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Abstract

As hedge fund replication based on factor models has encountered growing interest among professionals and academics, and despite the launch of numerous products (indexes and mutual funds) in the past year, it faced many critics. In this paper, we consider three of the main critiques, namely the lack of reactivity of hedge fund replication and its deficiency in capturing tactical allocations; its failure to apprehend non-linear positions of the underlying hedge fund industry and higher moments of hedge fund returns; and, finally, the lack of access to the alpha of hedge funds. To address these problems, we consider hedge fund replication as a general tracking problem which may be solved by means of Bayesian filters. Using the linear Gaussian model as a basis for discussion, we provide the reader with an intuition for the inner tenets of the Kalman filter and illustrate the results' sensitivity to the algorithm specification choices. This part of the paper includes considerations on the type of strategies which can be replicated, as well as the problem of selecting factors. We then apply more advanced Bayesian filters' algorithms, known as particle filters, to capture the non-normality and non-linearities documented on hedge fund returns. Finally, we address the problem of accessing the pure alpha by proposing a core/satellite approach of alternative investments between high-liquid alternative beta and less liquid investments.

Keywords: Tracking problem, hedge fund replication, alternative beta, global tactical asset allocation, Bayes filter, Kalman filter, particle filter, non-linear exposure, alpha.

JEL classification: G11, C60.

1 Introduction

Over the past decade, hedge-fund replication has encountered a growing interest both from an academic and a practitioner perspective. Recently, Della Casa, Rechsteiner and Lehmann [10] reported the results of an industry survey showing that, even though only 7% of the surveyed institutions had invested in hedge fund replication products in 2007, three times as many were considering investing in 2008. Despite this surge in interest, the practice still faces many critics. If the launch of numerous products (indexes and mutual funds) by several investment banks in the past year can be taken as proof of the attraction of the "clones" of hedge funds (HF) as investment vehicles, there remain nonetheless several shortcomings which need to be addressed.

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For instance, according to the same survey cited above, 13% of the potential investors do not invest for they do not believe that replicating Hedge Funds' returns was possible; 16% deplore the lack of track record of the products; another 16% consider the products as black boxes. Finally, 25% of the same investors do not invest for a lack of understanding of the methodologies employed, while 31% of them were not interested for they see the practice as only replicating an average performance, thus failing to give access to one of the main attractive features of investing in one hedge fund, namely its strategy of management.

As a whole, the reasons put forth by these institutions compound different fundamental questions left unanswered by the literature. Since the seminal work of Fung and Hsieh [13], most of the literature [1, 3, 4, 15, *inter alia*] has focused on assessing and explaining the characteristics of HF returns in terms of their (possibly time-varying) exposures to some underlying factors. Using linear factor models, these authors report the incremental progress in the explanatory power of the different models proposed. Yet, for now, the standard rolling-windows OLS regression methodology, used to capture the dynamic exposures of the underlying HF's portfolio has failed to show consistent out-of-sample results, stressing the difficulty of capturing the tactical asset allocation (TAA) of HF's managers. More recently, more advanced methodologies, in particular Markov-Switching models and Kalman Filter (KF), have been introduced [5, 30] and show superior results to the standard rolling-windows OLS approach. From the point of view of investors, however, the complexity of these algorithms certainly does not alleviate the lack of understanding in the replication procedure. Furthermore, despite superior dynamic procedures and an ever expanding set of explanatory factors, some nonlinear features of HF returns [11] as well as a substantial part of their performance remain unexplainable, unless surmising ultrahigh frequency trading and investments in illiquid assets or in derivative instruments by HF managers. To our knowledge, while commonly accepted by most authors, because of practical difficulties, these explanations have not led to a systematic assessment nor have been subject to systematic replication procedures.

In this paper, we address three of the main critiques formulated on hedge fund replication. First, using the notion of tracking problems and Bayesian filters and their associated algorithms, we address the alleged failure of HF replication to capture the tactical allocations of the HF industry. Using the linear Gaussian model as a basis for the discussion, we provide the readers with an intuition for the inner tenets of the Kalman Filter. We illustrate how one can obtain sensible results, in terms of alternative betas, taking the time to address some specification choices which need to be made. This part of the paper includes considerations on the type of strategies which can be replicated, as well as the problem of selecting factors. Secondly, we apply more advanced Bayesian filters' algorithms, known collectively as particle filters, to capture the nonlinearities documented on HF returns. We consider what type of nonlinearities can arise, and what models can be used to explain them. Finally, we address the problem of accessing the part of the HF performances attributed to un-captured dynamic strategies or investment in illiquid assets, i.e. the alpha of HF.

This paper is thus divided into five main sections. In section two, we provide the framework in which this paper is inscribed, providing a formal definition of a tracking problem, and casting our problem into a tracking problem. Section three deals with Bayesian filters and the methodologies which we use here to solve tracking problems. In section four, we consider the Gaussian linear case, before in section five to extend our work to a non-Gaussian non-linear framework. Finally, in section six, we consider the problem of access to the alpha of the HF industry.

2 Framework

We start by giving an overview of the literature on Hedge Fund replication, bringing forth both the accomplishments of the field and the critiques it still faces. Although Hedge Fund replication is at the core of this paper, we would like to inscribe our contribution in a larger framework, albeit limited to a few financial perspectives. Thus, this section further introduces the notion of tracking problems. After a brief and succinct formal definition, we show how this construct indeed underpins many different practices in finance, including some Hedge Fund replication techniques and some investment strategies such as, for example, Global Tactical Asset Allocation (henceforth, GTAA). It is armed with this construct and the tools associated to it that we tackle three of the main critiques heard in the context of Hedge Fund Replication in subsequent sections.

2.1 Hedge fund replication: overview of the factor approach

With the growing interest for hedge fund replication over the last decade, it is not surprising to find that there exists a rich literature which is almost impossible to cover extensively. Hence, the following overview only provides the main steps and the main directions the field has taken. If one goal is to provide a novice with a working knowledge of the field, it is also to establish the rationale behind this paper. Our exposé, inspired from [4], stresses on the factor approach, by opposition to the pay-off distribution approach introduced recently in the work of Kat in particular [6, 21, 22] or the systematic quantitative replication of strategies proposed by many investment banks as hedge funds' clones products and advocated by Fung and Hsieh [17] (see also Jaeger [19]).

2.1.1 Investing in the HF industry

Over the last 15 years, the hedge fund industry delivered higher Sharpe ratios than Buy and Hold strategies on traditional asset classes (cf. Table 2.1.1). During the equity bear market of 2000-2003, the HF industry managed to keep a relatively stable level of wealth as the equity markets dived (cf. Figure 1). This episode is a particular example of the moderate or time-varying correlation that HF returns display with standard asset classes. Starting in 2003, the industry saw a sharp increase in the creation of new hedge-funds, reflective of the augmentation of the size of the assets under management (AUM) in the hedge fund industry.

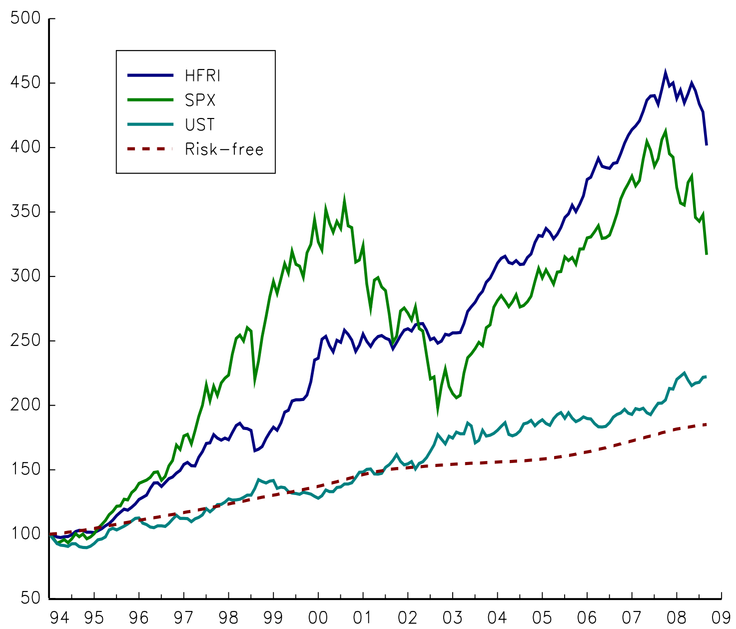
	HFRI	SPX	UST
Annualized Return	9.94%	8.18%	5.60%
1Y Volatility	7.06%	14.3%	6.95%
Sharpe	0.77	0.26	0.18

Table 1: Sharpe Ratios of HF, S&P500 (SPX) and 10y US Treasury Bond

2.1.2 Rationale behind HF replication

Even though, HF returns' characteristics make them an attractive investment, investing in hedge funds is limited for many investors due to regulatory or minimum size constraints, in particular for retail and institutional investors. Hedge funds as an investment vehicle have also suffered from several criticisms: lack of transparency of the management's strategy making it difficult

Figure 1: Performance between January 1994 and September 2008



to conduct risk assessment for investors; poor liquidity, particularly relevant in period of stress; and the problem of a fair pricing of their management fees. It is probably the declining average performance of the hedge fund industry coupled with a number of interrogations on the levels of fees [17] which led many major investors [4, page 5] to seek means of capturing hedge fund investments strategies and performance without investing directly into these alternative investment vehicles. Hence, the idea of replicating hedge funds' portfolios, already common in the context of equity portfolios, gained momentum.

2.1.3 Factor models

Starting with the work of Fung and Hsieh [13] as an extension of Sharpe's style regression analysis [33] to the world of hedge funds, factor-based models were first introduced as tools for performance analysis. The underlying assumption of Sharpe's style regression is that there exist, as in standard Arbitrage Pricing Theory (APT), a Return-Based Style (RBS) factor structure for the returns of all the assets that compose the investment world of the fund's manager [13, 33]. Factor-based models for hedge fund replication make a similar assumption but use Asset-Based Style (ABS) factors. While RBS factors describe risk factors, and are used to assess performance, ABS factors are directly selected with the purpose of being directly transposable into investment strategies. ABS factors have been used to take into account dynamic trading strategies with possibly nonlinear pay-off profiles [1, 15]. The idea of replicating a hedge fund's portfolio is therefore to take long and short positions in a set of ABS factors suitably selected so as to minimize the error with respect to the individual hedge fund or the hedge fund index.

A generic procedure for HF replication using factor models can therefore be decomposed in two steps. At step 1, one estimates a model of the HF returns as:

$$r_k^{\mathrm{HF}} = \sum_{i=1}^{m} w^{(i)} r_k^{(i)} + \varepsilon_k$$

Given the estimated positions $\hat{w}^{(i)}$ on ABS factor $r^{(i)}$ resulting from step 1, step 2 simply constructs the "clone" of the hedge fund by:

$$r_k^{\mathrm{Clone}} = \sum_{i=1}^{m} \hat{w}^{(i)} r_k^{(i)}$$

The factor-based approach is thus very intuitive and natural. There are however several caveats to this exercise.

Contrary to the passive replication of equity indices, the replication of hedge funds returns must take into account key unobservable determinants of hedge funds investment strategies such as the returns from the assets in the manager's portfolio; dynamic trading strategies; or the use of leverage [13, 15]. Recall that hedge funds returns do not share the characteristics of more classic investment vehicles, e.g. mutual funds, and are relatively uncorrelated to the main asset classes — see, e.g., [13, Figures 1 and 2, page 280] and [14]. Broadly, the factor models approach is subject to two types of difficulties. One is the in-sample explanatory power of step 1 described above being extremely low. Possible explanations for this are either an absence of systematic risk exposure of the HF industry, or the occurrence of model specification risk due to a faulty selection of the set of factors. Two, the out-of-sample replication is of poor quality. This last difficulty can result, for example, from the presence of noise in the calibration process in step 1, or from a violation of the implicit stationarity assumption of the time series in the model.

One avenue which has been extensively illustrated along the past decade [13, 4, 18, *inter alia*] was to work on the set of factors to include in the model. The underlying tenet of this stream of literature is that both in-sample and out-of-sample poor performances of the factor model are linked to the choice of the factors. Several rationales for different factor selections, including economic arguments and statistical methodologies, have been tested throughout the literature. For example, to different types of HF strategies (Convertible Arbitrage, Fixed Income Arbitrage, Event Driven, Long/Short Equity, etc.) different sets of factors have been proposed. Arguably, one possible reason behind the poor performance of linear factor models is the presence of non-observable dynamic trading strategies producing nonlinear HF return profiles which will not be captured in a linear framework. Thus, in complement to observable factors[1], generally corresponding to asset classes, one proposed methodology [1, 15] is to build synthetic factors corresponding to known trading strategies, including for example the writing of options on equity indices. By construction, these synthetic factors can exhibit nonlinear returns. This methodology thus attempts to render, by means of linear factor models, nonlinearities in the HF returns by modeling the nonlinearities in the synthetic factors. The use of these synthetic factors has been shown to improve the performance of the replicating factor models. There are, however, practical and theoretical difficulties to the use of such a methodology, as explained by Amin and Kat [6]:

> *"First, it is not clear how many options and which strike prices should be included [...] Second, since only a small number of ordinary puts and calls can be included, there is a definite limit to the range and type of non-linearities that can be captured."*

[1] One set of factors often used can be found in [18].

Nonlinearities in HF returns have also been assessed in a more direct approach. Recently, factor models including an option factor have been used to assess the nature and the extent of the presence of nonlinearities in hedge fund returns [11]:

$$r_k^{\mathrm{HF}} = \sum_{i=1}^{m} w^{(i)} r_k^{(i)} + \delta \max\left(r_k^{(1)} - s,\, 0\right) + \varepsilon_k$$

where $r_k^{(1)}$ is an equity index, and s represents the strike (moneyness) of the option. As noticed in [5], this methodology has so far not been implemented as a direct replication process, but rather as an assessment tool for HF investors. As such, the results in [11] are interesting. For the global index, they cannot reject the null hypothesis of linear returns, while at the category index level, they reject the hypothesis of linearity of returns only for the event-driven and managed futures categories at the 5% level, and for fixed-income arbitrage at the 10% level. Furthermore, testing fund by fund and correcting for possible data snooping, they demonstrate that the hypothesis of linear returns can be rejected for only one fifth of the whole universe of hedge funds reported in the Lipper/TASS database. Breaking down the universe into hedge-fund following arbitrage-based strategies (convertible arbitrage, fixed-income arbitrage, and event-driven), equity-market neutral and long-short strategies, and finally directional strategies (global macro, emerging markets, and managed futures), their results indicate that respectively only 20%, $10-15\%$, and 20% of these three groups exhibit significant nonlinearity with respect to the market return. As the whole, these results suggest that looking at the indexes can be misleading. Moreover, theoretically, dynamic trading of standard assets results in nonlinear return profiles for perfect market timers [23] suggesting that rather than nonlinear factors, one should focus on models capable of capturing the dynamic allocation of HF managers. And, while not ruling out the necessity to model nonlinearities in factor models in some cases, they underline the fact that linear models are most of the time appropriate.

Besides the work on the set of factors, the literature also examined other issues whose results can be summarized in the following way. Overall, on a general basis, linear factor models fail the test of robustness — for a good review see [4] — giving poor out-of-sample results. It seems, however, that an economic selection of the factors provides significant improvement of the out-of-sample tracking error of the clone hedge fund over other statistical methodologies.

While fairly recent, attempts to capture the dynamic nature of the HF portfolio allocation have been explored in the literature using several methods. One such method, used extensively [16, 18, *inter alia*], is to use rolling-windows OLS where the coefficients $\left\{w_k^{(i)}\right\}$ at time t_k are estimated by running the OLS regressions of $\left\{r_\ell^{\mathrm{HF}}\right\}_{\ell=k-L}^{k-1}$ on the set of factors $\left\{r_\ell^{(i)}\right\}_{\ell=k-L}^{k-1}$ for $i = 1, \ldots, m$. A common choice for the window length L is 24 months, even though one could consider a longer time-span trading-off the dynamic character of the coefficients for more stable and more robust estimates. By means of an example, Roncalli and Teiletche [30] have demonstrated however that such methodology captures poorly the dynamic allocation, in particular in comparison with the Kalman filter (KF). As we extensively expose the use of KF and similar methodologies in the rest of the paper, we will say little for now, except to point out that despite its superiority, the use of KF estimation requires caution in its implementation, making the estimation of the positions $\left\{w_k^{(i)}\right\}$ a non-trivial affair. Markov Regime-Switching models have also been considered — see, e.g. [5]. The idea therein is that HF managers switch from a type of portfolio exposure to another depending on some state of the world, assumed to be discrete in nature. One possible interpretation is to consider that the active management consists of changing the asset allocation depending on two states of the economy (high and low). Justifying the number of states or their interpretation is however tricky.

Finally, one needs to say a word on the meaning and the purpose of replication especially as it pertains to the academic or the practical dimensions of this exercise. From a practical point-of-view, hedge fund replication cannot compete with the single best hedge funds. Instead, it can provide a significant part of the performance of the industry. Hence, if the academic interest in replication is foremost to assess performance — particularly with the goal of assessing the quality of management — and understand the structure of risk behind specific hedge funds, replication as a process to create investment vehicles will have better chances of succeeding if it aims at replicating an aggregate of funds, where the idiosyncratic management styles — the "talent"— are averaged out, letting instead emerging investment decisions made on a macro scale[2]. From this perspective, it is more important to focus on the feasibility of the replication portfolios rather than ever improving the in-sample explanatory power. If one can contest this last point on the basis that the other side of replication is also to help HF investors to understand the risks they are exposed to, it answers to the possible urge of always augmenting the number of factors by pointing at the fact that there exists only a limited number of asset classes which can be used as instruments. We certainly do not want to diminish the value of the academic exercise, rather we point here that our purpose is perhaps better understood as an answer to the practitioner's problem of capturing the better part of the HF industry's performance. Nonetheless, our hope is that a correct answer to the practitioner's problem in turn will lead to a better decomposition of what can and what cannot be replicated. From there, it will be time again to pick up the mantel of academia.

The rest of this section provides the different concepts that we use to address this problem, introducing first the notion of a tracking problem. Then, we show that the problem of HF replication can be seen as a particular instance of a tracking problem of which there exist other instances in finance, like replicating Global Tactical Asset Allocation (GTAA) strategies.

2.2 Definition of the tracking problem

We follow [7] and [28] in their definition of the general tracking problem. We note $\mathbf{x}_k \in \mathbb{R}^{n_x}$ the vector of states and $\mathbf{z}_k \in \mathbb{R}^{n_z}$ the measurement vector at time index k. In our setting, we assume that the evolution of $\mathbf{x}_k$ is given by a first-order Markov model:

$$\mathbf{x}_k = f\left(t_k, \mathbf{x}_{k-1}, \nu_k\right)$$

where f is a non-linear function and ν_k a noise process. In general, the state $\mathbf{x}_k$ is not observed directly, but partially through the measurement vector $\mathbf{z}_k$. Thus, it is further assumed that the measurement vector is linked to the target state vector through the following measurement equation:

$$\mathbf{z}_k = h\left(t_k, \mathbf{x}_k, \eta_k\right)$$

where h is a non-linear function, and η_k is a second noise process independent from ν_k. Our goal is thus to estimate $\mathbf{x}_k$ from the set of all available measurements $\mathbf{z}_{1:k} = \{\mathbf{z}_i, i = 1, \ldots, k\}$. The

[2] We will come back to this idea in greater details in the course of the paper.

goal in a tracking problem is to estimate the state variable $\mathbf{x}_k$, the current state of the system at time t_k, using all available measurement $\mathbf{z}_{1:k} = \{\mathbf{z}_\ell\}_{\ell=1:k}$.

Remark 1 *In the rest of the paper, the following system will be referred to as a tracking problem (henceforth TP):*

$$\begin{cases} \mathbf{x}_k = f(t_k, \mathbf{x}_{k-1}, \nu_k) \\ \mathbf{z}_k = h(t_k, \mathbf{x}_k, \eta_k) \end{cases} \tag{1}$$

2.3 GTAA and Hedge Fund Replication: similar exercises

In this section, we cast hedge fund replication into a larger framework. In this new framework, problems can be formulated as TPs, thus enabling the use of tools that have been developed to solve tracking problems. We start by introducing the notion of Global Tactical Asset Allocation. The point is to show that HF replication can be seen as part of a broader class of problems which pertains to uncovering the tactical allocation of a manager.

2.3.1 What is GTAA?

GTAA[3], or Global Tactical Asset Allocation, is an investment strategy that attempts to exploit short-term market inefficiencies by establishing positions in an assortment of markets with a goal to profit from relative movements across those markets. This top-down strategy focuses on general movements in the market rather than on performance of individual securities. For instance, as opposed to bottom-up managers who must decide which individual securities to overweight and underweight, a GTAA manager decides which country indexes to overweight and underweight.

GTAA is similar to other strategies like global macro hedge funds and balanced funds. Generally, the fund manager takes positions on liquid instruments (equity indexes, bond markets and currencies). Given a strategic diversified allocation, the fund manager tries to enhance the performance by overweighting and under-weighting asset classes, depending on the expected future performance over relatively short time periods. The asset allocation may change largely at a rebalancing date but generally it presents some persistence because we observe strong momentum in the performance of asset classes in medium term periods.

Contrary to GTAA, hedge fund managers may invest in a larger universe. A part of the universe is composed by the asset classes we found in GTAA strategy and another part of the universe is composed by other asset classes and strategies:

- stock picking strategies (which may be found in equity market neutral, long/short event driven hedge funds);
- high frequency trading;
- non-linear exposures using derivatives;
- illiquid assets (corresponding to distressed securities, real estate or private equity).

As we show in the following, both GTAA and hedge fund replication can be described as TPs.

[3] This definition is excerpted from wikipedia at `http://en.wikipedia.org/wiki/Global_Tactical_Asset_Allocation`.

2.3.2 Global Tactical Asset Allocation as a tracking problem

Let $r_k^{(\mathrm{F})}$ and $r_k^{(i)}$ be respectively the returns of the GTAA strategy ("the fund") and the returns of the i^{th} asset class ("the factor") at time t_k. We assume that:

$$r_k^{(\mathrm{F})} = \sum_{i=1}^{m} w_k^{(i)} r_k^{(i)}$$

and:

$$w_{(t)}^{(i)} = w_k^{(i)} \quad \text{for} \quad t \in [t_k, t_{k+1}]$$

$w_{(t)}^{(i)}$ is the weight of the i^{th} asset class or factor in the GTAA portfolio. The first equation is just a decomposition of the return of the fund into the individual returns of the factors times their weights. The second equation implies that the portfolio is not rebalanced between two observed dates. Thus, if the rebalancing dates are weekly, the sampling interval $\mathrm{d}t_k = t_k - t_{k-1}$ must be weekly.

To recover the time-varying allocation $w_k^{(i)}$, we may consider the following tracking problem:

$$\begin{cases} \mathbf{w}_k = f\left(t_k, \mathbf{w}_{k-1}, \nu_k\right) \\ r_k^{(\mathrm{F})} = \mathbf{r}_k^\top \mathbf{w}_k \end{cases}$$

where the vector of returns $\mathbf{w}_k = \left(w_k^{(1)}, \ldots, w_k^{(m)}\right)^\top$ is the state vector and $r_k^{(\mathrm{F})}$ is the measurement. If we assume that the weights change in a smooth way, we may specify the first-order Markov equation in the following way:

$$\mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k$$

It means that the weight $w_k^{(i)}$ of the i^{th} asset class at time t_k is equal to the previous weight $w_{k-1}^{(i)}$ plus a noise. The noise corresponds to the active bet of the asset manager between two observed dates t_{k-1} and t_k. This relationship is the more general formula one may consider if no other information about the GTAA strategy is available. From a tracking point of view, ν_k is stochastic. It may be difficult to model ν_k. For simplicity, we will assume the noise to be Gaussian: $\nu_k \sim \mathcal{N}(0, Q_k)$ Without other information about the GTAA strategy, it is obvious that $\mathbb{E}[\nu_k] = 0$. It means that we do not have any idea or information about the expected change in weights. In this case, we obtain the following linear state-space model:

$$\begin{cases} \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k \\ r_k^{(\mathrm{F})} = \mathbf{r}_k^\top \mathbf{w}_k \end{cases} \tag{2}$$

which may be easily solved by Kalman filtering (see Appendix A.1). The algorithm is then the following:

1. specify the covariance matrix Q_k of the noise process ν_k;
2. define the initial conditions $\hat{\mathbf{w}}_0$ and $\hat{\mathbf{P}}_0$ of the system;
3. compute recursively the Kalman Filter described by the system of equations (A-1);
4. obtain the estimates $\hat{\mathbf{w}}_{k|k}$ of the dynamic allocation.

Even if the algorithm is very simple to run and implement, it requires some fine tuning to perform well. In Appendix B.1, we provide an example and we discuss the parametrization of the matrices $\hat{\mathbf{w}}_0$, $\hat{\mathbf{P}}_0$ and Q_k and their impacts on Kalman filtering.

2.3.3 Link Between GTAA and HF replication

We may decompose the return of a hedge fund into two components:

$$r_k^{(\mathrm{HF})} = \underbrace{\sum_{i=1}^{m} w_k^{(i)} r_k^{(i)}}_{\text{GTAA ABS factors}} + \underbrace{\sum_{i=m+1}^{p} w_k^{(i)} r_k^{(i)}}_{\text{HF ABS factors}}$$

The idea of HF replication is to replicate the first part. Let's note $\eta_k = \sum_{i=m+1}^{p} w_k^{(i)} r_k^{(i)}$. The TP system becomes:

$$\begin{cases} \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k \\ r_k^{(\mathrm{HF})} = \mathbf{r}_k^\top \mathbf{w}_k + \eta_k \end{cases} \tag{3}$$

The similarity between (2) and (3) is obvious, and explains in part why we consider the two problems of replicating a GTAA strategy and HF replication as belonging to the same class of approaches. This is not the only similarity. To develop this point, we shall come back to a point we made earlier without much explanation. As we pointed above, one can distinguish between the academic and the practitioner's points-of-view on replication where the former is more inclined than the latter to stress on explanation over implementation issues. We contained that HF replication as a process to develop investment vehicles has better chances of success if it aims at replicating an aggregate of hedge funds rather than single hedge funds. The argument rests on the efficiency of liquid markets. An example is often worth more than a lengthy argument, so let's take a closer look to the HF industry.

