:** This lemma is implied by equations (30) and (31). **Q.E.D.**

**Lemma 15:** *A one-time decrease in  $L/N$  leads to factor reallocation from the junior-sector to the senior-sector. That is, along the new PBGP  $l_s$  is higher in comparison to the  $l_s$  of the old PBGP.*

**Proof:** By using equation (A.23) from APPENDIX A, it can be shown that the employment share along the PBGP is given by  $l_s^* = (1 - \lambda_m^*) \frac{\chi\beta}{\chi\beta - \alpha\nu}$ . Since

equation (36e) implies that  $\frac{\partial \lambda_m^*}{\partial \left(1 - \frac{L}{N}\right)} < 0$ , a decrease in  $L/N$  leads to a decrease in

$\lambda_m^*$ . Therefore,  $l_s^*$  increases due to a decrease in  $L/N$ . (Remember that we assume that  $\chi\beta - \alpha\nu > 0$ .) **Q.E.D.**

Now, we discuss the intuition behind Lemma 13: We know that output is produced by using labour and capital. Ageing shifts demand (and, thus, production factors) towards senior-sectors (as implied by Lemmas 14 and 15). The key feature of the senior-sectors is that capital is less productivity-enhancing

in comparison to the (junior-sectors). This is reflected by the fact that optimal capital intensity in the senior-sector is lower in comparison to the junior-sector (see Proposition 2). Hence, the ageing-induced (one-time) demand-shift implies that aggregate capital becomes less productive when looking at the economy-wide-averages. Therefore, at the aggregate level a one-time decrease in  $L/N$  acts similarly like a negative productivity-shock (a decrease in the productivity of capital).<sup>10</sup> This leads to the negative impacts on aggregate net-output-growth, aggregate capital-growth and aggregate consumption-expenditures-growth (and of course, the savings rate decreases, since savings which are invested in capital become less rentable, i.e. the opportunity costs of consumption decrease). This adjustment-process occurs during the transition period. Since a one-time productivity-*level*-shock has no impacts on productivity-*growth rates*, the economy converges to a growth path (PBGPs) where the growth rate is the same as before. (Remember that steady state growth rates are determined only by productivity-growth and not by productivity-levels within the standard (one-sector) Ramsey-model; in this respect our aggregate model is the same as the standard Ramsey model.)

A further interesting question is about the effects of ageing on the senior-budget-to-the-net-output-ratio ( $E_S / Y_N$ ). Equations (31), (35b) and (35d) imply the following derivative (consider also equation (36e)):

$$(37) \quad \frac{\partial(\hat{E}_S^* / \hat{Y}_N^*)}{\partial\left(\frac{N-L}{N}\right)} = \frac{\hat{E}^*}{\hat{Y}_N^*} - \frac{\partial\lambda_m^*}{\partial\left(\frac{N-L}{N}\right)} \frac{N-L}{N} \frac{\alpha}{(\alpha + \beta\lambda_m^*)^2} s\left(\delta + g_L + \frac{g_G}{1-c}\right) > 0$$

Hence, we can see that ageing increases the senior-budget-to-output-ratio. The first term on the right-hand-side of equation (37) may be regarded as the direct effect of ageing (i.e. an increase in the old-to-young ratio, increases the share of the seniors in the *overall* consumption-to-output-share).<sup>11</sup> The second term may be regarded as an indirect effect: the increase in the old-to-young ratio leads to an

---

<sup>10</sup> This fact is reflected by the ageing-induced decrease in  $\lambda_m^*$ , which is implied by equation (36e); as discussed in section three  $\lambda_m^*$  can be interpreted as a productivity indicator, which reflects the aggregate impacts of cross-sector factor-reallocation.

<sup>11</sup> since  $E_S^* / Y_N^* = (N-L) / N * E^* / Y_N^*$ .

increase in the *overall* consumption-to-output-share<sup>12</sup> (i.e. as noted above the savings-rate decreases due to lower productivity level).

We can see from equations (36) and (37) that a rich “portfolio” of parameters determines the strength of the impact of ageing. (Note that this portfolio would not change, if we calculated elasticities instead of derivatives in equations (36) and (37).) These parameters are:

- a) technology parameters: input-elasticities of sectoral production functions (including labour, capital and intermediates elasticities), TFP-growth-rates (via  $g_G$ ) and the depreciation rate
- b) time preference rate
- c) old-to-young ratio and the growth rate of labour.

The reason for the fact that so many parameters determine the impact of ageing is the following: The demand-shift across technologically distinct sectors makes it necessary to change the (average) aggregate structure of the economy, especially the ratios between aggregate *capital*, labour and aggregate intermediates. The sectoral technology parameters (especially the input-elasticities) determine how strong this change has to be. Furthermore, since changes in *capital* in general require an adjustment of the savings rate  $(1 - \frac{\hat{E}^*}{\hat{Y}^*})$ , all the variables which determine the savings rate come into account, especially the parameters captured by the auxiliary variable  $s$ ; see equations (35), e.g. the time-preference rate. The portfolio of parameters which determine the savings rate includes all those parameters which are already known to determine the savings-rate of the standard Ramsey-model (see the auxiliary variable “ $s$ ”). However, our model provides a sector foundation of those parameters: especially,  $g_G$  and  $c$  are assumed to be exogenous in the standard Ramsey-model, while in our model these two variables are functions of sectoral parameters.

---

<sup>12</sup> since  $\frac{\partial(\hat{E}^* / \hat{Y}^*)}{\partial((N-L) / N)} = -\frac{\partial\lambda_m^*}{\partial((N-L) / N)} \frac{\alpha}{(\alpha + \beta\lambda_m^*)^2} s \left( \delta + g_L + \frac{g_G}{1-c} \right) > 0$ .

### 4.3.2 Additional impacts on GDP: The price-effect

Remember that we have shown in the previous section that a one-time increase in  $L/N$  leads to a transition from the old PBGP to a new PBGP. Due to this fact, the effect of ageing on real GDP-growth can be divided into transitional effects and PBGP-effects. Transitional effects have an impact on the real GDP-growth-rate during the transition between two PBGPs, while PBGP-effects of ageing have an impact on the growth rate along the new (PBGP). In fact we have shown that the effects from the previous section are transitional. In this section we will introduce a new effect which affects real GDP-growth (but not the growth rate of other aggregate variables). We name this effect price effect, and we show that this effect is not only transitional.

#### 4.3.2.1 Transitional effects of ageing on GDP

To show these facts we have to calculate the derivative of equation (35f):

(38)

$$\frac{\partial \hat{GDP}^*}{\partial \left( \frac{N-L}{N} \right)} = \overbrace{\frac{\beta s^{1-c} (\lambda_m^*)'}{(\bar{p}^*)^2 (\alpha + \beta \lambda_m^*)}}^{(-)} \left[ \overbrace{\alpha + \beta - \left( \beta - \frac{\alpha v}{\chi} \right) l_s^*}_{(+)} - (1 - p_s^*) (1 - \mu) \overbrace{\left( \frac{1 - \gamma}{\chi \beta - \alpha v} + \frac{l_s^*}{\chi} \right)}^{(+)} \right]$$

where  $(\lambda_m^*)' \equiv \frac{\partial \lambda_m^*}{\partial \left( \frac{N-L}{N} \right)}$ ,  $\bar{p}^* = 1 - \frac{\alpha \beta (1 - \mu)}{\chi \beta - \alpha v} \frac{1 - \lambda_m^*}{\alpha + \beta \lambda_m^*} (1 - p_s^*)$  and

$$l_s^* = (1 - \lambda_m^*) \frac{\chi \beta}{\chi \beta - \alpha v}.$$

and where  $p_s^*$  is given by equation (35g).

(For an explicit proof see APPENDIX D.) A “(+)” (a “(-)”) above a term denotes that this term is positive (negative).

**Theorem 4:** *A one-time decrease in  $L/N$  has a negative impact on the growth rate of GDP during the transition between the old and the new PBGP, provided that*

senior goods are “more expensive” in comparison to junior-goods; i.e. provided that  $p_s^* > 1$ , where  $p_s^*$  is given by equation (35g).

**Proof:** This theorem is implied by equation (38). If  $p_s^* > 1$ , an increase in the old-to-young-ratio has a negative impact on the  $\hat{GDP}^*$ -level (and hence a negative impact on the GDP-growth-rate during the transition period; see also the argumentation in the proof of Lemma 13). Note that  $p_s^*$  is always positive and determined by exogenous parameters. Furthermore, note that the relative price of senior goods is given by  $p_s^*$  (see proposition 1) and the price of junior goods is given by 1. The latter comes from the fact that sector m is numéraire (see equation (15a)) and belongs to the junior-sector and all junior sub-sectors have identical production functions (see also equations (A.5) and (A.6) in APPENDIX A). **Q.E.D.**

If  $p_s^* < 1$ , the effect of an increase in the old-to-young ratio may be positive or negative, depending on the parameter constellation, where the effect can be positive provided that  $p_s^*$  is relatively small (i.e. relatively close to zero). To isolate the set of parameter-values which ensures that the GDP-effect of ageing is positive when  $p_s^*$  is relatively close to zero, we have to calculate the limit-value of the term within the squared brackets of equation (38), i.e.

$$(39) \quad \lim_{p_s^* \rightarrow 0} \left[ \alpha + \beta - \left( \beta - \frac{\alpha v}{\chi} \right) l_s^* - (1 - p_s^*) (1 - \mu) \left( \frac{1 - \gamma}{\chi \beta - \alpha v} + \frac{l_s^*}{\chi} \right) \right]$$

$$= (\alpha + \beta) \left( \frac{\chi(\beta - \alpha) - 2\alpha v}{(\chi \beta - \alpha v)} - l_s^* \right)$$

where  $l_s^* = (1 - \lambda_m^*) \frac{\chi \beta}{\chi \beta - \alpha v}$ .

If (39) is negative, equation (38) implies that for small values of  $p_s^*$  the effect of ageing is positive regarding GDP-growth. Equation (39) implies that, e.g.,  $\beta < \alpha$  is a stronger than necessary condition for this. (Remember that we assume that  $\alpha v < \chi \beta$ .)



Now, the question is what parameter constellations ensure that  $p_s^*$  is relatively small.

