Copulas for finance

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Copulas for Finance
A Reading Guide and Some Applications

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Abstract

Copulas are a general tool to construct multivariate distributions and to investigate dependence structure between random variables. However, the concept of copula is not popular in Finance. In this paper, we show that copulas can be extensively used to solve many financial problems.

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Contents

1 Introduction

The problem of modelling asset returns is one of the most important issue in Finance. People generally use gaussian processes because of their tractable properties for computation. However, it is well known that asset returns are fat-tailed. Gaussian assumption is also the key point to understand the modern portfolio theory. Usually, efficient portfolios are given by the traditional mean-variance optimisation program (Markowitz [1987]):

$$\sup_{\alpha} \alpha^{\top}\mu \quad \text{u.c.} \quad \begin{cases} \alpha^{\top}\Sigma\alpha \leq \mathrm{s} \\ \alpha^{\top}\mathbf{1} = 1 \\ \alpha \geq \mathbf{0} \end{cases} \tag{1}$$

with μ the expected return vector of the N asset returns and Σ the corresponding covariance matrix. The problem of the investor is also to maximize the expected return for a given variance. In the portfolio analysis framework, **the variance corresponds to the risk measure, but it implies that the world is gaussian** (Schmock and Straumann [1999], Tasche [1999]). The research on value-at-risk (and capital allocation) has then considerably modified the concept of risk measure (Artzner, Delbaen, Eber and Heath [1997,1999]).

Capital allocation within a bank is getting more and more important as the regulatory requirements are moving towards economic-based measures of risk (see the reports [1] and [3]). Banks are urged to build sound internal measures of credit and market risks for all their activities (and certainly for operational risk in a near future). Internal models for credit, market and operational risks are fundamental for bank capital allocation in a *bottom-up* approach. Internal models generally face an important problem which is the modelling of joint distributions of different risks.

These two difficulties (gaussian assumption and joint distribution modelling) can be treated as a problem of *copulas*. A copula is a function that links univariate marginals to their multivariate distribution. Before 1999, copulas have not been used in finance. There have been recently some interesting papers on this subject (see for example the article of Embrechts, McNeil and Straumann [1999]). Moreover, copulas are more often cited in the financial litterature. Li [1999] studies the problem of default correlation in credit risk models, and shows that "*the current CreditMetrics approach to default correlation through asset correlation is equivalent to using a normal copula function*". In the Risk special report of November 1999 on Operational Risk, Ceske and Hernández [1999] explain that copulas may be used in conjunction with Monte Carlo methods to aggregate correlated losses.

The aim of the paper is to show that copulas could be extensively used in finance. The paper is organized as follows. In section two, we present copula functions and some related fields, in particular the concept of dependence. We then consider the problem of statistical inference of copulas in section three. We focus on the estimation problem. In section four, we provide applications of copulas to finance. Section five concludes and suggests directions for further research.

2 Copulas, multivariate distributions and dependence

2.1 Some definitions and properties

Definition 1 (Nelsen (1998), page 39) [1] *A N-dimensional copula is a function* $\mathbf{C}$ *with the following properties*[2]*:*

[1] The original definition given by Sklar [1959] is (in french)

> Nous appelerons copule (à n dimensions) tout fonction $\mathbf{C}$ continue et non-décroissante — au sens employé pour une fonction de répartition à n dimensions — définie sur le produit Cartésien de n intervalles fermés $[0,1]$ et satisfaisant aux conditions $\mathbf{C}(0,\ldots,0) = 0$ et $\mathbf{C}(1,\ldots,1,u,1,\ldots,1) = u$.

[2] We will note $\mathcal{C}$ the set of copulas.

1. *Dom* $\mathbf{C} = \mathbf{I}^N = [0,1]^N$*;*

2. $\mathbf{C}$ *is grounded and* N*-increasing*[3]*;*

3. $\mathbf{C}$ *has margins* $\mathbf{C}_n$ *which satisfy* $\mathbf{C}_n(u) = \mathbf{C}(1,\ldots,1,u,1,\ldots,1) = u$ *for all* u *in* $\mathbf{I}$.

A copula corresponds also to a function with particular properties. In particular, because of the second and third properties, it follows that $\operatorname{Im}\mathbf{C} = \mathbf{I}$, and so $\mathbf{C}$ is a multivariate uniform distribution. Moreover, it is obvious that if $\mathbf{F}_1,\ldots,\mathbf{F}_N$ are univariate distribution functions, $\mathbf{C}(\mathbf{F}_1(x_1),\ldots,\mathbf{F}_n(x_n),\ldots,\mathbf{F}_N(x_N))$ is a multivariate distribution function with margins $\mathbf{F}_1,\ldots,\mathbf{F}_N$ because $u_n = \mathbf{F}_n(x_n)$ is a uniform random variable. Copula functions are then an adapted tool to construct multivariate distributions.

