Contributions to oligopoly theory

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Abstract

In the context of the Cournot model a demand function parameter is treated as the dual of the firm's profit. Following the demand theory's duality approach it's possible to introduce the concept of the *compensated* reaction function (or *compensated* best-response function), as well as the concepts of *net* strategic complementarity/substitutability. The firm's reaction function is analysed into a type-1 and a type-2 effect, which are the counterparts, respectively, of the demand theory substitution and income effects. Further, new results are obtained regarding Cournot equilibrium in the case of profit functions which are homogeneous in their arguments.

JEL classification: D43; L13

Keywords: Cournot model; *compensated* reaction function; *net* strategic complementarity/substitutability

1.1. Introduction

Consider a Cournot duopoly of firms 1, 2, where the profit function is postulated as follows:

$\pi^i = p(a, q^i, q^j)\cdot q^i - c^i(q^i)$, $i = 1, 2$, $i \neq j$

where $p(a, q^i, q^j)$ is the inverse market demand function of the levels of output q^i, q^j of firms i, j and $c^i(q^i)$ is the cost function of firm i. A change in firm j's level of output, q^j, has an effect on the profit of firm i, π^i, which can be offset by a compensating change in the value of the inverse market demand parameter 'a'. This implies that the profit of firm i remains unchanged whereas the value of 'a' varies with q^j. Solving the profit function above for 'a' shows that the latter is a function of q^i, q^j and π^i, ($a = a(q^i, q^j, \pi^i)$). For any output profile (q^i, q^j) the function $a(q^i, q^j, \pi^i)$ determines the value of the parameter 'a' which is necessary for firm i to obtain a profit π^i. In the general case the value of $a(q^i, q^j, \pi^i)$ is different than the

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actual value of ‘a’, thus indicating the need for a positive or negative compensation.

We pose the question: Given the rival’s level of output q^j which level of output q^i allows firm i to earn a given profit π^i under the minimum value of the parameter ‘a’? In other words:

$$\min_{\{q^i\}} a(q^i, q^j, \pi^i)$$

The solution to this problem is given by the *compensated* reaction function

$$\tilde{q}^i = h^i(q^j, \pi^i)$$

On the other hand, firm i’s profit–maximisation problem can be posed as follows:

$$\max_{\{q^i\}} (p(a, q^i, q^j)\cdot q^i - c^i(q^i))$$

yielding i’s reaction function

$$q^{i*} = b^i(q^j, a)$$

and the maximum level of profits

$$\pi^{i*} = p(a, q^{i*}, q^j)\cdot q^{i*} - c^i(q^{i*})$$

A change in the rival’s level of output to $q^{j\prime}$ causes firm i to react, adjusting its own profit-maximising level of output to q^{i**} with corresponding profit of π^{i**}. The reaction $q^{i**} - q^{i*}$ can be analysed to two constituent effects:

1) The value of ‘a’ can be compensated so that i’s profit is restored to π^{i*} with a level of output determined by the compensated reaction function:

$$\tilde{q}^i = h^i(q^{j\prime}, \pi^{i*})$$

The difference $(\tilde{q}^i - q^{i*})$ is called the ‘type-1 effect’.

2) Withdrawing the compensation causes a further change in i’s level of output equal to:

$$q^{i**} - \tilde{q}^i$$

This is called ‘type-2 effect’.

The central idea is to use established demand theory methods for the analysis of firm behaviour in a Cournot setting. This approach is based on a recognition of the fact that duality finds application on the Cournot model. An exploitation of this relation allows us to gain insight into the structure of firm interaction.

Whereas demand theory treats the problem of the consumer from the viewpoint of both utility maximisation and expenditure minimisation, the problem of the firm is treated only from the viewpoint of profit maximisation. Oligopoly theory studies the questions of existence, uniqueness and stability of the Cournot equilibrium, as well as its comparative statics properties. In particular, Novshek [1985] furnishes an existence theorem and Kolstad and Mathiesen [1987] provide necessary and sufficient conditions for uniqueness of equilibrium. Hahn [1962] and Seade [1980] study the question of stability. Dixit [1986] studies comparative statics in a conjectural variations setting, whereas Bulow et al. [1985] introduced the concept of strategic complementarity/substitutability to study comparative statics in a multimarket setting. The present study extends the latter line of analysis by introducing the concepts of net strategic complementarity/substitutability.

Concise presentations of Cournot theory are made by Martin [2002] and Mahzabeen [1993], among others. A classic presentation of the Hicks decomposition is made by Hicks [1939].

The basic framework of analysis is presented in section 1.2. The type-1 effect is analysed and the concepts of net strategic complementarity/substitutability are presented in section 1.3. The type-2 effect is analysed in section 1.4. In section 1.5. we deal with the question of choosing which parameter to compensate. The Cournot equilibrium is defined in terms of compensated reaction functions in section 1.6. Comparative statics are studied in 1.7. In section 1.8. we deal with the analysis of price-setting duopoly with differentiated products. The analysis is extended to two-stage models in section 1.9. In the second part of the study homogeneous profit functions are introduced in section 2.1. and basic results are obtained in 2.2. The Cournot equilibrium is determined in 2.3. and the two-stage Cournot equilibrium with homogeneous profit functions is determined in 2.4. Finally, conclusions are presented in 2.5.

1.2. Duality approach in the Cournot oligopoly model.

Consider two firms i = 1, 2, with profit function

$$\pi^i = p(q^i + q^j, a_0)\cdot q^i - c^i\cdot q^i \ , \ i \neq j, \qquad (1.1)$$

where $p(Q, a_0)$ stands for the inverse market demand function in industry output Q. The firms compete in output q^i, taking the decision of the rival as given when making their own decision about output level. Decisions are taken simultaneously.

For convenience we represent the profit function of firm 1 in generic form as

$$\pi^1 = f(q^1, q^2, a_0, c^1)$$

Where a_0 is considered to be the actual value of an inverse market demand parameter a. The profit function can then be written as

$$\pi^1 = f(q^1, q^2, a, c^1)\ , \ \text{where } a = a_0. \qquad (1.2)$$

The following assumptions are imposed on the profit function:

I) Twice continuous differentiability in q^1, q^2, a, c^1.

II) Strict concavity w.r.t. q^1, i.e. the second derivative $f_{11}<0$.

III) Maximisation w.r.t. q^1 yields interior solutions.

IV) The partial derivative w.r.t. ‘a’ is strictly positive, $f_a > 0$.

V) There are two possibilities about the partial derivative w.r.t. q^2 , f_2.

a) $f_2<0$, for every vector (q^1, q^2, a, c^1).

b) A saddle point (q^{1*}, q^{2*}) exists, $f_2(q^{1*}, q^{2*}, a, c^1) = 0$.

VI) The inverse of the profit function

$$a = f^{-1}(q^1, q^2, \pi^1, c^1) \qquad (1.3)$$

is a well-defined function.

For a given q^2, firm 1 chooses a level of output q^{1*} which maximises its profit. q^{1*} thus satisfies the first order condition

$$\partial\pi^1 / \partial q^1 = f_1(q^{1*}, q^2, a, c^1) = 0, \qquad (1.4)$$

Equation (1.4) determines 1's reaction (best-response) function

$$q^{1*} = b^1(q^2, a, c^1) \qquad (1.5)$$

Substituting the reaction function from (1.5) into (1.2) yields the indirect profit function $F(q^2, a, c^1)$ which expresses 1's maximum profits in terms of the rival's output q^2 and of the demand and cost parameters a, c^1.

$$\pi^{1*} = f(b^1(q^2, a, c^1), q^2, a, c^1) = F^1(q^2, a, c^1) \qquad (1.6)$$

By the envelop theorem, assumptions (V), (IV) and eq.(1), π^{1*} is decreasing (nonincreasing) in q^2, increasing in 'a' and decreasing in c^1.

Substituting the inverse of the profit function (1.3) into (1.2), leads to the identity

$$\pi^1 \equiv f(q^1, q^2, f^{-1}(q^1, q^2, \pi^1, c^1), c^1) \qquad (1.7)$$

The following Lemmas will be used to prove the main results.

Lemma.

Under assumptions (I)-(VI) it holds that:

1. $d\pi^1/dq^1 \equiv f_1(q^1, q^2, a, c^1) + f_a(q^1, q^2, a, c^1) \cdot f_1^{-1}(q^1, q^2, \pi^1, c^1) \equiv 0$ (1.8)

where $f_1(q^1, q^2, a, c^1)$ is the partial derivative of $f(q^1, q^2, a, c^1)$ w.r.t. q^1 and $f_1^{-1}(q^1, q^2, \pi^1, c^1)$ is the partial derivative of $f^{-1}(q^1, q^2, \pi^1, c^1)$ w.r.t. q^1.

2. $d^2\pi^1/d(q^1)^2 \equiv f_{11}(q^1, q^2, a, c^1) + f_{1a}(q^1, q^2, a, c^1) \cdot f_1^{-1}(q^1, q^2, \pi^1, c^1) + (f_{a1}(q^1, q^2, a, c^1) + f_{aa}(q^1, q^2, a, c^1) \cdot f_1^{-1}(q^1, q^2, \pi^1, c^1)) \cdot f_1^{-1}(q^1, q^2, \pi^1, c^1) + f_a(q^1, q^2, a, c^1) \cdot f_{11}^{-1}(q^1, q^2, \pi^1, c^1) \equiv 0,$ (1.9)

3. $d^2\pi^1/(dq^1 \cdot dq^2) \equiv f_{12}(q^1, q^2, a, c^1) + f_{1a}(q^1, q^2, a, c^1) \cdot f_2^{-1}(q^1, q^2, \pi^1, c^1) +$

$(f_{a2}(q^1, q^2, a, c^1) + f_{aa}(q^1, q^2, a, c^1) \cdot f_2^{-1}(q^1, q^2, \pi^1, c^1)) \cdot f_1^{-1}(q^1, q^2, \pi^1, c^1) +$

$f_a(q^1, q^2, a, c^1) \cdot f_{12}^{-1}(q^1, q^2, \pi^1, c^1) \equiv 0,$ (1.10)

where $f_2^{-1}(q^1, q^2, \pi^1, c^1)$ is the partial derivative of $f^{-1}(q^1, q^2, \pi^1, c^1)$ w.r.t. q^2.

4. $d\pi^1/dq^2 \equiv f_2(q^1, q^2, a, c^1) + f_a(q^1, q^2, a, c^1) \cdot f_2^{-1}(q^1, q^2, \pi^1, c^1) \equiv 0$ (1.11)

where $f_2(q^1, q^2, a, c^1)$ is the partial derivative of $f(q^1, q^2, a, c^1)$ w.r.t. q^2.

5. $1 \equiv f_a(q^1, q^2, a, c^1) \cdot f_\pi^{-1}(q^1, q^2, \pi^1, c^1)$ (1.12)

6. $(f_{a1}(q^1, q^2, a, c^1) + f_{aa}(q^1, q^2, a, c^1) \cdot f_1^{-1}(q^1, q^2, \pi^1, c^1)) \cdot f_\pi^{-1}(q^1, q^2, \pi^1, c^1) +$

$f_a(q^1, q^2, a, c^1) \cdot f_{\pi 1}^{-1}(q^1, q^2, \pi^1, c^1) \equiv 0,$ (1.13)

7. $d\pi^1/dc^1 \equiv f_c(q^1, q^2, a, c^1) + f_a(q^1, q^2, a, c^1) \cdot f_c^{-1}(q^1, q^2, \pi^1, c^1) \equiv 0$ (1.14)

Proof.