**Lemma 16:** *In the limit  $p_s^* > 1$  ( $p_s^* < 1$ ), provided that the growth rate of labour-augmenting technological progress in the junior-sector is higher (lower) in comparison to the growth rate of labour-augmenting technological progress in the senior-sector, i.e. provided that  $\frac{1}{\alpha} g_A > \frac{1}{\chi} g_B$  ( $\frac{1}{\alpha} g_A < \frac{1}{\chi} g_B$ ).*

**Proof:** We know from equation (35g) that the actual level of  $p_s^*$  is determined by a time-variant term  $(A^\chi / B^\alpha)$  and by a constant term  $\left( \left( \frac{\alpha}{\chi} \right)^{\alpha\chi} \left( \frac{\beta}{\nu} \right)^{\alpha\nu} \left( \frac{\gamma}{\mu} \right)^{\alpha\mu} \left( \gamma \prod_{i=1}^n \varepsilon_i^{\varepsilon_i} \right)^{\gamma\chi - \mu\alpha} s^{\chi\beta - \alpha\nu} \right)$ .  $A^\chi / B^\alpha$  approaches infinity (zero), provided that  $\frac{1}{\alpha} g_A > \frac{1}{\chi} g_B$  ( $\frac{1}{\alpha} g_A < \frac{1}{\chi} g_B$ ). Thus, in the limit  $p_s^*$  approaches infinity (zero) as well, i.e.  $p_s^*$  becomes larger (smaller) than 1. ***Q.E.D.***

Hence, depending on the parameter setting, several cases can exist: (1)  $p_s^*$  can be relatively close to zero in the beginning, but approach to infinity with time; (2)  $p_s^*$  can be relatively close to zero in the beginning and approach to zero with time; (3)  $p_s^*$  can be relatively large in the beginning but approach to zero with time; (4)  $p_s^*$  can be relatively large in the beginning and approach to infinity with time.

These cases and the discussion above (about equations (35g) and (38)) imply that ageing may have positive and negative impacts on GDP-growth (during the transition period) depending on the exact constellation of parameters from equations (35g) and (38). Moreover, the effect of ageing may change with time (in cases (1) and (3)), i.e. in the beginning the effect on GDP-growth may be positive (negative) but later negative (positive).

Nevertheless, in the limit only the term  $A^\chi / B^\alpha$  (together with equation (38)) determines whether a future increase in the old-to-young ratio leads to an increase or to a decrease in GDP(-growth). Hence, from the today's point of view the growth rate of this term (namely  $\chi g_A - \alpha g_B$ ) is deciding for the question about

the (distant) future impacts of ageing: If  $\chi g_A - \alpha g_B > 0$  (or:  $\frac{1}{\alpha} g_A > \frac{1}{\chi} g_B$ ) we know that  $p_s^*$  approaches infinity. Hence, we know that sooner or later ageing will have negative (transitional) effects on GDP-growth. Otherwise, if  $\chi g_A - \alpha g_B < 0$  (or:  $\frac{1}{\alpha} g_A < \frac{1}{\chi} g_B$ ) we know that sooner or later ageing could have positive (transitional) effects on GDP-growth. This seems to be a quite convenient rule of thumb. Especially, since in this way the effects of ageing are related to two quite comprehensible and estimable parameters: in fact, our production functions imply that  $\frac{1}{\alpha} g_A$  and  $\frac{1}{\chi} g_B$  are the growth rates of labour-augmenting technological progress in the senior sector and junior sector respectively. Nevertheless, this is only a rule of thumb, since the other variables from equation (35g) may be dominant for a long period of time, if  $|\chi g_A - \alpha g_B|$  is not very large (i.e. if  $A^\chi / B^\alpha$  changes slowly).

**Theorem 5:** *In the limit, a one-time decrease in  $L/N$  has a negative impact on the growth rate of GDP during the transition between the old and the new PBGP, provided that the growth rate of labour-augmenting technological progress in the junior-sector is higher in comparison to the growth rate of labour-augmenting technological progress in the senior-sector, i.e. provided that  $\frac{1}{\alpha} g_A > \frac{1}{\chi} g_B$ .*

**Proof:** This theorem is implied by Theorem 4 and Lemma 16. **Q.E.D.**

To understand why it is important for the GDP-effects of ageing whether  $p_s^* < 1$  or  $> 1$ , we have to remember that we have shown in the proof of Theorem 4 that  $p_s^*$  is the price of senior-sector-goods and that the price of junior-sector-goods is equal to unity. Hence,  $p_s^* < 1$  ( $> 1$ ) means that senior-goods are less (more) expensive than junior-goods. Furthermore, with respect to GDP-growth ageing has two types of effects:

- a) The “**productivity effect**” has already been discussed in section 4.3.1. We stated there that ageing acts like a negative productivity shock, i.e. it leads to a decrease in net-output ( $Y_N$ ), provided that capital intensity in the senior sector

is lower in comparison to the junior-sector. This effect affects the GDP measure, since  $GDP \equiv \frac{Y_N}{\bar{p}}$  (see equation (16c)).

- b) **“Price effect”**: Remember that we divide our net-output ( $Y_N$ ) by the price-index ( $\bar{p}$ ) to obtain GDP. Hence, the changes in  $\bar{p}$  have an impact on GDP as well. Ageing leads to changes in  $\bar{p}$ , since the ageing-induced demand-shift leads to changes in output-shares which have been used to weight the prices of the price index (see equation (15b)). Hence, if the price of the senior sector is lower (higher) in comparison to the price of the junior-sector, ageing induced demand-shifts lead to a decrease (increase) of  $\bar{p}$  (since the relatively inexpensive senior-goods become a stronger weight in  $\bar{p}$ ). The price effect increases (decreases) GDP, provided that the senior-sector price ( $p_s^*$ ) is lower (higher) in comparison to the junior-sector price ( $= 1$ ). Note that the change in the weights of the price-index has (permanent) growth impacts as well, as will be of importance in the next section: since relative prices ( $p_i$ ) are changing over time, a change in the weighting causes permanent growth effects; e.g., if the weight is shifted towards relatively strongly growing prices, the price index increases more strongly over time.

Hence, if  $p_s^* > 1$ , both effects (the productivity effect and the price effect) point to the same direction, i.e. GDP-growth decreases. On the other hand, if  $p_s^* < 1$ , the productivity effect has a negative impact on GDP-growth, but the price effect increases GDP-growth. Hence, it is deciding which of those effects is stronger.

**Summary:** If model parameters (in equation (35g)) are such that the price of senior-sector-goods is relatively low, ageing may have positive transitional impacts on GDP. For example, if parameters from equation (35g) are such that  $p_s^*$  is close to zero and if  $\beta < \alpha$ , ageing has a (temporary) positive effect on GDP-growth, since in this case the positive price effect is stronger than the negative productivity effect. However, whether the transitional effects of a future decrease in  $L/N$  will be positive depends on the growth rate of  $A^\lambda / B^\alpha$ , which determines the growth rate of the senior-goods-price, and, hence, the price effect. On the other hand, if the model parameters are such that the price of the senior

sector ( $p_s^*$ ) is higher than the price of the junior sector ( $=1$ ), ageing has a negative transitional impact on GDP-growth, since the productivity effect and the price effect point to the same direction. Whether future ageing will have negative (transitional) effects in this case depends on the development of the term  $A^z / B^\alpha$  (and on the parameters of equation (35g)). Equations (35g) and (38) imply that the parameter-portfolio which determines the strength and direction of the ageing-impact comprises:

- a) sectoral labour-, capital- and intermediates-elasticities of output ( $\alpha, \beta, \gamma, \chi, \nu, \mu, \varepsilon_i$ )
- b) the parameters which determine the steady-state savings rate in neoclassical growth models (e.g. the time-preference rate, depreciation rate) (via parameter “s” in equation (35g))
- c) the relative level and growth rate of labour-augmenting technological progress in junior-sector in comparison to the senior sector (via the term  $A^z / B^\alpha$ )
- d) population-parameters (the old-to-young ratio ( $\frac{N-L}{N}$ ) via  $l_s^*$  and the growth-rate of labour ( $g_L$ ) via parameter “s”).

#### 4.3.2.2 PBGP-effects of ageing

In this subsection we show that ageing has not only transitional effects on GDP, but it affects the growth rate of GDP along the PBGP. That is, we show that the GDP-growth rate along the old PBGP is not the same as the GDP-growth rate along the new PBGP, where the new PBGP arises due to an increase in the old-to-young-ratio. This permanent effect is due to the price effect, which has been introduced in the previous section (after Theorem 5). Note that in contrast to the previous section the price effect in this section is permanent: A change in the weighting of prices induces permanent growth-effects, since prices are changing over time. Hence, a shift towards senior goods, which’s prices increase more strongly than the prices of junior-goods (in the limit), increases the growth rate of the price index permanently and, thus, reduces the growth rate of GDP.

**Theorem 6:** *A one-time decrease in  $L/N$  reduces (increases) the PBGP-growth rate of GDP, provided that the growth rate of labour-augmenting technological*

progress in the junior-sector is higher (lower) in comparison to the growth rate of labour-augmenting technological progress in the senior-sector, i.e. provided that

$\frac{1}{\alpha}g_A > \frac{1}{\chi}g_B$  ( $\frac{1}{\alpha}g_A < \frac{1}{\chi}g_B$ ). That is, the growth rate of GDP along the new

PBGP is lower (higher) in comparison to the growth rate of GDP along the old

PBGP, provided that  $\frac{1}{\alpha}g_A > \frac{1}{\chi}g_B$  ( $\frac{1}{\alpha}g_A < \frac{1}{\chi}g_B$ ).

**Proof:**  $\hat{GDP}$  along the PBGP is given by equation (35f). Along the PBGP all terms of equation (35f) are constant beside of  $p_s^*$ , which is given by equation (35g). Therefore, we obtain the following growth-rate:

$$\frac{\dot{\hat{GDP}}^*}{\hat{GDP}^*} = \frac{-(1-\lambda_m^*)\frac{\alpha\beta(1-\mu)}{\chi\beta-\alpha\nu}\dot{p}_s^*}{\alpha + \beta\lambda_m^* - (1-\lambda_m^*)\frac{\alpha\beta(1-\mu)}{\chi\beta-\alpha\nu}(1-p_s^*)}$$

Calculating the derivative of this growth rate implies:

$$(40) \quad \frac{\partial\left(\frac{\dot{\hat{GDP}}^*}{\hat{GDP}^*}\right)}{\partial(L/N)} = \frac{\partial\lambda_m^*}{\partial\left(\frac{L}{N}\right)}\dot{p}_s^* \frac{\alpha\beta(1-\mu)}{\chi\beta-\alpha\nu} \frac{\alpha + \beta}{\left[\alpha + \beta\lambda_m^* - (1-\lambda_m^*)\frac{\alpha\beta(1-\mu)}{\chi\beta-\alpha\nu}(1-p_s^*)\right]^2}$$

Equation (36e) implies that  $\frac{\partial\lambda_m^*}{\partial\left(\frac{L}{N}\right)} > 0$ . Furthermore, remember that we assume

$$\chi\beta - \alpha\nu > 0.$$

Equation (35g) implies  $\dot{p}_s^* > 0$ , if  $\frac{1}{\alpha}g_A > \frac{1}{\chi}g_B$ , and  $\dot{p}_s^* < 0$ , if  $\frac{1}{\alpha}g_A < \frac{1}{\chi}g_B$ .