Theorem 2 (Sklar's theorem) *Let* $\mathbf{F}$ *be an* N*-dimensional distribution function with continuous margins* $\mathbf{F}_1,\ldots,\mathbf{F}_N$. *Then* F *has a unique copula representation:*

$$\mathbf{F}(x_1,\ldots,x_n,\ldots,x_N) = \mathbf{C}(\mathbf{F}_1(x_1),\ldots,\mathbf{F}_n(x_n),\ldots,\mathbf{F}_N(x_N)) \tag{3}$$

The theorem of Sklar [1959] is very important, because it provides a way to analyse the dependence structure of multivariate distributions without studying marginal distributions. For example, if we consider the Gumbel's bivariate logistic distribution $\mathbf{F}(x_1,x_2) = (1+e^{-x_1}+e^{-x_2})^{-1}$ defined on $\mathbb{R}^2$. We could show that the marginal distributions are $\mathbf{F}_1(x_1) \equiv \int_{\mathbb{R}} \mathbf{F}(x_1,x_2)\,dx_2 = (1+e^{-x_1})^{-1}$ and $\mathbf{F}_2(x_2) = (1+e^{-x_2})^{-1}$. The copula function corresponds to

$$\begin{aligned} \mathbf{C}(u_1,u_2) &= \mathbf{F}\left(\mathbf{F}_1^{-1}(u_1),\mathbf{F}_2^{-1}(u_2)\right) \\ &= \frac{u_1 u_2}{u_1+u_2-u_1 u_2} \end{aligned} \tag{4}$$

However, as Frees and Valdez [1997] note, *it is* <u>*not*</u> *always obvious to identify the copula*. Indeed, for many financial applications, the problem is not to use a given multivariate distribution but consists in finding a convenient distribution to describe some stylized facts, for example the relationships between different asset returns. In most applications, the distribution is assumed to be a multivariate gaussian or a log-normal distribution for tractable calculus, even if the gaussian assumption may not be appropriate. Copulas are also a powerful tool for finance, because the modelling problem can be splitted into two steps:

- the first step deals with the identification of the marginal distributions;
- and the second step consists in defining the appropriate copula in order to represent the dependence structure in a good manner.

In order to illustrate this point, we consider the example of assets returns. We use the database of the London Metal Exchange[4] and we consider the spot prices of the commodities Aluminium Alloy (AL), Copper (CU), Nickel (NI), Lead (PB) and the 15 months forward prices of Aluminium Alloy (AL-15), dating back to January 1988. We assume that the distribution of these asset returns is gaussian. In this case, the corresponding ML estimate of the correlation matrix is given by the table 1.

Figure 1 represents the scatterplot of the returns AL and CU, the corresponding gaussian 2-dimensional covariance ellipse for confidence levels 95% and 99%, and the implied probability density function. Figure 2 contains the projection of the hyper-ellipse of dimension 5 for the asset returns. The gaussian assumption is

[3] $\mathbf{C}$ in N-increasing if the $\mathbf{C}$-volume of all N-boxes whose vertices lie in $\mathbf{I}^N$ are positive, or equivalently if we have

$$\sum_{i_1=1}^{2}\cdots\sum_{i_N=1}^{2}(-1)^{i_1+\cdots+i_N}\,\mathbf{C}\left[u_{1,i_1},\ldots,u_{N,i_N}\right] \geq 0 \tag{2}$$

for all $\left[u_{1,1},\ldots,u_{N,1}\right]$ and $\left[u_{1,2},\ldots,u_{N,2}\right]$ in $\mathbf{I}^N$ with $u_{n,1} \leq u_{n,2}$.

[4] In order to help the reader to reproduce the results, we use the public database available on the web site of the LME: http://www.lme.co.uk.

	AL	AL-15	CU	NI	PB
AL	1.00	0.82	0.44	0.36	0.33
AL-15		1.00	0.39	0.34	0.30
CU			1.00	0.37	0.31
NI				1.00	0.31
PB					1.00

Table 1: Correlation matrix ρ of the LME data

generally hard to verify because *rare* events occur more often than planned (see the outliers of the covariance ellipse for a 99.99% confidence level on the density function in figure 1). Figure 3 is a QQ-plot of the theoretical confidence level versus the empirical confidence level of the error ellipse. It is obvious that the gaussian hypothesis fails.

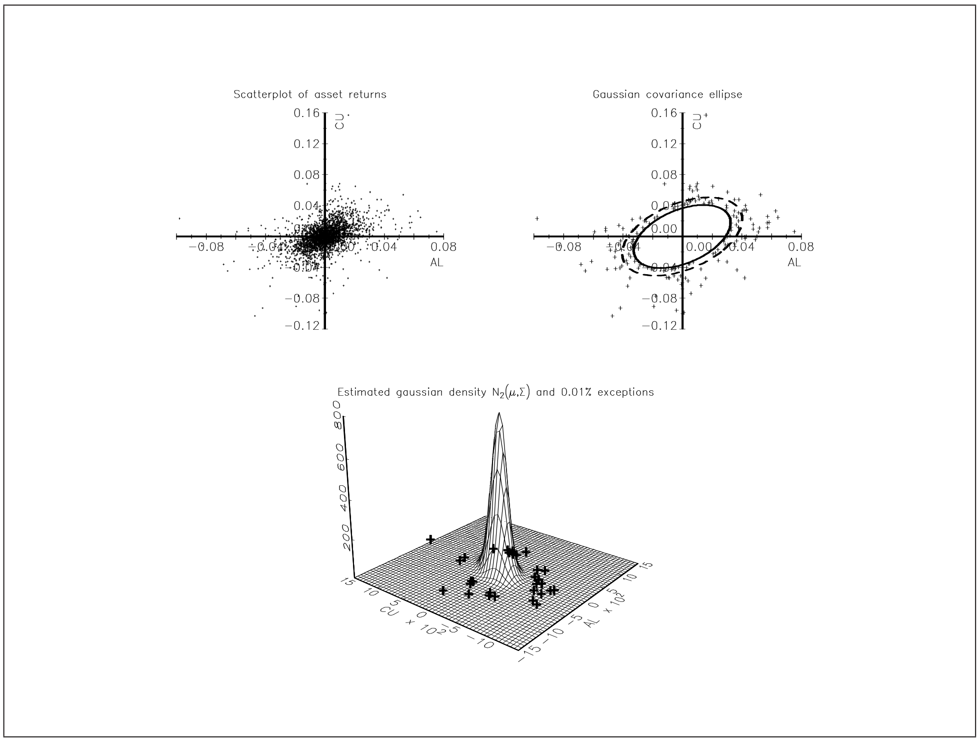

Figure 1: Gaussian assumption (I)

