A Generalization of Sen's Unification Theorem: Avoiding the Necessity of Pairs and Triplets

He, Junnan

Australian National University

10 November 2011

A Generalization of Sen's Unification Theorem: Avoiding the Necessity of Pairs and Triplets

Junnan He[1]

Abstract

This paper is concerned with the axiomatic foundation of the revealed preference theory. Many well-known results in literature rest upon the ability to choose over budget sets that contains only 2 or 3 elements, the situations which are not observable in real life. In order to give a more realistic approach, this paper shows that many of the famous consistency requirements, such as those proposed by Arrow, Sen, Samuelson etc., are equivalent if the domain of choice functions satisfy some set theoretical properties. And these properties, unions and inclusions for example, are proposed in a way that gives observability.

1 Introduction

Revealed preference theory started off as an approach to explain consumers' behaviors by the revealed preference through their actions (Samuelson 1938). Defining preference relations on the bundles rather than specific goods, revealed preference theoriests have been able to avoid notions such as marginal utilities, and to construct a theory based only on a notion of preference. They also have been trying to pin down necessary and sufficient conditions for the relations to be rationalizable and the interlinks between the conditions. In Sen's (1971) paper "Choice Functions and Revealed Preference", the famous theorem that unites many rationality conditions for a choice function has limitations on to what extent we can believe in our assumptions or axioms. The dispute lies in the assumptions on the domain of choice functions and it was first suggested by Arrow (1959). The assumptions of choice function implied that the axioms on rationality (Sen 1971) should be trusted over all budgets of pairs and triplets, situations that are unlikely to be observed. This leads to a philosophical problem. Despite the effort in justifying the issue (Sen 1973), the theory remains unsatisfying for some (See Suzumura 1976). Suzumura attempted in adopting a domain of choice functions with no restriction, and his result was recognized as the more general result (Sen

[1] Email: u4575078@anu.edu.au/. I would like to thank Jose Rodrigues-Neto and Blair Alexander for proof-reading. All errors are my responsibility.

1997 pp.776). However, because no assumptions were made, the results were not as strong and unifying.

In this paper, we seek to largely avoid the philosophical problems by changing Arrow's and Sen's assumptions. Instead of adopting a domain of budgets that includes all the pairs and triplets, we would prove a result similar to Sen's, based on other general set theoretical properties of the domain and of the choice function. The merit of the approach is to realize that the equivalency of the rationality requirements rests on the more subtle set theoretical properties of the domain of choice functions, other than the inclusion of all the pairs and triplets. Hence, we hope this generalized version would enhance our confidence in consumer choice theory because the requirements of previously mentioned properties would be much observable in real life, and hence that concerns about the extent of belief in the axioms would be dispelled.

2 Notations and Definitions.

This paper, we will adopt the following notations. Consider $X \neq \emptyset$ as the set of all bundles, the choice function $C(.)$ is defined on a nonempty subset of the powerset of X, $\mathbb{B}$, called the set of budgets. For any budget $B \in \mathbb{B}$, we require that $C(B) \subset B$ and $C(B)$ is not empty, so $\emptyset \notin \mathbb{B}$ (i.e. the empty set cannot be a budget set). Throughout the paper, we use the symbol $\neg$ for mathematical negation.

Based on the choice function, the following definitions for a relation on X have been much discussed in the literature. For the first ones, we should interpret relations R as "at least as good as", P as "strictly preferred to".

Definition 1 For any $x, y \in X$, xRy if and only if $\exists B \in \mathbb{B}$ such that $x \in C(B)$ and $y \in B$.

Definition 2 For any $x, y \in X$, xPy if and only if xRy and $\neg yRx$.

Another definition for a relation was given by Arrow's (1959) "revealed preference". It was denoted as $\widetilde{P}$.

Definition 3 For any $x, y \in X$, $x\widetilde{P}y$ if and only if $\exists B \in \mathbb{B}$ such that $x \in C(B)$ and $y \in B - C(B)$.

Definition 4 For any $x, y \in X$, $x\widetilde{R}y$ if and only if $\neg y\widetilde{P}x$.

Other than the above definitions, there are also some notations about the "wide sense" relations, or, the "transitive closures" of R and $\widetilde{P}$ defined as follows (Ritchter 1966):

Definition 5 For any $x, y \in X$, $n \in \mathbb{N}^+$, xWy if and only if there exists a sequence of $x_1, x_2, \ldots, x_n$ such that $x_1 = x$, $x_n = y$ and $x_{i-1}Rx_i$.

