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Abstract

Extreme Value Theory is increasingly used in the modelling of financial time series. The non-normality of stock returns leads to the search for alternative distributions that allows skewness and leptokurtic behavior. One of the most used distributions is the Pareto Distribution because it allows non-normal behaviour, which requires the estimation of a tail index.

This paper provides a new method for estimating the tail index. We propose an automatic procedure based on the computation of successive normality tests over the whole of the distribution in order to estimate a Gaussian Distribution for the central returns and two Pareto distributions for the tails. We find that the method proposed is an automatic procedure that can be computed without need of an external agent to take the decision, so it is clearly objective.

Keywords: Tail Index; Hill estimator; Normality Test

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1. Introduction

Extreme price movements are a common fact during the normal functioning of financial markets and during highly volatile periods corresponding to financial crises, like for example stock market crashes.

In the last decades, researches has been focused in modelling and estimating financial time series. However, most of this studies are concerned about expected returns, volatility or correlations, and not so much attention has been paid to extreme movements. Previous works in the use of extreme value to explain the fluctuations of the financial time series are Rothschild and Stiglitz (1970), who used the weight of the tails of two random variables in order to suggest a best definition of the risk increase than the usual variance, Parkinson (1980) discovered that extreme values offer an useful information in order to estimate volatility more efficiently, Haan et al. (1989) showed that the maximum of a distribution will be a Frechet one if the change in the stock price follows an ARCH process, Jansen and de Vries (1991) used extreme values in order to research the fat of the distribution tails.

It is a common conclusion in financial literature that the distribution of stock returns shows heavy tails, it means that there are more realizations in the tails than is to be expected if it had a normal distribution. In other words, stock return data shows more extreme realizations than can be accounted for by the normal distribution. Moreover round the mean value, the very small movements, there are more likelihood than expected. So the medium values are going to show a lower likelihood than the normal behavior. Originally Mandelbrot (1963), and later Fama (1965), pointed out that the distribution of the empirical returns is often leptokurtic and frequently positively skewed, which implies that it is peaked and fat-tailed.

This observation is of vital importance to risk management, and in particular to Value-at-Risk analysis, because it is the behavior of extremely low returns that causes large losses. EVT is a useful supplementary risk measure in risk management as a method for modelling and measuring this extreme risks. The seminal work of EVT is the one of Gnedenko (1943) who establishes three types of non-degenerated distributions for the standardized maximum: Frechet, Weibull and Gumbel. Galambos (1978) gives a rigorous account of the probability aspects of extreme value theory. Longin (1996) analysis extreme movements in the U.S. stock market, and obtains empirically that the extreme returns has a Frechet distribution. Moreover, applications of extreme value theory in insurance and finance can be found in Embrechts et al. (1997) and Reiss and Thomas (1997).

The structure of this article is as follows, in section 2 Extreme Value Theory is introduced. Then we brought up the Generalized Pareto Distribution in section 3 and the Pareto Distribution in section 4. Finally, in section 5 is proposed a new methodology for the estimation of the threshold value of the tail index.

2. Introduction to Extreme Value Theory

There are two classes of extreme value distributions who are used to find the correct limit distributions for maxima and minima. The first class was proposed by Jenkinson (1955), a Generalized Extreme Value (GEV) distribution that includes the three standard extreme value distributions established by Gnedenko (1943): Frechet, Weibull, and Gumbel.

The second class includes the distribution of excess over a given threshold, what it is interesting in modelling the behavior of the excess loss once a high threshold (loss) is reached. A more modern group of models are the Peaks-Over-Threshold (POT) models, that can be used to estimate the excess distribution with respect to a threshold level a, and to estimate the tail shape of the original distribution.

Within the POT class of models we can find two sorts of models, one of them is the semi-parametric model family built around the Hill estimator and its relatives Beirlant et al. (1996); Danielsson et al. (1998) and other are the fully parametric models based on de generalized Pareto distribution or GDP Embrechts et al. (1999).