In the next paragraph, we will present the concept of dependence and how it is linked to copulas. Now, we present several properties that are necessary to understand how copulas work and why they are an attractive tool. One of the main property concerns **concordance ordering**, defined as follows:

Definition 3 (Nelsen (1998), page 34) *We say that the copula* $\mathbf{C}_1$ *is smaller than the copula* $\mathbf{C}_2$ *(or* $\mathbf{C}_2$ *is larger than* $\mathbf{C}_1$*), and write* $\mathbf{C}_1 \prec \mathbf{C}_2$ *(or* $\mathbf{C}_1 \succ \mathbf{C}_2$*) if*

$$\forall\,(u_1,\ldots,u_n,\ldots,u_N) \in \mathbf{I}^N, \quad \mathbf{C}_1\,(u_1,\ldots,u_n,\ldots,u_N) \leq \mathbf{C}_2\,(u_1,\ldots,u_n,\ldots,u_N) \tag{5}$$

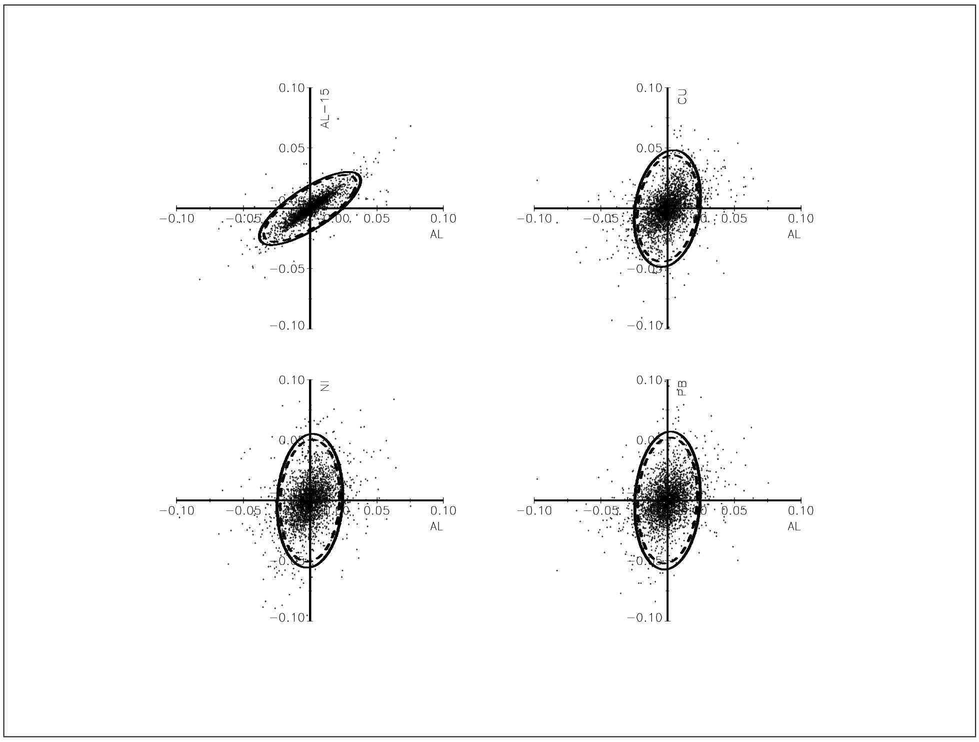

Figure 2: Gaussian assumption (II)

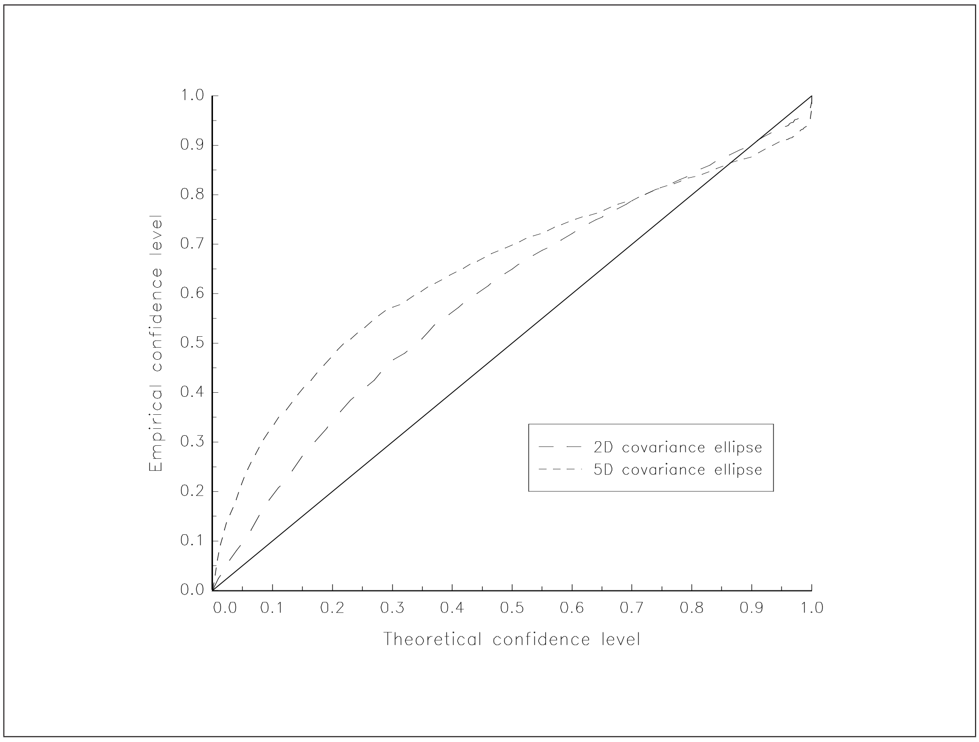

Figure 3: QQ-plot of the covariance ellipse

Two specific copulas play an important role[5], the lower and upper Fréchet bounds $\mathbf{C}^-$ and $\mathbf{C}^+$:

$$\begin{aligned}\mathbf{C}^-(u_1,\ldots,u_n,\ldots,u_N) &= \max\left(\sum_{n=1}^{N} u_n - N + 1, 0\right)\\ \mathbf{C}^+(u_1,\ldots,u_n,\ldots,u_N) &= \min(u_1,\ldots,u_n,\ldots,u_N)\end{aligned} \tag{6}$$

We could show that the following order holds for any copula $\mathbf{C}$:

$$\mathbf{C}^- \prec \mathbf{C} \prec \mathbf{C}^+ \tag{7}$$

The concept of concordance ordering can be easily illustrated with the example of the bivariate gaussian copula $\mathbf{C}(u_1,u_2;\rho) = \Phi_\rho\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right)$ (Joe [1997], page 140). For this family, we have

$$\mathbf{C}^- = \mathbf{C}_{\rho=-1} \prec \mathbf{C}_{\rho<0} \prec \mathbf{C}_{\rho=0} = \mathbf{C}^\perp \prec \mathbf{C}_{\rho>0} \prec \mathbf{C}_{\rho=1} = \mathbf{C}^+ \tag{8}$$

with $\mathbf{C}^\perp$ the *product* copula[6]. We have represented this copula and the Fréchet copulas in the figure 4. Level curves $\left\{(u_1,u_2) \in \mathbf{I}^2 | \mathbf{C}(u_1,u_2) = C\right\}$ can be used to understand the concordance ordering concept. Considering formula (7), the level curves lie in the area delimited by the lower and upper Fréchet bounds. In figure 5, we consider the Frank copula[7]. We clearly see that the Frank copula is *positively* ordered by the parameter α. Moreover, we remark that the lower Fréchet, product and upper Fréchet copulas are special cases of the Frank copula when α tends respectively to $-\infty$, 0 and $+\infty$. This property is interesting because a parametric family could cover the entire range of dependence in this case.

Remark 4 *The density c associated to the copula is given by*

$$c(u_1,\ldots,u_n,\ldots,u_N) = \frac{\partial\,\mathbf{C}(u_1,\ldots,u_n,\ldots,u_N)}{\partial\,u_1\cdots\partial\,u_n\cdots\partial\,u_N} \tag{11}$$

To obtain the density f of the N-dimensional distribution $\mathbf{F}$, we use the following relationship:

$$f(x_1,\ldots,x_n,\ldots,x_N) = c(\mathbf{F}_1(x_1),\ldots,\mathbf{F}_n(x_n),\ldots,\mathbf{F}_N(x_N)) \prod_{n=1}^{N} f_n(x_n) \tag{12}$$

where f_n is the density of the margin $\mathbf{F}_n$.

2.2 Dependence

2.2.1 Measure of concordance

Definition 5 (Nelsen (1998), page 136) *A numeric measure κ of association between two continuous random variables X_1 and X_2 whose copula is $\mathbf{C}$ is a measure of concordance if it satifies the following properties:*

1. *κ is defined for every pair X_1, X_2 of continuous random variables;*
2. *$-1 = \kappa_{X,-X} \le \kappa_{\mathbf{C}} \le \kappa_{X,X} = 1$;*

[5] $\mathbf{C}^-$ is not a copula if $N > 2$, but we use this notation for convenience.

[6] The product copula is defined as follows:

$$\mathbf{C}^\perp = \prod_{n=1}^{N} u_n \tag{9}$$

[7] The copula is defined by

$$\mathbf{C}(u_1,u_2) = \frac{1}{\alpha}\ln\left(1 + \frac{(\exp(\alpha u_1) - 1)(\exp(\alpha u_2) - 1)}{(\exp(\alpha) - 1)}\right) \tag{10}$$

with $\alpha \in \mathbb{R}^\star$ (Frees and Valdez [1997]).