Definition 6 For any $x, y \in X$, $n \in \mathbb{N}^+$, $x\widetilde{V}y$ if and only if there exists a sequence of $x_1, x_2, \ldots, x_n$ such that $x_1 = x$, $x_n = y$ and $x_{i-1}\widetilde{P}x_i$.

Given that we have the above definitions on relations, we can start stating definitions and axioms on rationality and consistency.

Definition 7 a choice function $C(.)$ is *normal* if and only if $\forall B \in \mathbb{B}, C(B) = \{x \in B | xRy\ \forall y \in B\}$.

Definition 8 a binary relation on X is called an *ordering* if and only if it is transitive and complete (or some author referrs to as connected).

If a choice function is normal with respect to some ordering R, we say $C(.)$ is *rational* or *rationalizable.*

Definition 9 A choice function is said to satisfy the *Weak Axiom of Revealed Preference* (WARP, Samuelson 1938) if for every $x, y \in X$: $x\widetilde{P}y \Rightarrow \neg yRx$ (or equivalently $yRx \Rightarrow \neg x\widetilde{P}y$).

Definition 10 A choice function is said to satisfy the *Strong Axiom of Revealed Preference* (SARP, Houthakker 1950) if for every $x, y \in X$: $x\widetilde{V}y \Rightarrow \neg yRx$.

Definition 11 A choice function is said to satisfy the *Weak Congruence Axiom* (WCA, Sen 1971) if for every $x, y \in X$: suppose xRy, then for any $B \in \mathbb{B}$, $(x \in B \textit{ and } y \in C(B)\ ) \Rightarrow x \in C(B)$.

Definition 12 A choice function is said to satisfy the *Strong Congruence Axiom* (SCA, Richter 1966) if for every $x, y \in X$: suppose xWy, then for any $B \in \mathbb{B}$, $(x \in B \textit{ and } y \in C(B)\ ) \Rightarrow x \in C(B)$.

3 Motivation

We hope that, from proving the results, we would be able to prove Sen's unification theorem (1971) as an direct application. Therefore we want to capture the essence of Sen's assumption. Sen included in $\mathbb{B}$ all the budgets with exactly 2 elements in order to compare each pair, and all the budgets with exactly 3 elements in order to avoid cycles. This suggests that first, if we are able to compare each two elements of X, not necessarily in a pair but in any sets, then we should be able to achieve completeness. Secondly, if we are also able to avoid cycles by choosing from more than 3 bundles, we should be able to reach a satisfying result.

To compare every two bundles a and b, it would be intuitive that when a is chosen from the set A, and b is chosen from the set B, then we would have implicitly compared a and b by choosing from the set $A \cup B$. Similarly, for a, b and c, we can try to choose from the set $A \cup B \cup C$. This seems promising because if one let the domain $\mathbb{B}$ to include all the singletons, then every element should be chosen in some singleton (by the assumption that $C(.)$ is never the empty set). Now by including the union of 2 or 3 singletons into the domain of $C(.)$, we would have included all the pairs and triplets and Sen's theorem can be proved. As it turns out, this union condition is almost enough, but not sufficient as will be seen in Theorem 2.

In the context of consumer theory, the benefit of the union condition is that it seems more plausible than requiring all the pairs and triplets. The usual setting is that given disposible income I and the price $P_i, 1 \leq i \leq n$ of n number of goods in the market, the consumer is facing a polyhedron shape budget under the constraint of $\sum_{i=1}^{n} P_i Q_i = I$, where Q_i is the quantity of the ith goods. Here, the union of budget sets would become some concave polyhedra, which can be observed in price cut or whole sale situations.

4 Construction and Main Results

Under the above motivation, we make the following assumption on $\mathbb{B}$ as below:

Assumption 1 For any domain $\mathbb{B}$ for a choice function $C(.)$, there exists $\mathbb{B}_1 \subseteq \mathbb{B}$ such that $\mathbb{B}_3 = \{B_1 \cup B_2 \cup B_3 | B_1, B_2, B_3 \in \mathbb{B}_1\} \subseteq \mathbb{B}$; moreover, for any bundle $x \in X$, there exists a budget $B_x \in \mathbb{B}_1$ such that $x \in C(B_x)$.