See Appendix A.

We now turn to posing the dual problem:

$$\min_{\{q^1\}} \; f^{-1}(q^1, q^2, \pi^1, c^1) \qquad (1.15)$$

s.t. $\pi^1 = \pi^1_0$, where π^1_0 is a given level of profit.

The first order condition for this problem is

$$f_1^{-1}(q^1, q^2, \pi^1_0, c^1) = 0 \qquad (1.16)$$

By equation (1.8) and assumption (IV), this condition is satisfied when the first order condition of the corresponding maximisation problem is. Notice also that, by equation (1.9) and assumption (IV), the second order condition of this problem is satisfied when the corresponding condition of the maximisation problem is.

Equation (1.16) determines firm 1's compensated reaction (compensated best-response) function

$$\tilde{q}^1 = h^1(q^2, \pi^1_0, c^1) \qquad (1.17)$$

Substituting (1.17) into the objective function (1.15) determines the minimum value of ‘a’ which is needed for firm 1 to realise a profit of $\pi^1{}_0$. That is,

$$\tilde{a} = f^{-1}(h^1(q^2, \pi^1{}_0, c^1), q^2, \pi^1{}_0, c^1) = A(q^2, \pi^1{}_0, c^1) \qquad (1.18)$$

By the envelop theorem, assumptions (IV) and (V.a) ($f_a > 0$, $f_2 < 0$) and equations (1.11), (1.12), (1.1) and (1.14) the function $A(q^2, \pi^1{}_0, c^1)$ is increasing in q^2, $\pi^1{}_0$ and c^1.

The compensated reaction function determines the level of output which allows firm 1 to realise a given profit under the minimum value of the inverse market demand parameter ‘a’.

The relation of the solution of the maximisation problem to the solution of the minimisation problem is hereby presented (suppressing the term c^1, which we take to be a constant).

The profit-maximisation problem

$$\max_{\{q^1\}} f(q^1, q^2, a) \qquad (A)$$

has the solution $q^{1*} = b^1(q^2, a)$. Moreover, $\pi^{1*} = f(q^{1*}, q^2, a)$

The problem of minimising the value of the parameter ‘a’ is

$$\min_{\{q^1\}} f^{-1}(q^1, q^2, \pi^{1*}) \qquad (B)$$

is solved by $\tilde{q}^1 = h^1(q^2, \pi^{1*})$.

The following proposition states that the two solutions coincide.

Proposition.

Assumptions:

1) $f(q^1, q^2, a)$ is twice continuously differentiable in q^1 and continuously differentiable in ‘a’.

2) $f^{-1}(q^1, q^2, \pi^{1*})$ is a well-defined function.

3) Solutions to both maximisation and minimisation problems exist.

4) $f_a > 0$, for every (q^1, q^2, a).

5) $f_{11} < 0$, for every (q^1, q^2, a).

A. Profit maximisation implies 'a'- minimisation.

Suppose that the above assumptions hold. Then q^{1*} solves problem (B).

Proof.

Suppose not and let $q^{1\prime}$ solve (B). Hence

$a' = f^{-1}(q^{1\prime}, q^2, \pi^{1*}) < f^{-1}(q^{1*}, q^2, \pi^{1*}) = a$

thus $f(q^{1\prime}, q^2, a') = f(q^{1*}, q^2, a) = \pi^{1*}$ (i)

However,

$f(q^{1*}, q^2, a) > f(q^{1\prime}, q^2, a) > f(q^{1\prime}, q^2, a')$ (ii)

where the first inequality holds because q^{1*} solves problem (A) and the second is due to the assumption $f_a > 0$ and $a' < a$. (ii) contradicts (i).■

B. 'a'-minimisation implies profit maximisation.

Suppose that the above assumptions hold. Then $\tilde{q}^1$ solves problem (A).

Proof.

Suppose not, and let $q^{1\prime}$ solve (A):

$\pi^{1\prime} = f(q^{1\prime}, q^2, a) > f(\tilde{q}^1, q^2, a) = \pi^{1*}$, thus

$f^{-1}(q^{1\prime}, q^2, \pi^{1\prime}) = f^{-1}(\tilde{q}^1, q^2, \pi^{1*}) = a,$ (i)

However,

$f^{-1}(\tilde{q}^1, q^2, \pi^{1*}) < f^{-1}(q^{1\prime}, q^2, \pi^{1*}) < f^{-1}(q^{1\prime}, q^2, \pi^{1\prime})$ (ii)

where the first inequality results from the property of $\tilde{q}^1$ as solution of (B) and the second from eq. (1.12) and $f_a > 0$, resulting in $f_\pi^{-1} > 0$, and considering that $\pi^{1\prime} > \pi^{1*}$. (ii) contradicts (i). ■

Thus an optimally chosen quantity can be expressed either as the solution of the profit maximisation problem or as the solution of the 'a'-minimisation problem. The answer to these two problems is the same q^1. This observation leads to the following identities:

1) $A(q^2, F(q^2, a_0)) \equiv a_0$ (1.19)

2) $F(q^2, A(q^2, \pi^1)) \equiv \pi^1$ (1.20)

3) $b^1(q^2, a_0) \equiv h^1(q^2, F(q^2, a_0))$ (1.19′)

4) $h^1(q^2, \pi^1) \equiv b^1(q^2, A(q^2, \pi^1))$ (1.20′)

As in the general case $A(q^2, \pi^1) \neq a_0$, the actual value a_0 is positively or negatively compensated in order to allow the firm to obtain a given level of profits at the minimal value of 'a'. Accordingly,

$M(q^2, \pi^1, a_0) \equiv A(q^2, \pi^1) - a_0$ (1.21)

is called the compensation function.

An example is presented to clarify the relevant concepts.

Example.

Assume that the market demand function is:

$D(a, p) = a/p$,

and the cost function of duopolist 1 is:

$c(q^1) = c \cdot q^1$

The profit function of firm 1, then, is:

$\pi^1 = (a \cdot q^1)/(q^1 + q^2) - c \cdot q^1$, (i)

Maximising the profit function w.r.t. q^1 we get firm 1′ s reaction function:

$q^{1*} = [(a \cdot q^2)/c]^{1/2} - q^2$, (ii)

Turning now to the dual problem, we solve the profit function (i) for 'a':

$a = (\pi^1 + c\cdot q^1)\cdot(q^1+q^2)/q^1,$ (iii)

Minimising 'a' w.r.t. q^1 we get firm 1's compensated reaction function:

$\tilde{q}^1 = [(\pi^1\cdot q^2)/c]^{1/2},$ (iv)

Substituting (iv) in (iii) we get the minimum value of the demand parameter 'a' which allows firm 1 to obtain profits π^1 :

$\tilde{a} = [(\pi^1)^{1/2} + (c\cdot q^2)^{1/2}]^2$

Theorem 1.1.

Consider firm 1 as a Cournot duopolist with profit function (1.2), subject to assumptions (I) – (VI).

The derivative of the compensated reaction function (type-1 effect) is determined by the following equation:

$\partial\tilde{q}^1/\partial q^2 = - f_{12}/ f_{11}+(f_{1a}/ f_{11})\cdot(f_2/f_a)$ (1.22)

where the derivatives are evaluated at $(\tilde{q}^1 , q^2, \tilde{a} , c^1) = (q^{1*}, q^2, a_0, c^1)$.

Proof.

See Appendix B.

The above equation expresses the type-1 effect as the sum of the firm's reaction and the type-2 effect. The reaction $(\partial q^{1*}/\partial q^2)$, as expressed by the term $(-f_{12}/ f_{11})$, involves a change in profit. Consider the compensation $(\partial\tilde{a} /\partial q^2=(-f_2/f_a)\cdot dq^2)$ which is needed to restore firm 1's profit to its initial level. The compensation causes firm 1's profit-maximising level of output to change by $(-f_{1a}/ f_{11}\cdot)$ $(-f_2/f_a)$, thus arriving at (1.22).

Eq.(1.22) can be re-written as

$\partial\tilde{q}^1/\partial q^2 = (f_{1a}\cdot f_2 - f_{12}\cdot f_a)/(f_{11}\cdot f_a)$ (1.23)

Taking into account that $(-f_{12}/ f_{11}) = \partial q^{1*}/\partial q^2$ (see Appendix B), equation (1.22) can also be written as

$$\partial q^{1*}/\partial q^2 = \partial \tilde{q}^1/\partial q^2 - (f_{1a}/ f_{11})\cdot(f_2/f_a) \tag{1.24}$$

Equation (1.24) analyses a firm's reaction according to the Hicks decomposition and forms the counterpart of the Slutsky equation.

Equation (1.22) is derived from the equation

$$\partial \tilde{q}^1/\partial q^2 = \partial q^{1*}/\partial q^2 + (\partial q^{1*}/\partial a)\cdot(\partial \tilde{a}/\partial q^2)$$

(see Appendix B)

The LHS represents the type-1 effect, the first term of the RHS is the firm's reaction and the second term is the type-2 effect.

Corollary

If the type-2 effect has a larger algebraic value than the type-1 effect, the reaction curve will be downward slopping. If the type-2 effect has a smaller algebraic value than the type-1 effect, the reaction curve will be upward slopping.

Proof.

By reference to the above equation. ■

The RHS terms of the above equation can be observed or estimated. It's, thus, possible to arrive at an estimate of the compensated reaction term.

We can use the Hicks decomposition to analyse the firm's reaction to a change in its cost structure. Even though there is no strategic interaction involved, type-1 and type-2 effects can still be defined and analysed according to a Slutsky equation, as above. In particular, differentiating totally the equation

$$q^{1*} = b^1(q^2, A(q^2, \pi^{1*}, c^1), c^1) \equiv h^1(q^2, \pi^{1*}, c^1) = \tilde{q}^1 \tag{1.25}$$

w.r.t. c^1 we get

$$\partial q^{1*}/\partial c^1 + (\partial q^{1*}/\partial a)\cdot(\partial \tilde{a}/\partial c^1) = \partial \tilde{q}^1/\partial c^1 \tag{1.26}$$

Totally differentiating (1.4) w.r.t. c^1 while taking (1.5) into account yields

$f_{11}(q^{1*}, q^2, a, c^1) \cdot \partial q^{1*}/\partial c^1 + f_{1c}(q^{1*}, q^2, a, c^1) = 0$, thus

$\partial q^{1*}/\partial c^1 = - f_{1c}(q^{1*}, q^2, a, c^1)/ f_{11}(q^{1*}, q^2, a, c^1)$ (1.27)

Taking into account equations (1.14), (1.26), (1.27) and (iv) from Appendix B, we get

$\partial \tilde{q}^1/\partial c^1 = - f_{1c} / f_{11} + (f_{1a}/ f_{11}) \cdot (f_c/f_a)$ (1.28)

where the RHS derivatives are evaluated at (q^{1*}, q^2, a, c^1).