Diez de los Rios and Garcia [11, p.15] report that a large proportion of the HF industry, about 30%, follows Long/Short Equity strategies[4]. The performance of a single HF following an L/S Equity strategy is explained by its proprietary model of stock picking and its proprietary model to choose its beta, such that its portfolio will be long of a 100% of the selected stocks, and short of x% of its benchmark index. It is almost impossible to determine without inside information the portfolio of stocks picked by the HF manager as it depends on its targeted risk profile and the private views of the managers. However, because of the efficiency of liquid markets, as an aggregate the performance of all the L/S Equity HF will be proportional to $1 - \bar{x}$, where $\bar{x}$ is the average taken over all L/S Equity funds of their exposure. In other words, the performance of the aggregate will be proportional to the beta of the entire industry, and the idiosyncratic decisions of each managers are be averaged out. It is worth noting that in this case, as the underlying asset classes are standard, replicating an aggregate of L/S Equity HF is about the same exercise as replicating a GTAA strategy, and this point is all the more salient that other strategies followed by HF are not represented in a proportion equivalent to the L/S Equity HF, as Fung and Hsieh [16] pointed out.

Seemingly, one weakness of the approach we propose is that only the beta of HF strategies seems to matters while one could rightly argue that an attractive feature of investing in single HF is the promise of absolute performance. Even in the case of L/S equity strategies, Fung and Hsieh [16] further argued in their paper that they produce "portable" absolute over performances, which they termed "alternative alphas", that are not sensitive to traditional asset classes. We contend

[4] Fung and Hsieh [16] report further that in March 2003 about 40% of the HF reported in the TASS database list Long/Short Equity as their primary investment style. There are historical reasons for that. L/S Equity strategy was the strategy used by the first HF on record, created in 1949 by A.W. Jones.

however, as our decomposition above between GTAA ABS factors and HF ABS factors hinted at, that one must be realistic between what can and what cannot be replicated. If HF performances can be divided between a beta component and a non-replicable alpha component, it is because HF managers engage in trading at high-frequencies or in illiquid assets, thus benefiting from local and transient market inefficiencies or illiquidity premia. We provide in sections 5 and 6 material to substantiate those claims. Moreover, if, as we demonstrate later, considering these typical HF ABS factors is very useful to explain the performance of the HF industry, those items cannot in good measure be replicated from an investment perspective. Thus, we need already to point out (section 4.4 explores this with greater depth) that not all of the HF strategies can be successfully replicated using the method we advocate in this paper. This is perhaps the one good news to the HF industry, as even though we will demonstrate one can truly capture a substantial part of the performance of the industry as a whole, they individually still retain some edge, particularly those practicing true alternative strategies.

3 Capturing Tactical Allocation with Bayesian Filters

3.1 Bayesian filters

The prior density of the state vector at time k is given by the Chapman-Kolmogorov equation:

$$p(\mathbf{x}_k \mid \mathbf{z}_{1:k-1}) = \int p(\mathbf{x}_k \mid \mathbf{x}_{1:k-1})\, p(\mathbf{x}_{k-1} \mid \mathbf{z}_{1:k-1})\, \mathrm{d}\mathbf{x}_{k-1} \tag{4}$$

where we used the fact that our model is a first-order Markov model to write $p(\mathbf{x}_k \mid \mathbf{x}_{1:k-1}, \mathbf{z}_{1:k-1}) = p(\mathbf{x}_k \mid \mathbf{x}_{1:k-1})$. This equation is known as the *Bayes prediction step*. It gives an estimate of the probability density function of $\mathbf{x}_k$ given all available information until $k-1$. At time k, as a new measurement value $\mathbf{z}_k$ becomes available, one can update the probability density of $\mathbf{x}_k$:

$$p(\mathbf{x}_k \mid \mathbf{z}_{1:k}) \propto p(\mathbf{z}_k \mid \mathbf{x}_k)\, p(\mathbf{x}_k \mid \mathbf{z}_{1:k-1}) \tag{5}$$

This equation is known as the *Bayes update step*. The Bayesian filter corresponds to the system of the two recursive equations (4) and (5). In order to initialize the recurrence algorithm, we assume the probability distribution of the initial state vector $p(\mathbf{x}_0)$ to be known.

Using Bayesian filters, we do not only derive the probability distributions $p(\mathbf{x}_k \mid \mathbf{z}_{1:k-1})$ and $p(\mathbf{x}_k \mid \mathbf{z}_{1:k})$, but we may also compute the best estimates $\hat{\mathbf{x}}_{k|k-1}$ and $\hat{\mathbf{x}}_{k|k}$ which are given by:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbb{E}[\mathbf{x}_k \mid \mathbf{z}_{1:k-1}] = \int \mathbf{x}_k p(\mathbf{x}_k \mid \mathbf{z}_{1:k-1})\, \mathrm{d}\mathbf{x}_k$$

and:

$$\hat{\mathbf{x}}_{k|k} = \mathbb{E}[\mathbf{x}_k \mid \mathbf{z}_{1:k}] = \int \mathbf{x}_k p(\mathbf{x}_k \mid \mathbf{z}_{1:k})\, \mathrm{d}\mathbf{x}_k$$

3.2 Implementing Bayesian filters: Kalman and particle filters

There exist many sorts of Bayesian filters, in forms as diverse as one can define the different density functions. Notice that there is a direct correspondence between Bayesian filters and tracking problems. In this paper, we look only at a couple of methods to run Bayesian filters.

When looking at Bayesian filters, the first distinction should be between the type of state variables. On the one hand, in the case of a state variable with a finite number of discrete states, one can use Grid-based methods to get an optimal solution to the Bayesian filter, independently of the form of the density functions. On the other hand, if the state variable is continuous, then there exists no method in general providing an optimal solution, except for the normal case. Since the Gaussian family is its own conjugate, models with Gaussian densities have a particular attraction. If, furthermore, the functions f and h in (1) are linear, then the optimal solution of the Bayesian filter is given by the Kalman filter. In Appendix A.1, we give the recursion equations of the Kalman filter algorithm. Moreover, in the case where the noise densities are Gaussian but the functions f and h are nonlinear, one can use an approximate method called Extended Kalman filter (EKF) where the functions f and h are replaced by local linear approximation using their first derivatives at each recursion. In the more general case of non Gaussian densities, one has to resort to sub-optimal algorithms, called particle filters, to approximate the solution to the Bayesian filter. The idea behind particle filters is rather simple. Since no closed-form solution to the tracking problem can be found in general, one simply simulate at each step a sample of particles which will be used to provide a discrete estimation of the density function, the filtering density, $p(\mathbf{x}_k|\mathbf{z}_{1:k})$. We give a more formal description of particle filters (PF) in Appendix A.2. For further details on specific algorithms implementations of PF algorithms, we refer the reader to [7].

Remark 2 *All the computations done in this paper on particle filters have been done using the public domain Gauss library PF [31] with 50000 particles whereas we have used the Gauss library TSM [29] for Kalman filter.*

In the context of this paper, following the TP as defined in (3), we are in presence of continuous state variables. In a first step, to gain better understanding of the advantages of using the tracking problem's formalization as well as Bayesian filters to answer the problem at hand, we will examine HF replication in a Gaussian linear framework using KF. In a second time, to allow for more flexible specification of the density function, as well as discuss possible nonlinear effects, we will consider the use of the PF algorithms.

4 Hedge Fund Replication: The Gaussian Linear Case

In this section, we start by providing an intuition of the inner workings of the KF algorithm, before, in a second time, taking an example to demonstrate the capacity of KF to determine plausible weights for a replicating portfolio of a standard HF index. Further, we show that the replicating portfolio provides a qualitatively sensible explanation for the behavior of the HFRI index over the period 1994-2008, while enabling to capture a significant part of its performance[5]. Finally, we look into the type of strategies that one could consider replicating in the HF industry.

4.1 Understanding linear Gaussian approach and Bayesian filtering to replication strategies

While we present in Appendix A.1 the classic KF algorithm, the set of equations (A-1) provides little insight on how the estimated weights are dynamically changed by the algorithm. In the

[5] To be more precise, the study period for all the computations done in the rest of this paper begins in January 1994 and ends in September 2008.

following, we provide another representation of the KF algorithm in terms of innovations which then allow us to explain with finer details the dynamic adjustments of the recursion.

4.1.1 Innovation representation of linear state-space models

It is difficult to use the representation (A-1) to understand how the Kalman filter changes dynamically the weights. We, thus, introduce another representation based on the tracking error. By combining the expressions of the system (A-1), one may rewrite $\hat{\mathbf{x}}_{k+1|k}$ as follows:

$$\hat{\mathbf{x}}_{k+1|k} = \left(F_{k+1} - K_k H_k\right) \hat{\mathbf{x}}_{k|k-1} + K_k \mathbf{z}_k + \left(c_{k+1} - K_k d_k\right)$$

where $K_k = F_{k+1} \hat{\mathbf{P}}_{k|k-1} H_k^\top \hat{V}_k^{-1}$ is the gain matrix. It comes that:

$$\hat{\mathbf{x}}_{k+1|k} = F_{k+1} \hat{\mathbf{x}}_{k|k-1} + c_{k+1} + K_k \left(\mathbf{z}_k - H_k \hat{\mathbf{x}}_{k|k-1} - d_k\right)$$

Finally, the innovation representation is given by:

$$\left\{ \begin{array}{l} \mathbf{z}_k = \mathbf{d}_k + H_k \hat{\mathbf{x}}_{k|k-1} + \hat{\mathbf{e}}_k \\ \hat{\mathbf{x}}_{k+1|k} = \mathbf{c}_{k+1} + F_{k+1} \hat{\mathbf{x}}_{k|k-1} + K_k \hat{\mathbf{e}}_k \end{array} \right. \tag{6}$$

In this representation, the two noise processes ν_k and η_k have been replaced by the innovation process $\hat{\mathbf{e}}_k$, and the transition equation is defined on the estimate of the state vector $\hat{\mathbf{x}}_{k|k-1}$, and not directly on the state vector $\mathbf{x}_k$.

In the case of the tracking problem (3), it becomes:

$$\left\{ \begin{array}{l} r_k^{(\mathrm{HF})} = \mathbf{r}_k^\top \hat{\mathbf{w}}_{k|k-1} + \hat{e}_k \\ \hat{\mathbf{w}}_{k+1|k} = \hat{\mathbf{w}}_{k|k-1} + \hat{\mathbf{P}}_{k|k-1} \mathbf{r}_k \left(\hat{e}_k / \hat{V}_k\right) \end{array} \right.$$

At time index k, KF performs an update of the previous weights estimates $\hat{\mathbf{w}}_{k|k-1}$ by applying the correction term $\hat{\mathbf{P}}_{k|k-1} \mathbf{r}_k \hat{e}_k^\star$ where $\hat{e}_k^\star = \hat{e}_k / \hat{V}_k$ is the normalized tracking error. For the i^{th} factor, it comes that:

$$\Delta \hat{w}_{k+1|k}^{(i)} = \hat{w}_{k+1|k}^{(i)} - \hat{w}_{k|k-1}^{(i)} = \hat{e}_k^\star \sum_{j=1}^{m} \left(\hat{\mathbf{P}}_{k|k-1}\right)_{i,j} r_k^{(j)}$$

4.1.2 Interpretation of the Kalman filter algorithm

We are now in a position to explain how KF adjusts the weights between two rebalancing dates. Here are some facts to understand the statistical prediction-correction system behind KF.

1. First, notice that the larger the normalized tracking error $\hat{e}_k^\star$, the larger the change in the allocation:

   $$\hat{e}_k^\star \nearrow \Rightarrow \left| \Delta \hat{w}_{k+1|k}^{(i)} \right| \nearrow$$

   This remark compounds three smaller ones.

   (a) Suppose that the last tracking error $\hat{e}_k$ is 1 and that $\hat{V}_k$ is 1. KF will produce smaller correction on $\left| \Delta \hat{w}_{k+1|k}^{(i)} \right|$ than the case $\hat{e}_k = 2$ and $\hat{V}_k = 5$ because the former tracking error is relatively more important than the second (when taking into account the volatility of the tracking error).

(b) Suppose that in a recent past, the Kalman filter has faced a lot of large errors. For a given value of $\hat{e}_k$, the normalized tracking error $\hat{e}_k^\star$ will be smaller than when the Kalman filter has done smaller errors in the recent past since from a statistical point of view, the covariance matrix $\hat{V}_k$ of the tracking error is larger in this case. We have:

$$\hat{V}_k \nearrow \Rightarrow \left|\Delta\hat{w}_{k+1|k}^{(i)}\right| \searrow$$

Thus, the change in weights at time k depends not only on the size of the tracking error but also on the past behavior of the Kalman filter. *Ceteris paribus*, the smaller the recent past errors, the more the algorithm will react to the last observed tracking error.

(c) $\hat{e}_k^\star$ is a relative measure of the correction on $\left|\Delta\hat{w}_{k+1|k}^{(i)}\right|$, but it does not indicate the direction of changes:

$$\hat{e}_k^\star > 0 \Rightarrow \Delta\hat{w}_{k+1|k}^{(i)} > 0 \text{ or } \Delta\hat{w}_{k+1|k}^{(i)} < 0$$

2. Second, assume that $\hat{\mathbf{P}}_{k|k-1}$ is a diagonal matrix. The errors on the estimated weights are not correlated. The direction of change for the asset class i will then be given by the sign of $r_k^{(i)} \times \hat{e}_k$:

$$r_k^{(i)} \times \hat{e}_k > 0 \Rightarrow \Delta\hat{w}_{k+1|k}^{(i)} > 0$$

(a) If the replicating strategy has outperformed the fund's strategy ($\hat{e}_k < 0$), the Kalman filter will reduce the weights of the factors which have a positive return and will increase the weights of the factors which have a negative return:

$$\hat{e}_k < 0 \Rightarrow \left\{ \begin{array}{l} \Delta\hat{w}_{k+1|k}^{(i)} < 0 \text{ if } r_k^{(i)} > 0 \\ \Delta\hat{w}_{k+1|k}^{(i)} > 0 \text{ if } r_k^{(i)} < 0 \end{array} \right.$$

(b) The directions are then adjusted by the Kalman filter to take into account the volatility of the Kalman filter errors on the estimated weights. For the i^{th} factor, we have:

$$\Delta\hat{w}_{k+1|k}^{(i)} = \left(\hat{\mathbf{P}}_{k|k-1}\right)_{i,i} r_k^{(i)} \hat{e}_k^\star$$

If KF has made a lot of errors on the weight of one factor (which means that the weights have highly changed in the past), it will perform a large correction:

$$\left(\hat{\mathbf{P}}_{k|k-1}\right)_{i,i} \nearrow \Rightarrow \left|\Delta\hat{w}_{k+1|k}^{(i)}\right| \nearrow$$

3. Third, assume that $\hat{\mathbf{P}}_{k|k-1}$ is a not diagonal matrix. The correction done by KF takes into account of the correlations between the errors on the estimated weights:

$$\Delta\hat{w}_{k+1|k}^{(i)} = \hat{e}_k^\star \sum_{j=1}^{m} \left(\hat{\mathbf{P}}_{k|k-1}\right)_{i,j} r_k^{(j)}$$

Suppose that $\hat{e}_k < 0$ and $r_k^{(1)} > 0$. According to point 2 above, the weight of the first factor should be reduced. However, because of the correlations between the errors on the estimated, there may be an opposite correction $\Delta\hat{w}_{k+1|k}^{(1)} > 0$, because, for instance, the errors on the other factors are negatively correlated with the error on the first factor **and** the performance of the other factors is negative.

4. Finally, notice that when, at time index k, the replication strategy has the same performance as the fund's strategy, KF does not change the estimated weights:

$$\hat{e}_k = 0 \Rightarrow \hat{w}^{(i)}_{k+1|k} = \hat{w}^{(i)}_{k|k-1}$$

4.2 An example with a well-diversified Hedge Fund index

We consider the example of replicating the HFRI index as in [30]. In Appendix B.1, we discuss some of the nontrivial choices which must be taken when specifying the model. In our example, the model considered is[6]:

$$\begin{cases} r^{(\mathrm{HF})}_k = \sum_{i=1}^{m} w^{(i)}_k r^{(i)}_k + \eta_k \\ \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k \\ Q_k = \operatorname{diag}\left(\sigma_1^2, \ldots, \sigma_m^2\right) \end{cases} \tag{7}$$

As we chose to use the example presented in [30], the set of factors that served as a basis for this exercise is: an equity exposure in the S&P 500 index (SPX), a long/short position between Russell 2000 and S&P 500 indexes (RTY/SPX), a long/short position between DJ Eurostoxx 50 and S&P 500 indexes (SX5E/SPX), a long/short position between Topix and S&P 500 indexes (TPX/SPX), a bond position in the 10-years US Treasury (UST) and a FX position in the EUR/USD.

If the model's specification is an important issue, in order to build a HF tracker however, one also has to consider the set of factors. Thus, in a first step, we look at the problem of factors selection by considering as a possible alternative the factors used in [18]. We present in a second time the results of the model's estimation using a 6 factors (6F) model. In the next section, we then interpret those results introducing the useful concept of alternative beta.

4.2.1 Selection of the factors

If one compares the factors of [30] and [18], one can notice that the authors of these two papers used two different universes. Conversely to the set of factors presented above, the authors of this latter paper consider as replicating factors the spread between the Lehman BAA Corporate Bond Index and the Lehman Treasury Index (CREDIT), the Goldman Sachs Commodity Index total return (GSCI), the variation of the CBOE Volatility Index (VIX). Other authors consider a bond position in the long term German debt (BUND), FX position in the JPY/USD and USD/GBP or long/short position between MSCI EM and S&P 500 indexes (MXEF/SPX). One may wonder which set of factors is best suited for our exercise. A common way to compare two sets of factors is to define a factor selection process. Generally, a statistic built on the results of given by each set of factors is considered and the factors are chosen such that they optimize these statistics. If a formal statistical process is still an opened issue in that particular context, our ad hoc answer for now is inspired from the stepwise selection procedure.

Starting from the six factors model (6F) describe in [30], we introduced each of the supplementary factors listed above and estimated using KF and the model specified in (7). For each estimated replicating clone, we computed the annualized performance $\hat{\mu}_{1\mathrm{Y}}$ of the clone, the proportion π_{AB} of the HF performance explained by the clone, the corresponding yearly tracking σ_{TE}, and the linear correlation ρ, the Kendall tau τ and the Spearman rho ϱ between the monthly returns of

[6] In [30], the number of factors m is equal to 6.

the clone and the HF index. Results are reported in Table 2. We have further considered the case of deleting one of the original factors.

		$\hat{\mu}_{1Y}$	π_{AB}	σ_{TE}	ρ	τ	ϱ
	6F	7.55	75.93	3.52	87.35	67.10	84.96
+	CREDIT	7.35	73.91	3.51	87.46	67.30	85.11
+	GSCI	7.46	75.07	3.55	87.42	68.74	86.52
+	VIX	6.55	65.94	4.05	83.71	67.29	85.14
+	BUND	7.75	77.94	3.54	87.09	66.95	84.84
+	JPY/USD	7.37	74.18	3.56	87.02	66.42	84.23
+	USD/GBP	7.48	75.25	3.58	86.81	66.66	84.63
+	MXEF/SPX	7.56	76.06	3.03	90.68	72.92	89.94
−	SPX	6.42	64.56	6.31	47.51	32.19	45.82
−	RTY/SPX	7.08	71.20	4.66	75.92	54.02	73.55
−	SX5E/SPX	6.51	65.47	3.73	85.88	68.19	85.94
−	TPX/SPX	7.34	73.82	3.72	85.78	64.43	82.30
−	UST	7.86	79.13	3.50	87.47	66.92	84.79
−	EUR/USD	6.57	66.08	3.60	86.59	66.66	84.70
	7F	7.82	78.64	3.05	90.55	72.92	89.95

Table 2: Results of adding or deleting a factor when replicating the HFRI index

$\hat{\mu}_{1Y}$ is the annualized performance of the clone, π_{AB} the proportion of the HFRI index performance explained by the clone, σ_{TE} the corresponding yearly tracking error, ρ the linear correlation, τ the Kendall tau and ϱ the Spearman rho between the monthly returns of the clone and the HFRI index.

The first line presents the results for the model given in [30]. The lines preceded by + present the results for the original model plus the additional factor given in the second column. Similarly, the lines preceded by − present the results when one factor was deleted from the original (6F) model. Finally, the last line presents the results of the seven factors (7F) model containing SPX, RTY/SPX, SX5E/SPX, TPX/SPX, EUR/USD, BUND and MXEF/SPX.

When considering the results of this process, note first that this procedure is for now purely qualitative. Second, it is worth noticing that the original 6F model presents comparable results to any of the other considered models. Third, adding or deleting a factor is an important decision, and choosing to include a factor simply because it optimizes the historical backtest of the replicating model may not be the best criterion. In particular, performance of the clone should not be the only motivation. Other statistical criteria, like minimizing the volatility of the tracking error or maximizing the correlation measure between the returns of the clone and the benchmark, are choices possibly as valid as any. Note that following theses last two criteria, it appears that two factors may be valuable to add to the original model: the MXEF and the Bund factor. On the same ground, one may drop the UST factor. It is also worth noticing that in those particular cases, adding or deleting any of these three factors does not hurt the performance of the clone's historical backtest. Indeed, we estimated a final 7F model, using the previous remarks, and one can note that the results of this model are sensibly better in every aspect.

If one uses HF replication for performance analysis and risk management purposes, the previous selection process, even though it could gain in rigor, is probably enough as it provides a model with good characteristics. Conversely, it is hardly satisfactory from the point-of-view of building an investment vehicle. In that case, statistical characteristics of the model will sometimes have to give way to more practical considerations. Recall that a tenet of the philosophy behind HF replication for investment purposes is to alleviate some of the problems which confront some investors, such as illiquidity or simply implementation issues. Thus, factors such as CREDIT or VIX would certainly not be used for now because they are not liquid instruments. Similarly, factors like GSCI and MXEF are difficult to implement, because there exists no appropriate futures (MXEF) or their liquidity is not very good (GSCI). Replication strategies which incorporate these factors would certainly have to find other wrappers like ETF, or have to find other investment solutions. Because of this added complexity, we simply do not consider those solutions in the framework of this paper.

Finally, there is yet another modeling aspect to take into consideration. As we expose in Appendix B.3, it appears that to avoid identification problems during the estimation, it is better to keep the cross-correlations between the factors as low as possible. This is the reason why in the original model in [30], long positions on Russell 2000, Eurostoxx and Topix were not directly considered as explanatory factors because they would have had naturally high cross-correlation between them and with the long positions on the S&P500. Instead, it was preferred to use long/short positions between these indices and the S&P 500.

4.2.2 Results

For the purpose of this exposé, we thus choose to keep the original model described in [30]. In order to present realistic results, we assumed during the replication procedure that the exposure to each of the factors considered is done using futures[7] (hedged in USD) and that the sampling period is one month. The study period begins in January 1994 and ends in September 2008.

When estimating this model, there are several non trivial choices to make, concerning particularly the implementation of the estimation algorithm. We refer the reader to Appendix B.1 for a more detailed exposé of the difficulties one can encounter. We estimated the model described in (7), choosing to initialize the parameters $\mathbf{w}_0$ and $\mathbf{P}_0$ at

$$\begin{aligned} \mathbf{w}_0 &= 0 \\ \mathbf{P}_0 &= \mathbf{I}_{6\times 6}. \end{aligned}$$

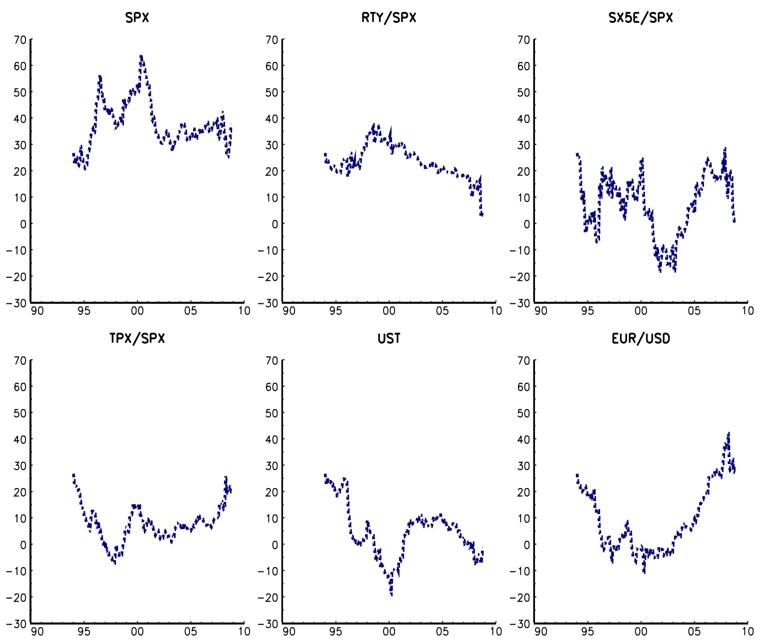
The estimates of the parameters are (in %) $\hat{\sigma}_\eta = 0.74$, $\hat{\sigma}_1 = 2.73$, $\hat{\sigma}_2 = 1.67$, $\hat{\sigma}_3 = 4.58$, $\hat{\sigma}_4 = 2.09$, $\hat{\sigma}_5 = 2.25$ and $\hat{\sigma}_6 = 2.52$. The resulting estimates weights are presented in Figure 2.

4.3 Interpretation of the results

In this section, we exploit the results of the previous estimation to demonstrate that replication using KF provides better replicators than traditional methods in the sense that it captures a better part of the performance of the HF benchmark while also providing estimated weights that possess a sensible explanation from an investment perspective. To do so, we first introduce the alternative Beta concept, before moving to an attribution performance of the replicating strategy.

[7] When the future does not exist, we approximate the monthly performance by the monthly return of the corresponding TR index minus the one-month domestic Libor and the hedging cost.