There has been in the last years several researches about the adaptation of the stable Pareto distribution in order to model the unconditional distribution of returns, the first considerations about it are the researches of Mandelbrot (1963) and Fama (1963, 1965). Not always the stable class of models are suitable to apply to leptokurtic and skewed returns, despite they are suitable in a theoretical approach.

Hill estimator (Hill, 1975) and other similar tail estimators, are known because they are not reliable in-even large-finite samples (cf. Mittnik and Rachev (1993), McCulloch (1997), Resnick (1997) and Mittnik et al. (1998)) and even worse for data with GARCH structures (Mittnik et al., 2000).

3. Generalized Pareto Distribution

The Generalised Pareto Distribution (GPD) is a two parameter distribution and its importance in extreme value theory was observed by Pickands (1975) who showed basically that the GDP offers a good approximation of the tail of the distribution of returns for some fixed ξ as a shape parameter and β as an additional parameter that is $\beta > 0$. Its distribution function is the following,

$$G_{\xi,\beta}(x) = \left\{ \begin{array}{lll} 1 - \left(1 + \xi \frac{X}{\beta}\right)^{\frac{-1}{\xi}} & if & \xi \neq 0 \\ 1 - e^{-\frac{X}{\beta}} & if & \xi = 0 \end{array} \right\} \tag{1}$$

where $x \geq 0$ when $\xi \geq 0$ and $0 \leq x \leq \frac{-\beta}{\xi}$ when $\xi < 0$. The sign of the shape parameter ξ determines its tail behavior and thus the tail behavior of the original distribution.

This distribution is generalized in the sense that it subsumes other distributions under a common parametric form. If $\xi > 0$ the tail of the distribution function $G_{\xi,\beta}(x)$ decays like a power function $x^{-\frac{1}{\xi}}$, in this case, the function $G_{\xi,\beta}$ belongs to a family of heavy-tailed distributions that includes: the Pareto, log-gamma, Cauchy and t-distributions and others. For $\xi = 0$ the tail for $G_{\xi,\beta}$ decreases exponentially, and belongs to a class of medium-tailed distributions that include the normal, exponential, gamma and lognormal distributions. Finally if, $\xi < 0$ is known as a Pareto Type II distribution, the underlying distribution $G_{\xi,\beta}$ is characterized by a finite right endpoint, this class of short-tailed distributions includes the uniform and beta distributions.

4. The Pareto Distribution

The first of the three previous groups of distributions, when $\xi > 0$, is the most relevant case for risk management purposes because implies a GDP with heavy tails. This is the reason why the Pareto distribution is the most used in fat-tailed distributions. It is considered as a truncated distribution, because the right tail of the distribution has values greater than an specific value a or threshold. It has this probability density distribution

$$f(x, a, \alpha) = \alpha \frac{a^{\alpha}}{x^{\alpha+1}} \tag{2}$$

with $x \geq a$ and α the tail index.

But we are interested in the extreme movements that lead to large losses, so we have to focus on the left tail where the values are below a certain threshold level. The cumulative distribution function of the Pareto distribution will be

$$F(-x) \approx a x^{-\alpha} \tag{3}$$

with $\alpha > 0$.

The issue is how we can obtain the value of a or threshold level and of α or tail index, if the threshold is known then α can be estimated by the maximum likelihood method. The maximum likelihood estimator of the reciprocal of the tail index, $\gamma \equiv 1/\alpha$, is obtained from the Hill estimator (Hill, 1975): (equation 4),

$$\widehat{\gamma}_H \equiv 1/\widehat{\alpha}_{ML} = \frac{\sum_{i=1}^{n} \log \frac{x_i}{a}}{n} \tag{4}$$

We usually not know the threshold value a from where the empirical distribution behaves as a Pareto distribution, the choice off this cut-off point is the more important of all the estimation results. In order to find it, the usual method consists in calculating and plotting the Hill estimator for different values of the threshold a (Drees et al., 2000), to search those value where the tail index is stable, so that the threshold a is selected from the hill plot for the stable areas of the tail index (In figure 1 we present the Hill plot for the FTSE 100 index). However, this choice is not always clear. In fact, this method applies well for a GPD or close to GPD type distribution. As stated by Bensalah (2000), the Hill estimator is the maximum likelihood for a GPD and since the extreme distribution converges to a GPD over a high threshold u its use is justified.