Theorem 1. Supposed $C(.)$ is defined on $\mathbb{B}$ and Assumption 1 is satisfied, then the following are equivalent:

(i) $C(.)$ satisfies Strong Axiom of Revealed Preference;

(ii) $C(.)$ satisfies Weak Axiom of Revealed Preference;

(iii) R is an ordering and $C(.)$ is normal;

(iv) $C(.)$ satisfies Strong Congruence Axiom;

(v) $C(.)$ satisfies Weak Congruence Axiom;

(vi) $R = \widetilde{R}$;

In the proof of theorem 1, we make use of the following lemma.

Lemma 1. Suppose that the WARP holds. Suppose that $x \in C(B_x),\ y \in C(B_y)$ for some budget sets $B_x, B_y \in \mathbb{B}$. If $B_x \cup B_y \in \mathbb{B}$, then at least one of x or y belongs to $C(B_x \cup B_y)$.

There are two equivalent conditions in Sen's 1971 paper that are not covered by theorem 1 because they do not fit in the context. They will be covered in the next result together with some other rationality conditions. However, additional definitions and notations are necessary. In order to be consistent with the motivation and the previous discussion, we mimic Uzawa (1956) and Arrow's (1959) definition of a relation "generated by comparison over all pairs" to be noted by $\overline{R}$:

Definition 13 For any $x, y \in X$, $x\overline{R}y$ if and only if there exists $B_x,\ B_y \in \mathbb{B}_1$ such that $x \in C(B_x),\ y \in C(B_y)$ and $x \in C(B_x \cup B_y)$.

Additionally, in the following generalization, we require some additional assumptions on a proper subset of $\mathbb{B}$ and the choice function $C(.)$ for some of the definitions to be meaningful or applicable.

Assumption 2 *Finiteness:* for each $B \in \mathbb{B}$, the number of elements in B is finite.
Assumption 3 *Pre-rationality:* $\forall B_x, B_y \in \mathbb{B}_1,\ C(B_x \cup B_y) \subseteq C(B_x) \cup C(B_y)$.
Assumption 4 *Closed Under Finite Union:* if $A, B \in \mathbb{B}$, then $A \cup B \in \mathbb{B}$.

Lemma 2. Assumption 3 is satisfied if and only if for some $1 \leq i \leq n$, $B_i \in \mathbb{B}$, then $\bigcup_{i=1}^{n} B_i \in \mathbb{B}$.

It can be seen that all the above assumptions can be implied from assuming that $\mathbb{B}$ consists of all finite subsets of X. In particular, Assumption 2 and 3 are required for the following consistency requirement to be meaningful in the context.

Axiom of Sequential Path Independence was a rationality condition originally proposed by Bandyopadhyay (1988). The idea was that if a choice function is rational, it should be necessary and sufficient that comparing each two bundles in a budget in different orders would give the same final choice. In order to fit in the above settings, we would use the following notations to give a modified version. For all $B \in \mathbb{B}$, let $\Omega(B)$ be the set of all permutations of elements of B, and $|B|$ denote the cardinality of B. Let B_r denote some budget in $\mathbb{B}_1$ such that $r \in C(B_r)$. Suppose assumption 2 holds, for any choice function $C(.)$ and any $\omega \in \Omega(B)$, define the following sets recursively:

1. $\hat{\omega}(1) = \{\omega(1)\}$

2. For any positive integer $i \leq |B|$, $\hat{\omega}(i+1) = B \cap \bigcup_{a \in \hat{\omega}(i)} C(B_a \cup B_{\omega(i+1)})$ for any $B_a,\ B_{\omega(i)} \in \mathbb{B}_1$.

Definition 14 If assumptions 2 and 3 are satisfied, a choice function $C(.)$ is said to satisfy the *Axiom of Sequential Path Independence* (ASPI, Bandyopadhyay 1988) if for any $B \in \mathbb{B}$ and for all $\omega \in \Omega(B)$, $C(B) = \hat{\omega}(|B|)$.