Figure 2: Estimated weights of the 6F model



4.3.1 The Alternative Beta concept

As mentioned in [18] and [30], we may compute performance attribution of the return $r_k^{(\mathrm{HF})}$ of hedge funds indices in severay ways. The first approach is to consider the traditional alpha/beta decomposition derived from the CAPM:

$$r_k^{(\mathrm{HF})} = \alpha_k + \beta_k$$

where β_k is the component return attributed to the benchmark. If we consider excess returns, we have:

$$r_k^{(\mathrm{HF})} - r_k^{(0)} = \left(r_k^{(\mathrm{HF})} - \left(\left(1 - \sum_{i=1}^{m} \hat{w}_{k|k-1}^{(i)} \right) r_k^{(0)} \right) \right) + \sum_{i=1}^{m} \hat{w}_{k|k-1}^{(i)} \left(r_k^{(i)} - r_k^{(0)} \right)$$

where $r_k^{(0)}$ is the return of the risk-free investment investment. In general, performance attribution is done directly on absolute returns[8]. In the traditional alpha/beta decomposition, we have:

$$\begin{cases} \alpha_k = r_k^{(\mathrm{HF})} - \beta_k \\ \beta_k = \left(1 - \sum_{i=1}^{m} \bar{w}^{(i)}\right) r_k^{(0)} + \sum_{i=1}^{m} \bar{w}^{(i)} r_k^{(i)} \end{cases}$$

where $\bar{w}^{(i)}$ are the fixed weights on the different asset classes. We may now consider another decomposition:

$$r_k^{(\mathrm{HF})} = \alpha_k^{\mathrm{AB}} + \beta_k^{\mathrm{TB}} + \beta_k^{\mathrm{AB}}$$

where β_k^{TB} is the traditional beta and β_k^{AB} is the alternative beta. We have:

$$\beta_k^{\mathrm{AB}} = \left(1 - \sum_{i=1}^{m} \hat{w}_{k|k-1}^{(i)} \right) r_k^{(0)} + \sum_{i=1}^{m} \hat{w}_{k|k-1}^{(i)} r_k^{(i)} - \beta_k^{\mathrm{TB}}$$

[8]In this case, we assume that the cash investment is part of the beta component.

Notice that the clone gives access to the sum of the traditional beta and the alternative beta:

$$r_k^{\text{Clone}} = \left(1 - \sum_{i=1}^{m} \hat{w}_{k|k-1}^{(i)}\right) r_k^{(0)} + \sum_{i=1}^{m} \hat{w}_{k|k-1}^{(i)} r_k^{(i)}$$

The term α_k^{AB} is called the alternative alpha. It is computed as follows:

$$\alpha_k^{\text{AB}} = r_k^{(\text{HF})} - r_k^{\text{Clone}}$$

Remark 3 *They are several ways to compute the fixed weights. One approach is to consider the mean of the dynamic weights:*

$$\bar{w}^{(i)} = \frac{1}{n} \sum_{k=1}^{n} \hat{w}_{k|k-1}^{(i)}$$

Another approach is to compute the OLS regression on the entire period $[t_0, t_n]$*:*

$$r_k^{(\text{HF})} = \sum_{i=1}^{m} \bar{w}^{(i)} r_k^{(i)} + \eta_k$$

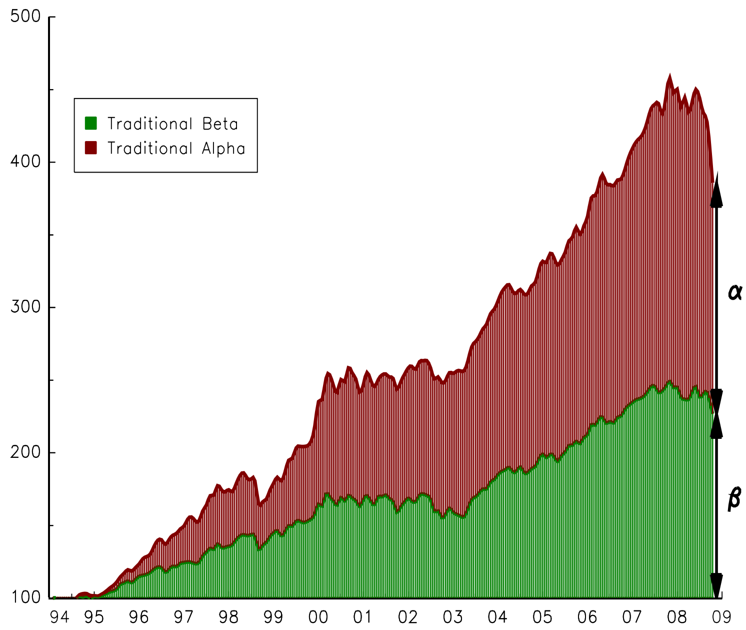
Finally, we may estimate the weights using the Kalman filter by imposing that $Q_k = \mathbf{0}_{m \times m}$*. In this case, the weights* $\bar{w}_k^{(i)}$ *correspond to the recursive OLS estimates.*

Table 3: Estimated yearly alpha (in %)

	Traditional		Alternative		Total
Period	Alpha	Beta	Alpha	Beta	
1994	0.43	1.13	0.68	0.88	1.56
1995	6.99	13.56	7.00	13.55	21.50
1996	11.77	8.35	12.18	7.95	21.10
1997	7.21	8.94	-2.61	19.93	16.79
1998	-3.98	6.87	-4.44	7.39	2.62
1999	15.56	13.62	7.96	21.61	31.29
2000	3.03	1.90	3.63	1.31	4.98
2001	4.08	0.53	2.11	2.47	4.62
2002	4.39	-5.59	0.74	-2.18	-1.45
2003	2.99	16.08	3.96	15.00	19.55
2004	1.23	7.71	1.83	7.08	9.03
2005	2.30	6.84	1.44	7.74	9.30
2006	2.32	10.33	1.10	11.67	12.89
2007	5.30	4.43	3.35	6.39	9.96
2008	-5.96	-5.13	-4.90	-6.19	-10.78
1994-2008	3.80	5.92	2.22	7.55	9.94
1997-2008	3.14	5.46	1.14	7.55	8.77
2000-2008	2.20	4.02	1.48	4.75	6.30

We have reported the performance attribution of α/β components in Figures 3 and 4. Notice that a large part of the HF returns are not explained by the traditional alpha but by the alternative beta. For the entire period, the alternative alpha explains about 23% of the HF returns whereas the alternative beta explains about 77%. In Table 3, we have computed the decomposition between alpha and beta for each years. Note that the alpha is overestimated using traditional beta.

Figure 3: Performance attribution between traditional alpha and beta



4.3.2 Performance attribution of the replicated strategy

In Figure 5, we have reported the performance attribution of the global asset classes (the performance of the individual exposures are given in Figure 6). It is obvious that the main contributor is the long equity exposure. However, it is interesting to remark that three other strategies have a good contribution. They are the two L/S equity strategies on small caps and Eurozone and the FX position EUR/USD. The last two other positions have a little performance: the L/S equity on Japan and the 10Y US Bond position. In a first approach, we may consider the elimination of factors which do not contribute to the performance of the clone. However, this point of view is not right because they may helps to track the volatility.

We can now explain the success of the HF industry between 2000 and 2003. In Figure 7, notice that the highest exposure of the HF industry to the directional Equity market was in March 2000 and represented more than 60%. After March 2000, the HF industry decreased the leverage on equity and modified the bets on L/S equity. In the bottom right graph, we compare the performance of the alternative beta strategy with respect to two other strategies. The first one uses the fixed allocation of March 2000 for all the asset classes and the second corresponds to the alternative beta, except for the directional equity exposure which is fixed and equal to the equity beta of March 2003. It appears that the relative good performance of the HF industry may be explained by two components:

(a) a first part which is the equity deleverage;

Figure 4: Performance attribution between alternative alpha and beta

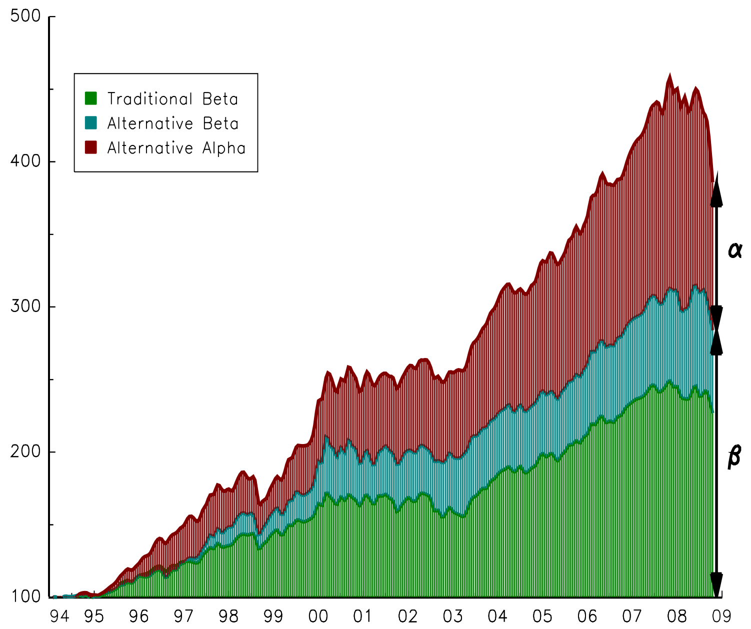


Figure 5: Performance attribution between Equity, L/S Equity, Bonds and FX asset classes

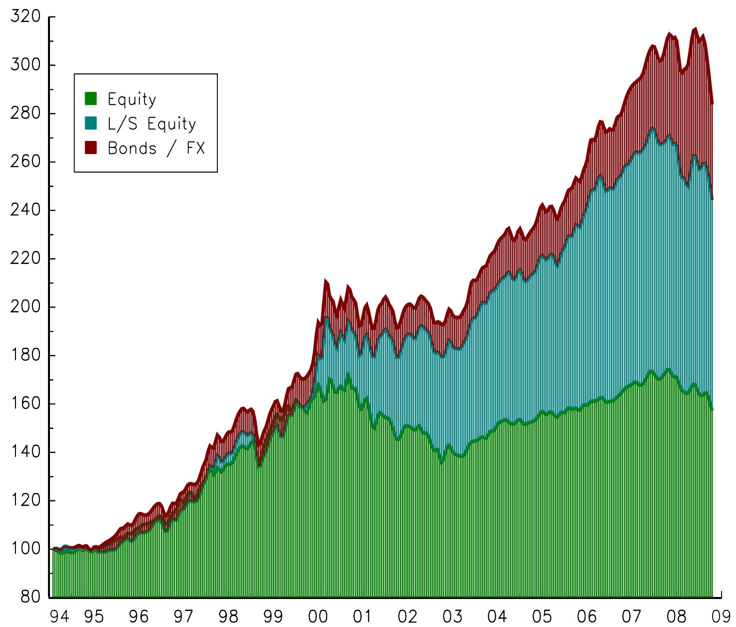
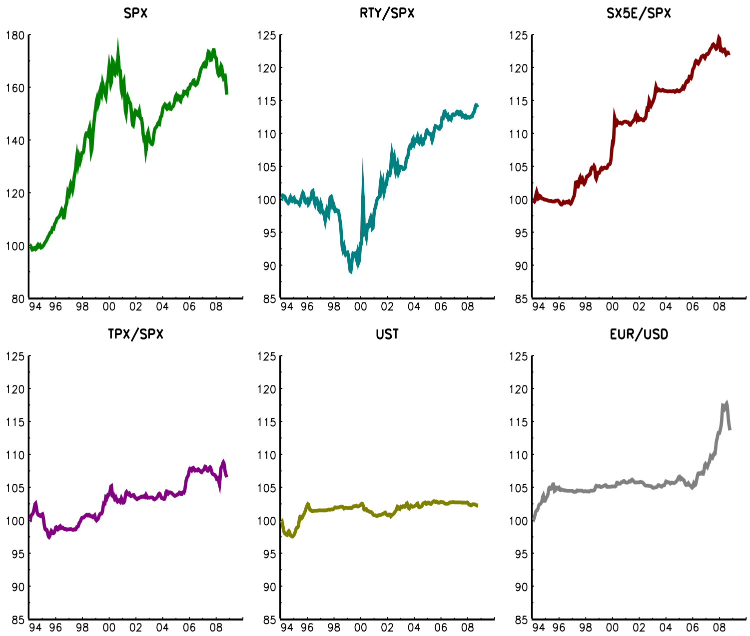


Figure 6: The performance of the individual exposures



(b) a second part which corresponds to the good bets on L/S equity on RTY/SPX and SX5E/SPX.

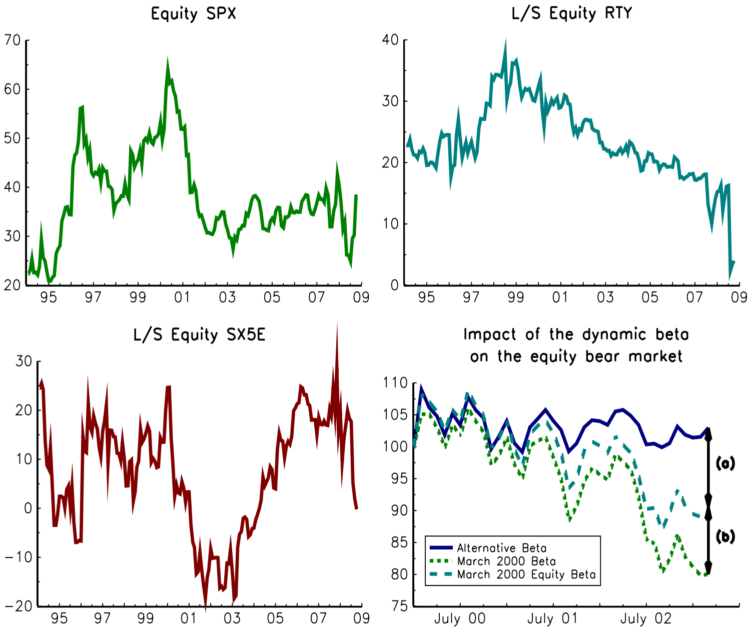
We estimate that with respect to the allocation of March 2000, the equity deleverage explains 40% of the outperformance whereas the reallocation of the L/S equity explains about 60% of the outperformance.

4.4 Which strategies may be replicated?

The example provided above is of course no proof that the methodology we have exposed so far is the panacea to the replication problem. Rather, the preceding example could almost be taken as a teaching case used to demonstrate the aptitudes to provide satisfying answers of the formulation we use of the replication problem. It is however important to understand better what types of strategies followed by the HF industry may subject themselves well to this replication process. To try to provide an answer to this problem, we thus estimated the 6F and 7F models on a series of HF indexes representing general categories of strategies.

In Tables 4, 5 and 6, we reported statistics about the HFRI strategy indices and their corresponding tracker using the 6F model. Results for the 7F model correspond to Tables 7 and 8. The different statistics reported are t_0 the beginning date of the study; $\hat{\mu}_{1\mathrm{Y}}$ the annualized performance; $\hat{\sigma}_{1\mathrm{Y}}$ the yearly volatility; s the sharpe ratio; γ_1 the skewness; γ_2 the excess kurtosis; $D_{1\mathrm{M}}$, $D_{3\mathrm{M}}$ and $D_{6\mathrm{M}}$ are respectively the drawdown for one, three and six months; $D_{\max}$ the maximum drawdown over the entire period; π_{AB} the proportion of the HF index performance explained by the tracker; σ_{TE} the yearly tracking error; ρ, τ and ϱ are respectively the linear correlation, the Kendall tau and the Spearman rho between the monthly returns of the HF index and the tracker. All the statistics are expressed in percents, except for the statistics s, γ_1 and γ_2.

Figure 7: Replication during the equity bear market



The key points of an analysis of our results can be summarized in the following way. Overall, HF trackers have smaller Sharpe ratios than their respective indexes, even though they generally exhibit lower volatilities. However, they also present a smaller risk if one measures risk as the maximum drawdown or as excess kurtosis of the returns. Finally, some strategies present low correlation with their respective trackers and one can thus conclude that they are difficult to replicate by the method employed here. This concerns mainly illiquid strategies (e.g. distressed securities), strategies with small betas (e.g. relative value) and strategies based on stock picking (like merger arbitrage or equity market neutral). Also of note, some tracker may not have a high correlation with their respective index, but may still exhibit similar performance. This is for example the case of funds of funds (FOF in the tables). One reason for this may be that part of the alternative betas of the underlying funds is captured by the fee structure of the FOF and thus do not appear in their performance, while the replicating process provides a direct access to this part of the performance.

It is also interesting to compare the statistics of the replicating strategies obtained using the two different models (6F) and (7F). As one could have expected after the results of the selection process above, the (7F) model performs in general better than the (6F) model on a number of accounts, providing better performances, lower volatility, lower volatility of the tracking error, better correlation of the returns of the tracker with its benchmark. One must however qualify these results. First, it is a well known fact that the inclusion of an additional factor leads to better fit results. This should be considered in light of the generally small increment in the statistics provided by the (7F) model over the (6F). Second, as we explained during the selection of the factors, from an investment point-of-view some of the factors in (7F) are not easily implementable, and any gain in performance may be offset by additional implementation costs these factors could involve. Third, the gain in the tracking performance is reflected, even if only slightly, by higher drawdowns of the (7F).

Table 4: Results by HF strategy indices — statistics of the HFRI index

t_0 is the beginning date of the study, $\hat{\mu}_{1Y}$ the annualized performance, $\hat{\sigma}_{1Y}$ the yearly volatility, s the Sharpe ratio, γ_1 the skewness and γ_2 the excess kurtosis. D_{1M}, D_{3M} and D_{6M} are respectively the drawdown for one, three and six months and D_{max} is the maximum drawdown over the entire period. All statistics are expressed in percents, except for the statistics s, γ_1 and γ_2.

Name	t_0	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2	D_{1M}	D_{3M}	D_{6M}	D_{max}
HFRI Event - Driven (Total)	01/94	11.27	6.51	1.03	-1.19	4.31	-8.90	-9.98	-10.50	-12.80
HFRI ED: Merger Arbitrage	01/94	8.96	3.83	1.17	-1.76	6.47	-5.69	-5.76	-3.84	-6.32
HFRI ED: Private Issue/Registered Daily	01/96	13.75	6.92	1.33	0.76	1.93	-3.97	-6.83	-9.34	-12.42
HFRI ED:Distressed / Restructuring	01/94	10.19	5.55	1.02	-1.53	7.04	-8.50	-12.43	-12.42	-12.83
HFRI Equity Hedge (Total)	01/94	11.79	9.03	0.80	0.04	1.99	-8.10	-12.65	-10.62	-17.96
HFRI EH: Energy / Basic Materials	01/95	20.55	18.61	0.84	0.26	1.21	-14.91	-25.22	-29.60	-37.08
HFRI EH: Equity Market Neutral	01/94	7.10	3.18	0.85	-0.09	1.38	-2.93	-5.36	-2.83	-5.36
HFRI EH: Quant. Directional	01/94	11.87	13.46	0.54	-0.45	0.62	-13.34	-16.89	-16.43	-28.17
HFRI EH: Short Bias	01/94	2.20	19.68	-0.10	0.28	2.54	-21.21	-29.51	-39.09	-52.01
HFRI EH: Technology / Healthcare	01/94	13.73	18.50	0.49	0.44	2.25	-15.16	-24.19	-29.07	-53.04
HFRI Emerging Markets (Total)	01/94	9.37	14.14	0.34	-0.93	4.31	-21.02	-25.99	-36.58	-43.37
HFRI Emerging Markets: Asia Excluding-Japan	01/94	5.88	12.80	0.12	0.01	0.23	-8.40	-13.87	-20.78	-33.21
HFRI Emerging Markets: Global	01/94	8.41	13.48	0.29	-2.15	14.55	-27.46	-31.20	-39.48	-44.45
HFRI Emerging Markets: Russia/E Europe	05/94	20.64	28.35	0.55	-0.23	4.30	-38.59	-49.15	-62.36	-70.29
HFRI Emerging Markets:Latin America	01/94	8.74	17.20	0.25	0.06	1.49	-15.63	-20.05	-25.37	-32.70
HFRI Macro (Total)	01/94	9.53	6.95	0.72	0.10	0.95	-6.40	-10.70	-7.90	-10.70
HFRI Macro:Syst. Diversified	01/94	11.72	7.52	0.95	0.18	-0.18	-4.41	-4.94	-4.26	-5.50
HFRI Relative Value (Total)	01/94	8.50	3.62	1.11	-2.76	15.22	-5.84	-6.74	-6.12	-8.03
HFRI RV: Yield Alternatives	01/94	7.95	6.89	0.51	-0.67	2.87	-7.90	-9.76	-9.70	-15.23
HFRI RV:Fixed Income - Asset Backed	01/94	8.58	4.25	0.97	-3.98	26.01	-9.24	-12.23	-13.11	-13.48
HFRI RV:Fixed Income - Conversion Arbeit	01/94	6.92	5.13	0.49	-3.90	28.25	-11.97	-14.36	-14.56	-20.96
HFRI RV:Fixed Income - Corporate	01/94	5.91	4.78	0.33	-1.80	6.15	-7.16	-12.41	-12.23	-13.23
HFRI RV:Multi - Strategy	01/94	6.57	3.65	0.60	-2.28	10.76	-6.28	-8.49	-8.06	-12.04
HFRI FOF: Conservative	01/94	6.36	3.80	0.52	-1.42	6.76	-5.76	-8.21	-6.67	-9.90
HFRI FOF: Diversified	01/94	5.95	6.38	0.25	-0.38	3.83	-7.75	-12.32	-14.51	-14.51
HFRI FOF: Market Defensive	01/94	8.29	5.59	0.69	-0.09	0.64	-5.42	-8.97	-8.30	-8.97
HFRI FOF: Strategic	01/94	6.88	9.01	0.28	-0.50	3.64	-12.11	-16.83	-19.35	-19.35
HFRI Fund of Funds Composite	01/94	6.42	6.08	0.34	-0.56	3.73	-7.47	-11.60	-13.08	-13.49
HFRI Fund Weighted Hedge Fund	01/94	9.94	7.06	0.77	-0.57	2.76	-8.70	-9.55	-9.95	-12.28

Table 5: Results by HF strategy indices — statistics of the 6F tracker

t_0 is the beginning date of the study, $\hat{\mu}_{1Y}$ the annualized performance, $\hat{\sigma}_{1Y}$ the yearly volatility, s the Sharpe ratio, γ_1 the skewness and γ_2 the excess kurtosis. D_{1M}, D_{3M} and D_{6M} are respectively the drawdown for one, three and six months and $D_{\max}$ is the maximum drawdown over the entire period. All statistics are expressed in percents, except for the statistics s, γ_1 and γ_2.

The factors used are an equity exposure in the S&P 500 index (SPX), a long/short position between Russell 2000 and S&P 500 indexes (RTY/SPX), a long/short position between DJ Eurostoxx 50 and S&P 500 indexes (SX5E/SPX), a long/short position between Topix and S&P 500 indexes (TPX/SPX), a bond position in the 10-years US Treasury (UST) and a FX position in the EUR/USD.

Name	t_0	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2	D_{1M}	D_{3M}	D_{6M}	$D_{\max}$
HFRI Event - Driven (Total)	01/94	8.24	6.00	0.63	-0.33	0.62	-5.07	-5.66	-6.48	-7.03
HFRI ED: Merger Arbitrage	01/94	5.62	2.92	0.44	-0.37	1.66	-2.45	-3.22	-3.70	-4.40
HFRI ED: Private Issue/Registered Daily	01/96	4.69	3.32	0.16	0.16	4.52	-3.16	-3.72	-3.90	-4.34
HFRI ED:Distressed / Restructuring	01/94	8.13	4.55	0.81	-0.18	0.72	-3.43	-4.49	-2.08	-4.49
HFRI Equity Hedge (Total)	01/94	6.36	8.83	0.23	0.30	3.21	-6.50	-10.51	-10.02	-17.52
HFRI EH: Energy / Basic Materials	01/95	8.51	13.37	0.30	-0.56	3.72	-15.90	-31.62	-33.61	-38.70
HFRI EH: Equity Market Neutral	01/94	5.01	1.69	0.41	0.04	0.63	-1.11	-0.80	-0.01	-1.11
HFRI EH: Quant. Directional	01/94	8.26	13.51	0.28	-0.32	1.55	-14.21	-18.50	-19.38	-27.46
HFRI EH: Short Bias	01/94	0.10	19.21	-0.21	0.53	2.59	-17.55	-25.00	-27.40	-54.74
HFRI EH: Technology / Healthcare	01/94	7.80	17.71	0.19	0.74	5.28	-14.96	-21.64	-23.00	-48.08
HFRI Emerging Markets (Total)	01/94	6.74	12.26	0.19	-0.30	1.63	-12.75	-21.49	-21.94	-31.31
HFRI Emerging Markets: Asia Excluding-Japan	01/94	4.85	9.82	0.06	-0.14	1.43	-9.53	-12.85	-12.41	-21.16
HFRI Emerging Markets: Global	01/94	5.07	11.70	0.06	0.10	2.14	-12.27	-20.61	-23.15	-32.56
HFRI Emerging Markets: Russia/E Europe	05/94	7.27	15.85	0.18	-0.52	2.62	-19.21	-23.91	-34.79	-58.56
HFRI Emerging Markets:Latin America	01/94	6.47	12.04	0.17	-0.25	1.16	-10.82	-18.88	-23.43	-42.94
HFRI Macro (Total)	01/94	6.67	5.97	0.38	0.36	2.06	-4.88	-6.87	-5.27	-6.87
HFRI Macro:Syst. Diversified	01/94	5.77	7.13	0.20	-0.92	2.20	-8.85	-11.81	-8.88	-17.78
HFRI Relative Value (Total)	01/94	5.77	2.75	0.52	0.16	1.18	-2.05	-2.31	-1.03	-3.40
HFRI RV: Yield Alternatives	01/94	6.50	4.73	0.45	-0.18	1.57	-3.82	-4.98	-2.87	-6.00
HFRI RV:Fixed Income - Asset Backed	01/94	4.36	1.01	0.07	0.91	3.77	-0.40	-0.12	0.14	-0.40
HFRI RV:Fixed Income - Conversion Arbeit	01/94	5.55	2.71	0.45	-0.43	3.28	-3.30	-3.13	-1.45	-4.20
HFRI RV:Fixed Income - Corporate	01/94	6.25	3.48	0.54	0.10	2.20	-3.00	-4.49	-4.59	-6.24
HFRI RV:Multi - Strategy	01/94	5.98	2.59	0.63	0.61	3.77	-2.31	-2.52	-0.52	-3.38
HFRI FOF: Conservative	01/94	5.41	2.83	0.38	0.20	1.70	-2.51	-3.51	-2.04	-4.30
HFRI FOF: Diversified	01/94	5.47	5.57	0.20	0.54	3.86	-4.85	-6.03	-4.96	-6.50
HFRI FOF: Market Defensive	01/94	5.68	3.31	0.40	0.13	1.61	-2.93	-4.82	-2.72	-5.19
HFRI FOF: Strategic	01/94	6.76	8.24	0.29	0.36	2.60	-6.10	-10.03	-8.37	-16.24
HFRI Fund of Funds Composite	01/94	5.92	5.36	0.29	0.57	3.45	-3.99	-5.65	-4.28	-7.01
HFRI Fund Weighted Hedge Fund	01/94	7.55	6.91	0.45	-0.02	2.25	-6.63	-8.61	-6.74	-9.44

Table 6: Results by HF strategy indices — statistics HFRI index / 6F tracker

t_0 is the beginning date of the study, π_{AB} the proportion of the HF index performance explained by the tracker and σ_{TE} the yearly tracking error. ρ, τ and ϱ are respectively the linear correlation, the Kendall tau and the Spearman rho between the monthly returns of the HF index and the tracker. All statistics are expressed in percents.