5. The distribution of financial returns. Identification of outliers

It is a common place now in the literature, since the initial works of Mandelbrot (1963) and Fama (1965) that the hypothesis of normality must be rejected. The three features that are commonly alleged as the causes of this rejection are:

1. Fat tails or extreme values in the distribution. The tails of the distributions concentrate more probability than is supposed on a Gaussian distribution.
2. Cluster of probability near the mean value of the distribution. Most of the movements in asset prices are relatively small.
3. The extreme movements are more frequent in the left side of the distribution, as a consequence of the higher sensibility of the market to bad

news.

The first two features lead to a leptokurtic distribution, and the third feature to the presence of skewness in the distribution. Figure 2 shows this features plotting the histogram of the returns in the case of four different stock indexes: S&P500 (NYSE), FTSE100 (London), CAC40 (Paris) and IBEX35 (Madrid). Table 1 shows the mean, standard deviation, skewness ($g_3 = \frac{m_3}{s^3}$) and kurtosis ($g_4 = \frac{m_4}{s^4} - 3$) coefficient. All of them have left skewness and an excess of kurtosis.

Extreme Value Theory as it is applied to financial markets assume that the tails of the distribution are governed by a different function than the rest of the distribution. The tails are then fitted with a Pareto distribution as presented in section 4. The Hill estimator (equation 4) is then used to split the returns in extreme and non-extreme returns. This estimation explains one of the three features of the asset returns and require a subjective selection of the threshold in the region where the hill estimator remains stable (Figure 1).

It is our aim to obtain a method for the estimation of the threshold that not only explains the fat tails, but also the skewness and kurtosis of the distribution. The distribution of the whole asset returns can be obtained as a mixture of a gaussian function for the center of the distribution and two Pareto laws for both tails as presented in figure 3.

If the tails are explained by two Pareto laws, the rest of the distribution will have a lower standard deviation. After diminishing the value of the variance, the gaussian function will be concentrated around the mean value as can be seen in figure 4. The mixture of normal distribution and Pareto laws would explain not only the fat tails but the cluster on the mean value and even the skewness if we consider a different Pareto law for each tail.

We propose to fit the lower price movements to a normal distribution instead of trying to fit the tails of the distribution. Our proposal is to identify the asset returns that cannot be considered gaussian, and will be treated as outliers that we will be modelled by a Pareto distribution.

Given a time series of asset returns, x_i, the process of identification of the values from both distribution will be as follows:

1. We define the absolute returns, $y_i = |x_i| \; \forall i = 1, \ldots, T$. A higher value will imply a strong movement of the asset price, doesn't matter if it is positive or negative.
2. We order the values of y_j, such as $y_1 \geq y_2 \geq y_3 \geq \ldots \geq y_T$, and save the ordination index.
3. We test the normality hypothesis for the subset $\{x_j\} \forall j = r, \ldots, T$.
4. We select the value of r that produce a better result in the normality test.

As we eliminate the extremal values we reduce the excess of kurtosis. The kurtosis coefficient for the returns of the FTSE 100 index is plotted in figure 5a. After eliminating a sufficient number of extreme value we pass from a leptokurtic to a platykurtic distribution. But also the skewness of the distribution (figure 5b) is reduced in the process. In Figure 6 can be seen the resulting distributions for the four indexes.

In order to test the normality of the distribution we have chosen three methods:

1. χ^2 goodness of fit. Also known as test χ^2, this test present as null hypothesis that the sample has been obtained from a variable with probability distribution equal to P, i.e. the normal distribution.