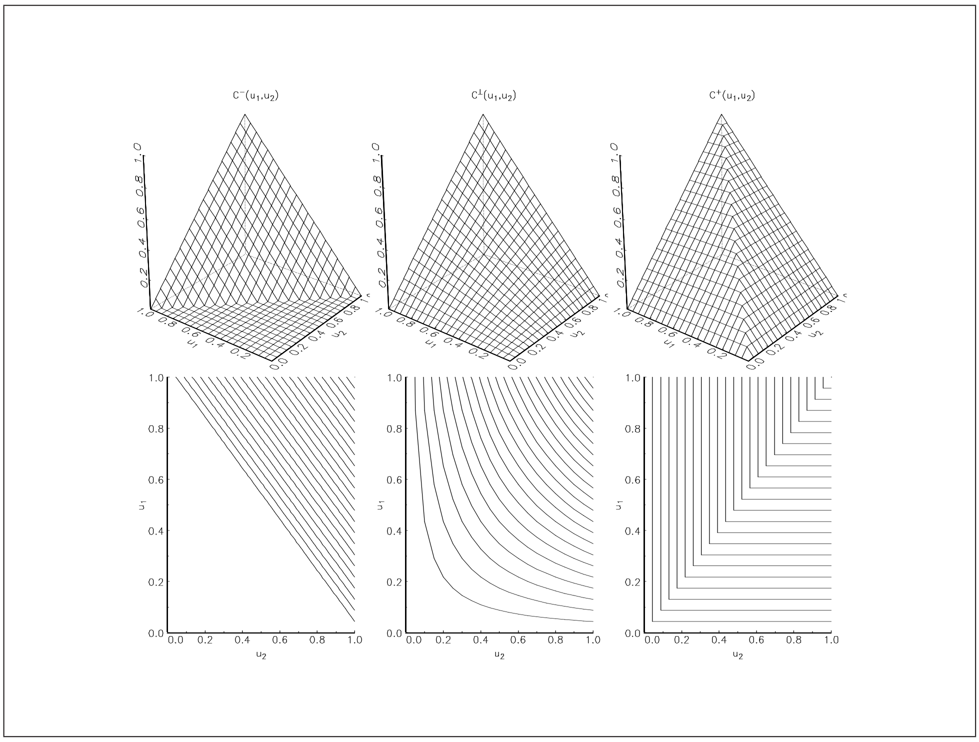

Figure 4: Lower Fréchet, product and upper Fréchet copulas

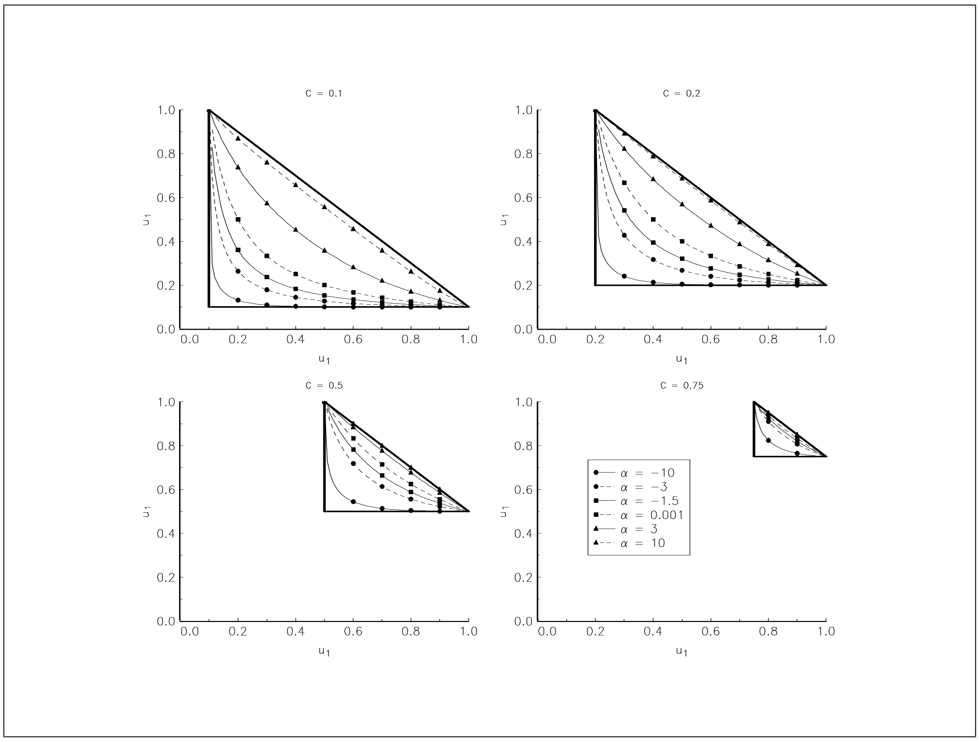

Figure 5: Level curves of the Frank Copula

3. $\kappa_{X_1,X_2} = \kappa_{X_2,X_1}$;

4. *if* X_1 *and* X_2 *are independent, then* $\kappa_{X_1,X_2} = \kappa_{\mathbf{C}^\perp} = 0$;

5. $\kappa_{-X_1,X_2} = \kappa_{X_1,-X_2} = -\kappa_{X_1,X_2}$;

6. *if* $\mathbf{C}_1 \prec \mathbf{C}_2$, *then* $\kappa_{\mathbf{C}_1} \leq \kappa_{\mathbf{C}_2}$;

7. *if* $\{(X_{1,n}, X_{2,n})\}$ *is a sequence of continuous random variables with copulas* $\mathbf{C}_n$, *and if* $\{\mathbf{C}_n\}$ *converges pointwise to* $\mathbf{C}$, *then* $\lim_{n\to\infty} \kappa_{\mathbf{C}_n} = \kappa_{\mathbf{C}}$.

Remark 6 *Another important property of* κ *comes from the fact the copula function of random variables* $(X_1, \ldots, X_n, \ldots, X_N)$ *is invariant under strictly increasing transformations:*

$$\mathbf{C}_{X_1,\ldots,X_n,\ldots,X_N} = \mathbf{C}_{h_1(X_1),\ldots,h_n(X_n),\ldots,h_N(X_N)} \quad \text{if} \quad \partial_x h_n(x) > 0 \tag{13}$$

Among all the measures of concordance, three famous measures play an important role in non-parametric statistics: the Kendall's tau, the Spearman's rho and the Gini indice. They could all be written with copulas, and we have (Schweitzer and Wolff [1981])

$$\begin{aligned}
\tau &= 4 \iint_{\mathbf{I}^2} \mathbf{C}(u_1, u_2)\, \mathrm{d}\mathbf{C}(u_1, u_2) - 1 && (14)\\
\varrho &= 12 \iint_{\mathbf{I}^2} u_1 u_2\, \mathrm{d}\mathbf{C}(u_1, u_2) - 3 && (15)\\
\gamma &= 2 \iint_{\mathbf{I}^2} (|u_1 + u_2 - 1| - |u_1 - u_2|)\, \mathrm{d}\mathbf{C}(u_1, u_2) && (16)
\end{aligned}$$

Nelsen [1998] presents some relationships between the measures τ and ϱ, that can be summarised by a bounding region (see figure 6). In Figure 7 and 8, we have plotted the links between τ, ϱ and γ for different copulas[8]. We note that the relationships are similar. However, some copulas do not cover the entire range $[-1, 1]$ of the possible values for concordance measures. For example, Kimeldorf-Sampson, Gumbel, Galambos and Hüsler-Reiss copulas do not allow negative dependence.