The factors used are an equity exposure in the S&P 500 index (SPX), a long/short position between Russell 2000 and S&P 500 indexes (RTY/SPX), a long/short position between DJ Eurostoxx 50 and S&P 500 indexes (SX5E/SPX), a long/short position between Topix and S&P 500 indexes (TPX/SPX), a bond position in the 10-years US Treasury (UST) and a FX position in the EUR/USD.

Name	t_0	π_{AB}	σ_{TE}	ρ	τ	ϱ
HFRI Event - Driven (Total)	01/94	73.08	4.16	78.20	59.58	78.55
HFRI ED: Merger Arbitrage	01/94	62.72	2.93	65.23	43.75	60.87
HFRI ED: Private Issue/Registered Daily	01/96	34.11	6.73	31.15	24.90	36.17
HFRI ED:Distressed / Restructuring	01/94	79.77	4.70	58.36	41.50	57.14
HFRI Equity Hedge (Total)	01/94	53.97	4.50	87.35	68.65	87.12
HFRI EH: Energy / Basic Materials	01/95	41.41	17.15	46.20	30.39	42.98
HFRI EH: Equity Market Neutral	01/94	70.59	2.84	45.77	32.31	44.52
HFRI EH: Quant. Directional	01/94	69.59	5.23	92.48	75.32	91.63
HFRI EH: Short Bias	01/94	4.48	10.18	86.34	70.69	87.18
HFRI EH: Technology / Healthcare	01/94	56.81	10.69	82.66	61.39	78.99
HFRI Emerging Markets (Total)	01/94	71.99	10.37	69.99	47.27	64.78
HFRI Emerging Markets: Asia Excluding-Japan	01/94	82.47	9.99	63.84	47.23	64.56
HFRI Emerging Markets: Global	01/94	60.29	10.52	65.94	46.14	62.47
HFRI Emerging Markets: Russia/E Europe	05/94	35.21	25.73	43.47	30.52	43.60
HFRI Emerging Markets:Latin America	01/94	73.96	13.61	61.68	39.54	55.36
HFRI Macro (Total)	01/94	69.97	5.71	61.82	44.21	62.28
HFRI Macro:Syst. Diversified	01/94	49.26	6.06	65.87	57.99	76.70
HFRI Relative Value (Total)	01/94	67.94	3.07	56.46	40.48	55.74
HFRI RV: Yield Alternatives	01/94	81.76	5.98	52.23	32.30	45.27
HFRI RV:Fixed Income - Asset Backed	01/94	50.81	4.29	7.53	4.75	6.77
HFRI RV:Fixed Income - Conversion Arbeit	01/94	80.26	4.36	52.48	30.30	43.74
HFRI RV:Fixed Income - Corporate	01/94	105.67	4.23	51.41	35.73	49.96
HFRI RV:Multi - Strategy	01/94	90.95	3.06	56.41	41.59	58.42
HFRI FOF: Conservative	01/94	85.13	2.98	63.09	47.63	65.09
HFRI FOF: Diversified	01/94	91.91	4.64	70.65	50.62	68.71
HFRI FOF: Market Defensive	01/94	68.56	5.09	44.05	29.13	41.17
HFRI FOF: Strategic	01/94	98.25	5.62	79.14	58.81	77.02
HFRI Fund of Funds Composite	01/94	92.31	4.23	73.40	53.74	71.94
HFRI Fund Weighted Hedge Fund	01/94	75.93	3.52	87.35	67.10	84.96

Table 7: Results by HF strategy indices — statistics of the 7F tracker

t_0 is the beginning date of the study, $\hat{\mu}_{1Y}$ the annualized performance, $\hat{\sigma}_{1Y}$ the yearly volatility, s the Sharpe ratio, γ_1 the skewness and γ_2 the excess kurtosis. D_{1M}, D_{3M} and D_{6M} are respectively the drawdown for one, three and six months and $D_{\max}$ is the maximum drawdown over the entire period. All statistics are expressed in percents, except for the statistics s, γ_1 and γ_2.

The factors used are an equity exposure in the S&P 500 index (SPX), a long/short position between Russell 2000 and S&P 500 indexes (RTY/SPX), a long/short position between DJ Eurostoxx 50 and S&P 500 indexes (SX5E/SPX), a long/short position between Topix and S&P 500 indexes (TPX/SPX), a long/short position between MSCI EM and S&P 500 indexes (MXEF/SPX), a bond position in the long term German debt (BUND) and a FX position in the EUR/USD.

Name	t_0	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2	D_{1M}	D_{3M}	D_{6M}	$D_{\max}$
HFRI Event - Driven (Total)	01/94	9.13	5.99	0.77	-0.35	0.81	-5.08	-5.68	-6.49	-7.31
HFRI ED: Merger Arbitrage	01/94	5.87	2.86	0.53	-0.52	1.82	-2.20	-3.35	-3.26	-4.59
HFRI ED: Private Issue/Registered Daily	01/96	4.18	3.12	0.01	-1.20	4.77	-4.35	-3.52	-3.78	-4.50
HFRI ED:Distressed / Restructuring	01/94	8.85	4.48	0.98	-0.18	2.17	-4.20	-4.68	-2.04	-4.89
HFRI Equity Hedge (Total)	01/94	6.99	8.93	0.29	-0.02	2.18	-6.86	-12.27	-9.65	-20.68
HFRI EH: Energy / Basic Materials	01/95	9.66	12.45	0.42	-0.24	2.31	-14.01	-26.12	-30.00	-34.80
HFRI EH: Equity Market Neutral	01/94	5.34	2.18	0.46	-0.25	0.77	-1.40	-1.82	-1.84	-2.09
HFRI EH: Quant. Directional	01/94	8.49	13.63	0.30	-0.51	0.93	-13.24	-19.16	-19.10	-33.05
HFRI EH: Short Bias	01/94	-0.19	19.12	-0.22	0.55	2.67	-17.34	-24.88	-27.11	-55.75
HFRI EH: Technology / Healthcare	01/94	7.22	17.39	0.16	0.03	2.36	-13.86	-24.89	-24.31	-55.40
HFRI Emerging Markets (Total)	01/94	8.30	13.41	0.29	-0.86	2.72	-18.13	-23.38	-28.31	-30.69
HFRI Emerging Markets: Asia Excluding-Japan	01/94	6.02	11.93	0.14	-0.64	1.19	-12.97	-16.88	-20.04	-28.48
HFRI Emerging Markets: Global	01/94	8.98	12.15	0.37	-0.56	2.05	-15.06	-20.69	-24.19	-24.63
HFRI Emerging Markets: Russia/E Europe	05/94	1.18	26.95	-0.11	-2.67	16.32	-56.89	-66.96	-73.60	-76.60
HFRI Emerging Markets:Latin America	01/94	5.92	13.60	0.11	-0.69	2.34	-17.49	-20.67	-23.50	-35.97
HFRI Macro (Total)	01/94	8.34	5.77	0.67	-0.29	0.71	-5.34	-5.18	-5.46	-6.04
HFRI Macro:Syst. Diversified	01/94	6.25	7.55	0.25	-1.04	2.77	-8.92	-14.17	-11.64	-19.50
HFRI Relative Value (Total)	01/94	6.82	2.77	0.88	0.25	1.73	-1.75	-2.52	-0.89	-3.60
HFRI RV: Yield Alternatives	01/94	7.48	4.27	0.72	-0.45	1.22	-3.86	-4.36	-2.91	-6.28
HFRI RV:Fixed Income - Asset Backed	01/94	5.58	1.94	0.64	0.06	7.04	-2.61	-1.64	-0.32	-2.61
HFRI RV:Fixed Income - Conversion Arbeit	01/94	5.67	2.67	0.50	-0.44	2.86	-3.08	-2.90	-1.41	-4.05
HFRI RV:Fixed Income - Corporate	01/94	6.97	3.56	0.72	0.07	3.16	-3.15	-4.40	-5.60	-7.00
HFRI RV:Multi - Strategy	01/94	6.59	2.62	0.84	0.37	2.51	-2.08	-2.54	-0.63	-3.57
HFRI FOF: Conservative	01/94	6.01	2.87	0.58	-0.08	1.82	-2.47	-3.57	-1.63	-4.16
HFRI FOF: Diversified	01/94	6.41	5.70	0.36	0.23	4.65	-6.43	-7.05	-6.22	-8.01
HFRI FOF: Market Defensive	01/94	6.65	3.65	0.62	-0.54	1.32	-3.10	-5.00	-2.79	-5.00
HFRI FOF: Strategic	01/94	7.79	8.18	0.41	0.05	1.92	-7.24	-9.98	-8.68	-14.05
HFRI Fund of Funds Composite	01/94	6.79	5.38	0.45	0.21	3.07	-4.67	-5.91	-4.05	-7.30
HFRI Fund Weighted Hedge Fund	01/94	7.82	6.94	0.49	-0.25	2.07	-7.23	-9.36	-8.71	-10.91

Table 8: Results by HF strategy indices — statistics HFRI index / 7F tracker

t_0 is the beginning date of the study, π_{AB} the proportion of the HF index performance explained by the tracker and σ_{TE} the yearly tracking error. ρ, τ and ϱ are respectively the linear correlation, the Kendall tau and the Spearman rho between the monthly returns of the HF index and the tracker. All statistics are expressed in percents.

The factors used are an equity exposure in the S&P 500 index (SPX), a long/short position between Russell 2000 and S&P 500 indexes (RTY/SPX), a long/short position between DJ Eurostoxx 50 and S&P 500 indexes (SX5E/SPX), a long/short position between Topix and S&P 500 indexes (TPX/SPX), a long/short position between MSCI EM and S&P 500 indexes (MXEF/SPX), a bond position in the long term German debt (BUND) and a FX position in the EUR/USD.

Name	t_0	π_{AB}	σ_{TE}	ρ	τ	ϱ
HFRI Event - Driven (Total)	01/94	80.98	4.11	78.71	60.87	79.83
HFRI ED: Merger Arbitrage	01/94	65.49	2.87	66.68	44.62	61.67
HFRI ED: Private Issue/Registered Daily	01/96	30.40	6.89	24.81	23.04	33.61
HFRI ED:Distressed / Restructuring	01/94	86.86	4.64	59.16	42.11	58.96
HFRI Equity Hedge (Total)	01/94	59.27	4.41	87.97	71.85	89.05
HFRI EH: Energy / Basic Materials	01/95	46.99	17.71	40.22	30.30	41.42
HFRI EH: Equity Market Neutral	01/94	75.32	2.79	50.96	35.32	49.15
HFRI EH: Quant. Directional	01/94	71.49	4.46	94.59	79.63	94.12
HFRI EH: Short Bias	01/94	-8.50	10.16	86.34	70.89	87.66
HFRI EH: Technology / Healthcare	01/94	52.59	10.60	82.74	62.46	80.38
HFRI Emerging Markets (Total)	01/94	88.66	6.88	87.66	64.97	83.23
HFRI Emerging Markets: Asia Excluding-Japan	01/94	102.30	7.42	82.21	61.65	79.86
HFRI Emerging Markets: Global	01/94	106.84	8.14	80.31	61.79	79.67
HFRI Emerging Markets: Russia/E Europe	05/94	5.72	24.85	59.57	47.34	63.79
HFRI Emerging Markets:Latin America	01/94	67.69	11.60	73.99	50.40	67.47
HFRI Macro (Total)	01/94	87.47	5.27	67.06	52.05	70.70
HFRI Macro:Syst. Diversified	01/94	53.37	6.02	68.07	59.93	77.71
HFRI Relative Value (Total)	01/94	80.22	3.12	55.06	39.97	55.38
HFRI RV: Yield Alternatives	01/94	94.05	5.67	57.04	33.45	47.00
HFRI RV:Fixed Income - Asset Backed	01/94	65.01	4.55	7.18	4.88	6.94
HFRI RV:Fixed Income - Conversion Arbeit	01/94	81.93	4.38	52.08	30.42	42.93
HFRI RV:Fixed Income - Corporate	01/94	117.97	4.20	52.51	36.50	52.91
HFRI RV:Multi - Strategy	01/94	100.24	3.00	58.61	45.47	62.45
HFRI FOF: Conservative	01/94	94.55	2.94	64.55	51.94	69.50
HFRI FOF: Diversified	01/94	107.86	4.16	76.81	56.56	73.60
HFRI FOF: Market Defensive	01/94	80.17	4.85	51.60	39.01	55.19
HFRI FOF: Strategic	01/94	113.15	5.07	83.00	65.57	82.42
HFRI Fund of Funds Composite	01/94	105.82	3.82	78.43	59.80	77.29
HFRI Fund Weighted Hedge Fund	01/94	78.64	3.05	90.55	72.92	89.95

Finally, on a more particular note, it is worth taking a look at two particular strategies. First, on the “Emerging Market: Russia/E. Europe” HFRI index, it is worth noting that both models perform particularly poorly, pointing at the fact that in our pool of factors, none had a strong relation with the economy of that region of the world. Second, the “Macro: Syst. Diversified” is the only case where both models produce a clone with higher drawdowns than the actual HFRI Macro: Syst. Diversified. In both of these two cases, it is probable that the first reason behind those poor results comes from the set of factors used. Another reason could be the inadequacy of factor models in those two cases, but one could ask why, if the concept of factor model is the underlying problem, our results do not show more results similar to these. These two cases are an illustration that the better results obtained with our replication methodology cannot replace a careful choice of the set of factors. It is also a sign that if a better selection methodology is found, it would still have to rely on some economic insight, echoing results found in the literature [4].

5 Hedge Fund Replication: The Non-Gaussian Non-Linear Case

As we have seen in the previous section, HF replication using the KF can provide good results, making it possibly the best method so far to estimate and implement HF clones. However, one may question the wisdom of using a Gaussian linear framework. Indeed, the distributions of HF returns are well known to exhibit skewness and excess kurtosis, and nonlinear effects have been documented in HF returns ever since the seminal paper [13] of Fung and Hsieh in 1997. In the following section, we relax the Gaussian and linear assumptions. Our goal here is less to provide an “off-the-shelf” solution to the problem of replication than to examine the impact of each assumptions on the quality of the replication. Note that some of the methods (especially those requiring particle filters) used in the following section require some careful implementation, as well as time to be carried out. The plan of the section is the following. We start by looking at the Gaussian distribution assumption. In a second time, we look at the problem of nonlinear assets. Our approach can then be decomposed into three main angles: replicating nonlinear assets; the use of option factors that are determined in a manner exogenous to the replication procedure; and finally, a very general and inclusive approach to the replication procedure using nonlinear assets.

5.1 The Gaussian distribution assumption

The Gaussian distribution is a fundamental assumption to the optimality of the use of KF for HF replication (or for rolling OLS regression as it is). It is however well known that return distributions of hedge funds exhibit negative skewness and positive kurtosis, rendering the use of a Gaussian framework faltering, and requiring at least inquiring into its adequacy. Moreover, one of the attractive features of the approach advocated by Kat [21, for a recent exposé] is to take into account in the replication process stylized facts — such as higher moments of the returns distribution, in particular skewness and kurtosis — in order to provide investors a more accurate exposition to the risk-return profile of the HF industry. Given the relative success of replicating hedge funds using the KF, it may not be necessary to introduce nonlinearities in the factors or in the model's structure to obtain a better replication process. A simple relaxation of the Gaussian assumption, particularly by taking into account the third and fourth moments of the distributions, may be enough to improve the results significantly.

To illustrate the departure from the Gaussian assumption, we reported in Figure 8 three comparative graphics using the results of the 6F model presented in the previous section. The top graphic compares the probability density function of the tracking errors $\hat{\mathbf{e}}_k$ obtained using KF (blue line) against a Gaussian approximation of the same density function (dashed green line). The bottom-left graphic compares the probability density function of the HFRI index returns $r_k^{(\mathrm{HF})}$ (blue line) against the probability density function of the returns of the replicating clone r_k^{Clone} (dashed green line). The bottom-right graphic compares the probability density function of the clone's returns (blue line) against a Gaussian approximation of the same distribution (dashed green line). One can make several comments on Figure 8. First, as illustrated, there is a clear violation of the Gaussian assumption for all three of the estimated distributions. However, not all departures are of the same magnitude. It is obvious that the Clone's distribution is the closest to a Gaussian distribution, probably as a consequence of the KF procedure. Most of the departure of the HFRI returns distribution seems to remain in the tracking error. Thus, in the following, we relax the Gaussian assumption on the distribution of the tracking errors, while keeping the rest of the model's structure (Gaussian innovations of the state variables and linear evolution equation).

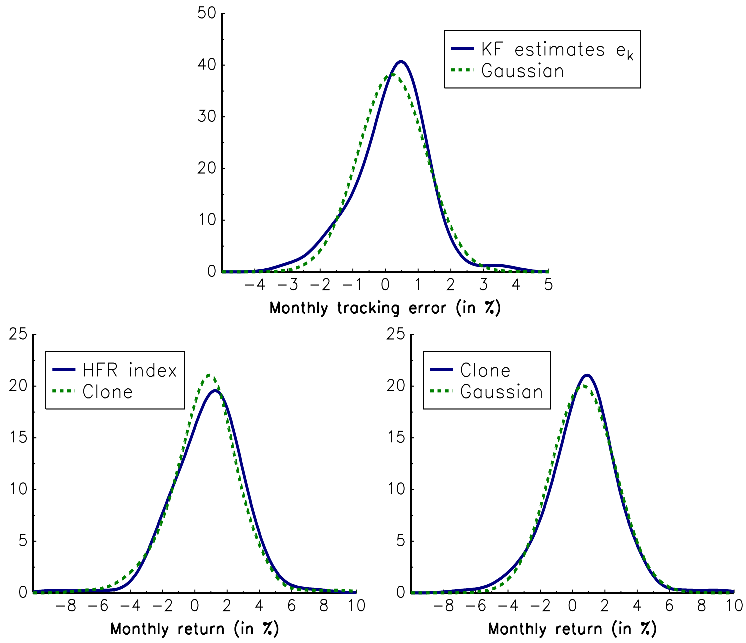


Figure 8: Departure from the Gaussian distribution assumption

Top: kernel estimate of the density function of the tracking error $\hat{\mathbf{e}}_k$ (blue line) vs. Gaussian approximation (dashed green line);

Bottom-left: kernel estimate of the density function of the HFRI index returns $r_k^{(\mathrm{RF})}$ (blue line) vs. kernel estimate of the density function of the replicating clone's returns r_k^{Clone} (dashed green line);

Bottom-right: kernel estimate of the density function of the replicating clone's returns r_k^{Clone} (blue line) vs. Gaussian approximation (dashed green line).

This extended tracking problem can be formalized as:

$$\begin{cases} r_k^{(\mathrm{HF})} = \sum_{i=1}^{m} w_k^{(i)} r_k^{(i)} + \eta_k \\ \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k \end{cases}$$

with η_k a general noise process with distribution $\mathcal{H}$. In the following, we assume that $\mathcal{H}$ is a Skew t distribution $\mathcal{ST}(\mu_\eta, \sigma_\eta, \alpha_\eta, \nu_\eta)$, obtained by perturbing a Student t distribution (for further details, cf. [8]). We hope to better capture the higher moments of the HFRI returns distribution. One could consider this methodology as one possible step toward incorporating some of the "sexiest" features of Kat's approach to the robust factor models approach. Note however that, since the returns are not normally distributed, we must resort to using particle filters to obtain the estimates $\hat{\mathbf{w}}_{k|k-1}$. The unknown parameters to estimate are $\theta = \{\sigma_1, \ldots, \sigma_m, \sigma_\eta, \alpha_\eta, \nu_\eta\}$. We consider two estimation methods.

(PF #1) A two-steps procedure consisting of a run of the KF algorithm to obtain ML estimates of $\{\hat{\sigma}_i, i = 1, \ldots, m\}$, followed by an ML estimation of the parameters σ_η, α_η, ν_η[9] of the Skew t distribution based on the tracking errors of the KF run.

(PF #2) A generalized method of moments (GMM) estimation procedure where the $m+3$ parameters are estimated together. The $m+3$ moments conditions are given by:

- the first moment considered is $m_{k,1} = e_k$ because we favor smaller tracking errors;
- the next moments are chosen to impose an orthogonality condition between the tracking error e_k and the return $r_k^{(j)}$ of the j^{th} asset: $m_{k,j+1} = e_k r_k^{(j)}$, $j = 1, \ldots, m$;
- The last two moments take into account the skewness and kurtosis of the hedge fund returns. They are defined as $m_{k,m+n-1} = \left(r_k^{\text{Clone}} - \bar{r}^{\text{Clone}}\right)^n - \mu_n$ $(n = 3, 4)$ where μ_n is the empirical n^{th} central moment of $r_k^{(\mathrm{HF})}$.

The statistics of the resulting clones obtained[10] are given in Table 9. Note that the estimation procedure using the GMM method is unfortunately extremely long and does not always converge to a solution. Compared to the KF results, notice that we obtain better results for the performance ($\hat{\mu}_{1\mathrm{Y}}$), but the volatility of the trackers' returns ($\hat{\sigma}_{1\mathrm{Y}}$) and the standard deviation of tracking errors (σ_{TE}) are also higher, providing only a small improvement in terms of the Sharpe ratios. The results on skewness and kurtosis are clearly disappointing as the sample values are comparable to those obtained by the KF estimation. One possible explanation for these poor results is that GMM makes a trade-off between the first-moment condition (maximizing π_{AB})

[9] By assumption, $\mu_\eta = 0$.

[10] The estimated values for the parameters are reported in the following table:

	PF #1	PF #2	PF #3
$\hat{\sigma}_1$	0.027	0.037	0.037
$\hat{\sigma}_2$	0.017	0.016	0.016
$\hat{\sigma}_3$	0.046	0.044	0.044
$\hat{\sigma}_4$	0.021	0.014	0.014
$\hat{\sigma}_5$	0.023	0.022	0.022
$\hat{\sigma}_6$	0.025	0.026	0.026
$\hat{\sigma}_\eta$	0.009	0.003	0.003
$\hat{\alpha}_\eta$	-1.131	-1.130	-10.00
$\hat{\nu}_\eta$	3.738	3.757	3.757

and the last two moment conditions (matching skewness and kurtosis). It does not mean however that building clones with more kurtosis and negative skewness is not possible. Let's consider for example a third set of estimates for the parameters of the Skew t distribution

(PF #3) The estimates are those of (PF #2) except for the parameter $\hat{\alpha}_\eta$ which is forced to -10.

As reported in Table 9, in this case, the tracker's returns present higher kurtosis but the tracking error's volatility is higher too. Other possible explanations for the poor success of these methods are the Gaussian dynamics of the state variables or the lack of non-linear exposures in the tracker. It is for now difficult to test for the first hypothesis as the number of parameters to estimate would grow significantly — it would be a 6-variate Skew t distribution on the state vector — and the execution time of the procedure would be absurdly long. As for the second hypothesis, we address it in the rest of this section on HF replication in the non-Gaussian non-linear case.

Table 9: Results with a Skew t distribution $\mathcal{ST}(0, \sigma_\eta, \alpha_\eta, \nu_\eta)$

	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2
HF	9.94	7.06	0.77	-0.57	2.76
LKF	7.55	6.91	0.45	-0.02	2.25
PF #1	7.76	7.44	0.45	-0.03	2.02
PF #2	7.57	7.28	0.43	-0.11	1.93
PF #3	6.90	7.99	0.31	-0.57	2.88
	π_{AB}	σ_{TE}	ρ	τ	ϱ
LKF	75.93	3.52	87.35	67.10	84.96
PF #1	78.09	4.03	84.71	63.49	81.94
PF #2	76.13	4.25	82.51	61.60	80.20
PF #3	69.43	5.11	77.62	54.75	73.55

5.2 Taking into account non-linear assets

Considering non-linear assets as factors in the replication model does not change the structure of the TP system. It suffices to notice that by considering a universe of factors composed of respectively m_1 and m_2 linear and nonlinear assets, the TP can still be written as:

$$\begin{cases} r_k^{(\mathrm{HF})} = \sum_{i=1}^{m_1} w_k^{(i)} r_k^{(i)} + \sum_{i=m_1+1}^{m_1+m_2} w_k^{(i)} r_k^{(i)} + \eta_k \\ \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k, \end{cases}$$

and even though some factors are "nonlinear" assets, the exposures $w_k^{(i)}$ are still linear and the TP system may be solved in the same way as in the previous section. The difficulty however with non-linear assets is to price the corresponding strategy. There are often only two possibilities:

1. Build ourselves the non-linear strategy. In this case, we have to calibrate the different parameters of the model, compute the backtest and use the backtest of the strategy as the non-linear factor.

2. Use custom indexes provided by investment banks like JP Morgan, Goldman Sachs, etc.

The second solution is often easier to implement because the first method assumes that we have the capacities to trade the strategy. It may however introduce biases because the performance of the index taken as factor depends on the proprietary strategy and on the market data of the index provider.