   We will have a sample called X with observations that could be classified in r classes (i.e. the intervals of an histogram). We could represent this categories by $A_1, A_2, \ldots, A_r$. In the sample X there are n_1 elements that belongs to category A_1, n_2 elements that belongs to A_2 and so on. Under the null hypothesis, we know the probability of each class, $P(A_i) = p_i$, where $p_1 + p_2 + \ldots + p_r = 1$. The probability of obtaining $n_1, n_2, \ldots, n_r$ elements of each class will have a multinomial distribution with probabilities,

$$P(n_1, n_2, \ldots, n_r) = \frac{n!}{n_1! \cdots n_r} p_1^{n_1} \cdots p_r^{n_r} \tag{5}$$

   where each n_i has a marginal binomial distribution $B(n, p_i)$ with an expected value, $E(n_i) = np_i = E_i$. This expected value called E_i represents the number of observations belonging to class A_i, that we expect if the null hypothesis is true. So we construct an statistic,

$$\sum_{i=1}^{r} \frac{(n_i - E_i)}{E_i} \; \chi^2_{r-k-1} \tag{6}$$

   under the null hypothesis, where k is the number of parameters estimated on the null hypothesis distribution (i.e. μ and σ in the case of a normal distribution)

   For a continuous distribution we obtain the classes and the n_i values from an histogram, and the correspondent p_i from the expression $F(x_i) - F(x_{i+1})$ where x_i and x_{i+1} are the extremes of the interval. In order to obtain not-empty intervals we select the extremes x_i that have equal values of n_i.

2. Kolmogorov Smirnov test. It compares the values of the theorized normal distribution function and the empirical sample distribution. The test compute the maximum difference between both distributions (equation 7).

$$D = \sup_x |F_t(x) - F_r(x)| \tag{7}$$

3. Jarque Bera test (Jarque and Bera, 1980). It is based on the kurtosis and skewness coefficients, comparing the results of both coefficients in the sample with the values of a gaussian distribution (equation 8).

$$JB = \frac{T-k}{6}\left(\left(\frac{m_3}{s_3}\right)^2 + \left(\frac{m_4}{s_4} - 3\right)^2 \frac{1}{4}\right) \sim \chi_2^2 \tag{8}$$

In figure 7 it is possible to see how as we eliminate the extreme values of the sample, the Jarque-Bera test gives increasing values, reaching a threshold in which the test accept the null hypothesis of normality. Figure 7 implies that for the FTSE 100 index it is possible to reach a pvalue of 0.999 in the Jarque-Bera test. If we continue eliminating values we pass from a problem from the leptokurtic shape of the distribution to the opposite problem of platykurtic distribution. This is clear if we observe figure 5 where the kurtosis and skewness of the distribution is presented against the number of extremal returns subtracted. In the same figure 5 it is possible to see the value of the threshold a for the three normality tests used. Jarque-Bera select a value that gives skewness and kurtosis coefficients closer to the ones expected in a normal distribution.

6. Conclusions

Fama (1963) and Mandelbrot (1965) proved empirically that stock returns do not have a Gaussian Distribution so it is necessary to select an alternative distribution that allows skewness and leptokurtic behaviour. The Extreme Value Theory is increasingly used in the modelling of financial time series and the Pareto Distribution is one of the most widely used because it allows non-normal behaviour.

The estimation of the tail index is usually obtained by plotting the Hill Estimator for different values of the threshold, choosing that value where this estimator becomes stable. However, this procedure requires a subjective choice on the part of the researcher and in addition it is not automatic.

This paper provides a new method for estimating the tail index of the distribution of stock returns.In this paper, successive normality tests are realized over the whole of the distribution in order to estimate a Gaussian Distribution for the central returns and two Pareto distributions for the tails.

It is possible to see that the threshold estimations obtained by the normality tests lies in the plateau of the Hill plot (figure 1). These threshold estimators are consistent with the results obtained with the Hill plot, but with the great advantage of been an automatic procedure that can be compute without the need of an external agent that takes the decision.