We have moderately adjusted the original notation. For any $\omega \in \Omega(B)$, $\hat{\omega}(i)$ can be interpreted as the i^{th} ω estimate for what is chosen from B. It is clear that purpose of having finiteness is so that the above version of Axiom of Sequential Path Independence adapted from Bandyopadhyay (1988) would fit in the context. Even though we require finiteness, this is very different from requiring $\mathbb{B}_3$ including all the pairs and triplets. In fact, these finite numbers can be of huge scale so that the condition does little harm in terms of realizability. In other words, the number of bundles in $B \in \mathbb{B}$ can be large enough to fill in a budget polyhedron of infinite many bundles in a way that a consumer cannot distinguish between choosing from each one of them. Assumption 3 is a restriction for the choice function on a proper subset of $\mathbb{B}$, which in itself is in some sense a weak consistency requirement. It is critical for determining what is chosen given the choice function satisfies ASPI. If put back into the original context where all pairs and triplets are included in the domain of the choice function and take all B_x to be sinigleton $\{x\}$, then the definition above gives the same meaning as those in Bandyopadhyay's (1988) paper.

Another consistency condition first given by Arrow (1959) is modified as below:

Definition 15 If assumption 4 is satisfied, a choice function $C(.)$ is said to satisfy the *Arrow's Condition* if for any $A, B \in \mathbb{B}$, when $A \subseteq B$ and $A \cap C(B) \neq \emptyset$, then $C(A) = C(B) \cap A$.

In Arrow's (1959) formulation, if we want to derive rationalizability from Arrow's Condition, it would heavily rely on the domain of choice function consisting of all finite subsets of X. Because for Arrow's Condition to be meaningful, for any $A \in \mathbb{B}$, there needs to be a proper superset or proper subset of A, and the intersection of their chosen elements is not empty. In the following result, we see that if $\mathbb{B}$ is closed under finite union, then Arrow's Condition is also equivalent to $C(.)$ being rational.

Theorem 2. Supposed $C(.)$ is defined on $\mathbb{B}$ and Assumption 1 is satisfied, then the following are equivalent:

(i) R is an ordering and $C(.)$ is normal;

(ii) $\overline{R}$ is an ordering and $C(B) = \{x \in B | x\overline{R}b \ \forall b \in B\}$;

(iii) $\overline{R} = \widetilde{R}$ and $C(B) = \{x \in B | x\overline{R}b \ \forall b \in B\}$;

(iv) $C(.)$ satisfies Axiom of Sequential Path Independence.

(v) $C(.)$ satisfies Arrow's Condition;

Remark 1. $(ii), (iii)$ are modified from Sen's (1971) formulations "$\overline{R}$ is an ordering, $C(.)$ is normal" and "$\overline{R} = \widetilde{R}$ and $C(.)$ is normal". But they are respectively equivalent if the domain of choice function consists of all finite subsets of X. To see this, $(ii), (iii)$ imply the corresponding formulations because they imply (i). And if $\mathbb{B}_1$ is all the singletons, then nomality would give $\overline{R} = R$ when considering choosing over the pairs. So they are respectively equivalent statements.

The above assumptions is enough to prove Sen's (1971) classic result, because it can be verified that choice functions defined on all pairs and triplets is a model of the theories in this paper. When every element $B_x \in \mathbb{B}_1$ is a singleton x, then obviously all of our assumptions are satisfied and all budgets of pairs and triplets are in $\mathbb{B}_3 \subseteq \mathbb{B}$, hence theorems 1 and 2 together give the desired equivalencies.

Sen′s Corollary : Suppose $\mathbb{B}$ consists of all finite budgets, then all the equivalent conditions in theorem 1 and 2 are equivalent.

Moreover, it can also be seen that this is not the only model for the theory. There are many different models, for the purpose of this paper, we would consider the following model in the context of consumer theory:

Example1 : Let $X = \{a, b, c, d\}$, $\mathbb{B}_1 = \{\{a, b, d\}, \{b, c, d\}, \{a, c, d\}, \{d\}\}$. Let $\mathbb{B} = \mathbb{B}_3$. So Assumptions 2, 4 are satisfied. If Assumptions 1 and 3 are also satisfied, then all the above equivalencies hold. For example, for any $C(.)$ such that $a \in C(\{a, b, d\}), b \in C(\{b, c, d\}), c \in C(\{a, c, d\}), \{d\} = C(\{d\})$, Assumption 1 would force one to choose from $\{a, b, c, d\} = X$. If one of a, b, c is chosen from X, then WARP would force this chosen one to be the chosen in every budget, and so R would be an ordering. If, on the other hand, d is chose, then WARP forces everything to be chosen at every budgets, so R is also an ordering.
Example2 : Fix a positive integer n, let $X = \mathbb{N}^n$ be the n-cartesian product of natural numbers. Fix the price vector $(p_1, p_2, \ldots, p_n) \in (\mathbb{R}^+)^n$. Let $\mathbb{B}_1$ consists of the budgets of the form $\{(q_1, q_2, \ldots, q_n) \,| q_i \in \mathbb{N}, \sum_{i=1}^n p_i q_i \leq I\}$ for any I such that $\min_i\{p_i\} \leq I$. I.e. $\mathbb{B}_1$ is the set of so called budget triangles. Let $\mathbb{B}$ be the smallest superset of $\mathbb{B}_1$ that is closed under finite union and intersection. Define the choice function $C(.)$ restricted on $\mathbb{B}_3$ to be for every $B \in \mathbb{B}_3$,