One must say that this methodology is certainly not new, and has been used, under one form or another by various authors [1, 2, 15, *inter alia*], with a relative success in increasing the explanatory power of the replication model. However, considering the difficulty of pricing such nonlinear assets, the question of whether the inclusion of a nonlinear asset can susceptibly provide a better replication methodology is of particular value. Indeed, the argument has been made that the component of HF returns due to non-linear assets in their portfolios can be partially replicated using the alternative beta methodology presented above since options may be replicated by delta hedging, i.e. taking linear positions in standard assets. We examine this claim in the following section before considering the introduction of option factors in the model.

5.2.1 Replicating non-linear assets

As argued above, since options may be replicated by delta hedging, the component of HF returns due to non-linear assets could theoretically be partially replicated by alternative beta. The argument however is more relevant as a marketing strategy for brokers of HF replicators than truly robust. To illustrate our claim, we provide below an example of the replication of a non-linear asset whose underlying strategy is well known using a Kalman filter and the methodology presented above. We consider the replication of the CBOE S&P 500 BuyWrite index more commonly known under the name BXM. The description of the BXM is the following[11]:

> *The BXM is a passive total return index based on buying an S&P 500 stock index portfolio, and selling the near-term S&P 500 Index call option, generally on the third Friday of each month. The SPX call written will have about one month remaining to expiration, with an exercise price just above the prevailing index level (i.e., slightly out of the money). The SPX call is held until expiration and cash settled, at which time a new one-month, near-the-money call is written.*

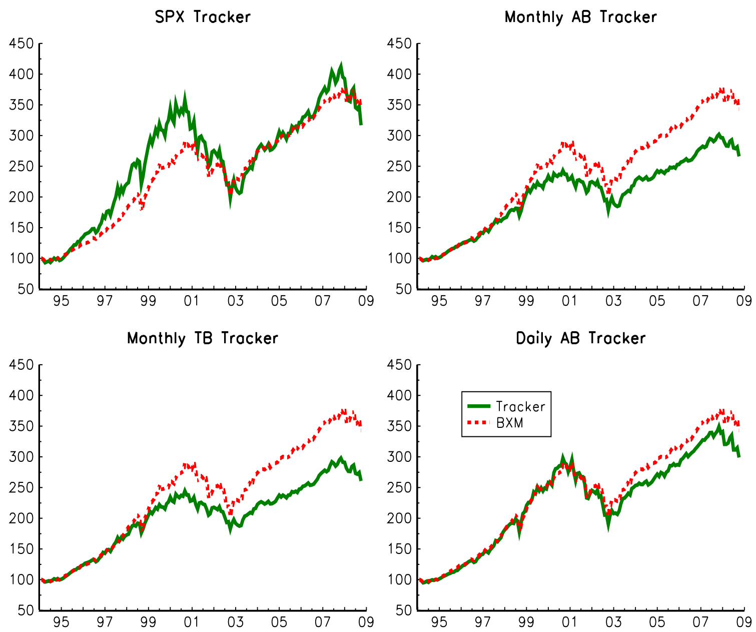
One may wonder if the replication of this non-linear asset with linear exposures on the S&P 500 index provides satisfying results, which would support the alternative beta argument for nonlinear asset replication. We consider 4 replicating portfolios (trackers):

1. A long position on the SPX index.
2. An alternative beta (AB) tracker using a monthly rebalacing method.
3. A portfolio consisting of a position of 57.9% on SPX futures and a position of 100% in cash. The 57.9% figure corresponds to the average value of the dynamic alternative beta. This tracker provides the traditional beta (TB) tracker as a benchmark for the AB trackers.
4. Finally, we consider an alternative beta (AB) tracker using a daily rebalacing method. This replicating portfolio uses the same methodology as the second tracker presented but at a higher frequency for purpose of comparing the two.

[11] This definition is taken from the website of the Chicago Board Options Exchange (CBOE) at `http://www.cboe.com/micro/bxm/introduction.aspx`.

The results of the replicating portfolios are reported in Figure 9 and Table 10. Notice that the use of a monthly rebalancing dynamic portfolio does not provide better results than using a constant beta portfolio. In order to improve the results, we have to use a higher frequency, a daily rebalancing period in our example. What happens? Let's call δ_k the *delta* of the hedging portfolio of the written call. The estimated weight $\hat{w}_k$ at time k may be approximated by $1-\delta_{k-1}$ where δ_{k-1} is the delta at time $k-1$. When one writes the call ATM option with one month of remaining to expiration, $\hat{w}_k$ is close to 50%. During the life of the option, the change in δ_k, and thus in $\hat{w}_k$, is relatively smooth and at the expiration date of the option, δ_k is equal to 1 or 0 depending if one exercises the option or not. If the frequency is daily, the estimated weight $\hat{w}_k$ will reflect this behavior varying smoothly everyday. If the frequency is monthly, δ_k is independent from δ_{k-1}. In this case, it is more difficult to replicate the BXM index and that explains that the monthly AB tracker has a comparable volatility of tracking errors than the TB tracker (with fixed weights $\hat{w}_k = 0.579$).

Figure 9: Tracking the BXM index



Now, there are several lessons to learn from this example. First, and it seems to provide support to the claim hereby tested, it is true that one can replicate option-based strategy by means of a dynamic alternative beta replication procedure. However, one must not forget that at best, in most cases, HF returns are only available on a monthly basis. As such, as we have seen, nonlinearities are not amenable to be replicated using only linear positions in standard assets. This explains why, even if its implementation is sometimes questionable because of its inherent difficulty, we believe that the presence of nonlinearities, when attested, often calls for the use of nonlinear asset factors. Before exploring this avenue in the rest of this section, there is one more comment to make which will be useful in the last section of this paper when we consider the access to the alpha of the HF strategy. Recall that we defined in section 4.3.1 the alternative alpha as the unexplained residual of the replicated strategy. Recall also that when we examined above the Gaussian assumption, most of the departure was captured by the distribution of the tracking

Table 10: Statistics of BXM trackers

	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2
BXM	8.69	9.92	0.43	-1.17	3.53
SPX	8.18	14.29	0.26	-0.58	0.62
AB (monthly)	6.90	8.55	0.29	-0.49	1.09
TB	6.75	8.15	0.29	-0.57	0.59
AB (daily)	7.75	9.91	0.34	-0.75	2.11
	π_{AB}	σ_{TE}	ρ	τ	ϱ
SPX	94.09	7.44	87.21	64.86	82.91
AB (monthly)	79.40	5.10	85.81	63.79	82.06
TB	77.69	4.88	87.20	64.96	83.02
AB (daily)	89.22	3.40	94.13	69.29	85.46

error. The point here is simply that, it appears if there are non-linear assets in the HF portfolio, a substantial part of the introduced nonlinearities will remain uncaptured and will appear in the alternative alpha. It thus prompts the thought that some of the alpha's performance is not accessible because its replication requires trading at high frequencies.

5.2.2 Using option factors with exogenous strikes

We now consider the linear factors (henceforth LF) model to which we add one non-linear asset factor. The tracking problem becomes:

$$\begin{cases} r_k^{(\mathrm{HF})} = \sum_{i=1}^{m} w_k^{(i)} r_k^{(i)} + w_k^{(m+1)} r_k^{(m+1)}(s_k) + \eta_k \\ \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k \end{cases}$$

where $r_k^{(m+1)}(s_k)$ is the return of a systematic one-month option selling[12] strategy on S&P 500 and s_k is the (exogenous) strike of the option at time k. Different values of the strike (moneyness) were implemented by taking the arbitrary values 95%, 100%, 105%. Note that, in this case, the TP system remains linear with respect to the state vector and we may solve it using Kalman filter. To price the option strategy, we used the Bloomberg's implied volatility data[13]. Results are reported in Table 11. It is interesting to note that even if we find non-linear factors (henceforth NLF) trackers with higher performance and Sharpe ratios than LF's, we do not obtain better results in terms of the volatility of the tracking error and the correlation between the HF index and the clone is not higher than the LF tracker.

Agarwal and Naik [2] find evidence that some hedge fund strategies exhibit the non-linear payoff structure described above. Diez de los Rios and Garcia [11] further find that there is statistical support for rejecting linearity only for a few categories (emerging markets, short bias and managed futures). If we consider the HFRI Macro and Relative Value indices, considering the results and the improvement in the proportion of HF returns explained, one may consider, in a first approach, that they exhibit this non-linear structure. However, these results are highly dependent

[12] It is not necessary to consider an option buying strategy because we do not constrain the weights $w_k^{(m+1)}$ to be positive. So, a negative weight on the selling strategy is equivalent to a long position on the buying strategy.

[13] The corresponding Bloomberg's functions are HIST_CALL_IMP_VOL and HIST_PUT_IMP_VOL.

Table 11: Results of replicating the HFRI index using call or put options

	s_k	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2
HF		9.94	7.06	0.77	-0.57	2.76
LKF		7.55	6.91	0.45	-0.02	2.25
	95%	7.61	6.93	0.46	-0.21	2.91
Call	100%	7.92	6.94	0.50	-0.22	2.92
	105%	8.14	6.88	0.54	-0.06	2.40
	95%	7.77	6.98	0.48	-0.22	3.35
Put	100%	8.15	6.97	0.53	-0.20	3.29
	105%	8.27	6.92	0.55	-0.04	2.60
		π_{AB}	σ_{TE}	ρ	τ	ϱ
LKF		75.93	3.52	87.35	67.10	84.96
	95%	76.62	3.55	87.12	65.48	83.56
Call	100%	79.74	3.58	86.95	65.53	83.61
	105%	81.90	3.55	87.07	66.81	84.57
	95%	78.20	3.49	87.68	66.61	84.33
Put	100%	81.98	3.53	87.40	66.60	84.22
	105%	83.21	3.48	87.62	67.68	85.21

Table 12: Results of replicating the HFRI Macro index using call or put options

	s_k	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2
HF		9.53	6.95	0.72	0.10	0.95
LKF		6.67	5.97	0.38	0.36	2.06
	95%	6.70	5.39	0.43	0.03	1.07
Call	100%	6.85	5.58	0.44	0.20	1.06
	105%	7.47	5.68	0.54	0.03	1.45
	95%	6.81	5.77	0.42	-0.13	2.17
Put	100%	7.22	5.81	0.48	-0.11	1.83
	105%	7.99	5.95	0.60	-0.06	1.43
		π_{AB}	σ_{TE}	ρ	τ	ϱ
LKF		69.97	5.71	61.82	44.21	62.28
	95%	70.26	5.69	60.13	43.00	60.35
Call	100%	71.86	5.69	60.71	43.35	60.57
	105%	78.31	5.55	63.14	43.95	61.89
	95%	71.42	5.61	62.54	43.74	61.65
Put	100%	75.73	5.59	63.07	43.86	61.63
	105%	83.79	5.45	65.40	45.47	63.30

Table 13: Results of replicating the HFRI Relative Value index using call or put options

	s_k	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	s	γ_1	γ_2
HF		8.50	3.62	1.11	-2.76	15.22
LKF		5.77	2.75	0.52	0.16	1.18
	95%	6.44	2.97	0.69	-0.48	3.29
Call	100%	6.71	3.02	0.77	-0.46	3.69
	105%	7.13	2.97	0.92	0.12	1.97
	95%	6.75	4.28	0.55	-6.49	66.57
Put	100%	6.97	3.07	0.84	-0.84	6.06
	105%	6.81	2.85	0.85	0.24	1.80
		π_{AB}	σ_{TE}	ρ	τ	ϱ
LKF		67.94	3.07	56.46	40.48	55.74
	95%	75.76	3.01	59.68	38.38	53.10
Call	100%	78.99	3.05	58.74	37.10	51.05
	105%	83.88	3.20	54.06	38.58	53.30
	95%	79.49	3.38	64.42	38.94	54.77
Put	100%	82.00	3.03	59.74	37.60	52.40
	105%	80.13	3.15	54.63	39.26	54.44

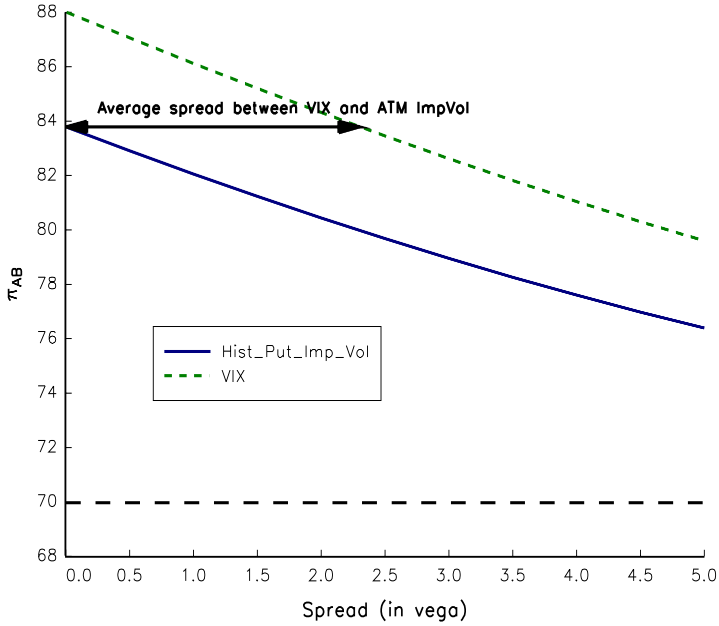
on the quality of the data. If we take into account a spread (explained by the volatility skew and the bid/ask effect) between the published implied volatility and the traded volatility, the results are less favorable. Figure 13 presents the impact of the trading spread on the replication performance of the HFRI Macro index with selling put options at 105%. It is obvious that at this strike, the impact of the skew is very high and that we have to consider a high spread. In this case, the difference between the LKF model and the NLF model is not so important. This suggests that HF replication including nonlinear assets as an investment vehicle still remains difficult to implement as any replicating portfolio would be confronted to implementation noise and distortions.

So far, we have considered that the non-linear assets included in the replication model were priced independently of the strategy followed by the HF fund manager. In the next section, we relax this assumption by considering that the parameters of the nonlinear strategy are dependent on the HF manager's decisions. One word of caution must be given here. We do necessarily hope that the results of the following section will be directly applicable in a systematic replication procedure, but we do hope however that it will give us further insight into the risk structure of the replicating strategy as well as the manager's outlook on available investments.

5.2.3 Using option factors with endogenous strikes

One of the drawbacks of the previous model is that the strikes of the options are exogenous, and thus do not depend on the decisions of the manager. From the point of view of pure replication of the performance and the strategy of a HF manager, it could be seen as an obvious deficiency. Indeed, a HF manager can adapt option strikes to his or her tactical bets and macroeconomic views. To our knowledge, there exist few academic works dealing with this problem directly, and when they do, strikes are considered constant over the period of study. In the spirit of the alternative beta methodology presented previously, it is more realistic to assume that the option

Figure 10: Impact of the trading spread on the replication performance of the HFRI Macro index with selling 105% put options



strikes are time-varying, and are part of the manager's general strategy. Thus, we propose in this section to consider option factors with endogenous strikes.

Before considering this difficult exercise, we must point out one necessary assumption that we need to make in our context. One could content that the argument above in favor of the use of endogenous strikes is dubious when applied to an aggregate of HF, as in the case of indices for example, for the idiosyncratic tactical bets of a particular manager are lost in the aggregation process. We must therefore make the assumption that the aggregation process results in an average strike reflective of the entire position of the underlying aggregate. This is not dissimilar to the argument we made earlier about the better fitness of replication methods to replicate aggregates over single HF. While the nonlinear character of the underlying panel of option exposures with different parameters (e.g. call and put, selling or buying positions, in-the-money or out-of-the-money strikes) does not easily lead to aggregation within the same class of nonlinear factors, for liquidity and trading reasons, we think it plausible that the time-varying strikes of a limited number of option factors on general asset classes can provide a good proxy for replication.

They are two possible ways to estimate these strikes. One possibility is to consider an econometric method to estimate the option strikes separately from the tracking problem. For example, we may first consider a macroeconomic model to estimate the strikes, then build the option factors using the time varying estimated strikes and finally estimate the linear exposures using the Kalman filter. In this case, the option strikes are endogenous in the sense that they have been estimated, but they are also exogenous in the sense that their estimation is independent from the TP system. Another possibility is to consider that the option strike belongs to the state vector of a nonlinear TP system. Thus, we obtain:

$$\left\{\begin{array}{l} \begin{pmatrix} \mathbf{w}_k \\ s_k \end{pmatrix} = \begin{pmatrix} \mathbf{w}_{k-1} \\ s_{k-1} \end{pmatrix} + \begin{pmatrix} \nu_k \\ \varepsilon_k \end{pmatrix} \\ r_k^{(\mathrm{HF})} = \sum_{i=1}^{m} w_k^{(i)} r_k^{(i)} + w_k^{(m+1)} r_k^{(m+1)}\left(s_k\right) + \eta_k \end{array}\right. \tag{8}$$

where the measurement equation is nonlinear with respect to the strike state variable s_k. As in the non-Gaussian case, the nonlinearity of the system prevents us to use the KF, and we must resort to using PF. One of the difficulties is to estimate the unknown parameters of (8). As in the previous sections, we assume that $\eta_k \sim \mathcal{N}\left(0, \sigma_\eta^2\right)$ and:

$$\begin{pmatrix} \nu_k \\ \varepsilon_k \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & \sigma_s^2 \end{pmatrix}\right)$$

with $Q = \operatorname{diag}\left(\sigma_1^2, \ldots, \sigma_m, \sigma_{m+1}^2\right)$. The vector of unkown parameters to estimate is then $\theta = \{\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \sigma_s, \sigma_\eta\}$.

The estimation of parameters in the context of nonlinear TP is not a trivial problem, and the literature while expanding is still relatively scarce (see for example [12, 9, 32, 24, 34]). Because of the nonlinearities included in the TP, general methods are overwhelmingly based on an extension of the Expectation-Maximization principle using discrete approximations of the different densities by means of particles. Moreover, the examples generally considered consist of a small number of parameters to estimate and small number of particles. For example, Wills, Schön and Ninness [34] consider estimating the parameters of a stochastic volatility model, comprising only 3 parameters and using 50 particles over 10000 time periods. In our case however, we have $m+3$ parameters (ie. 9 parameters if we use six linear factors) and a period of 177 observations. Moreover, the number of particles we need to use to satisfyingly duplicate the results of the KF in a linear framework is very high (generally more than 10000). Thus, with one additional nonlinear factor and three more parameters we cannot expect satisfying results for a lesser number of particles. We tried the ML methods described in the cited papers above. However, the task is still extremely difficult. The optimization step remains time-consuming and sensitive to the specified initial values and the number of particles. For this reason, we preferred the use a grid-based method to estimate the parameters, although it didn't solve all the implementation problems. For instance, let's note d_i the number of discretized values used for the parameter θ_i. A grid-based estimation requires that we run the PF algorithm d times with $d = \prod_{i=1}^{m+3} d_i$. Running a PF with 50000 particles takes about 30 seconds on our computers. So, with $d_i = 5$ and $m = 6$, it would take a little less than 2 years (678 days) to run until the end. This curse of dimensionality required of us to proceed yet again differently, and we decided to follow a two-step procedure:

1. the parameters σ_η and σ_i for $i = 1, \ldots, m$ are estimated using the method of maximum likelihood considering the linear factor model;
2. the last two parameters σ_{m+1} and σ_s are estimated using the grid-based method conditionally on the previous estimates. If f denotes the statistic of interest in the maximization (or minimization) of and if Ω denotes the set of grid points, we have:

$$\{\hat{\sigma}_{m+1}, \hat{\sigma}_s\} = \arg\max f\left(\sigma_{m+1}, \sigma_s \mid \hat{\sigma}_1, \ldots, \hat{\sigma}_m, \hat{\sigma}_\eta\right) \quad \text{u.c.} \quad \left(\sigma_{m+1}, \sigma_s\right) \in \Omega.$$

This approach is obviously biased compared to a full estimation of all the parameters. However, with respect to the linear factor model, it provides consistent results which could help explain the remaining alpha.

In what follows, we present two examples using the HFRI index and the HFRI Relative Value index. Both examples considered a put option on the S&P500 index. In the second step of our procedure, we chose to minimize the volatility σ_{TE} of the tracking errors. In Figure 11, we report the statistic of interest for the HFRI index with respect to σ_{m+1} and σ_s. Notice that the surface does not present an obvious minimum. Moreover, we remark that the volatility of the tracking error is above 3.52% which is the corresponding statistic for the linear model. Thus, using endogenous strikes does not improve the volatility of the tracking errors. For the HFRI Relative Value index, we obtain more convincing results. First, notice that the surface in Figure 12 presents a more convex function profile. And we estimate that the minimum is reached for $\sigma_{m+1} = 2.5\%$ and $\sigma_s = 1\%$.

We reported the exposures $w_k^{(i)}$ and the strike s_k of the put option in Figures 13 and 14. In the case of a fixed strike, notice that the exposure on the put option is very volatile. This is not the case when the strike is endogenous. The results suggest that HF managers are globally selling out-of-the-money put options. However, one major difficulty which is not taken into account here is the effect of the volatility's smile, and possible liquidity limitations.

6 Alpha Considerations

In the previous sections, we have developed and demonstrated that the use of Bayesian filters to answer the question of HF replication can provide both a practical procedure as well as a methodological tool enabling results and insights which often go beyond previous results. In this section, we focus on the part of the HF performance that is left unexplained by the methods exposed above. We thus look into the alternative alpha component, and look for possible explanations of its origin. In the previous sections, we already suggested possible sources including high frequency trading and investments in illiquid assets. To these two, we add here another component which stems not from specific strategies but from the fact that, by construction, a replicating portfolio implements its exposure with a time lag with respect to the replicated HF profile. Section 5.2.1 exposed the inability of the alternative beta method to capture non-linearities at too low frequencies. As this inability disappears as the frequency of rebalancing the replicating portfolio augments, we suggested that this was substantial evidence that part of the uncaptured performance was the product of either of high frequency trading or of holding nonlinear assets. As we already studied these points, we focus here on the remaining two and study these claims, namely implementation lag and illiquid investments, in this respective order.

Let's start by studying the impact of the implementation lag. Recall that the return of the clone is given by:

$$r_k^{\mathrm{Clone}} = \left(1 - \sum_{i=1}^{m} \hat{w}_{k|k-1}^{(i)}\right) r_k^{(0)} + \sum_{i=1}^{m} \hat{w}_{k|k-1}^{(i)} r_k^{(i)}$$

A time index k, implementing the replicating strategies requires to use the weights estimated using the information until time index $k-1$. So, the right allocation is used with a one-month

Figure 11: Grid approach applied to the HFRI index

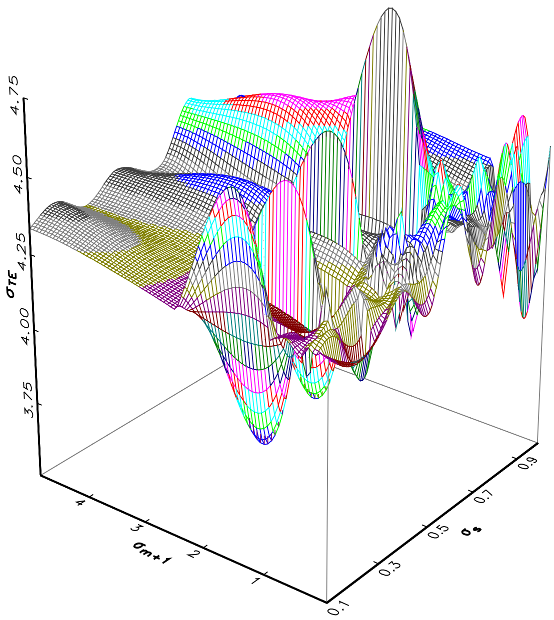


Figure 12: Grid approach applied to the HFRI RV index

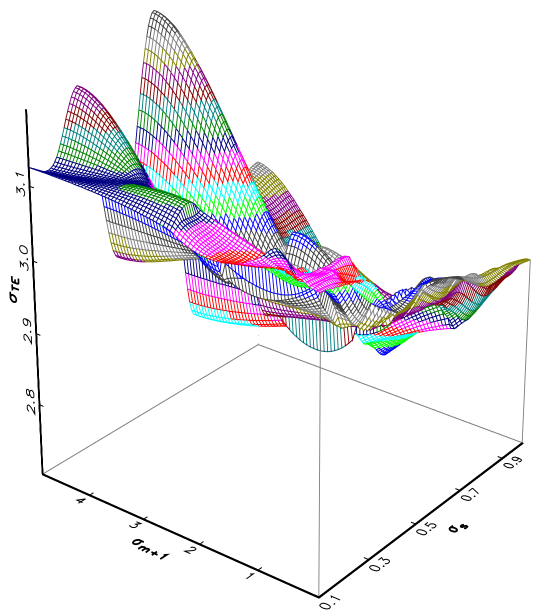


Figure 13: Exposures of the linear assets for the HFRI RV index

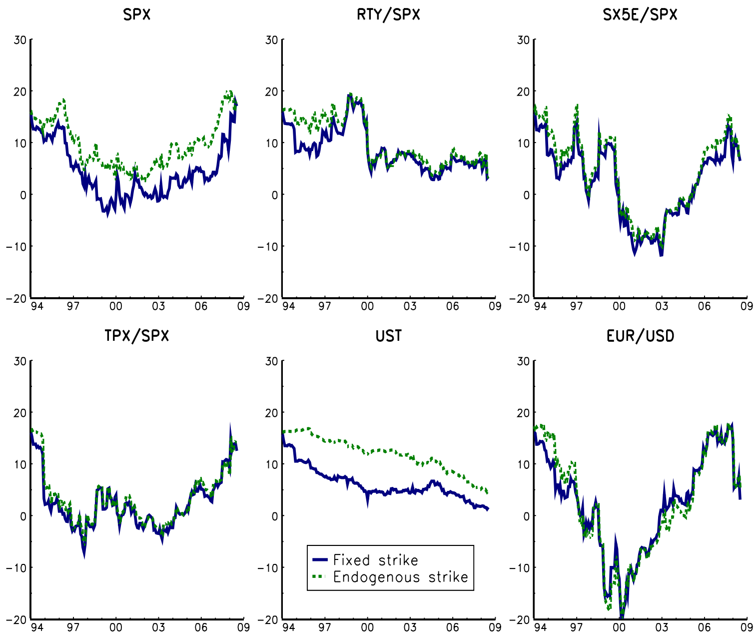


Figure 14: Option exposures and strikes for the HFRI RV index

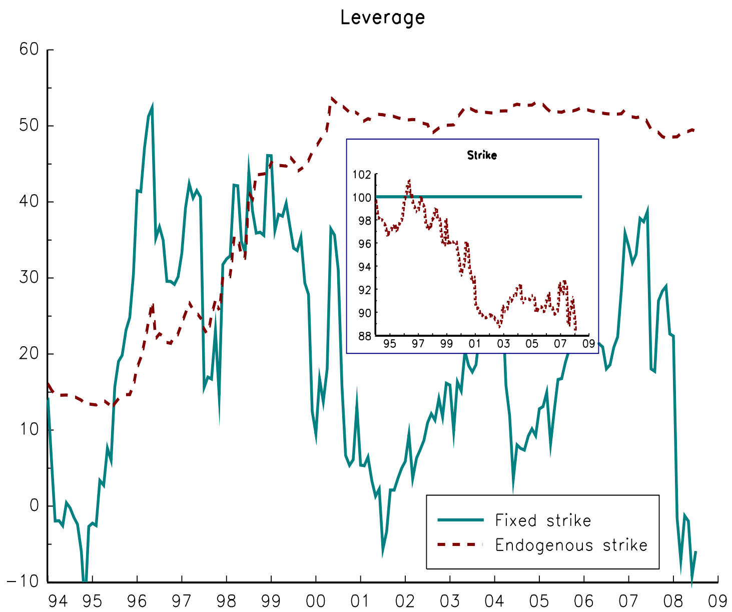


Table 14: Results of time lags implementation on the replicating portfolios

d	$\hat{\mu}_{1Y}^{HF}$	$\hat{\mu}_{1Y}^{Clone}$	π_{AB}	σ_{TE}	ρ	τ	ϱ
-1	9.94	7.55	75.93	3.52	87.35	67.10	84.96
0	9.94	8.39	84.45	1.94	96.17	80.18	94.55
1	9.94	8.42	84.77	2.05	95.71	80.09	94.42
2	9.94	8.26	83.11	2.22	94.96	78.42	93.59

delay. For the purpose of this analysis let's consider the following formula:

$$r_k^{\text{Clone}}(d) = \left(1 - \sum_{i=1}^{m} \hat{w}_{k+d+1|k+d}^{(i)}\right) r_k^{(0)} + \sum_{i=1}^{m} \hat{w}_{k+d+1|k+d}^{(i)} r_k^{(i)}$$