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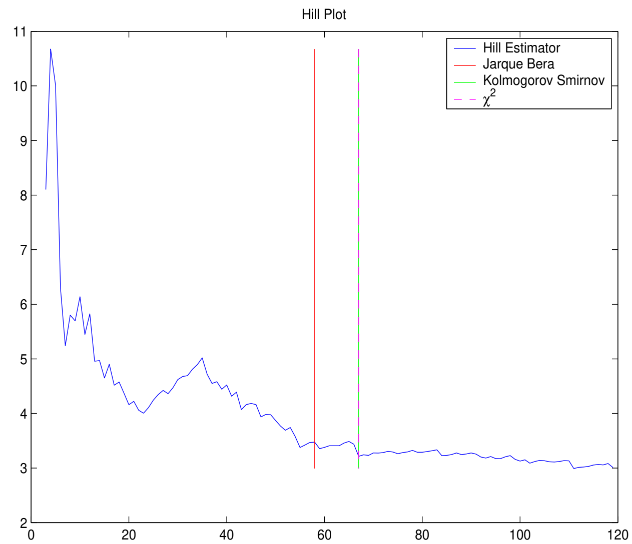


Figure 1: Hill plot of the FTSE 100 index and the values of the tail threshold with the normality tests

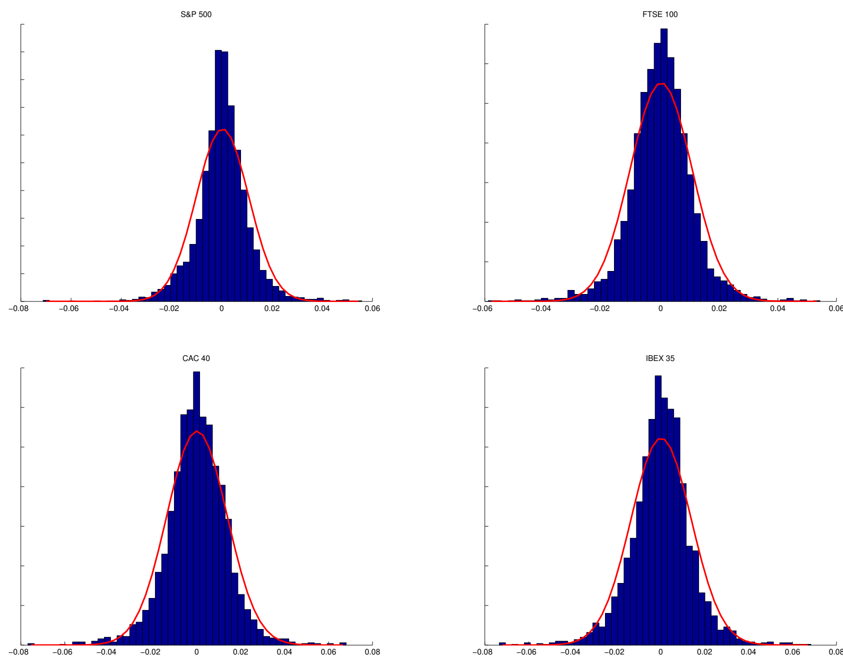


Figure 2: Histogram of the returns of the indexes S&P 500, FTSE 100, CAC 40 and IBEX 35