It's thus pointed out that it is possible to use the Hicks decomposition in order to analyse shifts of the reaction curve which are due to changes in parameters like the per unit costs.

To acquire a better understanding of the analysis we turn to the study of each one of the two effects.

1.3. Type-1 effect.

It follows from equation (1.23) that the type-1 effect is equal to zero iff the following condition is met:

$$f_2/f_a = f_{12}/f_{1a} \quad (1.29)$$

where the derivatives are evaluated at (q^{1*}, q^2, a, c^1). The compensation (da) which restores 1's profit to its initial level also restores the optimality of q^{1*}. In particular, the type-1 effect is equal to zero iff the following system of equations holds:

$f_2 \cdot dq^2 + f_a \cdot da = 0$,

so that 1's profit does not change, and

$f_{12} \cdot dq^2 + f_{1a} \cdot da = 0$,

so that q^{1*} remains optimal. Under eq. (1.29) both equations hold, ensuring that the type-1 effect is equal to zero.

Theorem 1.2.

A class of profit functions which satisfy condition (1.29) is

$f(q^1, q^2, a) = g(q^1) \cdot s(r(a, q^2), q^1)$ (1.30) where the function $f(q^1, q^2, a)$ is twice continuously differentiable in all the arguments and the function $r(a, q^2)$ is such that

$r_2(a, q^2) < 0$ and, $r_a(a, q^2) > 0$, for every (a, q^2).

Proof.

Checking whether the class of profit functions described by (1.30) satisfies condition (1.29):

$f_2(q^1, q^2, a) = g(q^1)\cdot s_1(r(a, q^2), q^1)\cdot r_2(a, q^2)$

$f_a(q^1, q^2, a) = g(q^1)\cdot s_1(r(a, q^2), q^1)\cdot r_a(a, q^2)$, thus,

$f_2 = (r_2/r_a)\cdot f_a$.

Considering that the $r(a, q^2)$ function is independent of q^1, differentiating the above equation w.r.t. q^1, yields

$f_{21} = (r_2/r_a)\cdot f_{a1}$

thus, condition (1.29) is satisfied by the profit function (1.30).■

Corollary

The class of profit functions described by (1.30) has type-1 effect equal to zero.

An important example is the following:

A duopolist facing linear inverse market demand and constant per unit costs has a profit function which satisfies eq.(1.30).:

Let the inverse market demand be: $P = a - b\cdot Q$,

where $Q = q^1 + q^2$. Firm 1's profit function is:

$$\pi^1 = (a - b\cdot(q^1 + q^2))\cdot q^1 - c\cdot q^1 = q^1\cdot[(a - b\cdot q^2) - b\cdot q^1 - c] \quad \text{(A.1)}$$

which can be seen to have the structure of eq. (1.30).

Maximising π^1 w.r.t. q^1 yields the reaction function

$$q^{1*} = (a-c)/(2\cdot b) - q^2/2 \quad \text{(A.2)}$$

Solving equation (A.1) for 'a', we get:

$$a = \pi^1/q^1 + c + b\cdot(q^1 + q^2) \tag{A.3}$$

Minimising 'a' w.r.t. q^1 we get 1's compensated reaction function $h^1(\pi^1)$

$$\tilde{q}^1 = (\pi^1/b)^{1/2} \tag{A.4}$$

$\tilde{q}^1$ does not depend on q^2, consequently the type-1 effect is zero.

1.3.1. The type-1 effect and the market demand function.

It should be pointed out that the non-existence of the type-1 effect is due to the fact that the parameter 'a' does not affect the slope dQ/dP of the market demand curve. In other words, a market demand function D(P, a) will not give rise to a type-1 effect if the slope $dQ/dP = D_p(P, a) = v(P)$ is independent of the parameter 'a' for any price P. In such a case, the compensation of the parameter 'a' causes a parallel shift of the market demand curve at any price P, thus exactly off-setting the effect of dq^2 on 1's demand curve, i.e. $D_a(P, a)\cdot da - dq^2 = 0$, for any P. The compensated demand curve of firm 1 coincides with its initial demand curve. Firm 1, facing the same demand conditions as before the change in q^2, chooses the same level of output as initially. Thus the type-1 effect is zero.

This insight leads to the following generalisation:

Consider Cournot duopolists 1, 2, facing market demand D(P, a), which satisfies tha following assumptions: 1) D(P, a) has a well-defined inverse demand function P(Q, a), where Q is industry output. 2) Both D(P, a) and P(Q, a) are assumed to be twice continuously differentiable in their arguments. 3) $D_p<0$, $D_a>0$ for any (P, a), where D_p and D_a are the derivatives of the market demand function w.r.t. price P and the parameter 'a', respectively. 4) The cost function of firm 1, $c(q^1)$, is assumed to be twice continuously differentiable and satisfying the marginal cost non-negativity condition. Consequently, firm1's profit function is: $f(q^1, q^2, a) = P(q^1+q^2, a)\cdot q^1 - c(q^1)$. 5) The profit function is assumed to be strictly concave in q^1 and to always admit interior solutions of the maximization problem.

Theorem 1.3.

Under the above assumptions, the type-1 effect is

$\partial\tilde{q}^1/\partial q^2 = -[D_{pa}(P(Q, a), a)\cdot(q^1)^2]/[(D_p(P(Q, a), a))^3\cdot f_{11}(q^1, q^2, a)\cdot f_a(q^1, q^2, a)]$, (1.31)

where D_{pa} is the market demand function's second derivative w.r.t. P and 'a'. The functions of eq. (1.31) are evaluated at $(\tilde{q}^1, q^2, \tilde{a}) = (q^{1*}, q^2, a_0)$, $Q = \tilde{q}^1 + q^2 =$

$q^{1*} + q^2$.

Proof.

See Appendix C.

Corollary 1.

If parameter 'a' does not affect the slope of the market demand ($D_{pa} = 0$), the type-1 effect is equal to zero.

Corollary 2.

If $D_{pa} < 0$, $\partial\tilde{q}^1/\partial q^2 > 0$ i.e. the compensated reaction (best-response) curve is upward sloping. If $D_{pa} > 0$, $\partial\tilde{q}^1/\partial q^2 < 0$, the compensated reaction (best-response) curve is downward sloping.

Proof.

Since $D_p < 0$, $f_a = P_a \cdot q^1 > 0$ and $f_{11} < 0$, according to the above assumptions, it turns out that

$-[(\tilde{q}^1)^2]/[(D_p)^3\cdot f_{11}\cdot f_a] < 0$. ■

1.3.2. Net strategic complementarity/substitutability

In this section we study the determinants of the slope of the compensated reaction curve and the relation they bear to the determinants of the corresponding reaction curve's slope. In the course of studying these relations the concepts of net strategic complementarity/substitutability are introduced.

In particular, substituting the expression of eq.(1.17) into eq.(1.16) we get:

$f_1^{-1}(h^1(q^2, \pi^1_0, c^1), q^2, \pi^1_0, c^1) \equiv 0$ (1.32)

Totally differentiating throughout w.r.t. q^2, we get:

$f_{11}^{-1} \cdot h_2^1 + f_{12}^{-1} \equiv 0$, or $f_{11}^{-1} \cdot (\partial\tilde{q}^1/\partial q^2) + f_{12}^{-1} \equiv 0$, thus

$$\partial\tilde{q}^1/\partial q^2 = -\, f_{12}^{-1}/\, f_{11}^{-1} \qquad (1.33)$$

The above equation determines the slope of the compensated reaction curve and is the counterpart of the equation for the slope of the reaction curve ($\partial q^{1*}/\partial q^2 = -\, f_{12}/f_{11}$). In order to express $\partial\tilde{q}^1/\partial q^2$ in terms of the derivatives of the profit function we start with the relation of f_{12} to f_{12}^{-1}. Considering eq. (1.10) and eq. (1.16), we get a relation of f_{12} to f_{12}^{-1} :

$f_{12}(q^{1*}, q^2, a, c^1) + f_{1a}(q^{1*}, q^2, a, c^1)\cdot f_2^{-1}(\tilde{q}^1, q^2, \pi^1, c^1) + f_a(q^{1*}, q^2, a, c^1)\cdot f_{12}^{-1}(\tilde{q}^1, q^2, \pi^1, c^1) \equiv 0$,

thus $f_{12}^{-1} = -\,(f_{12} + f_{1a}\cdot f_2^{-1})\,/\,f_a$

f_{12} represents the effect of a change in q^2 on f_1. However, the change in q^2 causes a compensation ($f_2^{-1}\cdot dq^2$) to be contributed, in order for the firm's profits to be preserved. The compensation affects f_1 through f_{1a}, thus resulting in the numerator ($f_{12} + f_{1a}\cdot f_2^{-1}$) of the expression above, which represents the compensated value of f_{12}.

Taking eq.(1.11) into account, the above expression becomes

$$f_{12}^{-1} = -(f_{12} - f_{1a}\cdot (f_2/\, f_a))\,/\,f_a \qquad (1.34)$$

Turning to the relation of f_{11}^{-1} to f_{11}:

From eq.(1.9) and the condition (1.16), we get:

$$f_{11}^{-1} = -f_{11}/f_a \qquad (1.35)$$

evaluated at $(\tilde{q}^1, q^2, a, c^1)$.

Considering eqs. (1.33), (1.34), (1.35) we get:

$\partial\tilde{q}^1/\partial q^2 = -f_{12}^{-1}/\, f_{11}^{-1} = (f_{12}^{-1}\cdot f_a)\,/\,f_{11} = (f_{1a}\cdot(f_2/f_a) - f_{12})/f_{11} =$

$=(f_{1a}/f_{11})\cdot(f_2/f_a) - f_{12}/f_{11}$

which is the equivalent of the Slutsky equation, as expressed in eq.(1.22).

Solving the above equation for f_{12}^{-1} we get:

$f_{12}^{-1} = [(f_{1a}/f_{11})\cdot(f_2/f_a) - f_{12}/f_{11}]\cdot(f_{11}/f_a)$ (1.36)

which shows that the basic difference of f_{12} w.r.t. f_{12}^{-1} is due to the type-2 effect.

Definition.

The product of firm 1 is a *net* strategic substitute to the product of firm 2, iff an increase in q^2 increases $f_1^{-1}(q^1, q^2)$. In other words, the firm 1 product is a net strategic substitute to firm 2 product iff $f_{12}^{-1}(q^1, q^2) > 0$. Conversely, the firm 1 product is a *net* strategic complement to firm 2 product, iff $f_{12}^{-1}(q^1, q^2) < 0$.

Eq.(1.36) shows that it is possible for the two products to be simultaneously both strategic substitutes and net strategic complements and vice-versa, depending on the algebraic value of $f_{1a.}$ If $f_{1a} = 0$ (in which case the type-2 effect is eliminated) the same strategic complementarity/substitutability relationship is preserved in both the primal and the dual form.

According to eq. (1.33), if the product of firm 1 is a *net* strategic substitute to the firm 2 product, firm 1's compensated reaction curve is downward sloping, and if it is a net strategic complement to the firm 2 product the compensated reaction curve is upward sloping. In other words, in the case of net strategic substitutability, a more aggressive strategy by firm 2 makes it necessary for firm 1 to follow a less aggressive strategy in order to preserve a given level of profits under the minimum value of 'a'. Conversely, 1's compensated reaction to a more aggressive strategy by 2 is a more aggressive strategy of its own, in case of net strategic complementarity.