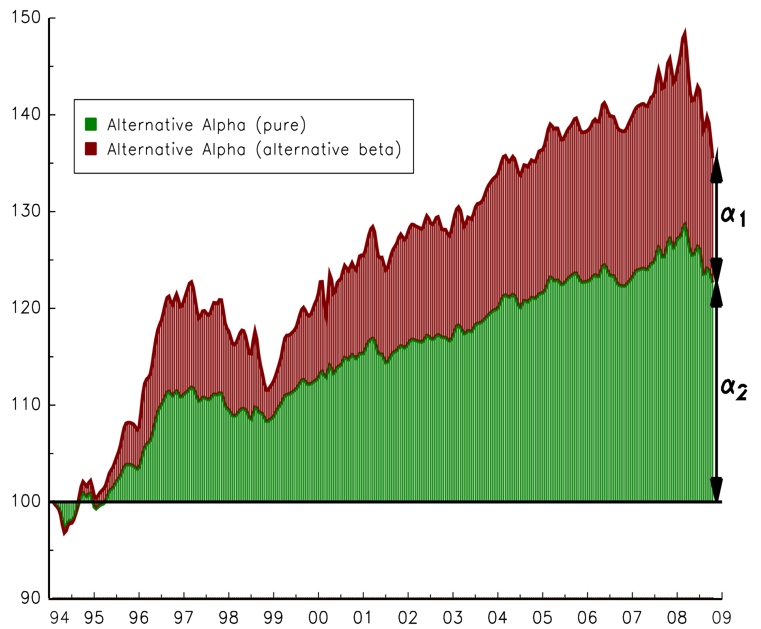
The previous formula is obtained for $d = -1$. If one uses $d = 0$, one assumes that one can implement at time k the true exposures of the period $[k, k+1[$ and for $d > 0$, the implemented exposures are those estimated for the period $[k+d, k+d+1[$. Putting to test our claim that the implementation lag contributes to the alpha, we computed the backtests of the portfolios obtained for $d = 0, 1, 2$ using the 6F model presented above and the HFRI Fund Weighted Composite Index and compared them with the case $d = -1$. We then calculated, as we have done previously, the statistics $\hat{\mu}_{1Y}$, π_{AB}, σ_{TE}, ρ, τ, and ϱ for each of the track records. The results obtained are provided in Table 14. Unsurprisingly, with the added information, the results are substantially better, with the best results for $d = 0$. In this case, the part of the HF performance explained by the alternative beta clone jumps by about 10% to 85%, reducing the alpha component from around 25% to about 15%. In other words, in our example, 40% of the alternative alpha is explained by the implementation delay. In this particular case, we thus propose a new breakdown on the HF performance where around:

- 75% of the performance corresponds to alternative beta which may be reproduced by the tracker;
- 25% is the alternative alpha of which:
  - 10% corresponds in fact to alternative beta which may not be implemented and are lost due to the dynamic allocation;
  - 15% makes up a component that we call the *pure* alternative alpha.

It is also interesting to note that the volatility of the pure alpha component (σ_{TE} for $d = 0, 1, 2$) is lower and is half of the volatility of the alternative alpha. We represented in Figure 15 the evolution of the two components of the alternative alpha, with α_1 representing the contribution of the implementation lag to the alternative alpha and α_2 the pure alternative alpha.

We now turn to our second claim, that the alternative alpha stems out of the illiquidity premia associated with investment in illiquid assets. Using the results of our previous experiment on implementation delay, we focus on explaining the pure component of the alternative alpha. One possible way to substantiate this claim would be, for example, to extract the pure alpha component and run an analysis in the same fashion as it was done at first for HF replication using regressions to determine whether factors representing different illiquid assets, e.g. distressed securities or private equity, are able to explain the returns of the pure alternative alpha. We

Figure 15: The decomposition of the alternative alpha

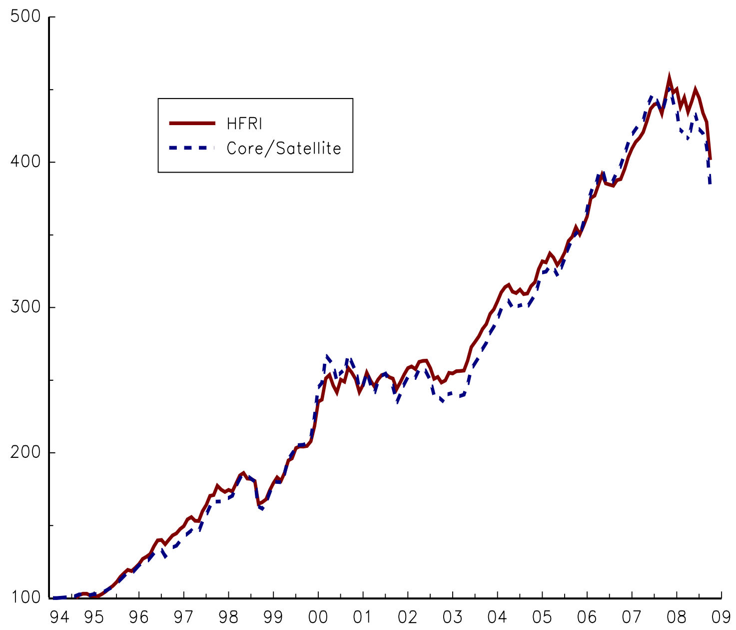


proceed differently here, with in mind the idea to demonstrate that it is possible to access the performance of this pure component from an investment perspective.

One idea then is to build a core/satellite portfolio where the core is the alternative beta and the satellite is a basket of illiquid or optional strategies. The previous construction of alternative investments has some important advantages. For example, one could consider a portfolio with 70% of alternative beta, 10% of optional or quantitative strategies, 10% of real estate and 10% of private equity. The core/satellite approach permits to distinguish clearly liquid and illiquid investments, small term and long term investments. In our example, these three satellite strategies are respectively proxied by equally weighted portfolios of the SGI volatility premium index and JP Morgan carry max index, UK IPD TR all property index and NCREIF property index, and LPX buyout index and LPX venture index. The results of this approach are displayed in Figure 16.

After obtaining these results, there is no doubt in our mind that our claim is substantiated, and that, in this case at least, the pure alternative alpha component can be replicated by means of this core/satellite strategy. One may wonder however why there is apparently no need to take into account a high frequency factor. Beside the fact that it is rather good news from a practitioner point-of-view, one must point out that in our example, we replicated the HFRI Fund Weighted Composite Index, which is the most general industry aggregate provided by Hedge Fund Researh, Inc. As such, in light of the results presented by Diez de los Rios and Garcia [11], we surmise that the effect of high frequency trading which would in part appear as nonlinear is negligible.

Figure 16: The core/satellite approach of alternative investments



7 Conclusion

In this paper, we presented a formal framework for hedge fund replication by introducing the notion of tracking problems which may be solved using Bayesian filters. After considering the Gaussian linear case, we extended the methodology to non-Gaussian nonlinear cases using particle filters. These advanced tracking techniques were used first to capture some stylized facts of HF returns, like negative skewness and excess kurtosis. They further enabled us to estimate endogenous option strikes in an attempt to capture non-linear exposures. The results obtained using particle filters are to some extent disappointing. First, it seems that matching higher moments of HF returns implies a necessary trade-off with higher volatility of the tracking errors of the HF clone. Second, consistent with some recent findings in the literature, we found little evidence of the presence of nonlinearities in the distribution of the returns of the overall hedge fund strategies.

Nevertheless, we believe these results to be very interesting both for the practitioners and the academics. From the practitioners' point of view, by grounding all of our approaches into a general and coherent framework, and by meticulously adding complexity to the methodology, we demonstrated that a robust replication process can be obtained by means of mainstream statistical methods, such as the Kalman filter, provided that careful thought is given to the specification of the model and the type of instruments used in the replication process (particularly with respect to liquidity or other trading considerations). It is perhaps necessary to remind the reader again that as an investment toolbox to manage HF exposures (both long and short) and liquidity, the first quality of a HF clone should be not to be a hedge fund in itself. As such, and in line with this HF replication philosophy, our core/satellite approach showed that this robust approach (Kalman filter and liquid instruments) can still be supplemented by other illiquid investments to capture and reproduce more efficiently the risk profile of the hedge fund industry. Incidently, it also hints at the efficiency of the "core" method to capture the HF betas to classic asset classes. From the academics' point of view, introducing particle filters opens a door for a better understanding of HF returns and the underlying risks of the HF strategies. If it already has direct implications from a risk management perspective, we also surmise that particles filters are one of the main avenues toward a better monitoring of for now unaccounted risks, as they are contained in the higher moments of the returns' distribution — we have yet to explore the use of ML estimation procedures for particle filters in the nonlinear context.

References

[1] Vikas Agarwal and Narayan Y. Naik. Performance evaluation of hedge funds with option-based and buy-and-hold strategies. *London Business School, Working Paper*, August 2000.

[2] Vikas Agarwal and Narayan Y. Naik. Risks and portfolio decisions involving hedge funds. *Review of Financial Studies*, 17(1):63–98, 2004.

[3] Noël Amenc, Sina El Bied, and Lionel Martellini. Evidence of predictability in hedge fund returns. *Financial Analysts Journal*, 59:32–46, September 2003.

[4] Noël Amenc, Walter Géhin, Lionel Martellini, and Jean-Christophe Meyfredi. The myths and limits of passive hedge fund replication. *EDHEC Risk and Asset Management Research Centre, Working Paper*, June 2007.

[5] Noël Amenc, Lionel Martellini, Jean-Christophe Meyfredi, and Volker Ziemann. Passive hedge fund replication – Beyond the linear case. *EDHEC Risk and Asset Management Research Centre, Working Paper*, 2008.

[6] Gaurav S. Amin and Harry M. Kat. Hedge fund performance 1990-2000: Do the "money machines" really add value? *Journal of Financial and Quantitative Analysis*, 38(2):251–274, June 2003.

[7] Sanjeev Arulampalam, Simon Maskell, Neil J. Gordon, and Tim Clapp. A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking. *IEEE Transaction on Signal Processing*, 50(2):174–188, February 2002.

[8] Adelchi Azzalini and Antonnella Capitanio. Distributions generated by perturbation of symmetry with emphasis on a multivariate skew-t distribution. *JRSS Ser. B*, 65:367–389, 2003.

[9] Olivier Cappé and Eric Moulines. On the use of particle filtering for maximum likelihood parameter estimation. In *European Signal Processing Conference, Antalya, Turkey*, September 2005.

[10] Thomas Della Casa, Mark Rechsteiner, and Ayako Lehmann. Attack of the 'hedge fund' clones. *Man Investments Research, Working Paper*, 2008.

[11] Antonio Diez de los Rios and Rene Garcia. Assessing and valuing the non-linear structure of hedge fund returns. *Bank of Canada Working Paper*, October 2008.

[12] Arnaud Doucet and Vladislav B. Tadic. Parameter estimation in general state-space models using particle methods. *Annals of the Institute of Statistical Mathematics*, 55(2):409–422, 2003.

[13] William Fung and David A. Hsieh. Empirical characteristics of dynamic trading strategies: The case of hedge funds. *Review of Financial Studies*, 10(2):275–302, 1997.

[14] William Fung and David A. Hsieh. A primer on hedge funds. *Journal of Empirical Finance*, 6:309–331, September 1999.

[15] William Fung and David A. Hsieh. The risk in hedge fund strategies: Theory and evidence from trend followers. *Review of Financial Studies*, 14(2):313–341, 2001.

[16] William Fung and David A. Hsieh. The risk in hedge fund strategies: Theory and evidence from long/short equity hedge funds. *Working Paper*, 2004.

[17] William Fung and David A. Hsieh. Hedge fund replication strategies: Implications for investors and regulators. *Financial Stability Review*, 10:55–66, April 2007.

[18] Jasmina Hasanhodzic and Andrew Lo. Can hedge-fund returns be replicated?: The linear case. *Journal of Investment Management*, 5(2):5–45, 2007.

[19] Lars Jaeger. *Alternative Beta Strategies and Hedge Fund Replication.* John Wiley & Sons, New York, 1st edition, 2009.

[20] Michael S. Johannes and Nick Polson. Particle filtering and parameter learning. *University of Chicago, Working Paper*, March 2007.

[21] Harry M. Kat. Alternative routes to hedge fund return replication. *Journal of Wealth Management*, 10(3):25–39, 2007.

[22] Harry M. Kat and Helder P. Palaro. Replication and evaluation of funds of hedge funds returns. *In Fund of Hedge Funds: Performance, Assessment, Diversification and Statistical Properties (eds: Greg Gregoriou), Chapter 3, Elsevier Press*, 2006.

[23] Robert C. Merton. On market timing and investment performance. I. an equilibrium theory of value for market forecasts. *Journal of Business*, 54(3):363–406, July 1981.

[24] Jimmy Olsson, Olivier Cappé, Randal Douc, and Eric Moulines. Sequential monte carlo smoothing with application to parameter estimation in nonlinear state space models. *Bernoulli*, 14(1):155–179, 2008.

[25] Michael K. Pitt and Neil Shephard. Filtering via simulation: Auxiliary particle filters. *Journal of the American Statistical Association*, 94(446):590–599, June 1999.

[26] George Poyiadjis, Arnaud Doucet, and Sumeetpal S. Singh. Maximum likelihood parameter estimation in general state-space models using particle methods. In *Proceedings of the American Statistical Association, JSM 05*, August 2005.

[27] George Poyiadjis, Arnaud Doucet, and Sumeetpal S. Singh. Particle methods for optimal filter derivative: application to parameter estimation. In *Proceedings IEEE International Conference on Acoustics, Speech, and Signal Processing*, March 2005.

[28] Branko Ristic, Sanjeev Arulampalam, and Neil J. Gordon. *Beyond the Kalman Filter: Particle Filters for Tracking Applications.* Artech House, Boston, 1st edition, 2004.

[29] Thierry Roncalli. *TSM – Time Series and Wavelets for Finance.* Global Design, Paris, 1996.

[30] Thierry Roncalli and Jerome Teiletche. An alternative approach to alternative beta. *Journal of Financial Transformation, Cass-Capco Institute Paper Series on Risk*, 2008.

[31] Thierry Roncalli and Guillaume Weisang. *PF: A Gauss Library for Particle Filters.* 2008. `http://www.thierry-roncalli.com/#gauss_l17`.

[32] Thomas B. Schön, Adrian G. Wills, and Brett Ninness. Maximum likelihood nonlinear system estimation. In *Proceedings of the 14th IFAC Symposium on System Identification, Newcastle, Australia*, pages 1003–1008, March 2006.

[33] William F. Sharpe. Asset allocation: management style and performance measurement. *Journal of Portfolio Management*, 18(2):7–19, 1992.

[34] Adrian G. Wills, Thomas B. Schön, and Brett Ninness. Parameter estimation for discrete-time nonlinear systems using EM. In *Proceedings of the 17th IFAC World Congress on Automatic Control, Seoul, Korea*, July 2008.

A Solving Tracking Problems

In this appendix, we describe succinctly two algorithms, the Kalman filter and the particle filter (henceforth KF and PF), as specific implementations of Bayesian filters.

A.1 The Kalman filter

If one assumes the tracking problem to be linear and Gaussian, one may prove that the optimal algorithm to estimate the state vector is the Kalman filter. The state space model is then given by:

$$\left\{\begin{array}{l}\mathbf{x}_k = \mathbf{c}_k + F_k \mathbf{x}_{k-1} + \nu_k \\ \mathbf{z}_k = \mathbf{d}_k + H_k \mathbf{x}_k + \eta_k\end{array}\right.$$

with $\nu_k \sim \mathcal{N}(0, Q_k)$ and $\eta_k \sim \mathcal{N}(0, S_k)$. Moreover, the initial distribution of the state vector is $p(\mathbf{x}_0) = \phi\left(\mathbf{x}_0, \hat{\mathbf{x}}_0, \hat{\mathbf{P}}_0\right)$, where $\phi(x, m, P)$ is the Gaussian pdf with argument x, mean m and covariance matrix P. The Bayes filter is then described by the following recursive equations:

$$\begin{array}{rcl} p(\mathbf{x}_k \mid \mathbf{z}_{1:k-1}) & = & \phi\left(\mathbf{x}_{k-1}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{P}}_{k|k-1}\right) \\ p(\mathbf{x}_k \mid \mathbf{z}_{1:k}) & = & \phi\left(\mathbf{x}_k, \hat{\mathbf{x}}_{k|k}, \hat{\mathbf{P}}_{k|k}\right)\end{array}$$

with:

$$\left\{\begin{array}{l}\hat{\mathbf{x}}_{k|k-1} = \mathbf{c}_k + F_k \hat{\mathbf{x}}_{k-1|k-1} \\ \hat{\mathbf{P}}_{k|k-1} = F_k \hat{\mathbf{P}}_{k-1|k-1} F_k^\top + Q_k \\ \hat{\mathbf{z}}_{k|k-1} = \mathbf{d}_k + H_k \hat{\mathbf{x}}_{k|k-1} \\ \hat{\mathbf{e}}_k = \mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1} \\ \hat{V}_k = H_k \hat{\mathbf{P}}_{k|k-1} H_k^\top + S_k \\ \hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \hat{\mathbf{P}}_{k|k-1} H_k^\top \hat{V}_k^{-1} \hat{\mathbf{e}}_k \\ \hat{\mathbf{P}}_{k|k} = \hat{\mathbf{P}}_{k|k-1} - \hat{\mathbf{P}}_{k|k-1} H_k^\top \hat{V}_k^{-1} H_k \hat{\mathbf{P}}_{k|k-1}\end{array}\right. \tag{A-1}$$

The set of equations (A-1) describes the Kalman filter algorithm. The previous quantities can be interpreted as follows:

- $\hat{\mathbf{x}}_{k|k-1} = \mathbb{E}\left[\mathbf{x}_k \mid \mathbf{z}_{1:k-1}\right]$ is the estimate of $\mathbf{x}_k$ based on all available information until time index $k-1$;

- $\hat{\mathbf{P}}_{k|k-1}$ is the covariance matrix of the estimator $\hat{\mathbf{x}}_{k|k-1}$:

$$\hat{\mathbf{P}}_{k|k-1} = \mathbb{E}\left[\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}\right)\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}\right)^\top \mid \mathbf{z}_{1:k-1}\right]$$

- $\hat{\mathbf{z}}_{k|k-1} = \mathbb{E}\left[\mathbf{z}_k \mid \mathbf{z}_{1:k-1}\right]$ is the estimate of $\mathbf{z}_k \mid \mathbf{z}_{1:k-1}$;

- $\hat{\mathbf{e}}_k = \mathbf{z}_k - \mathbb{E}\left[\mathbf{z}_k \mid \mathbf{z}_{1:k-1}\right]$ is the estimated tracking error $\mathbf{e}_k = \mathbf{z}_k - \mathbf{z}_k \mid \mathbf{z}_{1:k-1}$;

- $\hat{V}_k$ is the covariance matrix of the tracking error: $\hat{V}_k = \mathbb{E}\left[\mathbf{e}_k \mathbf{e}_k^\top\right]$. We have:

$$p\left(\mathbf{e}_k \mid \mathbf{z}_{1:k-1}\right) = \phi\left(\mathbf{e}_k, \hat{\mathbf{e}}_k, \hat{V}_k\right)$$

- $\hat{\mathbf{x}}_{k|k} = \mathbb{E}\left[\mathbf{x}_k \mid \mathbf{z}_{1:k}\right]$ is an estimate of $\mathbf{x}_k$ based on all available information until time index k;

- Finally, $\hat{\mathbf{P}}_{k|k}$ is the covariance matrix of the estimator $\hat{\mathbf{x}}_{k|k}$:

$$\hat{\mathbf{P}}_{k|k} = \mathbb{E}\left[\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right)\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right)^\top \mid \mathbf{z}_{1:k}\right]$$

A.2 Particle filters

Particle filtering methods are techniques to implement recursive Bayesian filters using Monte-Carlo simulations. The key idea is to represent the posterior density function by a set of random samples with associated weights and to compute estimates based on these samples and weights [7, 20, 25, 26, 27, 28]. As the samples become very large $N_s \gg 1$, this Monte-Carlo approximation becomes an equivalent representation on the functional description of the posterior pdf. To clarify ideas[14], let $\left\{\mathbf{x}_k^i, w_k^i\right\}_{i=1}^{N_s}$ denotes a set of support points $\left\{\mathbf{x}_k^i, i = 1, \ldots, N_s\right\}$ and their associated weights $\left\{w_k^i, i = 1, \ldots, N_s\right\}$ characterizing the posterior density $p\left(\mathbf{x}_k \mid \mathbf{z}_{0:k}\right)$. The posterior density at time k can then be approximated as:

$$p\left(\mathbf{x}_k \mid \mathbf{z}_k\right) \approx \sum_{i=1}^{N_s} w_k^i \delta\left(\mathbf{x}_k - \mathbf{x}_k^i\right) \tag{A-2}$$

We have thus a discrete weighted approximation to the true posterior distribution. One common way of choosing the weights is by way of *importance sampling* — see for example [7, 20, 25, 28]. This principle relies on the following idea. In the general case, the probability density $p\left(\mathbf{x}_k \mid \mathbf{z}_k\right)$ is such that it is difficult to draw samples from it. Assume for a moment that $p(x) \propto \pi(x)$ is a probability density from which it is difficult to draw sample from, but for which $\pi(x)$ is easy to evaluate. Hence, up to proportionality, so is $p(x)$. Also, let $x^s \sim q(x)$ be samples that are easily drawn from a proposal $q(\cdot)$, called an *importance density*. Then, similarly to A-2, a weighted approximation of the density $p(\cdot)$ can be obtained by using:

$$p(x) \approx \sum_{i=1}^{N_s} w^i \delta\left(x - x^i\right)$$

[14] Note that the succinct presentation given here of particle filters is adapted to our first-order Markovian framework.

where:

$$w^i \propto \frac{\pi\left(x^i\right)}{q\left(x^i\right)}$$

is the normalized weight of the i-th particle. Thus, if the samples $\{\mathbf{x}_k^i\}$ were drawn from a proposal density $q\left(\mathbf{x}_k \mid \mathbf{z}_k\right)$, then the weights in (A-2) are defined to be:

$$w_k^i \propto \frac{p\left(\mathbf{x}_k^i \mid \mathbf{z}_k\right)}{q\left(\mathbf{x}_k^i \mid \mathbf{z}_k\right)} \tag{A-3}$$

The PF sequential algorithm can thus be subsumed in the following steps. At each iteration, one has samples constituting an approximation of $p\left(\mathbf{x}_{k-1}^i \mid \mathbf{z}_{k-1}\right)$ and wants to approximate $p\left(\mathbf{x}_k^i \mid \mathbf{z}_k\right)$ with a new set of samples. If the importance density can be chosen so as to factorize in the following way:

$$q\left(\mathbf{x}_k \mid \mathbf{z}_k\right) = q\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}, \mathbf{z}_k\right) \times q\left(\mathbf{x}_{k-1} \mid \mathbf{z}_{k-1}\right) \tag{A-4}$$

then one can obtain samples $\{\mathbf{x}_k^i\}$ by drawing samples from $q\left(\mathbf{x}_k^i \mid \mathbf{z}_k\right)$. To derive the weight update equation:

$$\begin{aligned}
p\left(\mathbf{x}_k \mid \mathbf{z}_k\right) &= \frac{p\left(\mathbf{z}_k \mid \mathbf{x}_k, \mathbf{z}_{k-1}\right) \times p\left(\mathbf{x}_k \mid \mathbf{z}_{k-1}\right)}{p\left(\mathbf{z}_k \mid \mathbf{z}_{k-1}\right)} \\
&= \frac{p\left(\mathbf{z}_k \mid \mathbf{x}_k, \mathbf{z}_{k-1}\right) \times p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}, \mathbf{z}_{k-1}\right)}{p\left(\mathbf{z}_k \mid \mathbf{z}_{k-1}\right)} \times p\left(\mathbf{x}_{k-1} \mid \mathbf{z}_{k-1}\right) \\
&= \frac{p\left(\mathbf{z}_k \mid \mathbf{x}_k\right) \times p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}\right)}{p\left(\mathbf{z}_k \mid \mathbf{z}_{k-1}\right)} \times p\left(\mathbf{x}_{k-1} \mid \mathbf{z}_{k-1}\right) \\
&\propto p\left(\mathbf{z}_k \mid \mathbf{x}_k\right) \times p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}\right) \times p\left(\mathbf{x}_{k-1} \mid \mathbf{z}_{k-1}\right)
\end{aligned} \tag{A-5}$$

By substituting (A-4) and (A-5) into (A-3), the weight equation can be derived to be:

$$w_k^i \propto w_{k-1}^i \frac{p\left(\mathbf{z}_k \mid \mathbf{x}_k^i\right) \times p\left(\mathbf{x}_k^i \mid \mathbf{x}_{k-1}^i\right)}{q\left(\mathbf{x}_k^i \mid \mathbf{x}_{k-1}^i, \mathbf{z}_k\right)} \tag{A-6}$$

and the posterior density $p\left(\mathbf{x}_k | \mathbf{z}_k\right)$ can be approximated using (A-2). We refer the reader to [7] for a more detailed but concise exposé of the differences between the different PF algorithms: sequential importance sampling (SIS), generic particle filter, sampling importance resampling (SIR), auxiliary particle filter (APF), and regularized particle filter (RPF). We provide a succinct exposé of the SIS, SIR algorithms as well as the generic particle filter's and the regularized particle filter's in Appendix C. One important feature of PF is that not one implementation is better than all the others. In different contexts, different PFs may have wildly different performances.