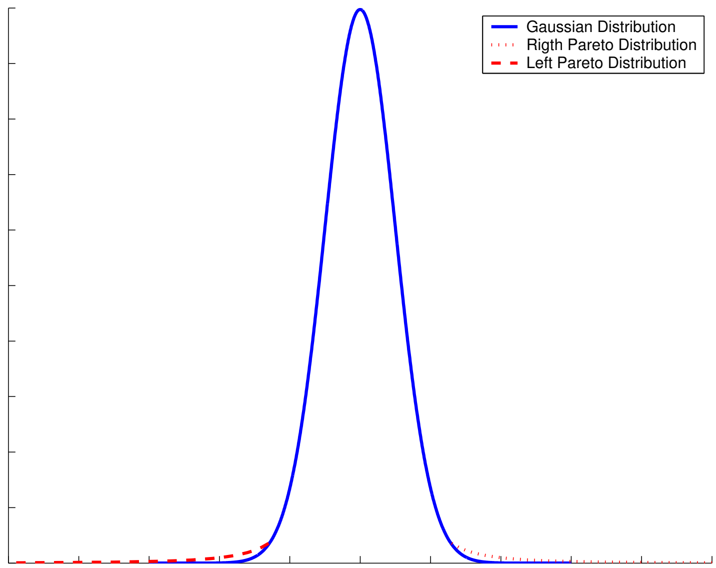


Figure 3: Gaussian distribution with two Pareto distribution added for the tails

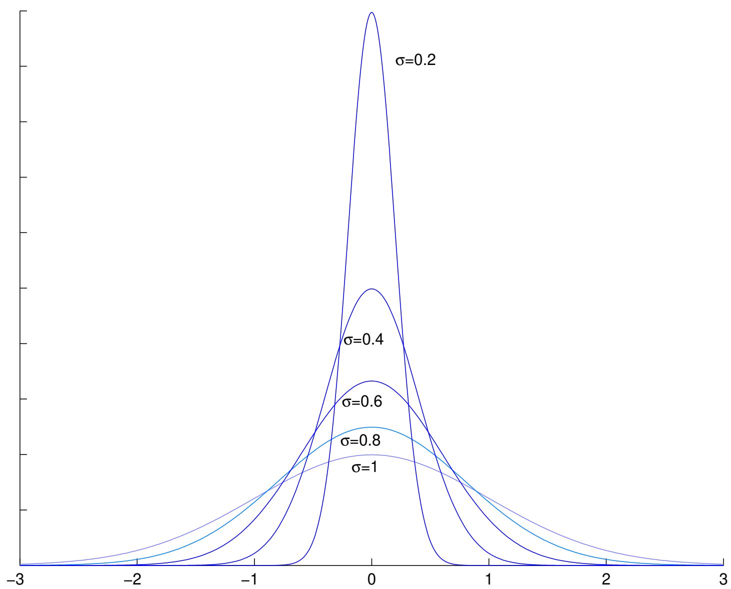


Figure 4: Gaussian distributions with different values of standard deviation.

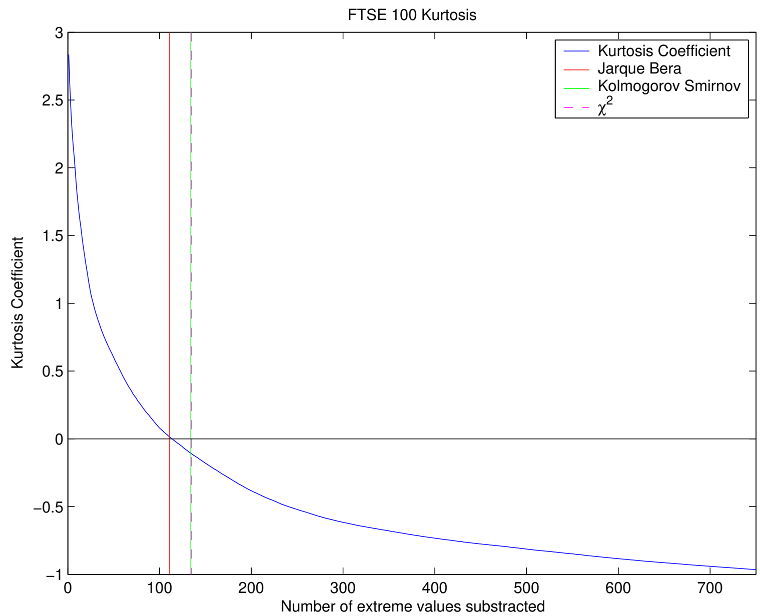
Table 1: Descriptive statistics from four stock index daily returns(S&P 500, FTSE 100, CAC 40, IBEX 35)