1.4. Type-2 effect.

If the type-2 effect is equal to zero throughout, the compensated reaction curve coincides with the reaction curve in the (q^1, q^2) space. According to equation (1.24)

$\partial q^{1*}/\partial q^2 = \partial \tilde{q}^1/\partial q^2 - (f_{1a}/ f_{11})\cdot(f_2/f_a)$

The type-2 effect is eliminated either if f_2=0, in which case the rival's actions do not affect 1's payoff, or in case f_{1a}=0.

In the case where $f_2=0$, there is no compensation needed to restore 1's profit to its initial level ($d\tilde{a} = -(f_2/f_a)\cdot dq^2 = 0$) and, thus, there is no type-2 effect:

$(f_{1a}/ f_{11})\cdot(f_2/f_a) = 0$

Player 1 nevertheless reacts to his rival's action, because the type-1 effect operates.

Theorem 1.4.

If the class of profit functions

$$\pi^1 = f(q^1, q^2, a) = g(t(q^1, q^2), a) \tag{1.37}$$

which can be re-written as

$$\pi^1 = g(w, a), \quad w = t(q^1, q^2) \tag{1.38}$$

satisfies the following assumptions:

1) $\partial t(q^1, q^2)/\partial q^1 = t_1(q^1, q^2) \neq 0$, for every (q^1, q^2)

2) $g_a(w, a) > 0$, for every (w, a)

3) $g_{ww}(w, a) < 0$, for every (w, a)

4) The inverse of the profit function, $a = g^{-1}(w, \pi^1)$, is a function,

then it exhibits the property $f_2(q^{1*}, q^2, a) = 0$.

Proof.

In maximising 1's profit, we have:

$\partial\pi^1/\partial q^1 = f_1(q^{1*}, q^2, a) = g_w(w^*, a) \cdot t_1(q^{1*}, q^2) = 0 \Rightarrow g_w(w^*, a) = 0$, since $t_1(q^{1*}, q^2) \neq 0$. Thus w* is a function of 'a':

$$w^* = z(a) = t(q^{1*}, q^2), \tag{1.39}$$

which determines 1's reaction function $q^{1*} = b^1(q^2, a)$.

Second order conditions: $f_{11} = g_{ww}\cdot (t_1)^2 + g_w \cdot t_{11} = g_{ww}\cdot (t_1)^2 < 0$, where the derivatives g_w, g_{ww} are evaluated at (w^*, a) and t_1, t_{11} are evaluated at (q^{1*}, q^2).

It is the case that:

$\partial\pi^1/\partial q^2 = f_2(q^{1*}, q^2, a) = g_w(w^*, a) \cdot t_2(q^{1*}, q^2) = 0$, as $g_w(w^*, a) = 0$.■

Alternatively, the type-2 effect is eliminated if $f_{1a} = 0$. In this case the firm's profit is affected by the rival's action and a compensation is necessary in order to restore it to its initial level. However, withdrawing the compensation does not affect the firm's choice. The type-2 effect is, thus, equal to zero.

Theorem 1.5.

The class of profit functions:

$\pi^1 = f(q^1, q^2, a, c^1) = u(a)\cdot g(q^1, q^2, c^1)$, $u > 0$, $u_a > 0$, for every a, (1.40)

$g > 0$, $g_{11} < 0$, $g_2 < 0$, $g_c < 0$, for every (q^1, q^2, c^1).

exhibits the property: $f_{1a}(q^{1*}, q^2, a, c^1) = 0$.

Proof.

At the profit-maximising value of q^1, q^{1*}, it is:

$f_1(q^{1*}, q^2, a, c^1) = 0 \Rightarrow u(a)\cdot g_1(q^{1*}, q^2, c^1) = 0 \Rightarrow g_1(q^{1*}, q^2, c^1) = 0$, $u(a) > 0$.

Thus,

$f_{1a}(q^{1*}, q^2, a, c^1) = u_a(a)\cdot g_1(q^{1*}, q^2, c^1) = 0$ ■

Proposition.

It holds that:

$f_{a1}(q^{1*}, q^2, a , c^1) = 0 \iff f_{\pi 1}^{-1}(\tilde{q}^1, q^2, \pi^1, c^1) = 0$, or,

$f_{1a}(q^{1*}, q^2, a , c^1) = 0 \iff f_{1\pi}^{-1}(\tilde{q}^1, q^2, \pi^1, c^1) = 0$

Proof.

According to the Lemma, eq.(1.13), and taking into account that $f_1^{-1}(\tilde{q}^1, q^2, \pi^1, c^1) = 0$, we have:

$f_{a1}(q^{1*}, q^2, a , c^1) \cdot f_\pi^{-1}(\tilde{q}^1, q^2, \pi^1, c^1) + f_a(q^{1*}, q^2, a , c^1) \cdot f_{\pi 1}^{-1}(\tilde{q}^1, q^2, \pi^1, c^1) \equiv 0$,

Taking the Lemma, eq.(1.12) into account, we have:

$f_{\pi}^{-1}(\tilde{q}^1, q^2, \pi^1, c^1) = 1/f_a(q^{1*}, q^2, a, c^1) > 0$, since $f_a(q^{1*}, q^2, a, c^1) > 0$

The two equations in conjunction yield (eliminating the arguments for simplicity):

$f_{a1}\cdot(1/f_a) + f_a\cdot f_{\pi 1}^{-1} \equiv 0,\ f_a > 0$

which proves the Proposition. ■

Accordingly, if the optimising condition of the primal problem, $f_1(q^{1*}, q^2, a, c^1) = 0$ is not affected by a change in the parameter 'a', then the optimising condition of the dual problem, $f_1^{-1}(\tilde{q}^1, q^2, \pi^1, c^1) = 0$ is not affected by a change in π^1. It follows that for profit functions described by eq.(1.40) the reaction curve is invariant to the parameter 'a' and the compensated reaction curve is invariant to π^1.

1.5. Choosing the parameter to be compensated.

An inverse demand function typically has a number of parameters. In choosing which parameter to compensate we are guided by the principle that an effect should be analysed in the way which involves the simplest structure. If, for ex., a given reaction is analysed into the type-1 effect only (type-2 effect = 0) when we compensate the demand parameter 'a', but both type-1 and type-2 effects arise as the result of compensating the parameter 'b', then we should choose to compensate the parameter 'a'.

In the case of linear demand function, for example, a duopolist's profit function is:

$$\pi^1 = f(q^1, q^2, a) = (a - b\cdot(q^1+q^2))\cdot q^1 - c\cdot q^1, \qquad a > c \qquad \text{(B.1)}$$

We can compensate parameter 'a', or alternatively, parameter 'b'. In order to find the type-1/type-2 effect structure resulting from the compensation of either parameter, we consider them in turn:

Solving the profit function for 'a', we get:

$$a = c + \pi^1/q^1 + b\cdot(q^1+q^2) \qquad \text{(B.2)}$$

Minimising 'a' w.r.t. q^1, for a given π^1, we have:

$$\partial a/\partial q^1 = -\pi^1/(\tilde{q}^1)^2 + b = 0 \Rightarrow \tilde{q}^1 = (\pi^1/b)^{1/2} \qquad \text{(B.3)}$$

S.O.C. can be checked to hold.

The compensated reaction function is independent of q^2. The type-1 effect, consequently, is equal to zero, $\partial\tilde{q}^1 / \partial q^2 = 0$.

Checking that the reaction is equal to the type-2 effect:

The reaction term is:

$\partial q^{1*}/ \partial q^2 = - f_{12}/f_{11}$, thus,

$\partial q^{1*}/ \partial q^2 = - (-b)/(-2\cdot b) = - ½$

According to eq.(1.24) the type-2 effect is:

$-(f_{1a}/f_{11})\cdot(f_2/f_a) = - (1/(2\cdot b))\cdot b = - ½$

Consequently, the reaction is equal to the type-2 effect.

The virtue of compensating the parameter 'a' is that it simplifies the type-1/type-2 effect structure by eliminating the type-1 effect and reducing the reaction to be equal to the type-2 effect.

Alternatively, compensating parameter 'b' generates both effects, thus complicating the analysis. In particular, we start by solving the profit function for (−b):

$-b = [\ \pi^1/q^1 + c - a]\cdot(1/(q^1 + q^2))$

where $q^1 > \pi^1 /(a - c)$ to guarantee the negativity of the expression. (We choose (−b), rather than 'b' as the parameter to compensate because π^1 is an increasing function of (−b), in accordance with ass. (IV).)

Minimising the function for (−b) w.r.t. q^1 we get the compensated reaction function:

$\tilde{q}^1 = [\ \pi^1 + ((\pi^1)^2 + (a - c)\cdot\pi^1 \cdot q^2)^{1/2}\ ]/(a - c)$

As q^2 affects $\tilde{q}^1$ the type-1 effect exists. The type -2 effect also exists as $f_2 = -b\cdot q^1 \neq 0$ and $f_{1a} = 1$.

It's, therefore, a more efficient line of analysis to compensate 'a', allowing us to explain the reaction in terms of only one effect (type-2).

1.6. Definition of Cournot equilibrium in terms of compensated reaction functions.

Consider a duopoly i = 1, 2. The Cournot equilibrium is defined as a pair of output levels (q^{1*}, q^{2*}), which satisfy the equations of both reaction functions:

$q^{1*} = b^1(q^{2*}, a_0)$,

$q^{2*} = b^2(q^{1*}, a_0)$,

the firms realising equilibrium profits $\pi^{i*} = F^i(q^{j*}, a_0)$, $i \neq j$.

According to eq.(1.19′):

$q^{i*} = b^i(q^{j*}, a_0) = h^i(q^{j*}, F^i(q^{j*}, a_0)) = \tilde{q}^i$, $i \neq j$.

This equation, along with eq.(1.19), makes it possible to define the Cournot equilibrium in terms of the compensated reaction function:

Definition.

The Cournot equilibrium is defined as a pair of output levels (q^{1*}, q^{2*}), which satisfy the equations of compensated reaction functions and the relevant restrictions:

$q^{1*} = h^1(q^{2*}, \pi^{1*})$,

$q^{2*} = h^2(q^{1*}, \pi^{2*})$,

$a_0 = A^1(q^{2*}, \pi^{1*})$,

$a_0 = A^2(q^{1*}, \pi^{2*})$.

where a_0 is the actual value of the parameter 'a'. ■

Equilibrium is determined by a system of 4 equations in 4 unknowns (q^{1*}, q^{2*}, π^{1*}, π^{2*}).

1.7. Comparative statics.

The duality approach lends itself for the study of comparative statics as it facilitates the analysis of the effects of an exogenous change. The approach proposed can be used to determine the effect of a change in per-unit costs or the effect of a demand shock on equilibrium firm profits.

Considering a Cournot duopoly as described in Section 1.2, we can determine the function $A(q^2, \pi^1, c^1)$, according to eq(1.18).The main result is summarized by the following Proposition.

Proposition.