B Some questions on the art of replication with Kalman filter

B.1 Specification of the parameters $\hat{\mathbf{w}}_0$, $\hat{\mathbf{P}}_0$ and Q

B.1.1 Parametrization of Q

To discuss the problem of the parametrization of the matrix Q, we consider the first example presented in [30]. Starting in 1990, a GTAA fund is built synthetically using two factors (the MSCI USA index and the MSCI EMU index). The weights are rebalanced on a monthly basis. In Figure 17, the two indexes, the weights[15] and the corresponding strategy are reported. On can

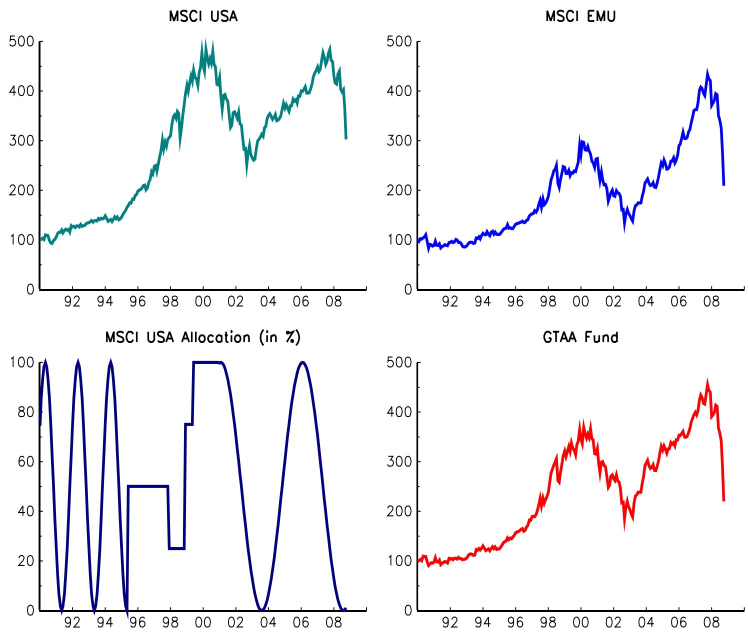
[15] We only report the weights $w_k^{(1)}$ of the MSCI USA index because we have $w_k^{(2)} = 1 - w_k^{(1)}$ for the MSCI EMU index.

then estimate the weights $\hat{w}^{(1)}_{k|k}$ and $\hat{w}^{(2)}_{k|k}$ of the dynamic allocation using only the measurements $r^{(F)}_k$, i.e. the monthly returns of the GTAA fund. In order to run a Kalman filter, the Q_k matrix must be specified. We set:

$$Q_k = Q = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$$

Q_k is constant and does not depend on the time sampling index k. Then, we have to define the initial conditions. We assume that $\hat{w}^{(1)}_0 = \hat{w}^{(2)}_0 = 0.5$ and $\hat{\mathbf{P}}_0 = \mathbf{0}_{2\times 2}$. We run the system (A-1) and obtain the estimates of $\hat{w}^{(1)}_{k|k}$ in Figure 18. One problem however is that the *ad hoc* manner the Q matrix is defined. Different initializations of the Q matrix will give very different estimates (see Figure 18 for several values of $Q_{(2,2)}$).

Figure 17: The GTAA example



One question then is how to define the Q matrix in an optimal way. One idea is to compute the covariance matrix[16] Σ of the difference between the true weights at two successive time steps, $w^{(1)}_k - w^{(1)}_{k-1}$ and $w^{(2)}_k - w^{(2)}_{k-1}$, and set $Q = \hat{\Sigma}$

$$Q = \hat{\Sigma} = \begin{pmatrix} 5.768 & -5.768 \\ -5.768 & 5.768 \end{pmatrix} \times 10^{-3}$$

We would then obtain the estimates $\hat{w}^{(1)}_{k|k}$ in Figure 19. Notice that the KF estimates are exactly the true values.

[16] We verify that the correlation between $w^{(1)}_k - w^{(1)}_{k-1}$ and $w^{(2)}_k - w^{(2)}_{k-1}$ is -1, because we have $w^{(1)}_k + w^{(2)}_k = 1$ and $w^{(2)}_k - w^{(2)}_{k-1} = -\left(w^{(1)}_k - w^{(1)}_{k-1}\right)$.

Figure 18: Kalman estimates (in %) of $\hat{w}^{(1)}_{k|k}$

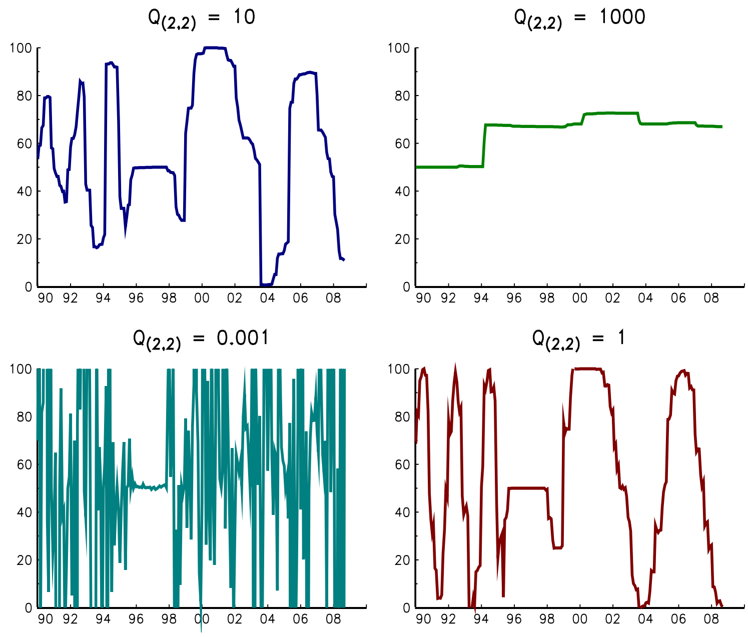
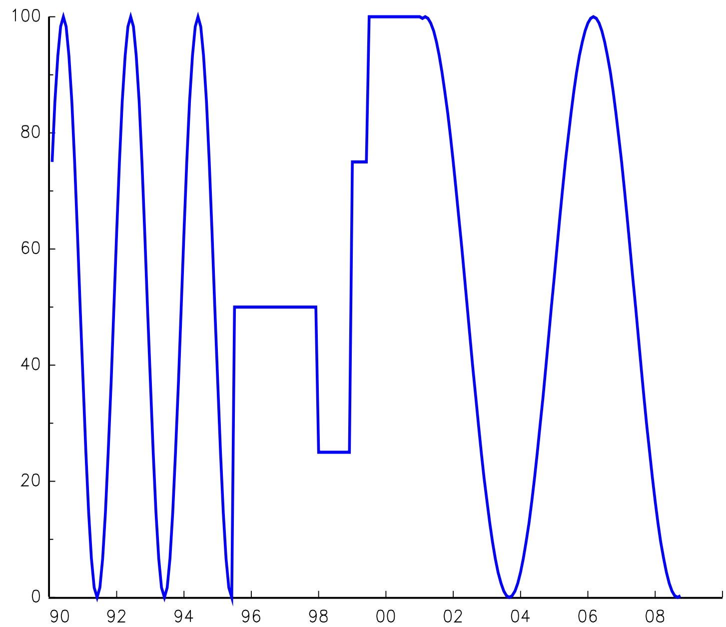


Figure 19: Kalman estimates (in %) of $\hat{w}^{(1)}_{k|k}$ with $Q_k = \hat{\Sigma}$



In reality, one does not know the true values of $w_k^{(1)}$ and $w_k^{(2)}$. Thus, one cannot compute the covariance matrix $\hat{\Sigma}$ and set $Q = \hat{\Sigma}$. In this case, a solution is to estimate some parameters θ of the state-space model. A very natural way is to use maximum likelihood estimation. Recall that $\mathbf{e}_k$ is the random variate denoting the tracking error. From a statistical point of view, it also represents the innovations of the measurement process $\mathbf{z}_k$. Because $\mathbf{e}_k$ is Gaussian with covariance matrix $\hat{V}_k$, the ML estimate $\hat{\theta}_{\mathrm{ML}}$ can be estimated by maximizing the sum of the following log-likelihood functions for each time t_k:

$$\begin{aligned} \ell_k(\theta) &= \ln p(\mathbf{z}_k \mid \mathbf{z}_{1:k-1}) \\ &= -\frac{n_z}{2}\ln(2\pi) - \frac{1}{2}\log\left|\hat{V}_k\right| - \frac{1}{2}\hat{\mathbf{e}}_k^\top \hat{V}_k^{-1}\hat{\mathbf{e}}_k \end{aligned}$$

and:

$$\hat{\theta}_{\mathrm{ML}} = \arg\max \sum \ell_k(\theta)$$

The matrix Q_k can be defined in several ways. One specification[17]:

$$Q_k = \begin{pmatrix} \theta_1^2 & 0 \\ 0 & \theta_2^2 \end{pmatrix}$$

assumes that the correlation between $w_k^{(1)} - w_{k-1}^{(1)}$ and $w_k^{(2)} - w_{k-1}^{(2)}$ is 0 even if we know that it is generally not true. When estimating the two parameters θ_1 and θ_2 by maximum likelihood in our example, we obtain:

$$Q_k = \begin{pmatrix} 8.198 & 0 \\ 0 & 6.514 \end{pmatrix} \times 10^{-3}$$

The corresponding KF estimates $\hat{w}_{k|k}^{(1)}$ are reported in Figure 20. Notice the true values of $w_k^{(1)}$ and $w_k^{(2)}$ are not tracked to perfection but the results are still very good. Yet, another way to specify $Q_k$is to consider $Q_k = AA^\top$ with A its Cholesky decomposition:

$$A = \begin{pmatrix} \theta_1 & 0 \\ \theta_2 & \theta_3 \end{pmatrix}$$

In this case, the ML estimates are $\hat{\theta}_1 = 0.07723$, $\hat{\theta}_2 = -0.07718$, $\hat{\theta}_3 \simeq 0$ and:

$$Q_k = \begin{pmatrix} 5.965 & -5.961 \\ -5.961 & 5.957 \end{pmatrix} \times 10^{-3}$$

The corresponding KF estimates $\hat{w}_{k|k}^{(1)}$ are not reported because we obtain results very close to those presented in Figure 19.

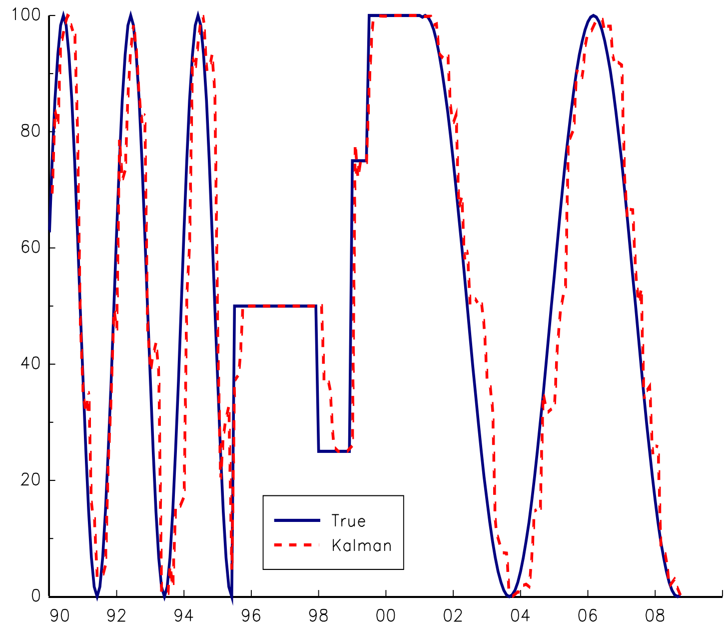
B.1.2 Choice of initialization parameters $\mathbf{w}_0$ and $\mathbf{P}_0$

In this appendix, we consider the problem of choosing the initial values $\mathbf{w}_0$ and $\mathbf{P}_0$ of the KF algorithm. We do so in the context of the model (7) estimated in [30].

Assume first that $\hat{\mathbf{P}}_0 = \mathbf{0}_{m\times m}$, i.e. $\mathbf{w}_0$ is not a random vector or equivalently, we know exactly the initial condition. In Figure 21, we report the KF estimates $\hat{w}_{k|k}^{(i)}$ when $\hat{w}_0^{(i)} = 0\%$ and $\hat{w}_0^{(i)} = 50\%$.

[17] We use this parametrization in order to be sure that Q_k is a positive definite matrix.

Figure 20: Kalman estimates (in %) of $\hat{w}^{(1)}_{k|k}$ with ML estimate of Q



Notice how the weights differ. One may think that we face an identification problem. However, the factors have been chosen in order to present small cross-correlations. For example, there is only one directional equity position (SPX) whereas all the other equity positions are done in a relative value (long/short exposure of the equity index versus the S&P 500 index). The problem comes from the fact that we impose $\hat{\mathbf{P}}_0 = \mathbf{0}_{m\times m}$. Indeed, the true positions of the initial exposures are not known exactly. Assume that $w^{(i)}_0 \sim \mathcal{N}\left(\hat{w}^{(i)}_0, 1\right)$. The initial state vector distribution is diffuse because its volatility is high[18], i.e. we have little confidence on the expected values $\hat{w}^{(i)}_0$. In this case, the results obtained are presented in Figure 22. Contrary to the previous case, the initial value of the state vector has now little impact on the KF estimates. In

Table 15: Difference between the weights (in %)

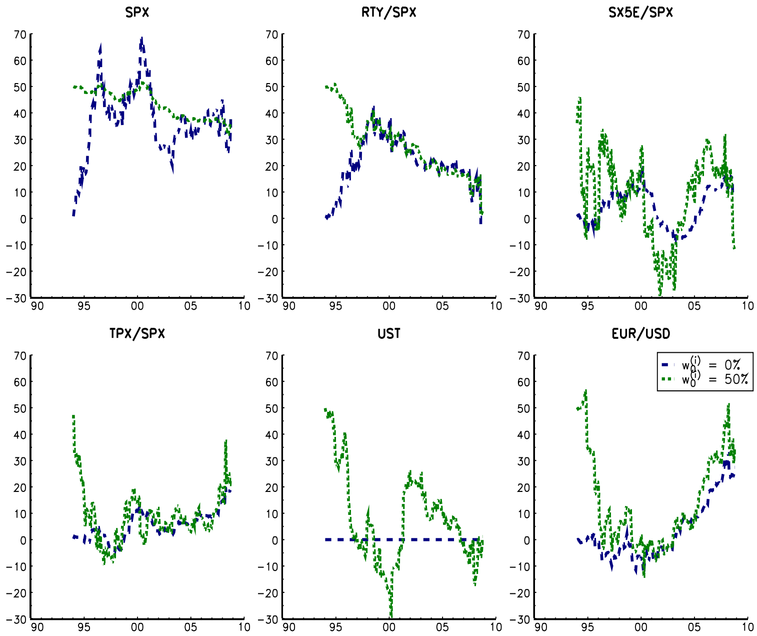
Factor	(a)	(b)	(c)	(d)
SPX	19.11	0.04	38.27	8.11
RTY/SPX	7.46	0.03	63.76	9.98
SX5E/SPX	26.39	0.05	73.77	19.51
TPX/SPX	17.44	0.03	13.23	10.16
UST	29.47	0.06	19.53	12.33
EUR/USD	17.73	0.04	29.37	16.44

table 15, we report the maximum difference between the estimates of the weights using $\hat{w}^{(i)}_0 = 0\%$ and $\hat{w}^{(i)}_0 = 50\%$. With $\hat{\mathbf{P}}_0 = \mathbf{0}_{6\times 6}$, we observe huge differences. In particular, the allocation in UST differs one month about 30%. With $\hat{\mathbf{P}}_0 = I_6$, these differences are largely reduced and are

[18] In Bayesian terms, we are choosing a very uninformative prior density function.

not significant (the larger difference is observed for the UST and is about 0.06%).

Figure 21: The case (a) with $\hat{\mathbf{P}}_0 = \mathbf{0}_{6\times 6}$



If we assume now that the measurement equation does not contain the noise process η_k meaning that $r_k^{(\mathrm{HF})} = \sum_{i=1}^m w_k^{(i)} r_k^{(i)}$, we obtain very different results. The reason is that in this model we have necessarily:

$$r_k^{(\mathrm{HF})} - \sum_{i=1}^m \hat{w}_{k|k}^{(i)} r_k^{(i)} = 0$$

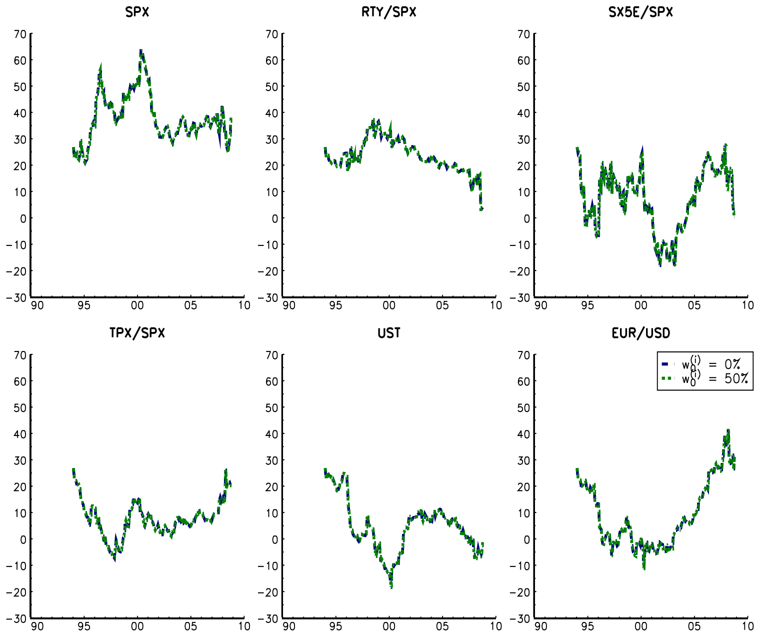
whereas in the previous approach:

$$r_k^{(\mathrm{HF})} - \sum_{i=1}^m \hat{w}_{k|k}^{(i)} r_k^{(i)} \neq 0$$

Without the noise process η_k, KF has to adjust the weights more frequently and produces larger changes. This is not the case in the previous model, because the prediction error defined by $\left(r_k^{(\mathrm{HF})} - \sum_{i=1}^m \hat{w}_{k|k-1}^{(i)} r_k^{(i)}\right)$ is partially absorbed by the noise process. Recall that:

$$\begin{aligned}
\left|\hat{\mathbf{w}}_{k|k} - \hat{\mathbf{w}}_{k-1|k-1}\right| &= \left|\hat{\mathbf{w}}_{k|k} - \hat{\mathbf{w}}_{k|k-1}\right| \\
&= \frac{\left|\hat{\mathbf{P}}_{k|k-1}\mathbf{r}_k \hat{e}_k\right|}{\mathbf{r}_k^\top \hat{\mathbf{P}}_{k|k-1}\mathbf{r}_k + S_k} \\
&\leq \frac{\left|\hat{\mathbf{P}}_{k|k-1}\mathbf{r}_k \hat{e}_k\right|}{\mathbf{r}_k^\top \hat{\mathbf{P}}_{k|k-1}\mathbf{r}_k}
\end{aligned}$$

Figure 22: The case (b) with $\hat{\mathbf{P}}_0 = I_6$



We have reported the KF estimates $\hat{w}_{k|k}^{(i)}$ when $w_0^{(i)} \sim \mathcal{N}(0,1)$ in Figure 23. We verify that the weights are very volatile without the noise process η_k with respect those estimated with the noise process η_k.

Finally, we have to decide whether we use a full or a diagonal covariance matrix Q_k. Let us consider the parametrization $\hat{S}_k = \hat{\sigma}_S^2$ and $\hat{Q}_k = \hat{\rho} \odot \hat{\sigma}_Q \odot \hat{\sigma}_Q^\top$. In the case where the Q_k matrix is diagonal, we obtain $\hat{\sigma}_S = 0.74 \times 10^{-2}$, $\hat{\rho} = I_6$ and:

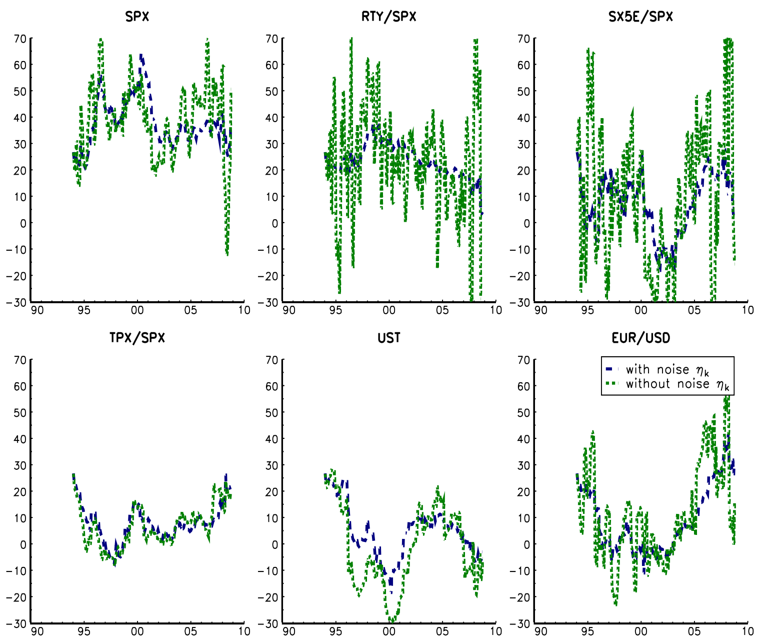
$$\hat{\sigma}_Q = \begin{pmatrix} 2.73 \\ 1.67 \\ 4.58 \\ 2.09 \\ 2.25 \\ 2.52 \end{pmatrix} \times 10^{-2}$$

In the case with cross-correlations[19], we obtain $\hat{\sigma}_S = 0.74 \times 10^{-2}$,

$$\hat{\rho} = \begin{pmatrix} 1.00 & & & & & \\ 0.53 & 1.00 & & & & \\ 0.19 & 0.16 & 1.00 & & & \\ -0.12 & -0.29 & 0.89 & 1.00 & & \\ -0.95 & -0.41 & -0.46 & -0.19 & 1.00 & \\ -0.38 & -0.71 & 0.57 & 0.88 & 0.09 & 1.00 \end{pmatrix}$$

[19]Following Appendix B.1, ML estimation is performed using the Cholesky parametrization of Q_k.

Figure 23: The case (c) without the noise process η_k



and:

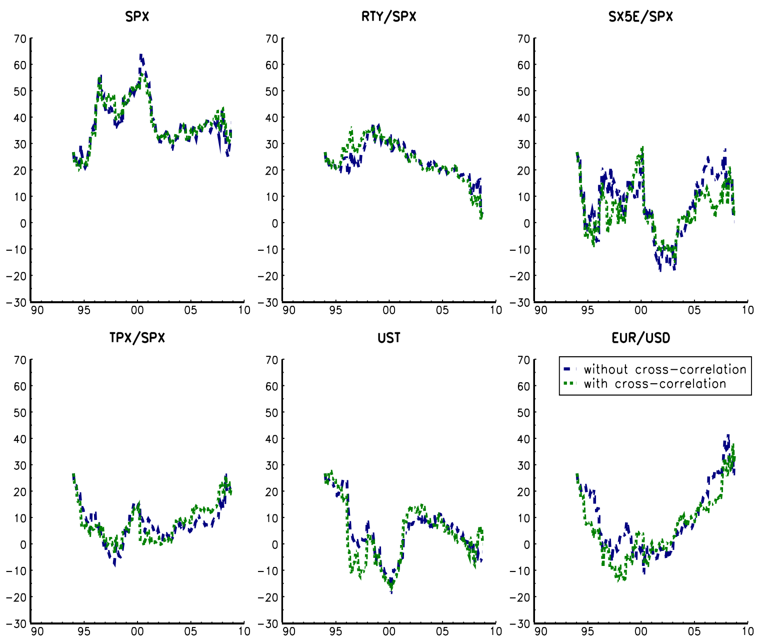
$$\hat{\sigma}_Q = \begin{pmatrix} 2.31 \\ 1.52 \\ 3.96 \\ 2.14 \\ 2.69 \\ 2.54 \end{pmatrix} \times 10^{-2}$$

We have reported the KF estimates $\hat{w}_{k|k}^{(i)}$ when $w_0^{(i)} \sim \mathcal{N}(0,1)$ in Figure 24. We notice that the weights are very close. In order to choose a model, we have to consider another statistics. For example, if we compare the two models in terms of tracking error volatility, we remark that the two models give similar results.