Hence, equation (40) is positive (negative), if  $\frac{1}{\alpha}g_A > \frac{1}{\chi}g_B$  ( $\frac{1}{\alpha}g_A < \frac{1}{\chi}g_B$ ). That

is, a decrease in L/N has a negative (positive) impact on the GDP-growth rate

along the PBGP, provided that  $\frac{1}{\alpha}g_A > \frac{1}{\chi}g_B$  ( $\frac{1}{\alpha}g_A < \frac{1}{\chi}g_B$ ). **Q.E.D.**

### 4.3.3 Dynamic aspects

By now, in this section we have analyzed the impacts of a one-time increase in the old-to-young ratio. If ageing is not regarded as a one-time increase but as a

sequence of (discrete) increases in the old-to-young ratio, our results still remain applicable: Since we have shown in Lemma 7 (Corollary 1) that the PBGP is saddle-path-stable, the economy will be on the converging path. The qualitative results remain the same. The overall magnitude of the change in the macro-variables (e.g. in GDP) is determined by the sum of the changes in the old-to-young-ratio (overall-change in the old-to-young ratio). Only the period of change (the transition period) is more prolonged, since the overall-change is dispersed over a sequence (i.e. the economy cannot reach the “final” PBGP before the sequence is finished).

## 5. Concluding remarks

In this paper we have specified how ageing affects the GDP-growth rate and the pension-to-GDP-ratio via factor-allocation-effects. In the following we summarize our results, compare them to previous literature, derive simple policy-rules and predictions, show the caveats and extensions of our model and discuss topics for further research.

### 5.1 The most important impact channels associated with factor-allocation-effects

In our model ageing has three effects regarding GDP-growth:

(1) *Direct productivity effect (structural change)*: The ageing-induced demand-shift alters the factor allocation across technologically distinct sectors, which yields a direct productivity effect (average factor-productivities change).

(2) *Indirect productivity effect (capital accumulation)*: The “direct productivity effect” has also an impact on GDP-growth via capital accumulation (change in the savings rate). This effect is similar to the effect of a productivity(growth)-increase in the standard one-sector Ramsey-model: a change in productivity leads to a change in the opportunity costs of consumption, since return on savings depends on productivity of capital (remember that savings are invested in capital). Note that the change in the savings rate in our model is not the same as the ageing-induced “savings-effect” in other ageing-models (e.g. in the models by Groezen et al. (2005) and Rausch (2006)). In the latter models the effect of ageing on the

savings rate is modeled as a direct effect (“more saving for retirement, since we live longer”).

(3) *Price effect*: Since the ageing-induced demand-shift leads to changes of sectoral output-shares, the average price-index, which is the weighted average of sector prices, changes as well. Hence, ageing leads to changes in GDP-deflator (average price index), which has an impact on (real-)GDP as well.

## **5.2 Parameters which determine the strength and direction of the effects**

We show that the strength and the direction of the factor-allocation-effects of ageing depend on the combination of several parameters, including (see also the *Summary* in section 4.3.2.1):

- technology parameters, e.g. sectoral TFP-growth-rates and (initial) TFP-levels as well as input-elasticities of sectoral production functions (including intermediates-elasticities, which supports Fougère et al. (2007) and Kuhn (2004)),
- parameters determining the savings rate (due to effect (2)), e.g. the time-preference rate, and
- population parameters (the old-to-young-ratio and the growth rate of working-population).

## **5.3 Three simple implications for policy making and trend prediction**

As shown in section 4.3.2, despite of its “complexity” our model provides quite easily interpretable results, which can be used in empirical research and policy making:

(1) *The present and past effects of ageing via structural change* can be assessed by analyzing the (market) prices of senior-goods and junior-goods. In fact, our model implies that ageing has a negative impact on real GDP-growth via structural change, if senior goods are “more expensive” than junior goods. Otherwise, if senior goods are cheaper, the effects of ageing can be positive or negative, depending on the exact constellation of model parameters which are summarized in section 5.2.

(2) For discussion of *future effects of ageing* analysis of parameters which determine the development of relative prices of senior and junior sector is necessary. Only in this way we can assess whether it makes sense to assume that

senior-goods will be more expensive than junior goods (see point (1)) in future. Our results imply that in the limit (or: in the very long run) ageing has negative impacts on real GDP-growth, provided that the growth rate of labour-augmenting technological progress in the senior sector is lower in comparison to the growth rate of labour-augmenting technological progress in the junior-sector. (In this case sooner or later senior goods become more expensive in comparison to junior goods.) Hence, when discussing the future effects of ageing it is important to know about the development of labour-augmenting technological progress in the senior and junior sector. (Note that labour-augmenting technological progress is a function of TFP-growth and output-elasticity of labour, as discussed in section 4.3.2.1.) However, as mentioned in section 4.3.2, this is only a rule of thumb, since “sooner or later” is a quite vague concept. That is, the exact parameter restrictions, which were derived in section 4.3.2, may be the key determinant of ageing-impact for a very long period of time; thus, the portfolio of parameters which has been summarized in section 5.2 may be an important determinant of ageing effects in reality.

(3) The portfolio of parameters which determine the effects of ageing via structural change (see section 5.2) provides a range of *policies to counteract the negative impacts of ageing* on real GDP-growth. Our model can help to isolate more or less efficient policies: For example, equation (40) implies that the relative price of senior goods can be influenced by policies which have an impact on

- the savings rate (via “s” in equation (40)),
- the sectoral output-elasticities of inputs,
- the growth rate of working population (via “s”) and
- the sectoral levels and rates of labour-augmenting technological progress.

For example, policies which influence the savings rate seem to be not effective in the very long run, since (as just explained) in the very long run the impacts of labour-augmenting technological progress are dominant regarding the development of the relative price of senior goods price. On the other hand, e.g., policies which increase the birth rate seem to be relatively effective: they do not only decrease the old-to-young ratio directly (hence reducing the rate of ageing) but also they reduce the relative price of senior goods (via  $g_L$  from “s” in equation (40)).



## 5.4 Further results and comparison to previous modelling-literature

It is unfair to compare our results to the results of previous modelling-literature, since we have a different focus of analysis.<sup>13</sup> Thus, the following discussion should not be understood as critique of the previous literature; in our opinion, the assumptions and model-choices which are made in previous literature are optimal regarding the goals which are set there. Rather, we aim to emphasize those of our results which are “*new*” in comparison to previous literature. Note that we omit here the comparison of technical aspects of modelling and their impacts on the results; furthermore, note that the discussion in sections 5.1 to 5.3 may be regarded as “novelty” for the most part.

In contrast to Groezen et al. (2005), our model does not imply that low capital-intensity in the senior sector is sufficient to constitute negative effects of ageing. That is, lower capital-intensity in the senior sector does not necessarily imply that senior goods are more expensive in comparison to junior goods. The reason is that, in contrast to the model by Groezen et al. (2005), in our model the “backward” sector is not completely stagnant, but features TFP-growth. Although it is reasonable to assume that senior-goods are and will “always” be relatively labour-intensive, it does not make sense to assume that they do not feature any TFP-growth.<sup>14</sup> Our model implies that such TFP-growth may offset the negative impacts of high labour-intensity; thus negative impacts of ageing may not arise in this case. Thus, the focus of analysis should be shifted to labour-augmenting technological progress, which takes account of both (output-elasticity of capital and TFP-growth); see also point (2) in section 5.3.

Anyway, as discussed in section 5.3 (points (2) and (3)), in most cases the impacts of ageing in reality may not be determined by the analysis of only few of such variables, but the exact parameter constellation of a relatively large parameter portfolio (see section 5.2) may be deciding. Therefore, the strength and direction of factor-allocation-effects of ageing may vary strongly across countries

---

<sup>13</sup> Groezen et al. (2005)-focus is on trade-off between “savings effect” and “factor-allocation-effect”, see section 1.3, while our focus is on detailed modeling of the “factor-allocation-effect”, see section 1.4.

<sup>14</sup> Let us take a very extreme example: The services of a psychologist seem to be very labour-intensive. That is, it is very difficult to imagine today, that even in far future a machine will be invented, which is able to substitute the human psychologist (or its ability to understand the emotions of the patient). Thus, we may assume that labour-intensity in this profession (psychologist) will always be very high. Nevertheless, it is reasonable to assume that psychology (as a science) will make some progress, and, thus, the productivity of the psychologist will increase, i.e. the TFP of psychological service provision will increase in future.

depending upon their values of these parameters. The fact that empirical studies were not able to identify an unambiguous effect of ageing on growth (see Groezen et al. (2005)) may come from the neglect of the importance of cross-country differences in this parameter portfolio.

The following result is related to this discussion as well: In contrast to the previous literature we show (in section 4.3.2) that the impacts of ageing on GDP-growth may be *non-monotonous* over time, i.e. in the beginning ageing may have a positive (negative) impact on GDP-growth and later the effect of ageing may be negative (positive).<sup>15</sup> This result, as well, implies that the strength of the ageing impact (and the necessary reforms of pension systems) may vary widely across countries and across time, depending on the parameters derived in our model.

Last not least, we introduced the “price effect” in our paper (see also section 5.1 point (3)). This effect is predominant in sections 4.2 and 4.3.2.2. In fact, it is the only “persistent” effect of a *one-time* increase in the old-to-young-ratio. Thus, this effect elucidates the theoretical importance of ageing for growth: even a one time-increase in the old-to-young ratio is associated with a permanent change in the (equilibrium) growth rate of real GDP.

Overall, our results imply that from the theoretical point of view factor-allocation-effects of ageing have significant impacts on the long-run GDP-growth rate. Projections of future GDP-growth and of future pension-system-challenges may be too optimistic. For example, the paper by the Economic Policy Committee of the EU Commission (2003) based on Mc Morrow and Röger (2003) (see there especially pp. 12 ff.) does not include factor-allocation-effects in its ageing-related projections. Hence, the negative impacts of ageing may be stronger than expected by now and the reforms of old-age-pension systems (and health-systems) may be too weak. Furthermore, theories which try to explain the past/historic development of aggregates (e.g. Kaldor-facts) without accounting for factor-allocation-effects of ageing seem to omit important determinants of

---

<sup>15</sup> This *non-monotonousity* comes from the fact that, although the senior sector has relatively low capital intensity, the price of senior-goods needs not necessarily being higher than the price of junior-goods. That is, the positive “price effect” of ageing can overweight the negative “productivity effects” of ageing. However, if the growth rate of labour-augmenting technological progress is relatively high in the junior sectors, the senior goods must become more expensive than junior goods at some point of time, i.e. the price effect becomes negative as well. (See also the explanations at the beginning of section 5.4 (second paragraph)).

aggregate development. In general, ageing causes unbalanced growth of sectors and aggregates in our model.