Under assumptions (I) - (V.a) and (VI), the change in equilibrium profits due to a change in per-unit costs is determined as follows:

$$d\pi^{1*}/dc^1 = -[A_2 \cdot (dq^{2*}/dc^1) + A_c]/A_\pi \ , \qquad (1.41)$$

where A_2, A_π, A_c are the derivatives of the function $A(q^2, \pi^1, c^1)$ w.r.t. q^2, π^1, c^1 respectively and they are strictly positive ($A_2 > 0$, $A_\pi > 0$, $A_c > 0$).

Firm 2's equilibrium reaction is:

$$dq^{2*}/dc^1 = (b_1^{\,2} \cdot b_c^{\,1})/(1 - b_1^{\,2} \cdot b_2^{\,1}) \qquad (1.42)$$

where $b_2^{\,1}$, $b_c^{\,1}$ are the derivatives of firm 1's reaction function $b^1(q^2, a_0, c^1)$ w.r.t. q^2 and c^1 respectively and $b_1^{\,2}$ is the derivative of firm 2's reaction function $b^2(q^1, a_0)$ w.r.t. q^1.

Proof.

According to the above definition of the Cournot equilibrium in terms of compensated reaction functions and taking per-unit costs into account, we have:

$$A(q^{2*}, \pi^{1*}, c^1) = a_0$$

Differentiating totally w.r.t. c^1 and rearranging terms yields eq.(1.41).

According to the Lemma, eq.(1.11), (1.12), (1.14), the envelop theorem, and assumptions (IV), (V.a) it holds:

$A_2 = - f_2/f_a > 0$, $A_\pi = 1/f_a > 0$, $A_c = - f_c/f_a > 0$.

At equilibrium it holds:

$q^{2*} = b^2(b^1(q^{2*}, a_0, c^1), a_0, c^2)$

Differentiating totally w.r.t. c^1 and rearranging terms yields eq.(1.42). ■

Stability requires that the denominator of eq.(1.42) be positive.

A Proposition similar to the above is the following one:

Proposition.

Under assumptions (I) - (V.a) and (VI) the change in equilibrium profits due to a change in a_0 is determined as follows:

$d\pi^{1*}/da_0 = [1 - A_2 \cdot (dq^{2*}/da_0)]/A_\pi$

Proof.

Differentiate

$A(q^{2*}, \pi^{1*}, c^1) = a_0$

totally w.r.t. a_0 and rearrange terms.■

1.8. Price-setting duopoly with differentiated products.

We consider a product which exists in two varieties 1, 2, produced by duopolists A, B respectively. The firms engage in price competition. Each firm chooses the price of its own variety which maximizes its profits, taking the rival's price as given. Decisions are taken simultaneously.

The demand structure is described by the system of inverse demand functions:

$$p_1 = a - b \cdot q_1 - \theta \cdot b \cdot q_2 \quad , \tag{1.43}$$

$$p_2 = a - \theta \cdot b \cdot q_1 - b \cdot q_2 \ , \qquad (1.44)$$

where p_i, q_i is, respectively, the price and quantity of variety $i = 1,2$. The parameter θ takes values between 0 and 1 and measures the degree of differentiation. $\theta = 0$ implies full differentiation, whereas $\theta = 1$ implies no differentiation.

The system of demand functions corresponding to inverse demand functions (1.43) – (1.44) is:

$$q_1 = (a \cdot (1 - \theta) - p_1 + \theta \cdot p_2)/(b \cdot (1 - \theta^2)) \qquad (1.45)$$

$$q_2 = (a \cdot (1 - \theta) - p_2 + \theta \cdot p_1)/(b \cdot (1 - \theta^2)) \qquad (1.46)$$

The quantity demanded of a variety is negatively related to its own price and positively related to the price of the other variety.

Both varieties are produced at a constant cost 'c' per unit of output.

The profit-maximisation problem of firm A is:

$$\underset{\{p_1\}}{\text{Max}}\ \pi^A = (p_1 - c) \cdot q_1 \qquad (1.47)$$

where q_1 is given by (1.45).

Firm A's reaction function is specified in terms of prices:

$$p_1^* = (½) \cdot (a \cdot (1 - \theta) + \theta \cdot p_2 + c) \qquad (1.48)$$

Firm A's best response is an increasing function of p_2.

Turning to the dual problem we start by considering firm A's profit function and solving for the demand parameter 'a'.

(1.45) combined with (1.47) yields:

$$\pi^A = [(p_1 - c) \cdot (a \cdot (1 - \theta) - p_1 + \theta \cdot p_2)]/(b \cdot (1 - \theta^2)) \qquad (1.49)$$

Solving for 'a' yields:

$$a = (1/(1 - \theta)) \cdot [\ \pi^A \cdot b \cdot (1 - \theta^2) \cdot (1/(p_1 - c)) + p_1 - \theta \cdot p_2] \qquad (1.50)$$

Minimising the above function w.r.t. p_1, we get firm A's compensated reaction function:

$$\tilde{p}_1 = c + (\pi^A \cdot b \cdot (1-\theta^2))^{1/2} \tag{1.51}$$

Notice that, as the compensated reaction of firm A is independent of p_2, the type-1 effect does not arise.

According to eq. (1.22) it should be:

$$f_{12}/f_{11} = (f_{1a}/f_{11})\cdot(f_2/f_a)$$

(The reaction, $(-f_{12}/f_{11})$, should be equal to the type-2 effect, $-(f_{1a}/f_{11})\cdot(f_2/f_a)$).

To check whether it holds, we differentiate the profit function (1.49) to calculate

$$f_{11} = \partial^2\pi^A/\partial p_1^2 = (-2)/(b\cdot(1-\theta^2))$$

$$f_{12} = \partial^2\pi^A/(\partial p_1\cdot\partial p_2) = \theta/(b\cdot(1-\theta^2))$$

Thus, $f_{12}/f_{11} = -\theta/2$, (1.52)

Furthermore,

$$f_{1a} = \partial^2\pi^A/(\partial p_1\cdot\partial a) = (1-\theta)/(b\cdot(1-\theta^2)),$$

$$f_2 = \partial\pi^A/\partial p_2 = ((p_1 - c)\cdot\theta)/(b\cdot(1-\theta^2)),$$

$$f_a = \partial\pi^A/\partial a = ((p_1 - c)\cdot(1-\theta))/(b\cdot(1-\theta^2)),$$

Consequently,

$$(f_{1a}/f_{11})\cdot(f_2/f_a) = ((1-\theta)\cdot\theta)/(-2)\cdot(1-\theta) = -\theta/2 \tag{1.53}$$

(1.52), (1.53) show that:

$$f_{12}/f_{11} = (f_{1a}/f_{11})\cdot(f_2/f_a)$$

In other words, only the type-2 effect exists. A change of p_2 is equivalent to a change in demand parameter 'a' so far as firm A's maximisation problem is concerned.

1.9. Cournot competition in two stages

It's possible to extend our analysis to cover competition which takes place in two-stages. Consider a Cournot duopoly where firms 1, 2 compete in variables (x^1, x^2) in the first stage and in variables (q^1, q^2) in the second stage. The profit function of firm i is:

$\pi^i = f^i(x^i, x^j, q^i, q^j, a), \; i = 1, 2, \quad i \neq j$

The following assumptions are imposed on the profit functions:

VII) Twice continuous differentiability in x^i, x^j, q^i, q^j, a.

VIII) Strict concavity w.r.t. q^i, for every (x^i, x^j) i.e. the second derivative $f^i_{qiqi} < 0$ for every (x^i, x^j), $i = 1,2$, $i \neq j$.

IX) For every (x^i, x^j) profit maximisation w.r.t. q^i yields interior solutions .

X) The partial derivative w.r.t. parameter ´a´ is positive ($f^i_a > 0$), for every vector (x^i, x^j, q^i, q^j, a).

XI) The partial derivative w.r.t. q^j, f^i_{qj}, $j \neq i$, is either

a) $f^i_{qj} < 0$, for every vector (x^i, x^j, q^i, q^j, a), or

b) a saddle point (q^{i*}, q^{j*}) exists, in which case $f^i_{qj}(x^i, x^j, q^{i*}, q^{j*}, a) = 0$.

XII) The inverse of the profit function

$a = f^{-1}(x^i, x^j, q^i, q^j, \pi^i)$

is a well-defined function. (The superscript ´ i ´ of the profit function f^i has been omitted for notational convenience).

We study firm i´s maximisation problem starting at the second stage, considering first-stage variables (x^i, x^j) as given.

$$\max_{\{q^i\}} \; f^i(x^i, x^j, q^i, q^j, a)$$

Assumptions (VII) – (IX) ensure the existence of a solution which is described by the reaction function

$q^i = b^i (q^j, x^i, x^j, a)$

The dual problem consists of the minimisation of the inverse profit function w.r.t. q^i, considering first stage variables (x^i, x^j) as given.

$$\min_{\{q^i\}} f^{-1}(x^i, x^j, q^i, q^j, \pi^i)$$

The second-stage compensated reaction solving this problem is given by the function

$\tilde{q}^i = h^i(q^j, x^i, x^j, \pi^i)$

The Slutsky equation holds for the second stage of a two-stage model in exactly the same manner it does for the single-stage Cournot competition. As the structure is the same in both cases the equation will be:

$\partial\tilde{q}^i/\partial q^j = - f_{qiqj}/ f_{qiqi} + (f_{qia}/ f_{qiqi})\cdot(f_{qj}/f_a)$

where the superscript ´ i ´ of the profit function f^i has been omitted for notational convenience.

The system of the reaction functions determines the second-stage equilibrium:

$q^{i*} = e^i(x^i, x^j, a), i = 1, 2, \quad i \neq j$

The second-stage equilibrium profit is given by the function $E^i(x^i, x^j, a)$, which is defined as follows:

$\pi^{i*} = f^i(x^i, x^j, e^i(x^i, x^j, a), e^j(x^i, x^j, a), a) \equiv E^i(x^i, x^j, a), \; i = 1, 2, \quad i \neq j$

We suppose that the functions $E^i(x^i, x^j, a)$, $i = 1,2$, $i \neq j$, satisfy assumptions (I)-(VI). At the first stage firm i chooses x^i, given x^j, to maximise $E^i(x^i, x^j, a)$. The corresponding reaction function is:

$x^{i*} = B^i(x^j, a)$, $i = 1,2$, $i \neq j$

P^i is the set of values of second-stage equilibrium profits which firm i can obtain under some profile (x^i, x^j) for some value of the parameter ‘a’. We can then define the inverse of the function E^i:

$a = E^{-1}(x^i, x^j, \pi^i)$, π^i in P^i

(The superscript ′ i ′ of the profit function E^i has been omitted for notational convenience).

Consequently, the first-stage compensated reaction function is determined by the solution of the problem:

$$\min_{\{x^i\}} E^{-1}(x^i, x^j, \pi^i)$$

which yields

$$\tilde{x}^i = H^i(x^j, \pi^i)$$

The Slutsky equation holds for the first stage of a two-stage model in exactly the same manner it does for the single-stage Cournot competition. As the structure is the same in both cases the equation will be:

$$\partial\tilde{x}^i / \partial x^j = - E_{xixj} / E_{xixi} + (E_{xia} / E_{xixi}) \cdot (E_{xj} / E_a)$$

To fix ideas the following example is presented, based on the d’Aspremont/Jacquemin model [1988]:

Example.