$$C(B) = \{(q_1, q_2, \ldots, q_n) \in B | \prod_{i=1}^n q_i \geq \prod_{i=1}^n q_i', \; \forall \left(q_1', q_2', \ldots, q_n'\right) \in B\}$$

It can be checked that all Assumptions 1 to 4 are satisfied. So theorems 1 and 2 ensure that the rationality conditions discussed are equivalent.

5 Discussion

We extend the equivalence results in Sen (1971) by allowing more flexibility in assumptions. In terms of the proofs given, some of them are similar to what there were in the literature because in some directions, no specific assumptions on the domain of choice functions are needed. However, in the other directions it is necessarily to check that the equivalencies still holds under the above settings. In particular, some proves are more technical then the original ones. It is as Arrow (1959) who first suggested that "the demand-function point of view would be greatly simplified if the range over which the choice functions are considered to be determined is brodened to include all finite sets".

However, there is a need to derive the theory based on some more observable assumptions. Resuming the discussion in Sen (1973) and Suzumura (1976), admittedly, no matter how general our theorems are, all the above conditions and axioms can never be verified, as there can be only finitely many data and situations are uncountably infinite. But this is a constraint of verifiability for all scientific theories, not limited to economics. And this is also the reason for theorizing, trying to give the simplest assumptions to explain and predict effectively in situations that are similar, but not exactly the same as those happened previously (Hempel, et al. 1948). Axioms and assumptions should also be abstraction of the physical world and be based on observable events and objects. Aligned with this perspective, it could be problematic if scope of our assumptions is not restricted to the more realizable circumstances. One of the examples was that the mathematical "axiom of choice" was intuitively true in considering only finite sets (so it is not called an axiom in this situation), but if it is believed to be true for arbitrary sets, many pathological objects arise (See Dudley 1989i). In addition, even if we do believe our conditions of choice functions to hold in the less realizable situations, it would be unsatisfying and problematic if the theorems *have to* be proven based on unrealistic assumptions that are made for the proof to work. In terms of the rationality conditions, our paper says they do not. In general, if it is possible to achieve the same theories basing on more realizable constructions, while the results are not invalidated by data, we can be more confident and certain about the implications of the theory.

Given that the theories can be tested in realizable situations, future research can be conducted on how the theories can be interpreted in heuristic documents, and whether the empirical data admits the consistency conditions.

6 Acknowledgements

I would like to thank Jose Rodrigues-Neto and Blair Alexander for their helps. I would not have finished this paper without the comments and suggestions from Jose. Also Blair has provided me with very useful comments during proofreading.

7 Appendix

7.1 Proof of Lemma 1

Proof. Suppose for contradiction, $x, y \notin C(B_x \cup B_y)$. Then by property of choice function, $\exists z \in C(B_x \cup B_y)$. Without loss of generality, suppose $z \in B_x$. Then by definition, we have $z\widetilde{P}x$. Since we also have xRz, this contradicts WARP. $\square$

7.2 Proof of Theorem 1

Proof. Theorem 1 will be proven in the following fashion ("$\overset{*}{\Rightarrow}$" indicates the proof requires Assumption 1 on $\mathbb{B}$):

$$(ii) \overset{*}{\Rightarrow} (iii) \Rightarrow (vi) \Rightarrow (ii)$$
$$(ii) \overset{*}{\Rightarrow} (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (ii)$$
$$(ii) \overset{*}{\Rightarrow} (iii) \Rightarrow (i) \Rightarrow (ii)$$

$(ii) \overset{*}{\Rightarrow} (iii)$: Suppose WARP holds, we want to show: 1,$C(.)$ is normal; 2, R is a complete binary relation; 3, R is transitive.