### **5.5 The usefulness of PBGPs**

From a theoretical point of view our model demonstrates the usefulness of partially balanced growth paths in dynamic analysis: In contrast to most literature, which searches for balanced growth paths of variables of interest, we used a different approach: Although we are interested in the development of GDP and sectoral employment shares, we did not analyze the balanced growth paths of these variables, since these growth paths are not interesting from a theoretical point of view (because they do not allow for structural change). Instead, we analyzed our variables of interest along the balanced growth path of an auxiliary dynamic system, which consisted of variables  $Y$ ,  $K$  and  $E$ . We named the balanced growth path of this auxiliary system “PBGp”. In section 4.2 we analyzed our variables of interest along the PBGP, while in section 4.3 we analyzed the transitional dynamics between two PBGPs and compared the PBGP-dynamics of two equilibriums. Overall, the usage of PBGPs helped us to analytically study a topic which otherwise would require simulations.

### **5.6 A word on challenges to pension-systems**

Needless to say that our paper has also implications for all the literature and predictions which deal with the challenges to old-age-pension and health-systems associated with ageing. If the ageing-induced change in GDP-growth is negative, there is *additional* upward-pressure on the pension-to-output ratio. (Remember that upward pressure on the pension-to-output-ratio comes from an increasing number of pension-recipients as well).

### **5.7 Extension of our model: Endogenous technology**

The existence and strength of “factor-allocation-effects” of ageing depends upon the existence and strength of technology-bias. “Technology-bias” means here cross-sector-difference in technology: we assume that senior-goods-sectors do not use the same technology as junior-goods-sectors. If this bias vanished, i.e. if both sectors had the same technology, the factor allocation across the two sectors would be irrelevant regarding GDP-growth. Thus, there would not be any factor-

allocation-effect of ageing. In the introduction of the paper we provided literature-references which imply that the technology-bias exists in reality. In our model the technology(-bias) is exogenous; ageing has no impact on the technology-bias.

Stijepic and Wagner (2011) provide an extension of our model, where the technology-bias is endogenized. They show that ageing-related demand-shifts can have an impact on the technology-bias (see section 5 in that paper). Nevertheless, their results do not imply that any of our results is incorrect. However, their results imply that additional forces have an impact on the strength of the factor-allocation-effects of ageing: in fact, all the forces which influence the technology-bias (e.g. sort of implementation of technological progress, dispersion of capital across sectors, etc) have also an impact on the factor-allocation-effects of ageing discussed in this essay. Note, however, that all these forces determine the strength of the factor-allocation-effects of ageing, but not the direction. An extensive discussion of these forces is provided by Stijepic and Wagner (2011).

### **5.8 Caveats and topics for further research**

Throughout the paper we assumed that parameter restrictions (5a,b) hold. In fact, these restrictions ensure that there are no other sources of demand-shifts between the senior and junior sector beside of ageing. However, the impact of social welfare parameters (especially, the question how the utility of the old and the young is weighted in a society) may be captured by deviation from these restrictions. The question is whether pension systems (and private “savings-for-retirement”-behaviour) change systematically (i.e. whether the weight is shifted to the old or to the young) with an increasing income. The answer to this question decides whether the budget-share of the old increases/decreases over the growth process. The higher the budget for senior-goods consumption (per old person), the stronger the factor-allocation-effects of ageing. Income-dependent changes in pension-systems could be modelled by a departure from the restrictions (5a,b). Furthermore, it should be mentioned here that, if the budget of the old ( $E_s$ ) was restricted by an exogenous force (e.g. by an “inefficient” pension system) in our model, the ageing impacts on GDP-growth would be weaker. However, from the social welfare point of view this would be suboptimal (i.e. the social welfare would be suboptimal).

These facts may as well have some explanatory power regarding differences in the strength of ageing impacts across countries.

In section 4.3 we modelled ageing like a shock (or series of shocks) and not like a smooth and perfectly foresighted process. The difference between these two approaches is that the latter is more difficult to model (We could not rely on the PBGP-results) and we would have to use simulations. Furthermore, if perfectly foresighted the effects of ageing would be smoother, i.e. dispersed over a longer period (i.e. even before the increase in the old-to-young ratio the effects of ageing would show up), which would affect our results quantitatively but not qualitatively (i.e. the impact channels would be the same). Furthermore, it should be questioned whether it makes sense to model ageing like a smooth perfectly foresighted process, especially when taking into account irrational or bounded rational behaviour of households in reality. Last but not least, since our model is aimed to postulate some crude qualitative relationships between macroeconomic parameters and variables, the question whether ageing is modelled as a series of shocks or as a perfectly foresighted smooth process seems to be less relevant.

Overall, our results imply that for assessing (the future) growth-impacts of ageing some further empirical research is necessary to estimate the exact technological properties of the junior and senior sector. Especially, it seems to be necessary to link the data on demand-differences across the old and the young with the data on technological properties of the sectors. Only in this way a general conclusion can be drawn about the past and future strength of the ageing impacts. Furthermore, the question whether the senior sector will have a lower growth rate of labour-augmenting technological progress in future seems to be interesting regarding the effects of ageing.

Last not least, as mentioned in Section 2.1, it could be interesting to endogenize the population development (e.g. old-to-young ratio as a function of income) and to analyze which factor-allocation-effects arise in such a model. However, the results of such an exercise are approximated by standard structural change literature, as discussed in Section 2.1.

Analyzing these questions is left for further research.

## References

- Acemoglu D (2009).** Introduction to Modern Economic Growth. Princeton University Press.
- Acemoglu D, Guerrieri V (2008).** Capital Deepening and Nonbalanced Economic Growth. *Journal of Political Economy*, 116(3): 467-498.
- Barro RJ, Sala-i-Martin X (2004).** Economic Growth. 2nd edition. MIT Press. Cambridge, Mass.
- Baumol WJ (1967).** Macroeconomics of Unbalanced Growth: The Anatomy of Urban Crisis. *American Economic Review*, 57(3): 415-426.
- Baumol WJ, Batey-Blackman SA, Wolf EN (1985).** Unbalanced Growth Revisited: Asymptotic Stagnancy and New Evidence. *American Economic Review*, 57(3): 415-426.
- Bernard AB, Jones CI (1996).** Comparing Apples to Oranges: Productivity Convergence and Measurement across Industries and Countries. *American Economic Review*, 86(5): 1216-1238.
- Berthélmey JC, Söderling L (1999).** The Role of Capital Accumulation, Adjustment and Structural Change for Economic Take-off: Empirical Evidence from African Growth Episodes, OECD Development Centre Working Paper No. 150.
- Börsch-Supan A (1993).** Aging in Germany and the United States: International Comparisons. NBER Working Paper 4530.
- Börsch-Supan A (2003).** Labor Market Effects of Population Aging. *LABOR*, 17(Special Issue): 5-44.
- Broadberry SN (1997).** The Long Run Growth and Productivity Performance of the United Kingdom, *Scottish Journal of Political Economy*, 44(4): 403-424.
- Broadberry SN (1998).** How Did the United States and Germany Overtake Britain? A Sectoral Analysis of Comparative Productivity Levels, 1870-1990", *The Journal of Economic History*, 58(2): 375-407.
- Broadberry SN, Irwin DA (2006).** Labor Productivity in the United States and the United Kingdom during the Nineteenth Century, *Explorations in Economic History*, 43: 257-279.

- Caselli F, Coleman WJII (2001).** The U.S. Structural Transformation and Regional Convergence: A Reinterpretation, *Journal of Political Economy*, 109(3): 584-616.
- Close FA, Shulenburger DE (1971).** Labor's Share by Sector and Industry, 1948-1965. *Industrial and Labor Relations Review*, 24(4): 588-602.
- Disney R, Haskel J, Heden Y. (2003).** Restructuring and Productivity Growth in UK Manufacturing, *The Economic Journal*, 113: 666-694.
- Dowrick S, Gemmel N (1991).** Industrialisation, Catching up and Economic Growth: A Comparative Study across the World's Capitalist Economies, *The Economic Journal*, 101: 263-275.
- Duarte M, Restuccia D (2010).** The Role of the Structural Transformation in Aggregate Productivity, *The Quarterly Journal of Economics*, 125(1): 129-173.
- Economic Policy Committee of the EU Commission (2003).** The Impact of Ageing Populations on Public Finances: Overview of Analysis Carried out at EU Level and Proposals for a Future Work Programme. EPC/ECFIN/435/03 final.
- Foster L, Haltiwanger J, Krizan CJ (1998).** Aggregate Productivity Growth: Lessons from Microeconomic Evidence, NBER Working Paper 6803.
- Fougère M, Mercenier J, Mérette M (2007).** A Sectoral and Occupational Analysis of Population Ageing in Canada Using a Dynamic CGE Overlapping Generations Model. *Economic Modelling*, 24(4): 690-711.
- Fuchs VR (1998).** Provide, Provide: The Economics of Aging. NBER Working Paper 6642.
- Gollin D (2002).** Getting Income Shares Right. *Journal of Political Economy*, 110(2): 458-474.
- Groezen van B, Meijdam L, Verbon HAA (2005).** Serving the Old: Ageing and Economic Growth. *Oxford Economic Papers*, 57(4): 647-663.
- Gruescu S (2007).** Population Ageing and Economic Growth. Physica. Heidelberg.
- International Monetary Fund (IMF) (2004).** World Economic Outlook: The Global Demographic Transition. Chapter III: How Will Demographic Change Affect the Global Economy? Available on the web at:  
<http://www.imf.org/External/Pubs/FT/weo/2004/02/pdf/chapter3.pdf>
- Irmen A (2009).** Population Aging and the Direction of Technical Change, CESIFO Working Paper, No. 2888.