Consider a two-stage Cournot duopoly where firms choose R&D levels in the first stage and production levels in the second. The profit function is:

$$\pi^i = [a - b \cdot Q] \cdot q^i - [A - x^i - \beta \cdot x^j] \cdot q^i - \gamma \cdot ((x^i)^2/2), \; i = 1,2, \; i \neq j \qquad \text{(C.1)}$$

$a, b \,\square\, 0, 0 \,\square\, A \,\square\, a, 0 \,\square\, \beta < 1, x^i + \beta \cdot x^j \leq A, Q \leq a/b$

where $Q = q^i + q^j$, i.e. represents industry output ,

$P = [a - b \cdot Q]$ is the inverse demand function·

$[A - x^i - \beta \cdot x^j]$ is the cost of firm i per unit of output, where

β is a spillover parameter which measures the effect of firm j's R&D on the unit production cost of firm i.

x^i is the R&D level of firm i, and

$\gamma\cdot(x^i)^2/2$ is the cost of firm i's R&D.

We start at the second stage. Conditional on x^i, x^j, firm i's profit function is maximised w.r.t. q^i, yielding the following reaction function:

$$q^{i*} = (a - A + x^i + \beta\cdot x^j)/(2\cdot b) - q^j/2, \tag{C.2}$$

Correspondingly, the reaction function of firm j is:

$$q^{j*} = (a - A + x^j + \beta\cdot x^i)/(2\cdot b) - q^i/2, \tag{C.3}$$

The reaction function system of equations determines the second-stage Cournot equilibrium:

$$q^{i*} = (a - A + (2-\beta)\, x^i + (2\cdot\beta - 1)\cdot x^j)/(3\cdot b) \tag{C.4}$$

$$q^{j*} = (a - A + (2-\beta)\, x^j + (2\cdot\beta - 1)\cdot x^i)/(3\cdot b) \tag{C.5}$$

Substituting the output equilibrium values into firm i's profit function (C.1) we get i's profit at the second-stage equilibrium:

$$\pi^{i*} = (a - A + (2-\beta)\, x^i + (2\cdot\beta - 1)\cdot x^j)^2/(9\cdot b) - \gamma\cdot((x^i)^2/2) \tag{C.6}$$

Turning to the dual problem:

The second-stage compensated reaction function of firm i determines the level of output which allows firm i to realise a given level of profits under the minimal value of the demand parameter 'a'. Thus, solving profit function (C.1) for the parameter 'a', we get:

$$a = [\pi^i + \gamma\cdot((x^i)^2/2)]\cdot(1/q^i) + A - x^i - \beta\cdot x^j + b\cdot Q,$$

Minimising the above expression w.r.t. q^i, keeping π^i constant, we get:

$$\partial a/\partial q^i = -[\pi^i + \gamma\cdot((x^i)^2/2)]\cdot(1/(q^i)^2) + b = 0 = \square$$

$$\tilde{q}^i = [[\pi^i + \gamma\cdot((x^i)^2/2)]\cdot(1/b)]^{1/2} \tag{C.7}$$

(S.O.C. can be checked to hold).

The second-stage compensated reaction function of firm i is given by eq. (C.7). The type-1 effect is equal to zero as $\tilde{q}^i$ is independent of q^j.

To find the first-stage compensated reaction function of firm i we solve the second-stage equilibrium profit function (eq. (C.6)) for the parameter 'a':

$$a = [9 \cdot b \cdot \pi^{i*} + \gamma \cdot ((x^i)^2/2)]^{1/2} + A - (2-\beta) \cdot x^i - (2 \cdot \beta - 1) \cdot x^j , \tag{C.8}$$

In order to find the level of R&D which allows firm i to realise a given level of profit under the minimum value of the demand parameter 'a' we mimimise the above expression of 'a' w.r.t. x^i:

$\partial a / \partial x^i = 0$,

which yields the first-stage compensated reaction function of firm i:

$$\tilde{x}^i = 6 \cdot (2 - \beta) \cdot (b \cdot \pi^{i*})^{1/2} / (\gamma^2 - 2 \cdot \gamma \cdot (2 - \beta)^2)^{1/2} \tag{C.9}$$

(S.O.C. can be checked to hold.)

It can be seen that the type-1 effect is equal to zero.

Homogeneous profit functions

2.1. Introduction

It is possible to get additional results by imposing the assumption that profit functions are homogeneous in (q^1, q^2, a). This assumption holds in the case of profit functions resulting from linear demand functions and quadratic costs:

$$\pi^1 = (a - b \cdot (q^1 + q^2)) \cdot q^1 - c \cdot (q^1)^2$$

Another class of profit functions which satisfy the homogeneity assumption is the one with unitary price elasticity demand functions and constant per-unit costs:

$\pi^1 = (a / (q^1 + q^2))\cdot q^1 - c\cdot q^1$

This assumption has, consequently, a substantial field of application. This restriction on the structure of the profit function allows us to obtain closed form expressions for the equilibrium variables (q^{1*}, q^{2*}).

2.2. Duality approach with homogeneous profit functions

Euler's formula allows us to present a homogeneous profit function of degree $L > 1$ in quadratic form.

Let the profit functions of firms 1, 2 be:

$\pi^1 = f(q^1, q^2, a)$,

$\pi^2 = g(q^1, q^2, a)$,

where $f(q^1, q^2, a)$, $g(q^1, q^2, a)$ satisfy assumptions (I)-(VI) and are homogeneous of degree $L_1, L_2 \geq 2$, in (q^1, q^2, a). By Euler's formula we get that for every (q^1, q^2, a):

$$L_1\cdot f(q^1, q^2, a) = f_1(q^1, q^2, a)\cdot q^1 + f_2(q^1, q^2, a)\cdot q^2 + f_a(q^1, q^2, a)\cdot a \qquad (2.1)$$

$$(L_1 - 1)\cdot f_1(q^1, q^2, a) = f_{11}(q^1, q^2, a)\cdot q^1 + f_{12}(q^1, q^2, a)\cdot q^2 + f_{1a}(q^1, q^2, a)\cdot a \qquad (2.2)$$

$$(L_1 - 1)\cdot f_2(q^1, q^2, a) = f_{21}(q^1, q^2, a)\cdot q^1 + f_{22}(q^1, q^2, a)\cdot q^2 + f_{2a}(q^1, q^2, a)\cdot a \qquad (2.3)$$

$$(L_1 - 1)\cdot f_a(q^1, q^2, a) = f_{a1}(q^1, q^2, a)\cdot q^1 + f_{a2}(q^1, q^2, a)\cdot q^2 + f_{aa}(q^1, q^2, a)\cdot a \qquad (2.4)$$

The profit function can be written in quadratic form:

$\pi^1 = (1/L_1)\cdot(1/(L_1-1))\cdot q\cdot D^2f(q^1, q^2, a)\cdot q^T$,

where $q = [q^1, q^2, a]$, and $D^2f(q^1, q^2, a)$ is the Hessian matrix of the profit function.

Properties of the Hessian matrix:

1) At the profit–maximising level of output it holds that $f_1 = 0$, thus the first row of the matrix is othrogonal to the vector q^{*T} and q^* is orthogonal to the first column when the matrix is evaluated at $q^* = [q^{1*}, q^2, a]$.

2) By ass.(IV) $f_a > 0$ for every $[q^1, q^2, a]$. Thus the product of the third row of the matrix times the vector q^T is strictly positive as is the product of q times its third column.

3) By ass. (II), $f_{11} < 0$.

4) According to assumption (V. a.), $f_2 < 0$, namely, the matrix evaluated at $q = [q^1, q^2, a]$, is such that the product of vector q times the second column is negative for every q and, correspondingly, the product of the second row times q^T is negative, for every q.

According to assumption (V. b.) the profit function $f(q^1, q^2, a)$ has a saddle point $q^{**} = [q^{1*}, q^{2*}, a]$, where $f_1 = 0$, $f_2 = 0$. The matrix, evaluated at q^{**}, is such that the product of q^{**} times the first or second column is zero and, correspondingly, the product of the first or second row times q^{**T} is zero.

At the saddle point q** the type-2 effect is eliminated as $f_2 = 0$.

Furthermore, the determinant of the submatrix with the derivatives w.r.t. (q^1, q^2), evaluated at q^{**}, is negative.

The profit-maximising condition referred to in property 1 of the Hessian matrix is:

$$f_{11}(q^{1*}, q^2, a)\cdot q^{1*} + f_{12}(q^{1*}, q^2, a)\cdot q^2 + f_{1a}(q^{1*}, q^2, a)\cdot a = 0$$

The strategic complementarity/substitutability of strategies q^1, q^2 is determined by the term f_{12}.

In particular, we can state the following proposition.

Proposition

Under homogeneity of the profit function $f(q^1, q^2, a)$ and assumptions (I) – (III), if $f_{1a}(q^{1*}, q^2, a) \leq 0$, then $f_{12}(q^{1*}, q^2, a) > 0$.

Proof.

It follows from assumption (II) ($f_{11}<0$) and the above equation.■

In other words, if an increase in 'a' lowers f_1, the relation of q^2 to q^1 is one of strategic complementarity.

Proposition

Under homogeneity of the profit function $f(q^1, q^2, a)$ and assumptions (I) – (III), for $f_{12}(q^{1*}, q^2, a)$ to be negative , $f_{1a}(q^{1*}, q^2, a)$ has to be positive.

Proof.

It follows from assumption (II) ($f_{11}<0$) and the above equation.■

Consequently, for q^1, q^2 to be strategic substitutes, f_{1a} has to be positive.

Proposition

Under assumptions (I) – (III), if the second row or column of the Hessian matrix is collinear to the third one, then the type-1 effect is eliminated.

Proof.

The type-1 effect is eliminated as the condition

$f_2/f_a = f_{12}/f_{1a}$

is satisfied. This can be seen by taking the collinearity relation into account in connection to the equations (2.3), (2.4) above.■

Theorem 2.1.

Consider firm 1 with a profit function $f(q^1, q^2, a)$ satisfying assumptions (I)-(VI) and assumed to be homogeneous of degree $L \geq 2$ in (q^1, q^2, a).

Then firm 1 has a reaction function implicitly determined by the equation:

$$q^{1*} = -\left(f_{12}(q^{1*}, q^2, a)\cdot q^2 + f_{1a}(q^{1*}, q^2, a)\cdot a\right)/f_{11}(q^{1*}, q^2, a) \qquad (2.5)$$

Proof.

The profit-maximising condition is:

$f_1(q^{1*}, q^2, a) = 0,$

combined with Euler's formula

$$(L-1)\cdot f_1(q^{1*}, q^2, a) = f_{11}(q^{1*}, q^2, a)\cdot q^{1*} + f_{12}(q^{1*}, q^2, a)\cdot q^2 + f_{1a}(q^{1*}, q^2, a)\cdot a$$

gives

$f_{11}(q^{1*}, q^2, a)\cdot q^{1*} + f_{12}(q^{1*}, q^2, a)\cdot q^2 + f_{1a}(q^{1*}, q^2, a)\cdot a = 0$

Solving the equation for q^{1*} proves the theorem.■

The function

$-(f_{12}(q^{1*}, q^2, a)\cdot q^2 + f_{1a}(q^{1*}, q^2, a)\cdot a)/f_{11}(q^{1*}, q^2, a)$

is homogeneous of degree 1 in (q^{1*}, q^2, a), irrespectively of L.