2.2.2 Measure of dependence

Definition 7 (Nelsen (1998), page 170) *A numeric measure* δ *of association between two continuous random variables* X_1 *and* X_2 *whose copula is* $\mathbf{C}$ *is a measure of dependence if it satifies the following properties:*

1. δ *is defined for every pair* X_1, X_2 *of continuous random variables;*

2. $0 = \delta_{\mathbf{C}^\perp} \leq \delta_{\mathbf{C}} \leq \delta_{\mathbf{C}^+} = 1$;

3. $\delta_{X_1,X_2} = \delta_{X_2,X_1}$;

4. $\delta_{X_1,X_2} = \delta_{\mathbf{C}^\perp} = 0$ ***if and only if*** X_1 *and* X_2 *are independent;*

5. $\delta_{X_1,X_2} = \delta_{\mathbf{C}^+} = 1$ ***if and only if*** *each of* X_1 *and* X_2 *is almost surely a strictly monotone function of the other;*

[8] If analytical expressions are not available, they are computed with the following equivalent formulas

$$\begin{aligned}
\tau &= 1 - 4 \iint_{\mathbf{I}^2} \partial_{u_1}\mathbf{C}(u_1, u_2)\, \partial_{u_2}\mathbf{C}(u_1, u_2)\, \mathrm{d}u_1\, \mathrm{d}u_2 && (17)\\
\varrho &= 12 \iint_{\mathbf{I}^2} \mathbf{C}(u_1, u_2)\, \mathrm{d}u_1\, \mathrm{d}u_2 - 3 && (18)\\
\gamma &= 4 \int_{\mathbf{I}} (\mathbf{C}(u, u) + \mathbf{C}(u, 1-u) - u)\, \mathrm{d}u && (19)
\end{aligned}$$

and a Gauss-Lengendre quadrature with 128 knots (Abramowitz and Stegun [1970]).

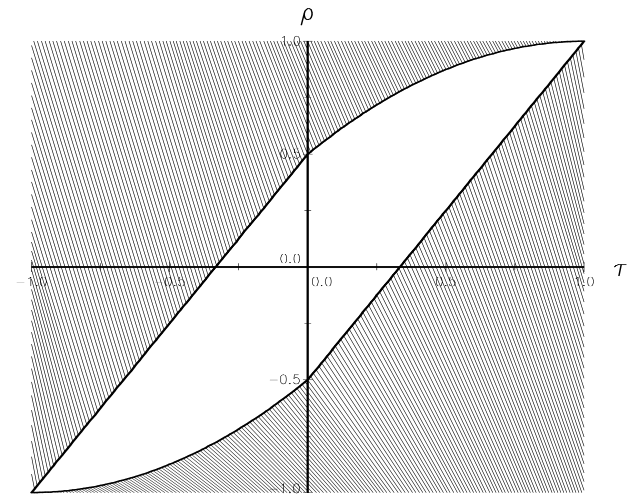

Figure 6: Bounding region for τ and ϱ

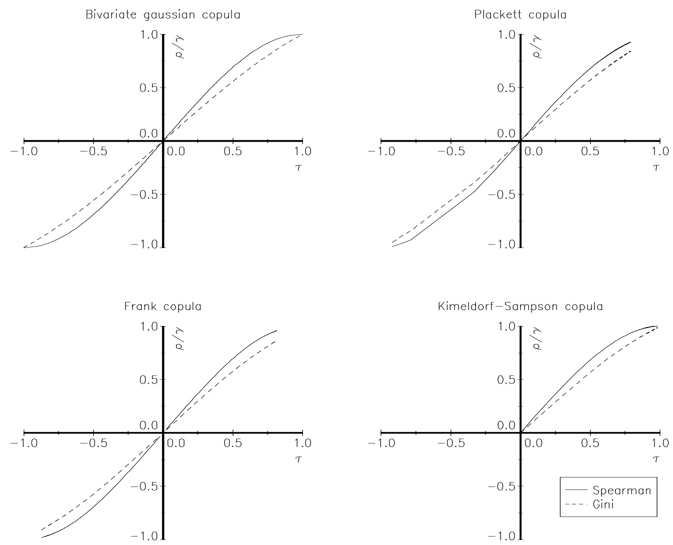

Figure 7: Relationships between τ, ϱ and γ for some copula functions (I)