Remark 4 *The case $Q_k = diag\left(\sigma_1^2, \ldots, \sigma_m^2\right)$ may be viewed as a special case of the general model. This constrained model imposes that the cross-correlations are zero. If we compute the empirical correlation matrix of the changes $\hat{\mathbf{w}}_{k|k} - \hat{\mathbf{w}}_{k-1|k-1}$, we obtain effectively:*

$$\hat{\rho} = \begin{pmatrix} 1.00 & & & & & \\ -0.06 & 1.00 & & & & \\ -0.02 & 0.05 & 1.00 & & & \\ 0.07 & -0.04 & 0.02 & 1.00 & & \\ -0.12 & 0.14 & 0.02 & 0.06 & 1.00 & \\ -0.11 & 0.06 & -0.05 & 0.01 & -0.02 & 1.00 \end{pmatrix}$$

Figure 24: The case (d) with cross-correlations in the Q_k matrix



For the general model, the empirical correlation matrix is very different:

$$\hat{\rho} = \begin{pmatrix} 1.00 & & & & & \\ 0.47 & 1.00 & & & & \\ 0.18 & 0.20 & 1.00 & & & \\ -0.13 & -0.23 & 0.87 & 1.00 & & \\ -0.88 & -0.22 & -0.35 & -0.05 & 1.00 & \\ -0.36 & -0.64 & 0.52 & 0.87 & 0.17 & 1.00 \end{pmatrix}$$

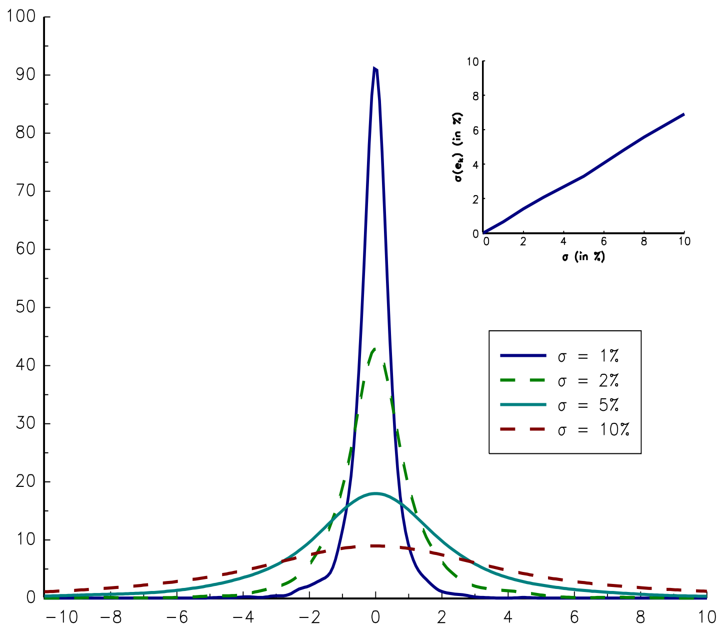
It is interesting to note that allocation changes in SPX are positively correlated to the allocation changes in RTY/SPX and negatively correlated to UST and EUR/USD.

Remark 5 *From a practical point of view, we prefer however to use a diagonal parametrization of the Q_k covariance matrix when the sample period is very long (like in our study), because cross-correlations are generally considered time varying. In this case, volatilities σ_i are higher in order to compensate the effect of cross-correlations on the Kalman filter adjustments.*

B.2 Probability distribution of the tracking error

The quality of the replication process depends on the tracking error $\mathbf{e}_k$. Consider the formulas of $\hat{\mathbf{e}}_k$ and $\hat{V}_k$, and note that different parameters influence these quantities. Among them, the most important parameters are the covariance matrices Q_k. Consider the previous example with MSCI indexes and assume that $w_k^{(i)} = w_k^{(i)} + \sigma\varepsilon_k^{(i)}$ with $\varepsilon_k^{(i)} \sim \mathcal{N}(0,1)$ *i.i.d.* processes. The density of the tracking error $\mathbf{e}_k$ is presented in Figure 25 for different values of σ. The relationship between σ and the volatility $\sigma(\mathbf{e}_k)$ is reported as well. We verify that the tracking error depends on the dynamics of the asset allocation. The more dynamic the asset allocation, the more difficult the replication.

Figure 25: Density of the tracking error (yearly basis and in percent)



B.3 Identification of the tracking system

Recall that the state-space model of the replication system for the GTAA strategy is:

$$\left\{ \begin{array}{l} r_k^{(\mathrm{F})} = \mathbf{r}_k^\top \mathbf{w}_k \\ \mathbf{w}_k = \mathbf{w}_{k-1} + \nu_k \\ \mathbf{w}_0 \sim \mathcal{N}\left(\hat{\mathbf{w}}_0, \hat{\mathbf{P}}_0\right) \end{array} \right.$$

The model is not completely specified until the three parameters $\hat{\mathbf{w}}_0$, $\hat{\mathbf{P}}_0$ and Q_k are defined. Most of the time, Q_k is considered time invariant. It has been shown above that $Q = Q_k$ may be estimated by the method of maximum likelihood once $\hat{\mathbf{w}}_0$ and $\hat{\mathbf{P}}_0$ have been specified.

Let $\hat{\mathbf{w}}_{k|k}\left(\hat{\mathbf{w}}_0, \hat{\mathbf{P}}_0\right)$ be the KF estimates depending on $\hat{\mathbf{w}}_0$ and $\hat{\mathbf{P}}_0$. Roncalli and Teiletche [30] explain with an example that $\hat{\mathbf{w}}_0$ and $\hat{\mathbf{P}}_0$ have in general little influence on the KF estimates $\hat{w}_{k|k}^{(i)}$. In other words,

$$\hat{\mathbf{w}}_{k|k}\left(\hat{\mathbf{w}}_0, \hat{\mathbf{P}}_0\right) \longrightarrow \hat{\mathbf{w}}_{k|k}\left(\hat{\mathbf{w}}_0', \hat{\mathbf{P}}_0'\right) \text{ when } k \to \infty.$$

This property is true when the factors are uncorrelated. In that case, the learning algorithm of the Kalman filter operates very fast and there is no identification problem. When the factors are correlated however, identification can be more difficult. Consider the case with three factors. One has:

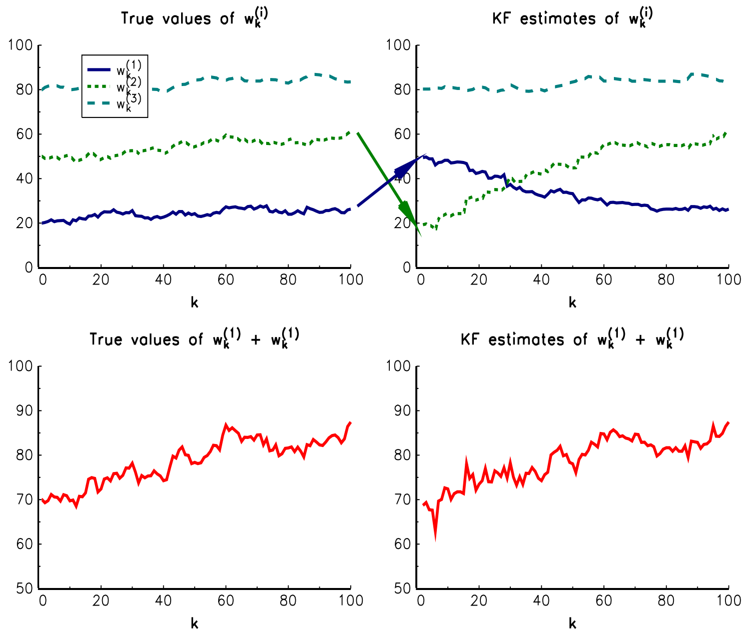
$$r_k^{(\mathrm{F})} = w_k^{(1)} r_k^{(1)} + w_k^{(2)} r_k^{(2)} + w_k^{(3)} r_k^{(2)}$$

If the first two factors are perfectly correlated[20], one has $r_k^{(1)} = r_k^{(2)} = r_k^{(1,2)}$ and $r_k^{(\mathrm{F})} = \left(w_k^{(1)} + w_k^{(2)}\right) r_k^{(1,2)} + w_k^{(3)} r_k^{(2)}$. In this case, only the sum $\left(w_k^{(1)} + w_k^{(2)}\right)$ of the first two weights

[20] We assume that the volatility of the two factors is the same.

and the third weight can be identified since there are several representations of the same state-space model. To illustrate this problem, we simulate the model with three factors whose returns are normally distributed with the same volatility of 10% and zero expected returns. The correlation of the returns of the first two factors is set equal to 99% whereas the returns of the third factors are not correlated to the returns of the other factors. The true weights are represented in the top left graph of Figure 26. The initial true values are $w_0^{(1)} = 20\%$, $w_0^{(2)} = 50\%$ and

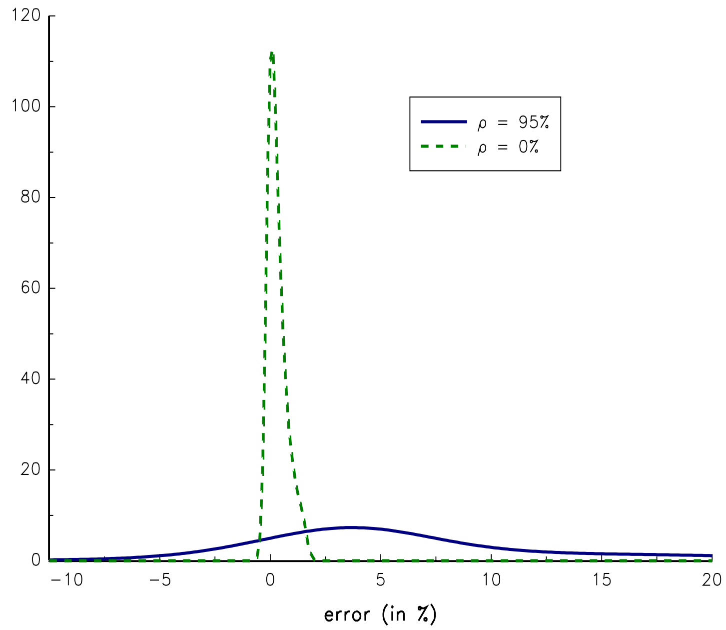
Figure 26: Illustration of the identification problem when the factors are correlated



$w_0^{(3)} = 80\%$. We set $\hat{\mathbf{w}}_0 = (20\%, 50\%, 80\%)$ and $\hat{\mathbf{P}}_0 = \mathbf{0}_{3\times 3}$. Then, we estimate the Q matrix by maximum likelihood and $\hat{\mathbf{w}}_{k|k}$ using KF. The values of $\hat{w}_{k|k}^{(i)}$ are reported in the top right graph of Figure 26. Notice how, at the start, the estimated weights of the first factor are closest to the true weights of the second factor and vice versa. By the end of the period, the Kalman filter has finally succeeded to distinguish the first two factors and we verify that the estimated weights of the first factor are closest to the true weights of the first factor than to the true weights of the second factor. Consider now the sum of the weights of the two first factors. Notice how in this case the Kalman filter has done a good job with the differences between $w_k^{(1)} + w_k^{(2)}$ and $\hat{w}_{k|k}^{(1)} + \hat{w}_{k|k}^{(2)}$ being small (see bottom graphs of Figure 26). Now, assume that the initial state is chosen randomly according to $\hat{w}_0^{(i)} \sim \mathcal{U}_{[0,1]}$. The density of the difference between $\hat{w}_{100|100}^{(1)}(\hat{\mathbf{w}}_0, \mathbf{0}_{3\times 3})$ and $w_{100}^{(1)}$ is reported in Figure 27 when the correlation $\rho\left(r_k^{(1)}, r_k^{(2)}\right)$ is respectively 95% and 0%. It is obvious that the case of highly correlated factors produces larger errors. Identification may thus be very difficult because, in this case, changing the initial conditions will modify the estimates.

One may think that the solution is to consider $\hat{\mathbf{w}}_0$ and $\hat{\mathbf{P}}_0$ as ML parameters $\theta = \left(\hat{\mathbf{w}}_0, \hat{\mathbf{P}}_0, Q\right)$. However, this solution is good only when the state-space is time-invariant and stationary. In our case, it is not relevant and numerical experiments show that the convergence of the ML optimization is difficult.

Figure 27: Density of the error $\hat{w}^{(1)}_{100|100}\left(\hat{\mathbf{w}}_0, \mathbf{0}_{3\times3}\right) - w^{(1)}_{100}$



C Numerical Algorithms for implementation of Particle Filters

In this appendix, we provide, in pseudo code, the algorithms for the particle filters implemented for the purpose of this study. In Appendix A.2, we presented the algorithm, known under the name Sequential Importance Sampling (SIS), which forms the basis for most sequential Monte Carlo filters developed over the past decade [7]. We start by providing its pseudo code in Algorithm 1, before exposing the more advanced algorithms we used: a generic Particle Filter (GPF), a Sampling Importance Resampling (SIR) algorithm, and a regularized Particle Filter (RPF).

The SIS algorithm is thus a very simple algorithm, easy to implement. However, it commonly suffers from a degeneracy phenomenon, where after only a few iterations, all but one particle will have negligible weights. This degeneracy problems implies that a large computational effort will be devoted to updating particles whose contribution to the approximation of the filtering density $p\left(\mathbf{x}_k \mid \mathbf{z}_{1:k}\right)$ is quasi null. In order to alleviate this problem, more advanced algorithm have been devised. One way to deal with degeneracy is to carefully choose the importance density function $q\left(\mathbf{x}_k \mid \mathbf{x}^i_{k-1}, \mathbf{z}_k\right)$. We leave to the reader to consult [7] for a discussion of the importance of the choice of the importance density. Another simple idea is to resample the particles when a certain measure of degeneracy becomes too large (or too small). For example, one could calculate the effective sample size N_{eff} defined as:

$$N_{\text{eff}} = \frac{N_s}{1 + \sigma\left(w^{*i}_k\right)^2}$$

where $w^{*i}_k = p\left(\mathbf{x}^i_k \mid \mathbf{z}_{1:k}\right) / q\left(\mathbf{x}^i_k \mid \mathbf{x}^i_{k-1}, \mathbf{z}_k\right)$ is referred to as the "true weight." As this cannot be

Algorithm 1 SIS Particle Filter

```
procedure SIS_PARTICLE_FILTER(z_{1:T}, N_s)                ▷ Runs a SIS Particle Filter
    {x_0^i, w_0^i}_{i=1:N_s} ~ p_0(.)                       ▷ Initialization
    k ← 1
    while k < T do
        {x_k^i, w_k^i}_{i=1:N_s} ← SIS_STEP(x_{k-1}^i, w_{k-1}^i, z_k)
        k ← k + 1
    end while
    return {x_{1:T}^i, w_{1:T}^i}_{i=1:N_s}
end procedure

procedure SIS_STEP(x_{k-1}^i, w_{k-1}^i, z_k)   ▷ Propagates the sample from state k − 1 to state k
    for i = 1 : N_s do
        Draw x_k^i ~ q(x_k | x_{k-1}^i, z_k)
        Assign the particle a weight, w_k^i, according to A-6
    end for
    return {x_k^i, w_k^i}_{i=1:N_s}
end procedure
```

Algorithm 2 Resampling Algorithm

```
procedure RESAMPLE({x_k^i, w_k^i}_{i=1:N_s})
    c_1 ← 0                                                 ▷ Initialise the CDF
    for i = 2 : N_s do                                      ▷ Construct the CDF
        c_i ← c_{i-1} + w_k^i
    end for

    i ← 1                                                   ▷ Start at the bottom of the CDF
    u_1 ~ U[0, N_s^{-1}]                                    ▷ Draw a starting point
    for j = 1 : N_s do
        u_j ← u_1 + N_s^{-1}(j − 1)                         ▷ Move along the CDF
        while u_j > c_i do
            i ← i + 1
        end while
        x_k^{j*} = x_k^i                                    ▷ Assign sample
        w_k^j = N_s^{-1}                                    ▷ Assign weight
        parent_j ← i                                        ▷ Assign parent
    end for
    return {x_k^{j*}, w_k^j, parent_j}_{j=1:N_s}
end procedure
```

Algorithm 3 Generic Particle Filter

```
procedure GENERIC_PARTICLE_FILTER(z_{1:T}, N_s)          ▷ Runs a Generic Particle Filter
    {x_0^i, w_0^i}_{i=1:N_s} ~ p_0(.)                      ▷ Initialization
    k ← 1
    while k < T do
        {x_k^i, w_k^i}_{i=1:N_s} ← PF_STEP(x_{k-1}^i, w_{k-1}^i, z_k)
        k ← k + 1
    end while
    return {x_{1:T}^i, w_{1:T}^i}_{i=1:N_s}
end procedure

procedure PF_STEP(x_{k-1}^i, w_{k-1}^i, z_k)
    for i = 1 : N_s do
        Draw x_k^i ~ q(x_k | x_{k-1}^i, z_k)
        Assign the particle a weight, w_k^i, according to A-6
    end for
    t ← Σ_{i=1}^{N_s} w_k^i                                ▷ Calculate total weight
    for i = 1 : N_s do
        w_k^i ← t^{-1} w_k^i
    end for
    Calculate N̂_eff using C-7
    if N̂_eff < N_s then
        {x_k^i, w_k^i, -}_{i=1:N_s} ← RESAMPLE({x_k^i, w_k^i}_{i=1:N_s})
    end if
end procedure
```

valued exactly, this quantity can be estimated using:

$$\hat{N}_{\text{eff}} = \frac{1}{\sum_{i=1}^{N_s} \left(w_k^i\right)^2} \tag{C-7}$$

We provide in Algorithm 2 and in Algorithm 3 respectively the resampling algorithm we used and the generic Particle Filter which is deduced from the SIS algorithm by adding this resampling step to avoid degeneracy.

In many particle filters implementations, one uses the prior density $p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}^i\right)$ as the importance density $q\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}^i, \mathbf{z}_k\right)$ for even though it is often suboptimal, it simplifies the weights update equation A-6 into:

$$w_k^i \propto w_{k-1}^i \times p\left(\mathbf{z}_k \mid \mathbf{x}_k^i\right)$$

Furthermore, if resampling is applied at every step — this particular implementation is called the Sampling Importance Resampling (SIR) of which we give the algorithm in pseudo code in Algorithm 4 — then we have $w_{k-1}^i = 1/N_s \, \forall i$, and so:

$$w_k^i \propto p\left(\mathbf{z}_k \mid \mathbf{x}_k^i\right) \tag{C-8}$$

The weights given in C-8 are normalized before the resampling stage.

Algorithm 4 SIR Particle Filter

```
procedure SIR_PARTICLE_FILTER(z_{1:T}, N_s)                ▷ Runs a SIR Particle Filter
    {x_0^i, w_0^i}_{i=1:N_s} ~ p_0(.)                       ▷ Initialization
    k ← 1
    while k < T do
        {x_k^i, w_k^i}_{i=1:N_s} ← SIR_STEP(x_{k-1}^i, w_{k-1}^i, z_k)
        k ← k + 1
    end while
    return {x_{1:T}^i, w_{1:T}^i}_{i=1:N_s}
end procedure

procedure SIR_STEP(x_{k-1}^i, w_{k-1}^i, z_k)
    for i = 1 : N_s do
        Draw x_k^i ~ p(x_k | x_{k-1}^i)
        w_k^i ← p(z_k | x_k^i)
    end for
    t ← Σ_{i=1}^{N_s} w_k^i                                  ▷ Calculate total weight
    for i = 1 : N_s do
        w_k^i ← t^{-1} w_k^i
    end for
    {x_k^i, w_k^i, −}_{i=1:N_s} ← RESAMPLE({x_k^i, w_k^i}_{i=1:N_s})   ▷ Systematic resampling
end procedure
```

The regularized Particle Filter is based on the same idea as the Generic Particle Filter, with the same resampling condition, but the resampling step provides an entirely new sample based on a continuous approximation of the posterior filtering density $p\left(\mathbf{x}_k \mid \mathbf{z}_k\right)$, such that we have the following approximation:

$$\hat{p}\left(\mathbf{x}_k \mid \mathbf{z}_k\right) = \sum_{i=1}^{N_s} w_k^i K_h\left(\mathbf{x}_k - \mathbf{x}_k^i\right) \tag{C-9}$$

where:

$$K_h\left(\mathbf{x}\right) = \frac{1}{h^{n_x}} K\left(\frac{\mathbf{x}}{h}\right)$$

is the re-scaled Kernel density $K\left(\cdot\right)$, $h > 0$ is the Kernel bandwidth, n_x is the dimension of the state vector $\mathbf{x}$, and w_k^i, $i = 1, \ldots, N_s$ are normalized weights. The Kernel $K\left(\cdot\right)$ and bandwidth h should be chosen to minimize the Mean Integrated Square Error (MISE), between the true posterior density and the corresponding regularized empirical representation in C-9, defined as:

$$\mathrm{MISE}\left(\hat{p}\right) = \mathbb{E}\left[\int \left[\hat{p}\left(\mathbf{x}_k \mid \mathbf{z}_k\right) - p\left(\mathbf{x}_k \mid \mathbf{z}_k\right)\right]^2 \mathrm{d}\mathbf{x}_k\right]$$

One can show that in the case where all the samples have the same weight, the optimal choice of the Kernel is the Epanechnikov Kernel:

$$K_{\mathrm{opt}} = \begin{cases} \frac{n_x+2}{2c_{n_x}}\left(1 - \|x\|^2\right) & \text{if } \|x\| < 1, \\ 0 & \text{otherwise} \end{cases}$$

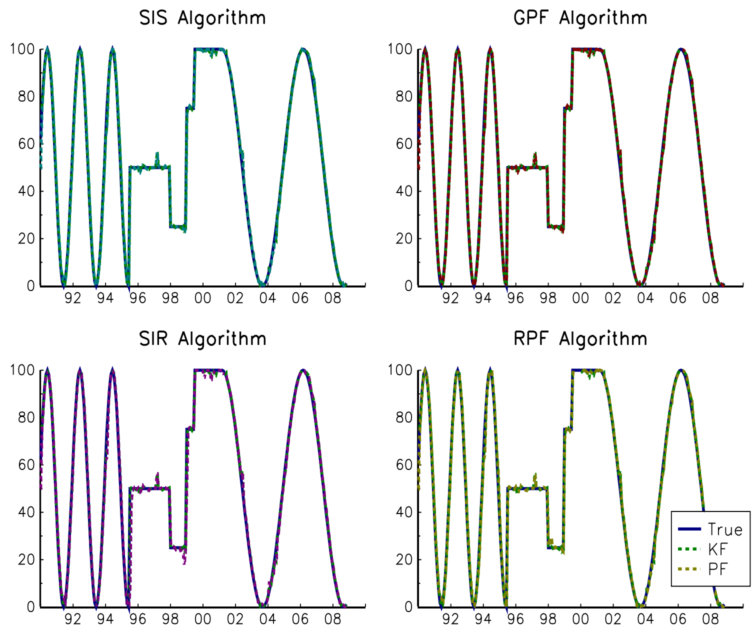
where c_{n_x} is the volume of the unit hypersphere in $\mathbb{R}^{n_x}$. Furthermore, when the underlying density is Gaussian with a unit covariance matrix, the optimal choice for the bandwidth is:

$$h_{\text{opt}} = AN_s^{-\frac{1}{n_x+4}}$$
$$A = \left[8c_{n_x}^{-1}\left(n_x+4\right)\left(2\sqrt{\pi}\right)^{n_x}\right]^{-\frac{1}{n_x+4}}$$

We can now provide the algorithm for the regularized Particle Filter in Algorithm 5.

We also illustrate these algorithms by reproducing the example given in Appendix B.1 using $N_s = 1000$ particles. Note that the parameters of the distributions in the particle filters were estimated using the Kalman filter. The results (sample means) are reported in Figure 28 with, from top to bottom and left to right, the SIS, the generic PF, the SIR and the RPF runs.

Figure 28: Solving example B.1 using particle filers — $N_s = 1000$.



Algorithm 5 Regularized Particle Filter

procedure REGULARIZED_PARTICLE_FILTER($\mathbf{z}_{1:T}$,N_s) ▷ Runs a Regularized Particle Filter
  $\{\mathbf{x}_0^i,\, w_0^i\}_{i=1:N_s} \sim p_0(.)$ ▷ Initialization
  $k \leftarrow 1$
  while $k < T$ **do**
    $\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s} \leftarrow$ RPF_STEP($\mathbf{x}_{k-1}^i$, w_{k-1}^i, $\mathbf{z}_k$)
    $k \leftarrow k+1$
  end while
  return $\{\mathbf{x}_{1:T}^i, w_{1:T}^i\}_{i=1:N_s}$
end procedure

procedure RPF_STEP($\mathbf{x}_{k-1}^i$, w_{k-1}^i, $\mathbf{z}_k$)
  for $i = 1 : N_s$ **do**
    Draw $\mathbf{x}_k^i \sim q\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}^i, \mathbf{z}_k\right)$
    Assign the particle a weight, w_k^i, according to A-6
  end for
  $t \leftarrow \sum_{i=1}^{N_s} w_k^i$ ▷ Calculate total weight
  for $i = 1 : N_s$ **do**
    $w_k^i \leftarrow t^{-1} w_k^i$
  end for
  Calculate $\widehat{N_{\text{eff}}}$ using C-7
  if $\widehat{N_{\text{eff}}} < N_s$ **then**
    Compute the empirical covariance matrix S_k of $\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s}$
    Compute $\mathbf{D}_k \leftarrow \text{Chol}(S_k)$ ▷ Cholesky decomposition of S_k: $\mathbf{D}_k\mathbf{D}_k^\top = S_k$
    $\{\mathbf{x}_k^i, w_k^i, -\}_{i=1:N_s} \leftarrow$ RESAMPLE($\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s}$)
    for $i = 1 : N_s$ **do**
      Draw $\epsilon^i \sim K_{\text{opt}}$ from the Epanechnikov Kernel
      $\mathbf{x}_k^{i*} \leftarrow \mathbf{x}_k^i + h_{\text{opt}}\mathbf{D}_k\epsilon^i$
    end for

    return $\{\mathbf{x}_k^{i*}, w_k^i\}_{i=1:N_s}$
  else
    return $\{\mathbf{x}_k^i, w_k^i\}_{i=1:N_s}$
  end if
end procedure