	Mean	Std. Deviation	Skewness	kurtosis
S&P 500	0,000284268	0,010479535	-0,123687694	3,922177894
FTSE 100	0,000164333	0,01055061	-0,130502363	2,8471159
CAC 40	0,000147307	0,013697291	-0,1205063	2,544084239
IBEX 35	0,000214679	0,013742123	-0,145496279	2,757789877

Table 2: Number of observation subtracted in each index according with the gaussian test used

	Jarque - Bera	Kolmogorov - Smirnov	χ^2
S&P 500	178	381	433
FTSE 100	111	134	135
CAC 40	84	111	95
IBEX 35	113	201	179

a.



b.

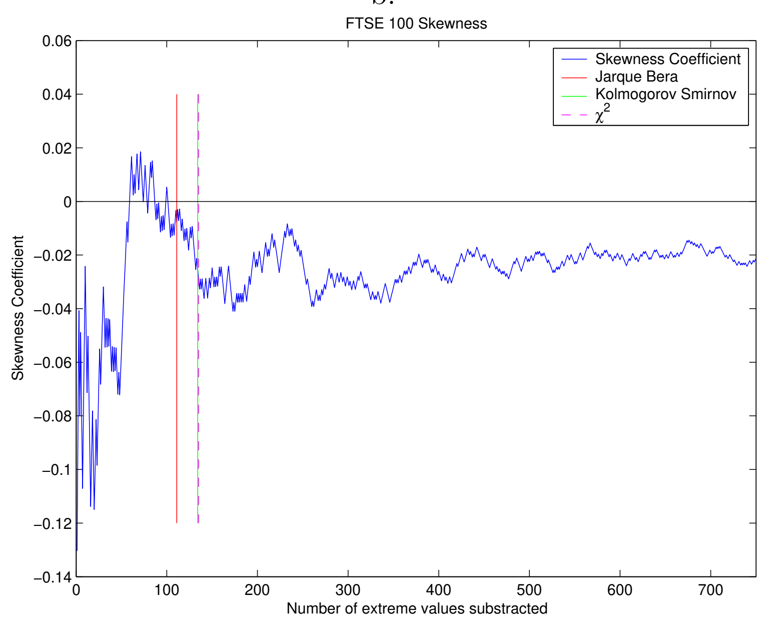


Figure 5: Returns of FTSE 100 (a. Skewness Coefficient and b. kurtosis Coefficient). In each case the number of extreme values eliminated is represented in x axis

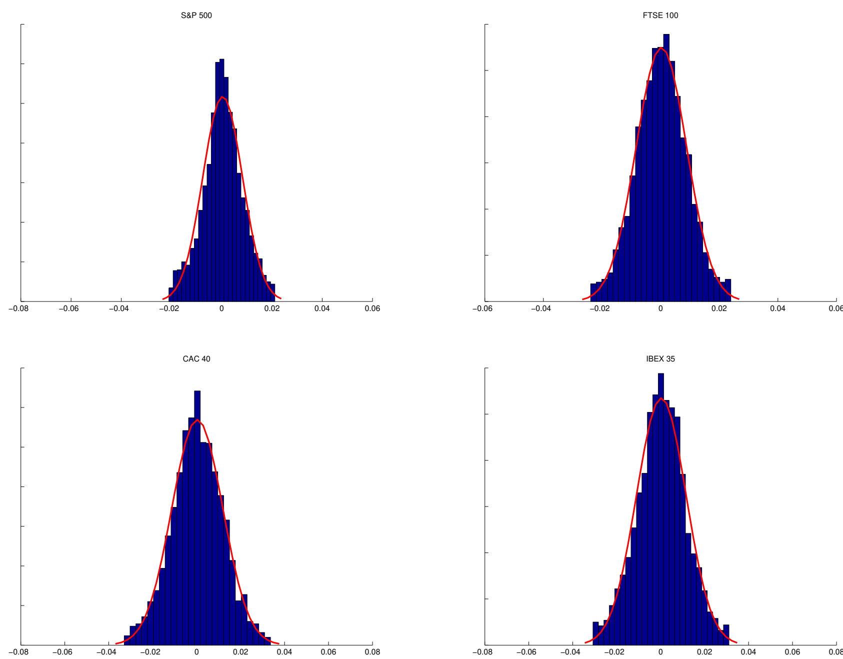


Figure 6: Histogram of the returns of the indexes S&P 500, FTSE 100, CAC 40 and IBEX 35 after eliminating the outliers