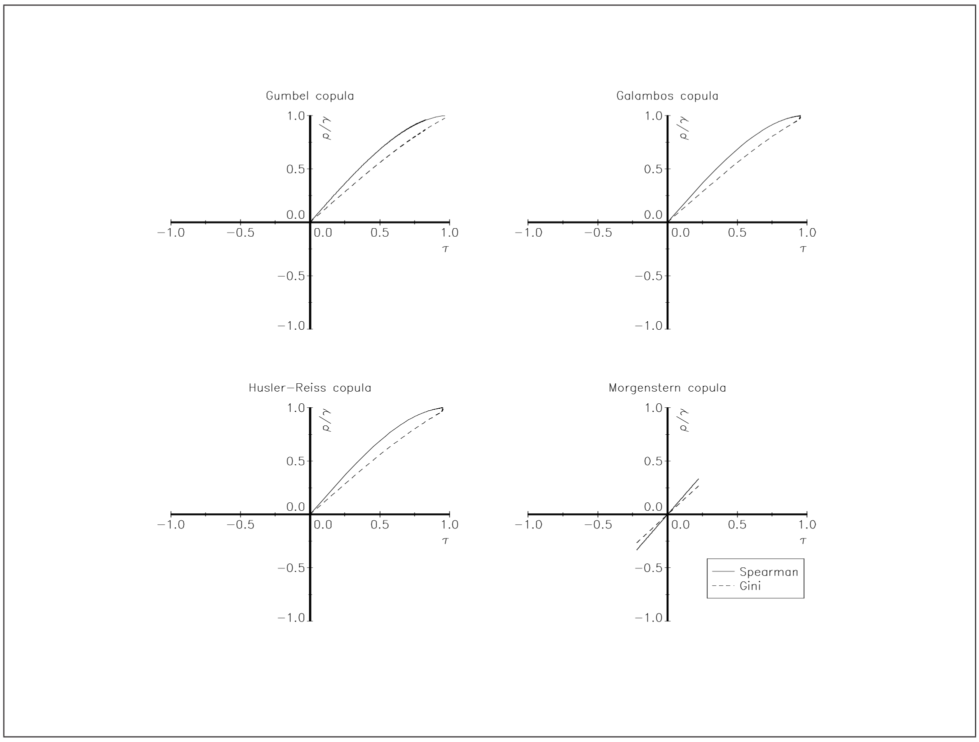

Figure 8: Relationships between τ, ϱ and γ for some copula functions (II)

6. *if h_1 and h_2 are almost surely strictly monotone functions on* $\operatorname{Im} X_1$ *and* $\operatorname{Im} X_2$ *respectively, then*

$$\delta_{h_1(X_1),h_2(X_2)} = \delta_{X_1,X_2}$$

7. *if $\{(X_{1,n}, X_{2,n})\}$ is a sequence of continuous random variables with copulas $\mathbf{C}_n$, and if $\{\mathbf{C}_n\}$ converges pointwise to $\mathbf{C}$, then $\lim_{n\to\infty} \delta_{\mathbf{C}_n} = \delta_{\mathbf{C}}$.*

Schweitzer and Wolff [1981] provide different measures which satisfy these properties:

$$\sigma = 12 \iint_{\mathbf{I}^2} \left| \mathbf{C}(u_1, u_2) - \mathbf{C}^{\perp}(u_1, u_2) \right| \mathrm{d}u_1 \, \mathrm{d}u_2 \tag{20}$$

$$\Phi^2 = 90 \iint_{\mathbf{I}^2} \left| \mathbf{C}(u_1, u_2) - \mathbf{C}^{\perp}(u_1, u_2) \right|^2 \mathrm{d}u_1 \, \mathrm{d}u_2 \tag{21}$$

where σ is known as the Schweitzer or Wolff's σ measure of dependence, while Φ^2 is the dependence index introduced by Hoeffding.

2.2.3 Other dependence concepts

There are many other dependence concepts, that are useful for financial applications. For example, X_1 and X_2 are said to be positive quadrant dependent (PQD) if

$$\Pr\{X_1 > x_1, X_2 > x_2\} \geq \Pr\{X_1 > x_1\} \Pr\{X_2 > x_2\} \tag{22}$$

Suppose that X_1 and X_2 are random variables standing for two financial losses.The probability of simultaneous large losses is greater for dependent variables than for independent ones. In term of copulas, relation (22) is equivalent to

$$\mathbf{C} \succ \mathbf{C}^{\perp} \tag{23}$$

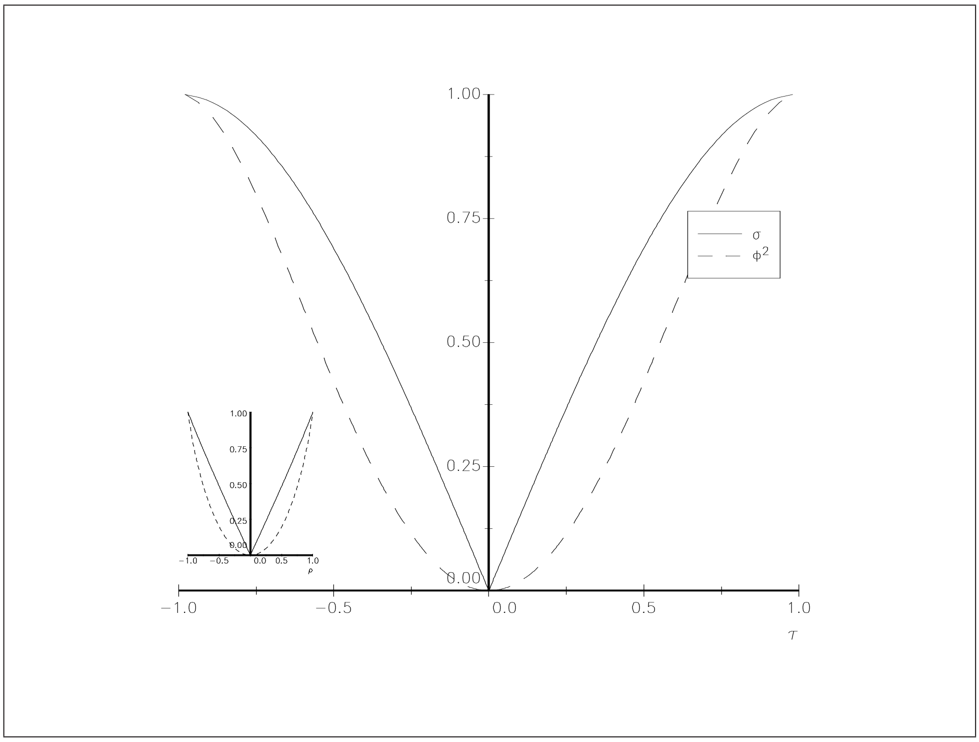

Figure 9: Dependence measures for the Gaussian copula

The notion of tail dependence is more interesting. JOE [1997] gives the following definition:

Definition 8 *If a bivariate copula* $\mathbf{C}$ *is such that*[9]

$$\lim_{u \to 1} \frac{\bar{\mathbf{C}}(u,u)}{1-u} = \lambda \tag{27}$$

exists, then $\mathbf{C}$ *has upper tail dependence for* $\lambda \in (0,1]$ *and no upper tail dependence for* $\lambda = 0$.