To show 1, on one hand, $x \in C(B) \Rightarrow xRy$ for every $y \in B$. Therefore $x \in \{x \in B | \forall y \in B, xRy\}$. On the other hand, $x \in \{x \in B | \forall y \in B, xRy\} \Rightarrow \neg y\widetilde{P}x$ for any $y \in B$ by WARP. If $x \notin C(B)$, then $z\widetilde{P}x$ for some $z \in C(B)$ leads to a contradiction. Hence $C(.)$ is normal.

To show 2, suppose $x, y \in X$, then there exists $B_x, B_y \in \mathbb{B}_1$ such that $x \in C(B_x), y \in C(B_y)$. By Lemma 1, we have x or y belonging to $C(B_x \cup B_y)$. So we have xRy or yRx.

To show 3, suppose xRy and yRz. By construction, we can consider the budgets $B_x, B_y, B_z \in \mathbb{B}_1$ such that

$$r \in C(B_r) \ \forall r \in \{x, y, z\}$$

Let $B = B_x \cup B_y \cup B_z \in \mathbb{B}_3 \subseteq \mathbb{B}$, lemma 1 implies either x, y or $z \in C(B)$. In the view of WARP, $z \in C(B) \Rightarrow y \in C(B)$ and $y \in C(B) \Rightarrow x \in C(B)$. So we must have xRz.

$(iii) \Rightarrow (vi)$: R is an ordering and $C(.)$ is normal, we want to show $R = \widetilde{R}$.

Suppose $x\widetilde{R}y$, so $\neg y\widetilde{P}x$. We know R is complete. So we have either yRx or xRy or both. So $\neg y\widetilde{P}x$ implies we must have xRy.

Suppose xRy, by the normality of $C(.)$, whenever $x, y \in B$ and $y \in C(B)$, we must have $x \in C(B)$ by transitivity of R. So it is impossible that $y\widetilde{P}x$. Hence we have $x\widetilde{R}y$.

$(vi) \Rightarrow (ii)$:

$xRy \Rightarrow x\widetilde{R}y \Rightarrow \neg y\widetilde{P}x$.

$(iii) \Rightarrow (iv)$: Assuming R is an ordering and $C(.)$ is normal, we want to show SCA.

Suppose we have $x_1Rx_2R\ldots Rx_n$, and for some $B \in \mathbb{B}$ we have $x_1, x_n \in B$ and $x_n \in C(B)$. R being an ordering implies x_1Rx_n. $x_n \in C(B)$ and $C(.)$ is normal implies $\forall z \in B,\ x_nRz$. Therefore x_1Rz by transitivity and $x_1 \in C(B)$ by normality.

$(iv) \Rightarrow (v)$: (trivial).

$(v) \Rightarrow (ii)$: want to show WCA $\Rightarrow$ WARP.

Suppose xRy, by WCA: $\forall B \in \mathbb{B}$, if $x, y \in B$ and $y \in C(B)$, then $x \in C(B)$. So it is impossible that $y\widetilde{P}x$. Hence WCA $\Rightarrow$ WARP.

$(iii) \Rightarrow (i)$: Assuming R is an ordering and $C(.)$ is normal, we want to show SARP.

Suppose $x_1\widetilde{P}x_2\ldots\widetilde{P}x_n$, by transitivity of R, we have x_1Rx_n. Now suppose for contradiction, x_nRx_1, then transitivity implies x_iRx_j for any $1 \leq i, j \leq n$. In particular, x_2Rx_1. Now, by $(iii) \Rightarrow (vi)$ we have $R = \widetilde{R}$. Hence $x_2\widetilde{R}x_1$ which contradicts $x_1 \widetilde{P} x_2$.

$(i) \Rightarrow (ii)$: (trivial).

The above completes the proof.

$\square$

7.3 Proof of Lemma 2

Trivial.

7.4 Proof of Theorem 2

When it is clear from the context, we will write B_x for the some budget in $\mathbb{B}_1$ where x is chosen. The result will be proven in the following sequence (The "$*$" on the arrow shows when Assumption 1 is used).

$$(i) \overset{*}{\Rightarrow} (iv) \overset{*}{\Rightarrow} (i)$$
$$(i) \Rightarrow (v) \overset{*}{\Rightarrow} (i)$$
$$(i) \overset{*}{\Rightarrow} (iii) \overset{*}{\Rightarrow} (ii) \overset{*}{\Rightarrow} (i)$$

Proof.
$(i) \overset{*}{\Rightarrow} (iv)$: this proof is adapeted from the one given by Bandyopadhyay.