- Kongsamut P, Rebelo S, Xie D (2001).** Beyond Balanced Growth. *Review of Economic Studies*, 68(4), 869-882.
- Kongsamut P, Rebelo S, Xie D (1997).** Beyond Balanced Growth. NBER Working Paper 6159.
- Kuhn P (2003).** Effects of Population Ageing on Labour Market Flows in Canada: Analytical Issues and Research Priorities. HRDC-IC Working Paper A-02.
- Landefeld JS, Seskin EP, Fraumeni BM (2008).** Taking the Pulse of the Economy: Measuring GDP. *Journal of Economic Perspectives*, 22(2): 193-216.
- Maddison, A (1987).** Growth and Slowdown in Advanced Capitalist Economies: Techniques of Quantitative Assessment, *Journal of Economic Literature*, 25: 649-698.
- Mc Morrow K, Röger W (2003).** Economic and Financial Market Consequences of Ageing Populations. European Commission Economic Paper 182.
- Ngai RL, Pissarides CA (2007).** Structural Change in a Multisector Model of Growth. *American Economic Review*, 97(1): 429-443.
- Ngai RL, Pissarides CA (2004).** Structural Change in a Multi-Sector Model of Growth. CEPR discussion paper 4763.
- Nordhaus DW (2008).** Baumol's Diseases: A Macroeconomic Perspective. *The B.E. Journal of Macroeconomics (Contributions)*, 8(1): Article 9.
- Penderer M (2003).** Industrial Structure and Aggregate Growth, *Structural Change and Economic Dynamics*, 14: 427-448.
- Poirson H (2000).** The Impact of Intersectoral Labor Reallocation on Economic Growth, IMF Working Paper No. WP/00/104.
- Rausch S (2006).** Aging, Trade and Welfare in a Heckscher-Ohlin Model with Overlapping Generations. Available on the web at: [http://www.rwi-essen.de/pls/portal30/docs/FOLDER/RGS/RESEARCH/PUBLICATIONS/RAUSCH\\_AGING,+TRADE+AND+WELFARE\\_ABSTRACT.PDF](http://www.rwi-essen.de/pls/portal30/docs/FOLDER/RGS/RESEARCH/PUBLICATIONS/RAUSCH_AGING,+TRADE+AND+WELFARE_ABSTRACT.PDF)
- Restuccia D, Yang DT, Zhua X (2008).** Agriculture and Aggregate Productivity: A Quantitative Cross-Country Analysis” *Journal of Monetary Economics*, 55: 234–250.
- Robinson, S (1971).** Sources of Growth in Less Developed Countries: A Cross-Section Study, *The Quarterly Journal of Economics*, 85(3): 391-408.



**Stijepic D (2011).** Structural Change and Economic Growth: Analysis within the "Partially Balanced Growth-Framework", Dissertation, Suedwestdeutscher Verlag für Hochschulschriften, Saarbrücken, also available on the web at:

<http://deposit.fernuni-hagen.de/2763/>

**Stijepic D, Wagner H (2011).** Implementation of technological breakthroughs at sector level and the technology-bias. Munich Personal RePEc Archive (MPRA) Working Paper, No. 33352. Also available on the web at: <http://mpa.ub.uni-muenchen.de/33352/>

**Temple J (2001).** Structural Change and Europe's Golden Age, CEPR (International Macroeconomics) Working Paper No. 2861.

**United Nations (2006).** Structural Change and Economic Growth, World Economic and Social Survey 2006, Chapter 2: 29-51.

**Valentinyi Á, Herrendorf B (2008).** Measuring Factor Income Shares at the Sectoral Level. *Review of Economic Dynamics*, 11(4): 820-835.

## APPENDIX A

Inserting equations (7) and (8) into equation (21) yields:

$$(A.1) \quad k_i = \frac{k_m}{l_m} l_i \text{ and } z_i = \frac{z_m}{l_m} l_i, \text{ for } i = 1, \dots, m$$

$$(A.2) \quad k_i = \frac{\alpha}{\chi} \frac{\nu}{\beta} \frac{k_m}{l_m} l_i \text{ and } z_i = \frac{\alpha}{\chi} \frac{\mu}{\gamma} \frac{z_m}{l_m} l_i, \text{ for } i = m+1, \dots, n$$

Inserting equations (A.1) and (A.2) into equation (7) yields

$$(A.3) \quad Y_i = A l_i L \kappa_m^\beta \zeta_m^\gamma, \text{ for } i = 1, \dots, m$$

$$(A.4) \quad Y_i = B l_i L \mathcal{G} \kappa_m^\nu \zeta_m^\mu, \text{ for } i = m+1, \dots, n$$

$$\text{where } \zeta_m \equiv \frac{z_m Z}{l_m L}, \mathcal{G} \equiv \left( \frac{\alpha \nu}{\chi \beta} \right)^\nu \left( \frac{\alpha \mu}{\chi \gamma} \right)^\mu \text{ and } \kappa_m \equiv \frac{k_m K}{l_m L}.$$

Inserting equations (A.3) and (A.4) into equation (21) yields:

$$(A.5) \quad p_i = 1, \text{ for } i = 1, \dots, m$$

$$(A.6) \quad p_i = \frac{\alpha}{\chi} \frac{A}{B} \frac{1}{\mathcal{G}} \kappa_m^{\beta-\nu} \zeta_m^{\gamma-\mu} \equiv p_s, \text{ for } i = m+1, \dots, n$$

It follows from equations (15a) and (21) that  $p_i = \frac{\partial Z / \partial h_i}{\partial Z / \partial h_m}, \forall i$ , which implies

that

$$(A.7) \quad h_i = \varepsilon_i \frac{h_m}{\varepsilon_m} \frac{1}{p_i}, \quad \forall i$$

Inserting equation (A.7) into equation (18) yields:

$$(A.8) \quad H = \frac{h_m}{\varepsilon_m}$$

Inserting equations (A.7), (A.5) and (A.6) into equation (13) yields:

$$(A.9) \quad Z = H \prod_{i=1}^n \varepsilon_i^{\varepsilon_i} \left( \frac{\chi}{\alpha} \frac{B}{A} \mathcal{G} \kappa_m^{\nu-\beta} \zeta_m^{\mu-\gamma} \right)^{\varepsilon_s}$$

$$\text{where } \varepsilon_s \equiv \sum_{i=m+1}^n \varepsilon_i$$

Inserting equations (A.3) and (A.5) into equation (21) yields for  $i = m$

$$(A.10) \quad 1 = \gamma \frac{A l_m L \kappa_m^\beta \zeta_m^\gamma}{z_m Z} \frac{Z}{H}$$

Solving this equation for H yields:

$$(A.11) \quad H = \gamma \frac{l_m}{z_m} AL \kappa_m^\beta \zeta_m^\gamma$$

Inserting equation (A.9) into equation (A.10) yields

$$(A.12) \quad \zeta_m = D \kappa_m^\psi$$

where 
$$\psi \equiv \frac{\beta - (\beta - \nu)\varepsilon_S}{1 + (\gamma - \mu)\varepsilon_S - \gamma}$$
 and

$$D \equiv \left\{ \gamma A \left[ \frac{\chi}{\alpha} \frac{B}{A} \left( \frac{\alpha \nu}{\chi \beta} \right)^\nu \left( \frac{\alpha \mu}{\chi \gamma} \right)^\mu \right]^{\varepsilon_S} \prod_{i=1}^n \varepsilon_i^{\varepsilon_i} \right\}^{\frac{1}{1 - \gamma(1 - \varepsilon_S) - \mu \varepsilon_S}}$$

Inserting equations (A.1) and (A.2) into equation (14) yields (remember: it

follows from equation (14) that  $\sum_{i=1}^m l_i = 1 - \sum_{i=m+1}^n l_i$ ):

$$(A.13) \quad \sum_{i=m+1}^n l_i = \left( 1 - \frac{l_m}{k_m} \right) \left( 1 - \frac{\alpha \nu}{\chi \beta} \right)^{-1}$$

$$(A.14) \quad \frac{l_m}{z_m} = 1 - \left( 1 - \frac{\alpha \mu}{\chi \gamma} \right) \sum_{i=m+1}^n l_i$$

It follows from equations (1)-(6) and (22) that

$$(A.15a) \quad C_i = \frac{\omega_i}{\omega_m} (C_m - \theta_m) + \theta_i \quad \text{for } i = 1, \dots, m$$

$$(A.15b) \quad C_i = \frac{N - L}{N} \frac{\omega_i}{\omega_m} \frac{C_m - \theta_m}{p_i} + \theta_i \quad \text{for } i = m + 1, \dots, n$$

Inserting equations (A.15), (A.5) and (A.6) into equation (17) yields

$$(A.16) \quad E = \frac{N}{L} \frac{C_m - \theta_m}{\omega_m}$$

Inserting equations (2)-(6) and (A.3) into equation (23) yields due to equation

(A.16):

$$(A.17) \quad \frac{\dot{E}}{E} = \beta A \kappa_m^{\beta-1} \zeta_m^\gamma - \delta - \rho$$

Inserting equations (A.3)-(A.6) into equation (16a) yields (remember:

$\sum_{i=1}^m l_i = 1 - \sum_{i=m+1}^n l_i$ ):

$$(A.18) \quad Y = AL \kappa_m^\beta \zeta_m^\gamma \left[ 1 - \left( 1 - \frac{\alpha}{\chi} \right) \sum_{i=m+1}^n l_i \right]$$

Inserting equation (A.13) into equation (A.18) yields:

$$(A.19) \quad Y = AL\kappa_m^\beta \zeta_m^\gamma \left( a + b \frac{l_m}{k_m} \right)$$

$$\text{where } a \equiv 1 - \frac{1 - \frac{\alpha}{\chi}}{1 - \frac{\alpha\nu}{\chi\beta}} \text{ and } b \equiv \frac{1 - \frac{\alpha}{\chi}}{1 - \frac{\alpha\nu}{\chi\beta}}$$

Inserting first equation (A.14) and then equation (A.13) into equation (A.11) yields:

$$(A.20) \quad H = \gamma AL\kappa_m^\beta \zeta_m^\gamma \left[ d_1 + d_2 \frac{l_m}{k_m} \right]$$

$$\text{where } d_1 \equiv 1 - \frac{1 - \frac{\alpha\mu}{\chi\gamma}}{1 - \frac{\alpha\nu}{\chi\beta}} \text{ and } d_2 \equiv \frac{1 - \frac{\alpha\mu}{\chi\gamma}}{1 - \frac{\alpha\nu}{\chi\beta}}.$$

It follows from equation (A.4) that

$$(A.21) \quad l_i = \frac{Y_i}{BL\vartheta\kappa_m^\nu \zeta_m^\mu} \text{ for } i = m+1, \dots, n.$$

Inserting first equation (11), then equations (A.15) and (A.6) and finally equation (A.19) into equation (A.21) yields

$$(A.22) \quad l_i = \frac{N-L}{N} \frac{\chi}{\alpha} \frac{\omega_i}{\omega_m} \frac{C_m - \theta_m}{\tilde{Y}} + \frac{\theta_i}{BL\vartheta\kappa_m^\nu \zeta_m^\mu} + \frac{\chi}{\alpha} \frac{h_i p_i}{\tilde{Y}} \text{ for } i = m+1, \dots, n$$

$$\text{where } \tilde{Y} \equiv \frac{Y}{a + b \frac{l_m}{k_m}}.$$

It follows from equation (A.13) that

$$(A.23) \quad \lambda_m = 1 - \left( 1 - \frac{\alpha\nu}{\chi\beta} \right) \sum_{i=m+1}^n l_i$$

$$\text{where } \lambda_m \equiv \frac{l_m}{k_m}$$

Inserting first equation (A.22), then equations (A.16) and (A.7) and finally equation (A.8) into equation (A.23) yields after some algebra (remember that

$$\sum_{i=m+1}^n \theta_i = 0):$$

$$(A.24) \quad \lambda_m = 1 - \left(1 - \frac{\alpha\nu}{\chi\beta}\right) \frac{\chi}{\alpha} \left(a + b \frac{l_m}{k_m}\right) \left(\frac{N-L}{N} \frac{E}{Y} + \frac{H}{Y} \varepsilon_s\right)$$

where  $\lambda_m \equiv \frac{l_m}{k_m}$ .