In case $f_{1a} = 0$, (the type-2 effect is eliminated) the reaction function yields

$q^{1*}/q^2 = -f_{12}/f_{11} = \partial q^{1*}/\partial q^2$

The compensated reaction function is determined by the following theorem.

Theorem 2.2.

Consider firm 1 with a profit function $f(q^1, q^2, a)$ satisfying assumptions (I)-(VI) and assumed to be homogeneous of degree $L \geq 2$ in (q^1, q^2, a).

Then firm 1 has a compensated reaction function implicitly determined by the equation:

$$\tilde{q}^1 = [(f_2\cdot f_{a1} - f_{21}\cdot f_a)\cdot q^2 - L\cdot\pi^1\cdot f_{a1}]/(f_{11}\cdot f_a) \qquad (2.6)$$

evaluated at $(\tilde{q}^1, q^2, f^{-1}(\tilde{q}^1, q^2, \pi^1))$.

Proof.

By Euler's formula we have:

$L\cdot\pi^1 = f_1(q^1, q^2, a)\cdot q^1 + f_2(q^1, q^2, a)\cdot q^2 + f_a(q^1, q^2, a)\cdot a$, for every (q^1, q^2, a).

Solving for 'a', we get:

$a = (L\cdot\pi^1 - f_1(q^1, q^2, a)\cdot q^1 - f_2(q^1, q^2, a)\cdot q^2)/ f_a(q^1, q^2, a)$

We take into account that:

$a = f^{-1}(q^1, q^2, \pi^1)$

and substitute it in the above equation, thus getting:

$a = (L\cdot\pi^1 - f_1(q^1, q^2, f^{-1}(q^1, q^2, \pi^1))\cdot q^1 - f_2(q^1, q^2, f^{-1}(q^1, q^2, \pi^1))\cdot q^2)/ f_a(q^1, q^2, f^{-1}(q^1, q^2, \pi^1))$

Differentiating throughout w.r.t. q^1, we get:

$\partial a/\partial q^1 =$

$[[-(f_{11}+f_{1a}\cdot f_1^{-1})\cdot q^1 - f_1 - (f_{21} + f_{2a}\cdot f_1^{-1})\cdot q^2] \cdot f_a - [L\cdot\pi^1 - f_1\cdot q^1 - f_2\cdot q^2]\cdot(f_{a1}+f_{aa}\cdot f_1^{-1})]/(f_a)^2$

Since at $\tilde{q}^1$ it is the case that $\partial a/\partial q^1 = f_1^{-1} = 0$, and by eq. (1.8), $f_1 = 0$, the above expression becomes:

$\partial a/\partial q^1 = [[-f_{11}\cdot\tilde{q}^1 - f_{21}\cdot q^2]\cdot f_a - [L\cdot\pi^1 - f_2\cdot q^2]\cdot f_{a1}]/(f_a)^2 = 0$

which yields

$[-f_{11}\cdot\tilde{q}^1 - f_{21}\cdot q^2]\cdot f_a - [L\cdot\pi^1 - f_2\cdot q^2]\cdot f_{a1} = 0$, thus,

$\tilde{q}^1 = [[f_2\cdot q^2 - L\cdot\pi^1]\cdot f_{a1} - f_{21}\cdot q^2\cdot f_a]/(f_{11}\cdot f_a) = [(f_2\cdot f_{a1} - f_{21}\cdot f_a)\cdot q^2 - L\cdot\pi^1\cdot f_{a1}]/(f_{11}\cdot f_a)$

Second order conditions are satisfied by assumptions (II), (IV) and Lemma 2■

The compensated reaction function in the general case ($L\neq 1$) is not homogeneous in (q^1, q^2, π^1).

In case $f_{1a} = 0$ (type-2 effect is eliminated) the compensated reaction function (2.6) yields

$\tilde{q}^1/q^2 = - f_{21}/f_{11} = \partial\tilde{q}^1/\partial q^2$,

where the second equality is given by eq.(1.23).

2.3. Cournot Equilibrium.

Theorem 2.3.

Consider Cournot duopolists 1, 2 with profit functions $f(q^1, q^2, a)$ and $g(q^1, q^2, a)$, respectively, satisfying assumptions (I)-(VI) and assumed to be homogeneous of degree $L_1, L_2 \geq 2$ in (q^1, q^2, a). Then the Cournot equilibrium is implicitly determined by the equations:

$$q^{1*} = \frac{D}{Z} \tag{2.7}$$

$$q^{2*} = \frac{K}{Z} \tag{2.8}$$

where,

$$D = -(f_{1a}/f_{11})\cdot a + (f_{12}/f_{11})\cdot(g_{2a}/g_{22})\cdot a \tag{2.9}$$

$$Z = 1 - (f_{12}/f_{11})\cdot(g_{21}/g_{22}) \tag{2.10}$$

$$K = -(g_{2a}/g_{22})\cdot a + (f_{1a}/f_{11})\cdot(g_{21}/g_{22})\cdot a \tag{2.11}$$

Proof.

By analogy to firm 1 (Theorem (2.1)), the reaction function of firm 2 is implicitly determined by the equation:

$$q^{2*} = -(g_{21}\cdot q^1 + g_{2a}\cdot a)/g_{22} \tag{2.12}$$

evaluated at (q^1, q^{2*}, a). The equilibrium is determined by the system of eqs. (2.5), (2.12), namely:

$$q^{1*} = -(f_{12}\cdot q^{2*} + f_{1a}\cdot a)/f_{11}$$

$$q^{2*} = -(g_{21}\cdot q^{1*} + g_{2a}\cdot a)/g_{22}$$

thus the equilibrium is:

$$q^{1*} = \frac{D}{Z}$$

$$q^{2*} = \frac{K}{Z}$$

where

$$D = -(f_{1a}/f_{11})\cdot a + (f_{12}/f_{11})\cdot(g_{2a}/g_{22})\cdot a$$

$$Z = 1 - (f_{12}/f_{11})\cdot(g_{21}/g_{22})$$

$$K = -(g_{2a}/g_{22})\cdot a + (f_{1a}/f_{11})\cdot(g_{21}/g_{22})\cdot a$$

■

For the equilibrium to exist it has to be $Z \neq 0$. The functions D/Z and K/Z are homogeneous of degree 1 in (q^1, q^2, a). Consequently, the equilibrium levels of output (q^{1*} , q^{2*}) are homogeneous of degree 1 in the parameter 'a'. The equilibrium profits of firms 1, 2 are homogeneous of degree L_1, L_2, respectively in the parameter 'a'.

2.4. Two-stage Cournot competition with homogeneous profit functions

Consider a two-stage Cournot duopoly model where firms i = 1,2, have profit functions of the form:

$$\pi^i = f^i(x^i, x^j, q^i, q^j, a),\ i = 1,2,\ i\neq j, \tag{2.13}$$

where x^i, x^j, are the first-stage control variables and q^i, q^j, are the second-stage ones.

We assume that the functions $f^i(x^i, x^j, q^i, q^j, a)$, i = 1,2, $i \neq j$, are homogeneous of degree $L_i \geq 2$ in (x^i, x^j, q^i, q^j, a).

Theorem 2.4.

Consider Cournot duopolists i =1, 2, with profit functions of the form described by eq.(2.13), assumed to be homogeneous of degree $L_i \geq 2$ in their arguments and satisfying assumptions (VII)-(XII).

The second-stage Cournot equilibrium is implicitly determined by the equation:

$$q^{i*} = [1/(1 - (f^i_{q^iq^j}/ f^i_{q^iq^i})\cdot(f^j_{q^jq^i}/f^j_{q^jq^j}))]\cdot[-R^i + (f^i_{q^iq^j}/ f^i_{q^iq^i})\cdot R^j], \tag{2.14}$$

where $R^i = (f^i_{q^ix^i}/ f^i_{q^iq^i})\cdot x^i + (f^i_{q^ix^j}/ f^i_{q^iq^i})\cdot x^j + (f^i_{q^ia}/ f^i_{q^iq^i})\cdot a$, (2.15)

$$R^j = (f^j_{q^jx^i}/ f^j_{q^jq^j})\cdot x^i + (f^j_{q^jx^j}/ f^j_{q^jq^j})\cdot x^j + (f^j_{q^ja}/ f^j_{q^jq^j})\cdot a, \tag{2.16}$$

i = 1,2, $i \neq j$.

Proof.

See APPENDIX D.

Notice that the above equations (2.14) – (2.16) determine that q^{i*} is homogeneous of degree 1 in (q^i, q^j, x^i, x^j, a) independently of L_i, L_j. Consequently, the second stage equilibrium values q^{i*}, q^{j*} are homogeneous of degree 1 in (x^i, x^j, a). By Euler's formula it holds:

$$q^{i*} = (\partial q^{i*}/\partial x^i)\cdot x^i + (\partial q^{i*}/\partial x^j)\cdot x^j + (\partial q^{i*}/\partial a)\cdot a$$

Comparing the above equation with the equations (2.14) – (2.16) allows us to calculate the intertemporal strategic term $\partial q^{i*}/\partial x^j$:

$$\partial q^{i*}/\partial x^j = [1/(1 - (f^i_{qiqj}/ f^i_{qiqi})\cdot(f^j_{qjqi}/f^j_{qjqj}))]\ [(f^i_{qiqj}/ f^i_{qiqi})\cdot(f^j_{qjxj}/f^j_{qjqj}) - (f^i_{qixj}/ f^i_{qiqi})]$$

Substituting the second-stage equilibrium functions $q^{i*} = q^i(x^i, x^j, a)$, i = 1,2, i ≠ j into the profit function (2.13) determines the second-stage equilibrium profits:

$$E^i(x^i, x^j, a) \equiv f^i(x^i, x^j, q^i(x^i, x^j, a), q^j(x^i, x^j, a), a) \qquad (2.17)$$

The function $E^i(x^i, x^j, a)$ can be checked to be homogeneous of degree L_i in (x^i, x^j, a)

Theorem 2.5.

Consider Cournot duopolists i =1, 2, with profit functions of the form described by eq.(2.17), homogeneous of degree $L_i \geq 2$ in their arguments and satisfying assumptions (I)-(VI).

The first-stage Cournot equilibrium is implicitly determined by the equation:

$$x^{i*} = [a/(1 - (E^i_{xixj}/E^i_{xixi})\cdot(E^j_{xjxi}/E^j_{xjxj}))]\cdot[(E^i_{xixj}/E^i_{xixi})\cdot(E^j_{xja}/E^j_{xjxj}) - (E^i_{xia}/E^i_{xixi})]$$

Proof.

See APPENDIX E.