Let R be an ordering and $C(.)$ be normal. Therefore $x \in C(B) \Rightarrow xRb\ \forall b \in B$. Choose an arbitrary $\omega \in \Omega(B)$ and that $\omega(i) = x$, then $\hat{\omega}(i) = B \cap \bigcup_{a \in \hat{\omega}(i-1)} C(B_a \cup B_x)$. By R being an ordering and $C(.)$ is normal, we have $x \in \hat{\omega}(i)$. Similarly, we see that $x \in \hat{\omega}(j)$ for every $j \geq i$. So $C(B) \subseteq \hat{\omega}(|B|)$.

Now suppose $x \notin C(B)$, then $\forall y \in C(B)$ transitivity and normality implies yPx. Moreover, normality ensures that $x \notin C(B)$ if and only if yPx. So it follows that $\omega(i) = y \Rightarrow x \notin \hat{\omega}(i)$. Similar reasoning gives $x \notin \hat{\omega}(j)\ \forall j \geq i$. Hence $x \notin \hat{\omega}(|B|)$ and $C(B) \supseteq \hat{\omega}(|B|)$.

$(iv) \overset{*}{\Rightarrow} (i)$:given that ASPI holds, try to show WCA, then by Theorem 1, the claim is proven.

In order to show WCA, suppose xRy, and for some $B \in \mathbb{B}$ there is $x \in B$, $y \in C(B)$, we will try to show $x \in C(B)$.

$xRy \Rightarrow \exists A \in \mathbb{B}$ such that $x \in C(A)$ and $y \in A$. But by ASPI, this means $\neg\,(\exists a \in A$ such that for some $B_a \in \mathbb{B}_1,\ x \notin C(B_a \cup B_x)\ \forall B_x \in \mathbb{B}_1)$. Because otherwise by letting $\omega(|A|) = a$, assumption 3 gives that $x \notin \hat{\omega}(|A|)$. In other words, $\forall a \in A, \forall B_a \in \mathbb{B}_1, \exists B_x \in \mathbb{B}_1$ such that $x \in C(B_x \cup B_a)$; in particular, $x \in C(B_y \cup B_x)$.

Similarly, by hypothesis $y \in C(B)$, if $b \in B$, then $\forall B_b \in \mathbb{B}_1, \exists B_y \in \mathbb{B}_1\ y \in C(B_b \cup B_y)$. So choose $\omega \in \Omega(B)$ such that $y = \omega(|B|-1), x = \omega(|B|)$. Then we have $y \in \hat{\omega}(|B|-1)$. By the previous paragraph, we can choose $B_y, B_x \in \mathbb{B}_1$ so that $x \in C(B_y \cup B_x)$ and hence $x \in \hat{\omega}(|B|) = C(B)$.

$(i) \Rightarrow (v)$: assuming R is an ordering and $C(.)$ is normal, we want to prove AC.

Suppose $A, B \in \mathbb{B}_3$, $A \subset B$ and $A \cap C(B) \neq \emptyset$. By normality and ordering, it is obvious that $C(A) \supset C(B) \cap A$.

Now suppose $y \in C(A)$ and $a \in A \cap C(B)$. $C(.)$ is normal implies $aRx\ \forall x \in B$ and yRa. Now transitivity implies $yRx\ \forall x \in B$. So normality gives $y \in C(B)$ and $C(A) \subset C(B) \cap A$.

$(v) \overset{*}{\Rightarrow} (i)$: Assuming Arrow's Condition, we want to show Weak Congruence Axiom, and by theorem 1 (where the "$*$" is used) we have (i).

Let xRy and $y \in C(B)$, $x \in B$ for some $B \in \mathbb{B}$, we want to show $x \in C(B)$.

Let $A \in \mathbb{B}$ such that $x \in C(A)$ and $y \in A$. Consider $C(A \cup B)$. If $C(A \cup B) \cap A = \emptyset$, then it is necessary that $C(A \cup B) \cap B \neq \emptyset$ and so $C(B) = C(A \cup B) \cap B$. But $y \in C(B)$, so $y \in C(A \cup B)$ and $y \in A$, which a contradiction. Hence we have to have $C(A \cup B) \cap A \neq \emptyset$. Then $x \in C(A) = C(A \cup B) \cap A$. Because $x \in B$, so $x \in C(B) = C(A \cup B) \cap B$.

$(i) \overset{*}{\Rightarrow} (iii)$: it suffices to show $\overline{R} = \widetilde{R} = R$.