Equations (11), (12), (16a), (17), (18) and (15a) imply:

$$(A.25) \quad Y = \dot{K} + \delta K + E + H$$

Inserting equation (A.12) into equation (A.19) yields

$$(A.26) \quad Y = G \lambda_m^{-c} (a + b \lambda_m) L^{1-c} K^c$$

where  $c \equiv \frac{\beta(1 - \varepsilon_s \mu) + \varepsilon_s \gamma \nu}{1 - \gamma(1 - \varepsilon_s) - \varepsilon_s \mu}$ ,

$$G \equiv A \left\{ \gamma A \left[ \frac{\chi}{\alpha} \frac{B}{A} \left(\frac{\alpha\nu}{\chi\beta}\right)^\nu \left(\frac{\alpha\mu}{\chi\gamma}\right)^\mu \right]^{\varepsilon_s} \prod_{i=1}^n \varepsilon_i^{\varepsilon_i} \right\}^{\frac{\gamma}{1 - \gamma(1 - \varepsilon_s) - \mu\varepsilon_s}}, \text{ and } \lambda_m \equiv \frac{l_m}{k_m}.$$

Inserting equation (A.12) into equation (A.17) yields

$$(A.27) \quad \frac{\dot{E}}{E} = \beta G \lambda_m^{1-c} K^{c-1} L^{1-c} - \delta - \rho$$

Inserting equation (A.19) into equation (A.20) yields equation

$$(A.28) \quad H = \gamma Y \frac{d_1 + d_2 \frac{l_m}{k_m}}{a + b \frac{l_m}{k_m}}$$

**Equations (A.24)-(A.28) can be transformed into equations (24), (25), (26a,b,d) and (26c). Q.E.D.**

Now we only have to derive equation (27). By using equations (14), (15c), (16a), (A.5) and (A.6) equation (15b) can be transformed as follows:

$$\bar{p} = \frac{1}{Y_N} \left[ Y_N - (1 - p_s) \left( \sum_{i=m+1}^n Y_i p_s - H \sum_{i=m+1}^n z_i \right) \right]$$

where  $p_s$  is given by equation (A.6). Now, inserting equations (A.2), (A.4) and (A.6) yields:

$$\bar{p} = \frac{1}{Y_N} \left[ Y_N - (1 - p_s) l_s \frac{\alpha}{\chi} \left( L A \kappa_m^\beta \zeta_m^\gamma - H \frac{\mu}{\gamma} \frac{z_m}{l_m} \right) \right].$$

where  $l_s$  is given by equation (19). Inserting equations (16b), (A.14), (A.19) and (A.20) yields

$$\bar{p} = 1 - (1 - p_s) l_s \frac{\alpha}{\chi} \frac{Y}{Y_N (a + b \lambda_m)} \left( 1 - \frac{d_1 + d_2 \lambda_m}{1 - \frac{\chi \gamma - \mu \alpha}{\chi \gamma} l_s} \mu \right)$$

where  $a \equiv 1 - \frac{1 - \frac{\alpha}{\chi}}{1 - \frac{\alpha \nu}{\chi \beta}}$ ,  $b \equiv \frac{1 - \frac{\alpha}{\chi}}{1 - \frac{\alpha \nu}{\chi \beta}}$ ,  $d_1 \equiv 1 - \frac{1 - \frac{\alpha \mu}{\chi \gamma}}{1 - \frac{\alpha \nu}{\chi \beta}}$  and  $d_2 \equiv \frac{1 - \frac{\alpha \mu}{\chi \gamma}}{1 - \frac{\alpha \nu}{\chi \beta}}$ . Now,

inserting equations (26a,b) and (A.23) yields

$$(A.29) \quad \bar{p} = \frac{\alpha + \beta \lambda_m - (1 - \lambda_m)(1 - p_s) \frac{\alpha \beta (1 - \mu)}{\chi \beta - \alpha \nu}}{\alpha + \beta \lambda_m}$$

where  $p_s$  is given by equation (A.6). Inserting equation (A.12) into (A.6) yields after some algebra:

(A.30)

$$p_s \equiv \left[ \left( \frac{\alpha}{\chi} \right)^{\alpha \chi} \left( \frac{\beta}{\nu} \right)^{\alpha \nu} \left( \frac{\gamma}{\mu} \right)^{\alpha \mu} \left( \gamma \prod_{i=1}^n \varepsilon_i^{\varepsilon_i} \right)^{\gamma \chi - \mu \alpha} \left( \frac{\hat{K}}{\lambda_m} \right)^{\frac{(\chi \beta - \alpha \nu)[\alpha + \varepsilon_s (\gamma \chi - \alpha \mu)]}{1 - \gamma + (\gamma - \mu) \varepsilon_s}} \frac{A^\chi}{B^\alpha} \right]^{\frac{1}{\alpha + \varepsilon_s (\gamma \chi - \alpha \mu)}}$$

**(Hint:** the following equations may be useful for obtaining (A.30):  $\alpha + \beta + \gamma = 1$  and  $\chi + \nu + \mu = 1$ ; these equations imply among others that  $\beta - \nu + \gamma \nu - \beta \mu = \beta \chi - \alpha \nu$  and  $\gamma - \mu + \mu \beta - \gamma \nu = \gamma \chi - \mu \alpha$ .)

**The rest of the proof is quite simple: equation (27) can be obtained by dividing the net-output (equation (26b)) by the price-index (equation (A.29)), where  $p_s$  is given by equation (A.30). Q.E.D.**

## **APPENDIX B**

Equations (A.7), (A.8), (A.16) and (A.22) from APPENDIX A and equations (5a,b) and (19) imply equation (29).

Equation (28) can be derived in the same way as equation (29).

Equations (A.5), (A.6), (A.15a,b), (A.16) from APPENDIX A and equations (5a,b) and (20) imply equations (30) and (31).

For a proof of equations (32) and (33) see APPENDIX A equations (A.1) and (A.2).

## APPENDIX C

First, we show by using linear approximation that the saddle-path-feature of the PBGP is given (Lemma 7a). Then we prove local stability by using a phase diagram (Lemma 7b).

### Existence of a saddle-path (Lemma 7a)

The study of local stability of the PBGP is analogous to the proof by Acemoglu and Guerrieri (2008) (see there for details and see also Acemoglu (2009), pp. 269-273, 926).

First, we have to show that the determinant of the Jacobian of the differential equation system (24)-(25) (where  $\lambda_m$  is given by equation (26c)) is different from zero when evaluated at the PBGP (i.e. for  $\hat{K}^*$ ,  $\hat{E}^*$ ,  $\lambda_m^*$  from equations (35a,b,e)). This implies that this differential equation system is hyperbolic and can be linearly approximated around  $\hat{K}^*$ ,  $\hat{E}^*$ ,  $\lambda_m^*$  (Grobman-Hartman-Theorem; see as well Acemoglu (2009), p. 926, and Acemoglu and Guerrieri (2008)). The determinant of the Jacobian is given by:

$$(C.1) \quad |J| = \begin{vmatrix} \frac{\partial \dot{\hat{K}}}{\partial \hat{K}} & \frac{\partial \dot{\hat{K}}}{\partial \hat{E}} \\ \frac{\partial \dot{\hat{E}}}{\partial \hat{K}} & \frac{\partial \dot{\hat{E}}}{\partial \hat{E}} \end{vmatrix} = \frac{\partial \dot{\hat{K}}}{\partial \hat{K}} \frac{\partial \dot{\hat{E}}}{\partial \hat{E}} - \frac{\partial \dot{\hat{E}}}{\partial \hat{K}} \frac{\partial \dot{\hat{K}}}{\partial \hat{E}}$$

The derivatives of equations (24)-(25) are given by:

(C.2)

$$\begin{aligned} \frac{\partial \dot{\hat{K}}}{\partial \hat{K}} &= c\hat{K}^{c-1}(\alpha\lambda_m^{-c} + \beta\lambda_m^{1-c}) + \hat{K}^c(-c\alpha\lambda_m^{-c-1} + (1-c)\beta\lambda_m^{-c})\frac{\partial \lambda_m}{\partial \hat{K}} - \left(\delta + g_L + \frac{g_G}{1-c}\right) \\ \frac{\partial \dot{\hat{K}}}{\partial \hat{E}} &= \hat{K}^c(-c\alpha\lambda_m^{-c-1} + (1-c)\beta\lambda_m^{-c})\frac{\partial \lambda_m}{\partial \hat{E}} - 1 \\ \frac{\partial \dot{\hat{E}}}{\partial \hat{K}} &= \beta\hat{E}\left((c-1)\hat{K}^{c-2}\lambda_m^{1-c} + (1-c)\hat{K}^{c-1}\lambda_m^{-c}\frac{\partial \lambda_m}{\partial \hat{K}}\right) \\ \frac{\partial \dot{\hat{E}}}{\partial \hat{E}} &= \left(\beta\lambda_m^{1-c}\hat{K}^{c-1} - \delta - \rho - g_L - \frac{g_G}{1-c}\right) + \beta\hat{E}\hat{K}^{c-1}(1-c)\lambda_m^{-c}\frac{\partial \lambda_m}{\partial \hat{E}} \end{aligned}$$

where the derivatives of equation (26c) are given by



(C.3)

$$\frac{\partial \lambda_m}{\partial \hat{E}} = - \frac{\frac{\chi\beta - \alpha\nu}{\alpha\beta} \frac{N-L}{N} \frac{\lambda_m^c}{\hat{K}^c}}{\left(1 + \varepsilon_s \frac{\chi\gamma - \alpha\mu}{\alpha}\right) \left(1 + \left(1 + \varepsilon_s \frac{\chi\gamma - \alpha\mu}{\alpha}\right)^{-1} \frac{\chi\beta - \alpha\nu}{\alpha\beta} \frac{N-L}{N} c \frac{\hat{E}}{\hat{K}^c \lambda_m^{1-c}}\right)}$$

$$\frac{\partial \lambda_m}{\partial \hat{K}} = \frac{\frac{\chi\beta - \alpha\nu}{\alpha\beta} \frac{N-L}{N} c \frac{\hat{E}}{\hat{K}^{c+1} \lambda_m^{-c}}}{\left(1 + \varepsilon_s \frac{\chi\gamma - \alpha\mu}{\alpha}\right) \left(1 + \left(1 + \varepsilon_s \frac{\chi\gamma - \alpha\mu}{\alpha}\right)^{-1} \frac{\chi\beta - \alpha\nu}{\alpha\beta} \frac{N-L}{N} c \frac{\hat{E}}{\hat{K}^c \lambda_m^{1-c}}\right)}$$