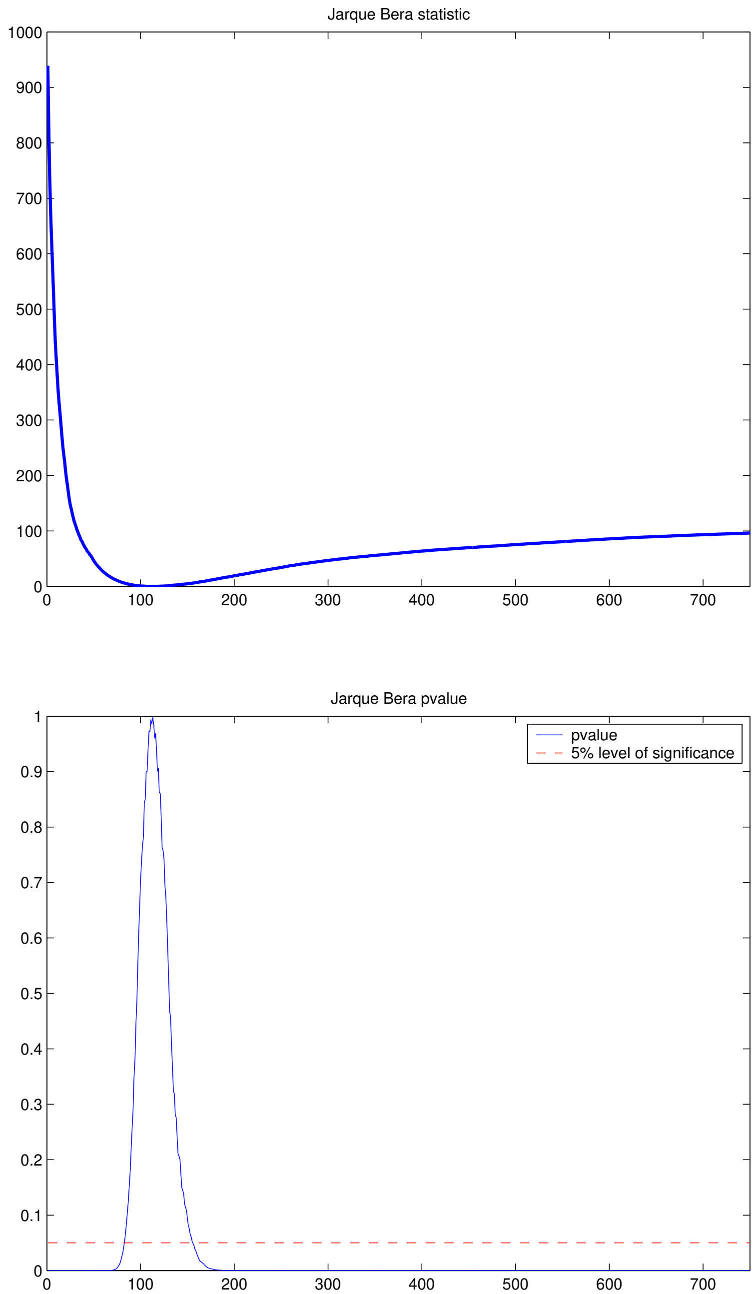


Figure 7: Jarque Bera test on the returns of FTSE 100 (a. Jarque-Bera test b. P-value). In each case the number of extreme values eliminated is represented in x axis