$x\overline{R}y \Rightarrow xRy$. Since R is an ordering and $C(.)$ is normal, theorem 1 says WARP holds. Therefore $xRy \Rightarrow \neg y\widetilde{P}x$ and hence $x\widetilde{R}y$.

$x\widetilde{R}y \Rightarrow \neg y\widetilde{P}x$. Because R is an ordering and $C(.)$ is normal, we must have x *or* $y \in C(B_x \cup B_y)$. So $\neg y\widetilde{P}x \Rightarrow x\overline{R}y$. Therefore $\overline{R} = \widetilde{R}$.

$x\overline{R}y \Rightarrow xRy$ by definition. For the other direction, suppose xRy, normality and transitivity implies $x \in C(B_x \cup B_y)$. Hence $x\overline{R}y$.

$(iii) \overset{*}{\Rightarrow} (ii)$: it suffices to show that $\overline{R}$ is an ordering.

To show $\overline{R}$ is complete, suppose $x, y \notin C(B_x \cup B_y)$ for contradiction. Let $z \in C(B_x \cup B_y)$. Since $x \in C(B_x)$, we have $\forall a \in B_x$, $x\overline{R}a$ and hence $\neg a\widetilde{P}x$. But $z\widetilde{P}x$, so $z \notin B_x$, similarly, $z \notin B_y$, which is impossible. So $\overline{R}$ is complete. This argument also shows that for any $B_a, B_b, B_c \in \mathbb{B}_1$, either a, b or $c \in C(B_a \cup B_b \cup B_c)$.

For transitivity, suppose $x\overline{R}y$, $y\overline{R}z$. It follws that $x\widetilde{R}y$, $y\widetilde{R}z$ and $\neg y\widetilde{P}x$, $\neg z\widetilde{P}y$. It follows from the above argument that we must have $x \in C(B_x \cup B_y \cup B_z)$. Therefore $\forall t \in B_x \cup B_y \cup B_z$, $x\overline{R}t$. So we have $x\overline{R}z$ as desired.

$(ii) \overset{*}{\Rightarrow} (i)$: It suffices to show that $R = \overline{R}$.

$x\overline{R}y \Rightarrow xRy$ by definition. Suppose xRy, then $\exists B \in \mathbb{B}$ such that $y \in B$, $x \in C(B)$. So $x\overline{R}y$.

This completes the proof.

□

8 References

Arrow, Kenneth J. (1959), "Rational Choice Functions and Orderings", *Economica*, N.S., Vol. 26, pp. 121-127.

Bandyopadhyay, Taradas. (April, 1988), "Revealed Preference Theory, Ordering and the Axiom of Sequential Path Independence", in *The Review of Economic Studies*, Vol. 55, No. 2, pp. 343-351.

Dudley, Richard M. (1989), *Real Analysis and Probability*, Cambridge Studies in Advanced Mathematics, Vol. 74, Cambridge University Press.

Hempel, Carl G. and Oppenheim, Paul. (April, 1948), "Studies in the Logic of Explanation", *Philosophy of Science*, Vol. 15, No. 2, pp. 135-75.

Houthakker, Hendrik S. (1950), "Revealed Preference and the Utility Function", *Economica*, Vol. 17, pp. 159-174.

Richter, Marcel K. (July, 1966), "Revealed Preference Theory", *Econometrica*, Vol. 34, No. 3, pp. 635-645.

Samuelson, Paul A. (1938), "A Note on the Pure Theory of Consumer's Behaviour", *Economica*, N.S., Vol. 5, pp. 61-67.

Sen, Amartya K. (July, 1971), "Choice Functions and Revealed Preference", *The Review of Economic Studies*, Vol. 38, No. 3, pp. 307-317.

Sen, Amartya K. (August, 1973), "Behaviour and the Concept of Preference", *Econometrica*, New Series, Vol. 40, No. 159, pp. 241-259.

Sen, Amartya K. (July, 1973), "Maximization and the Act of Choice", *Econometrica*, Vol. 65, No. 4, pp. 745-779.

Suzumura, Kotaro. (Feburary, 1976), "Rational Choice and Revealed Preference", *The Review of Economic Studies*, Vol. 43, No. 1, pp.149-158.

Uzawa, Hirofumi. (1956), "A Note on Preference and Axiom of Choice", *Annals of Institute of Statistical Mathematics*, Vol. 8, pp. 35-40.