Inserting the derivatives (C.2) and (C.3) into (C.1) and inserting the PBGP-values from equations (35a,b,e) yields after some algebra the following value of the determinant of the Jacobian evaluated at the PBGP:

(C.4)

$$|J|^* = - \frac{(1-c) \frac{\chi\beta - \alpha\nu}{\alpha} \frac{N-L}{N} \frac{\hat{E}^*}{\hat{K}^*} \left[ \rho + \frac{\alpha}{\chi\beta - \alpha\nu} \frac{N}{N-L} \left(1 + \varepsilon_s \frac{\chi\gamma - \alpha\mu}{\alpha}\right) \left( \delta + \rho + g_L + \frac{g_G}{1-c} \right) \right]}{\left(1 + \varepsilon_s \frac{\chi\gamma - \alpha\mu}{\alpha}\right) \left(1 + \left(1 + \varepsilon_s \frac{\chi\gamma - \alpha\mu}{\alpha}\right)^{-1} \frac{\chi\beta - \alpha\nu}{\alpha\beta} \frac{N-L}{N} c \frac{\hat{E}^*}{(\hat{K}^*)^c (\lambda_m^*)^{1-c}}\right)}$$

We can see that the determinant evaluated at PBGP is different from zero. Hence, the PBGP is hyperbolic. Furthermore, we can be sure that  $|J|^* < 0$ , provided that  $\chi\beta - \alpha\nu > 0$ . Since we assume in our paper that the capital intensity in the senior-sector is lower in comparison to the junior sector, the relation  $\chi\beta - \alpha\nu > 0$  holds (see section 4.3).

Our differential equation system consists of two differential equations ((24) and (25)) and of two variables ( $\hat{E}$  and  $\hat{K}$ ), where we have one state and one control-variable. Hence, saddle-path-stability of the PBGP requires that there exist one negative (and one positive) eigenvalue of the differential equation system when evaluated at PBGP (see also Acemoglu and Guerrieri (2008) and Acemoglu (2009), pp. 269-273). Since  $|J|^* < 0$  we can be sure that this is the case. ( $|J|^* < 0$  can exist only if one eigenvalue is positive and the other eigenvalue is negative. If both eigenvalues were negative or if both eigenvalues were positive, the determinant  $|J|^*$  would be positive.) Therefore, in the neighborhood of the PBGP there is a saddle-path along which the economy converges to the PBGP. **Q.E.D.**

**Local stability (Lemma 7b)**

In the following, we omit intermediates for simplicity, i.e. we set  $\gamma = \mu = 0$ . Furthermore, as noted above we study here only the case  $\chi\beta - \alpha\nu > 0$  (see also section 4.3). Since  $\chi\beta - \alpha\nu = \chi - \alpha$  if  $\gamma = \mu = 0$ , we can say as well that we study here only the case  $\chi - \alpha > 0$ . Note, however, that the qualitative stability results for the other case (i.e.  $\chi - \alpha < 0$ ) are the same.

To show the stability-features of the PBGP, the three-dimensional system (C.1)-(C.3) has to be transformed into a two dimensional system, in order to allow me using a phase-diagram. By defining the variable  $\kappa \equiv \frac{\hat{K}}{\lambda_m}$ , the system (24)-(25)-(26c) can be reformulated as follows (after some algebra):

$$(C.5) \quad \frac{\dot{\hat{E}}}{\hat{E}} = \beta\kappa^{\beta-1} - \left( \delta + \rho + g_L + \frac{g_G}{1-\beta} \right)$$

$$(C.6) \quad \frac{\dot{\kappa}}{\kappa} = \frac{\kappa^{\beta-1} - \left( \delta + g_L + \frac{g_G}{1-\beta} \right) - \frac{\hat{E}}{\kappa} \left( 1 - \frac{\alpha - \chi}{\alpha\beta} \frac{N-L}{N} \rho \kappa^{1-\beta} \right)}{1 + \frac{\alpha - \chi}{\beta} \frac{N-L}{N} \frac{\hat{E}}{\kappa^\beta}}$$

We can focus attention on showing that the stationary point of this differential equation system is stable: The discussion in section 4.1 (Definition 3 and Lemmas 1-4) implies that  $\kappa$  and  $\hat{E}$  are jointly in steady state only if  $\hat{K}$ ,  $\hat{E}$  and  $\lambda_m$  are jointly in steady state and that  $\hat{K}$ ,  $\hat{E}$  and  $\lambda_m$  are jointly in steady state only if  $\kappa$  and  $\hat{E}$  are jointly in steady state. Therefore, the proof of stability of the stationary point of system (C.5)-(C.6) implies stability of the stationary point of system (24)-(25)-(26c). Hence, in the following we will prove stability of the stationary point of system (C.5)-(C.6).

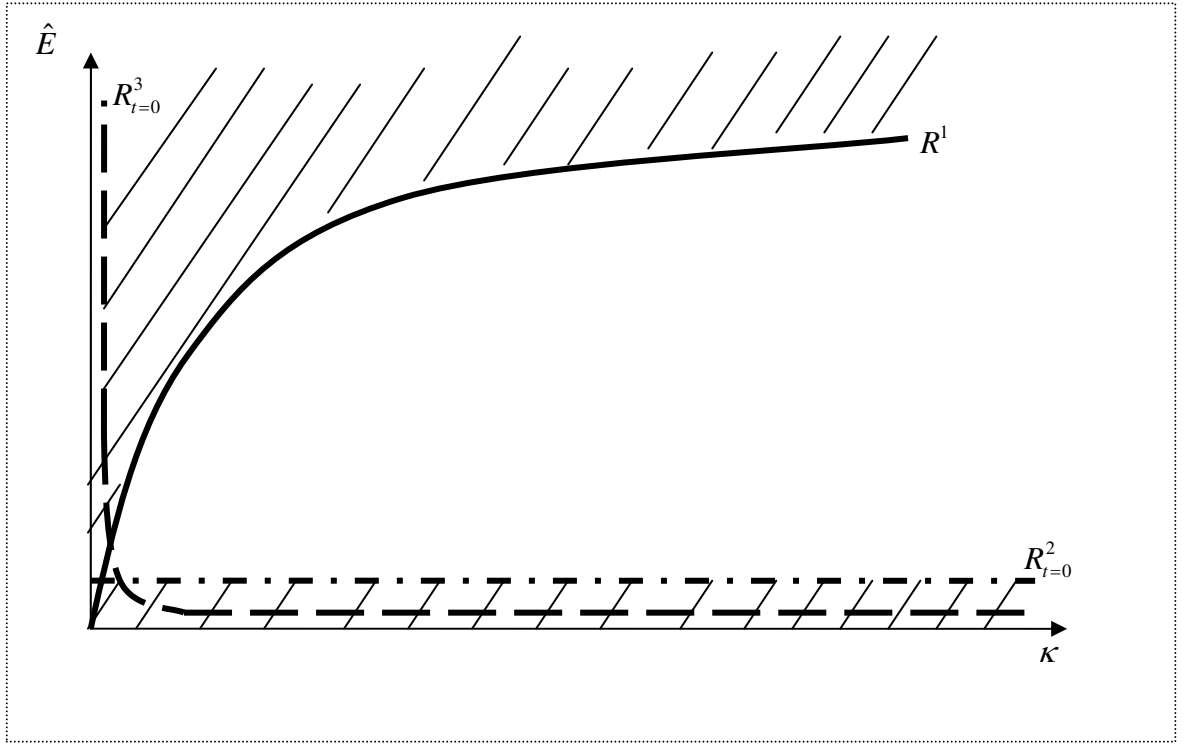
It follows from equations (C.5) and (C.6) that the steady-state-loci of the two variables are given by

$$(C.5a) \quad \frac{\dot{\hat{E}}}{\hat{E}} = 0: \quad \kappa^* = \left( \frac{\beta}{\delta + \rho + g_L + \frac{g_G}{1-\beta}} \right)^{\frac{1}{1-\beta}}$$

$$(C.6a) \quad \frac{\dot{\kappa}}{\kappa} = 0: \quad \hat{E}_{\dot{\kappa}=0} = \frac{\kappa^{\beta-1} - (\delta + g_L + \frac{g_G}{1-\beta})}{1 - \rho \frac{\alpha - \chi}{\alpha\beta} \bar{\omega}_n \kappa^{1-\beta}} \kappa$$

Now, we could depict the differential equation system (C.5)-(C.6) in the phase space  $(\hat{E}, \kappa)$ . Before doing so, we show that not the whole phase space  $(\hat{E}, \kappa)$  is economically meaningful. The economically meaningful phase-space is restricted by three curves  $(R^1, R_t^2, R_t^3)$ , as shown in the following figure and as derived below:

**Figure C.1: Relevant space of the phase diagram**



Only the space below the  $R^1$ -line is economically meaningful, since the employment-share of at least one sub-sector  $i$  is negative in the space above the  $R^1$ -line. This can be seen from the following fact:

As shown in APPENDIX A (see there equation (A.23)), the following relation is true

$$(C.7) \quad \lambda_m = 1 - \left( \frac{\chi\beta - \alpha\nu}{\chi\beta} \right) \sum_{i=m+1}^n l_i$$

Note that  $\chi\beta - \alpha\nu = \chi - \alpha$  when  $\gamma = \mu = 0$ .