2.5. Conclusions

The main point of this study is that demand theory methods find application in the Cournot model. In particular, firm behaviour can be analysed according to the substitution effect/income effect approach. The type-1 effect is the counterpart of the substitution effect and is motivated by the dual problem of minimising a market demand parameter subject to a given level of firm profits. This involves a positive or negative compensation in terms of the value of the market demand parameter. When withdrawing the compensation a further adjustment of firm output takes place which determines the type-2 effect. In more general terms, the type-1 effect is determined by the minimisation of a market demand parameter subject to a given level of firm profits, whereas the type-2 effect is associated to a change in firm profits.

In examining the structure of the type-1 effect, Theorem 1.3. points out that separability (and not linearity) of the market demand function is the property which determines whether the type-1 effect arises or not.

Net strategic complementarity/substitutability captures the effect of the rival's action on firm 1's marginal compensation ($\partial a/\partial q^1$) and determines the slope of the compensated reaction curve.

The proposed approach finds broad application, both for the analysis of the behaviour of price-setting duopolists and on Cournot competition in two stages.

Furthermore, results can be obtained under the assumption that the firms' profit functions are homogeneous in the market demand parameter and their levels of output. This property can be exploited to implicitly determine the reaction functions of the firms and, consequently, the Cournot equilibrium.

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APPENDIX A.

PROOF OF LEMMAS.

1. Differentiating (1.7) w.r.t. q^1 yields (1.8).

2. Differentiating (1.8) w.r.t. q^1 yields (1.9).

3. Differentiating (1.8) w.r.t. q^2 yields (1.10).

4. Differentiating (1.7) w.r.t. q^2 yields (1.11).

5. Differentiating (1.7) w.r.t. π^1 yields (1.12).

6. Differentiating (1.12) w.r.t. q^1 yields (1.13).

7. Differentiating (1.7) w.r.t. c^1 yields (1.14).

APPENDIX B.

By duality

$q^{1*} = b^1(q^2,\ A(q^2, \pi^{1*}, c^1),\ c^1) \equiv h^1(q^2, \pi^{1*}, c^1) = \tilde{q}^1$ (i)

where $A(q^2, \pi^{1*}, c^1) = \tilde{a}$

Differentiating (i) totally w.r.t. q^2 we get

$dq^{1*} / dq^2 = \partial q^{1*}/\partial q^2 + (\partial q^{1*}/\partial a)\cdot(\partial \tilde{a} /\partial q^2) = \partial \tilde{q}^1/\partial q^2$ (ii)

To express equation (ii) in terms of profit function derivatives, we proceed as follows:

Eq.(1.4) is totally differentiated w.r.t. q^2 while taking (1.5) into account,

$f_{11}(q^{1*}, q^2, a, c^1) \cdot \partial q^{1*}/\partial q^2 + f_{12}(q^{1*}, q^2, a, c^1) = 0$, thus

$\partial q^{1*}/\partial q^2 = - f_{12}(q^{1*}, q^2, a, c^1)/ f_{11}(q^{1*}, q^2, a, c^1)$ (iii)

Totally differentiating (1.4) w.r.t. a while taking (1.5) into account yields

$f_{11}(q^{1*}, q^2, a, c^1) \cdot \partial q^{1*}/\partial a + f_{1a}(q^{1*}, q^2, a, c^1) = 0$, thus

$\partial q^{1*}/\partial a = - f_{1a}(q^{1*}, q^2, a, c^1)/ f_{11}(q^{1*}, q^2, a, c^1)$ (iv)

Evaluating (1.11) at $\tilde{q}^1$ we get

$f_2(\tilde{q}^1, q^2, \tilde{a}, c^1) + f_a(\tilde{q}^1, q^2, \tilde{a}, c^1) \cdot f_2^{-1}(\tilde{q}^1, q^2, \pi^{1*}, c^1) = 0$,

where $\tilde{a} = f^{-1}(\tilde{q}^1, q^2, \pi^{1*}, c^1) = A(q^2, \pi^{1*}, c^1)$

thus, $f_2^{-1}(\tilde{q}^1, q^2, \pi^{1*}, c^1) = \partial \tilde{a} /\partial q^2 = - f_2(\tilde{q}^1, q^2, \tilde{a}, c^1) / f_a(\tilde{q}^1, q^2, \tilde{a}, c^1)$ (v)

By equations (i), (ii), (iii), (iv) and (v) we get

$\partial\tilde{q}^1/\partial q^2 = - f_{12}/ f_{11}+(f_{1a}/ f_{11})\cdot(f_2/f_a)$

where the derivatives are evaluated at $(\tilde{q}^1, q^2, \tilde{a}, c^1) = (q^{1*}, q^2, a_0, c^1)$, since π^{1*} is s.t.

$A(q^2, \pi^{1*}, c^1) = \tilde{a} = a_0$.

APPENDIX C.

According to eq. (1.23)

$\partial\tilde{q}^1/\partial q^2 = (f_{1a}\cdot f_2 - f_{12}\cdot f_a)/(f_{11}\cdot f_a)$

Considering firm 1's profit function

$f(q^1, q^2, a) = P(q^1+q^2, a)\cdot q^1 - c(q^1)$

the relevant derivatives are:

$f_{12} = P_{QQ} \cdot q^1 + P_Q$ (i)

$f_{1a} = P_{Qa} \cdot q^1 + P_a$ (ii)

$f_2 = P_Q \cdot q^1$ (iii)

$f_a = P_a \cdot q^1$ (iv)

where P_Q is the derivative of the inverse market demand function w.r.t. Q, P_{QQ} is the second derivative w.r.t. Q, P_{Qa} is the second derivative w.r.t. (Q, a) and P_a is the derivative w.r.t. a.

Taking eqs. (i) – (iv) into account,

$f_{1a}\cdot f_2 - f_{12}\cdot f_a = (P_{Qa} \cdot q^1 + P_a) \cdot (P_Q \cdot q^1) - (P_{QQ} \cdot q^1 + P_Q) \cdot (P_a \cdot q^1)$ (v)

Consider the identity

$Q \equiv D(P(Q, a), a)$

Differentiating throughout w.r.t. Q we get

$$1 = D_p(P(Q, a), a) \cdot P_Q(Q, a) \qquad \text{(vi)}$$

Differentiating eq. (vi) throughout w.r.t. Q we get

$$0 = D_{pp}(P(Q, a),a) \cdot (P_Q(Q, a))^2 + D_p(P(Q, a), a) \cdot P_{QQ}(Q, a) \qquad \text{(vii)}$$

Differentiating eq. (vi) throughout w.r.t. the parameter 'a' we get

$$0 = (D_{pp}(P(Q, a), a)\cdot P_a (Q, a)+D_{pa}(P(Q, a), a))\cdot P_Q(Q, a) + D_p(P(Q, a), a)\cdot P_{Qa}(Q, a) \qquad \text{(viii)}$$

Taking eqs. (vii)-(viii) into account, in substituting P_{QQ} , P_{Qa} into eq. (v) yields

$f_{1a}\cdot f_2 - f_{12}\cdot f_a =$

$$[(-(D_{pp} \cdot P_a + D_{pa}) \cdot P_Q /D_p) \cdot q^1+P_a]\cdot(P_Q\cdot q^1) - [(-D_{pp}\cdot(P_Q)^2 /D_p)\cdot q^1 + P_Q]\cdot(P_a\cdot q^1)$$

where the arguments are ignored to simplify the expression.

Thus, doing the algebraic calculations and taking (vi) into account, we have

$$f_{1a}\cdot f_2 - f_{12}\cdot f_a = - D_{pa} \cdot (P_Q \cdot q^1)^2/D_p = - D_{pa} \cdot (q^1)^2/(D_p)^3$$

evaluated at $(\tilde{q}^1, q^2, \tilde{a}) = (q^{1*}, q^2, a_0)$, $Q = \tilde{q}^1 + q^2 = q^{1*} + q^2$.

Eq. (1.23) in conjunction with the above equality proves the theorem.

APPENDIX D.

Firm i's profit function homogeneity implies, by Euler's formula, that:

$(L_i - 1)\cdot f_{qi} = f_{qixi}\cdot x^i + f_{qixj}\cdot x^j + f_{qiqi}\cdot q^i + f_{qiqj}\cdot q^j + f_{qia}\cdot a,$

At the profit maximizing value of q^i it holds that $f_{qi} = 0$. The above equation, thus, becomes:

$f_{qixi}\cdot x^i + f_{qixj}\cdot x^j + f_{qiqi}\cdot q^i + f_{qiqj}\cdot q^j + f_{qia}\cdot a = 0$

Solving for q^i, we get:

$q^i = -(1/f_{qiqi})\cdot(f_{qixi}\cdot x^i + f_{qixj}\cdot x^j + f_{qiqj}\cdot q^j + f_{qia}\cdot a)$

which implicitly determines firm i's reaction function.

Similarly for firm j:

$q^j = -(1/f_{qjqj})\cdot(f_{qjxj}\cdot x^j + f_{qjxi}\cdot x^i + f_{qjqi}\cdot q^i + f_{qja}\cdot a)$

which implicitly determines firm j's reaction function.

Solution of the system of the two reaction function equations yields:

$q^{i*} = [1/(1 - (f^i_{qiqj}/ f^i_{qiqi})\cdot(f^j_{qjqi}/f^j_{qjqj}))]\cdot[- R^i + (f^i_{qiqj}/ f^i_{qiqi})\cdot R^j]$,

where $R^i = (f^i_{qixi}/ f^i_{qiqi})\cdot x^i + (f^i_{qixj}/ f^i_{qiqi})\cdot x^j + (f^i_{qia}/ f^i_{qiqi})\cdot a$,

$R^j = (f^j_{qjxi}/ f^j_{qjqj})\cdot x^i + (f^j_{qjxj}/ f^j_{qjqj})\cdot x^j + (f^j_{qja}/ f^j_{qjqj})\cdot a$,

$i = 1,2,\ i \neq j$. ■

APPENDIX E

The homogeneity of $E^i(x^i, x^j, a)$ implies, by Euler's formula, that:

$(L_i - 1)\cdot E^i_{xi} = E^i_{xixi}\cdot x^i + E^i_{xixj}\cdot x^j + E^i_{xia}\cdot a$,

At the profit maximizing value of x^i it holds that $E^i_{xi} = 0$. The above equation, thus, becomes:

$E^i_{xixi}\cdot x^i + E^i_{xixj}\cdot x^j + E^i_{xia}\cdot a = 0$

Solving for x^i, we get:

$x^i = -(1/E^i_{xixi})\cdot(E^i_{xixj}\cdot x^j + E^i_{xia}\cdot a)$, $i =1,2,\ i \neq j$

which implicitly determines firm i's first stage reaction function.

Similarly, j's reaction function is implicitly determined by the equation:

$x^j = -(1/E^j_{xjxj})\cdot(E^j_{xjxi}\cdot x^i + E^j_{xja}\cdot a)$,

Solution of the system of reaction function equations for the two firms yields:

$x^{i*} = [a/(1 - (E^i_{xixj}/E^i_{xixi})\cdot(E^j_{xjxi}/E^j_{xjxj}))]\cdot[(E^i_{xixj}/E^i_{xixi})\cdot(E^j_{xja}/E^j_{xjxj}) - (E^i_{xia}/E^i_{xixi})]$

which implicitly determines the first-stage Cournot equilibrium. ■