Since,  $l_i$  cannot be negative (hence,  $0 \leq \sum_{i=m+1}^n l_i \leq 1$ ) this equation implies that

$$(C.8) \quad \frac{l_m}{k_m} < \frac{\alpha v}{\chi \beta}$$

Inserting equation (26c) into this relation yields

$$(C.9) \quad R^1: \hat{E} < \frac{\alpha}{\chi} \frac{N}{N-L} \kappa^\beta \quad (\text{remember that } \gamma = \mu = 0).$$

Hence, the space above  $R^1$  is not feasible. When the economy reaches a point on  $R^1$ , no labour is used in sub-sectors  $i=1, \dots, m$ . If we impose Inada-conditions on the production functions, as usual, this means that the output of sub-sectors  $i=1, \dots, m$  is equal to zero, which means that the consumption of these sectors is equal to zero. Our utility function implies that life-time utility is infinitely negative in this case. Hence, the household prefers not to be at the  $R^1$ -curve.

Now we turn to the  $R_i^2$  and  $R_i^3$ -curves. We have to take account of the non-negativity-constraints on consumption ( $C_i > 0 \forall i$ ), since our Stone-Geary-type utility function can give rise to negative consumption. By using equations (A.6), (A.15) and (A.16) from APPENDIX A and Definition 1 the non-negativity-constraints ( $C_i > 0 \forall i$ ) can be transformed as follows (remember that we assume here  $\gamma = \mu = 0$ ):

$$(C.10) \quad \hat{E} > \frac{-\theta_i}{\omega_i} \frac{N}{L} \frac{1}{LA^{\frac{1}{\alpha}}} \quad i = 1, \dots, m$$

$$(C.11) \quad \hat{E} > \frac{-\theta_i}{\omega_i} \frac{N}{N-L} \left( \frac{\alpha v}{\chi \beta} \right)^\chi \frac{\beta}{v} \frac{1}{LBA^{\frac{2v-\beta}{\alpha}}} \frac{1}{\kappa^{v-\beta}} \quad i = m+1, \dots, n$$

This set of constraints implies that at any point of time only two constraints are binding, namely those with respectively the largest  $\frac{-\theta_i}{\omega_i}$ . Hence, the set (C.10),

(C.11) can be reduced to the following set:

$$(C.12) \quad R_i^2: \hat{E} > \frac{-\theta_j}{\omega_j} \frac{N}{L} \frac{1}{LA^{\frac{1}{\alpha}}}$$

where  $\frac{-\theta_j}{\omega_j} > \frac{-\theta_i}{\omega_i} \quad i = 1, \dots, m$

and  $1 \leq j \leq m$ .

$$(C.13) \quad R_t^3: \quad \hat{E} > \frac{-\theta_x}{\omega_x} \frac{N}{N-L} \left( \frac{\alpha v}{\chi \beta} \right)^z \frac{\beta}{v} \frac{1}{LBA} \frac{1}{\alpha} \frac{1}{\kappa^{v-\beta}}$$

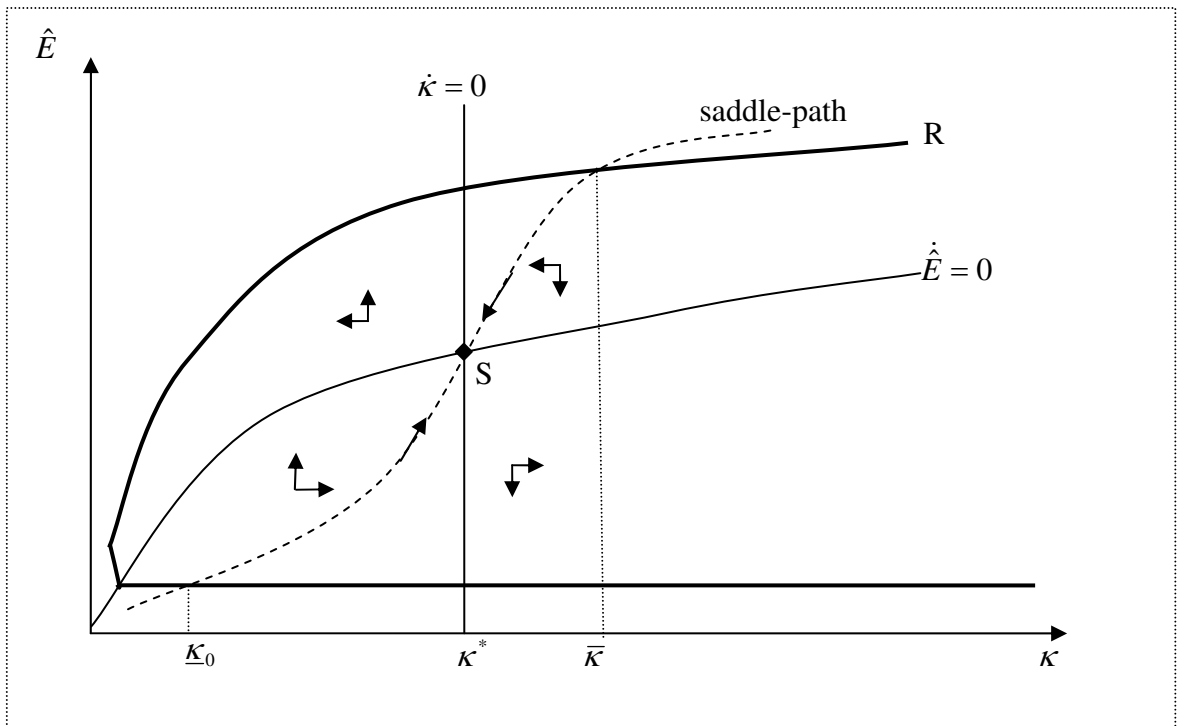
where  $\frac{-\theta_x}{\omega_x} > \frac{-\theta_i}{\omega_i} \quad i = m+1, \dots, n$

and  $m+1 \leq x \leq n$

These constraints are time-dependent. It depends upon the parameter setting whether  $R_t^2$  or whether  $R_t^3$  is binding at a point of time. In Figure C.1 we have depicted examples for these constraints for the initial state of the system. Only the space above the constraints is economically meaningful, since below the constraints the consumption of at least one good is negative. Last not least, note that equations (C.12)/(C.13) imply that the  $R_t^2$ -curve and the  $R_t^3$ -curve converge to the axes of the phase-diagram as time approaches infinity.

Now, we depict the differential equation system (C.5)-(C.6) in the phase space  $(\hat{E}, \kappa)$ .

Figure C.2: The differential equation system (C.5)-(C.6) in the phase-space for  $\chi\beta - \alpha v = \chi - \alpha > 0$



Note that we have depicted here only the relevant (or: binding) parts of the restriction-set of Figure C.1 as a bold line  $R$ .

The phase diagram implies that there must be a saddle-path along which the system converges to the stationary point S (where S is actually the PBGP). The length of the saddle-path is restricted by the restrictions of the meaningful space  $R^1, R_t^2, R_t^3$  (bold line). In other words, only if the initial  $\kappa$  ( $\kappa_0$ ) is somewhere between  $\underline{\kappa}_0$  and  $\bar{\kappa}$ , the economy can be on the saddle-path. Therefore, the system can be only *locally* saddle-path stable. Now, we have to show that the system will be on the saddle-path if  $\underline{\kappa}_0 < \kappa_0 < \bar{\kappa}$ . Furthermore, we have to discuss what happens if  $\kappa_0$  is not within this range.

All trajectories which start *above the saddle-path or left from*  $\underline{\kappa}_0$  reach the  $R^1$ -curve in finite time. As discussed above, the life-time utility becomes infinitely negative if the household reaches the  $R^1$ -curve. These arguments imply that the representative household will never choose to start above the saddle path if  $\underline{\kappa}_0 < \kappa_0 < \bar{\kappa}$ , since all the trajectories above the saddle-path lead to a state where life-time-utility is infinitely negative.

Furthermore, all initial points which are situated *below the saddle-path or right from*  $\bar{\kappa}$  converge to the point T. If the system reaches one of the constraints ( $R_t^2, R_t^3$ ) during this convergence process, it moves along the binding constraint towards T. However, the transversality condition is violated in T. Therefore, T is not an equilibrium. To see that the transversality condition is violated in T consider the following facts: The transversality condition in our model requires

that  $\lim_{t \rightarrow \infty} \beta \kappa^{\beta-1} - \delta - g_L - \frac{g_G}{1-\beta} > 0$ , which is equivalent to:

$\lim_{t \rightarrow \infty} \kappa < \left( \frac{\beta}{\delta + g_L + \frac{g_G}{1-\beta}} \right)^{\frac{1}{1-\beta}}$ . However, equation (C.6a) implies that in point T in

Figure C.2  $\kappa = \left( \frac{1}{\delta + g_L + \frac{g_G}{1-\beta}} \right)^{\frac{1}{1-\beta}}$ . Hence, the transversality condition is

violated if the system converges to point T.

Overall, we know that, if  $\underline{\kappa}_0 < \kappa_0 < \bar{\kappa}$ , the household always decides to be on the saddle-path. Hence, we know that for  $\underline{\kappa}_0 < \kappa_0 < \bar{\kappa}$  the economy converges to the PBGP. In this sense, **the PBGP is locally stable (within the range  $\underline{\kappa}_0 < \kappa_0 < \bar{\kappa}$ )**. **If the initial capital is too small ( $\kappa_0 < \underline{\kappa}_0$ )**, the economy converges to a state where some existence minima are not satisfied (curve  $R^1$ ) and, thus, utility becomes infinitely negative. This may be interpreted as a **development trap**. On the other hand, **if initial capital-level is too large ( $\kappa_0 > \bar{\kappa}$ )**, all trajectories violate the transversality condition. Therefore, in this case, **the representative household must waste a part of its initial capital** to come into the feasible area ( $\underline{\kappa}_0 < \kappa_0 < \bar{\kappa}$ ).

**Q.E.D.**

## APPENDIX D

Due to equation (16c) we know that

$$(D.1) \quad \frac{\partial GDP^*}{\partial \left( \frac{N-L}{N} \right)} = \frac{\frac{\partial Y_N^*}{\partial \left( \frac{N-L}{N} \right)} \bar{p}^* - \frac{\partial \bar{p}^*}{\partial \left( \frac{N-L}{N} \right)} Y_N^*}{\bar{p}^*}$$

Equation (38) can be obtained by inserting equation (A.29) from APPENDIX A and equation (35d) into equation (D.1). **Hints:** Equation (A.30) from APPENDIX

A and equation (35a) imply that  $\frac{\partial p_s^*}{\partial \left( \frac{N-L}{N} \right)} = 0$ . Furthermore, we used equation

(A.23) to transform  $\lambda_m^*$  into  $l_s^*$ .