The measure λ is extensively used in extreme value theory. It is the probability that one variable is extreme given that the other is extreme. Let $\lambda(u) = \Pr\{U_1 > u | U_2 > u\} = \frac{\bar{\mathbf{C}}(u,u)}{1-u}$. $\lambda(u)$ can be viewed as a "quantile-dependent measure of dependence" (COLES, CURRIE and TAWN [1999]). Figure 10 represents the values of $\lambda(u)$ for the gaussian copula. We see that extremes are asymptotically independent for $\rho \neq 1$, i.e $\lambda = 0$ for $\rho < 1$. EMBRECHTS, MCNEIL and STRAUMANN [1999] remarked that the Student's copula provides an interesting contrast with the gaussian copula. We have reported the values of $\lambda(u)$ for the Student's copula with one and

[9] $\bar{\mathbf{C}}$ is the joint survival function, that is

$$\bar{\mathbf{C}}(u_1,u_2) = 1 - u_1 - u_2 + \mathbf{C}(u_1,u_2) \tag{24}$$

Note that it is related to the survival copula $\breve{\mathbf{C}}$

$$\breve{\mathbf{C}}(u_1,u_2) = u_1 + u_2 - 1 + \mathbf{C}(1-u_1,1-u_2) \tag{25}$$

in the following way

$$\bar{\mathbf{C}}(u_1,u_2) = \breve{\mathbf{C}}(1-u_1,1-u_2) \tag{26}$$

five degrees of freedom[10] respectively. In this case, extremes are asymptotically dependent for $\rho \neq 1$. Of course, the strength of this dependence decreases as the degrees of freedom increase, and the limit behaviour as ν tends to infinity corresponds to the case of the gaussian copula.

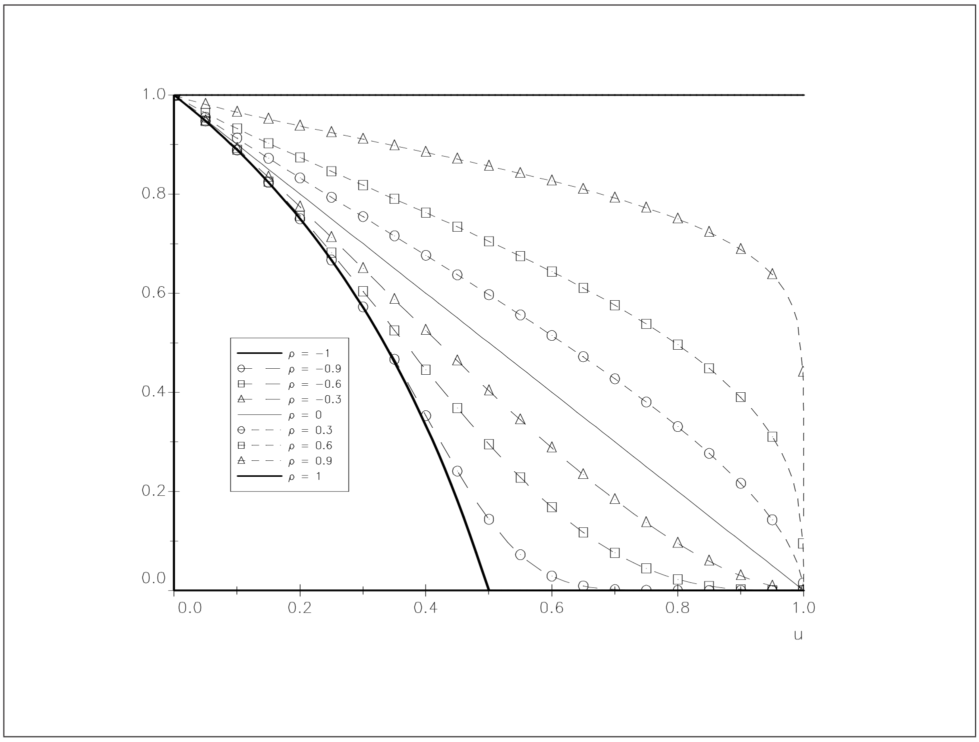

Figure 10: Quantile-dependent measure $\lambda(u)$ for the gaussian copula

2.3 A summary of different copula functions

2.3.1 Copulas related to elliptical distributions

Elliptical distributions have density of the form $f\left(\mathbf{x}^{\top}\mathbf{x}\right)$, and so density contours are ellipsoids. They play an important role in finance. In figure 13, we have plotted the contours of bivariate density for the gaussian copula and differents marginal distributions. We verify that the gaussian copula with two gaussian marginals correspond to the bivariate gaussian distribution, and that the contours are ellipsoids. Building multivariate distributions with copulas becomes very easy. For example, figure 13 contains two other bivariate densities with different margins. In figure 14, margins are the same, but we use a copula of the Frank family. For each figure, we have choosen the copula parameter in order to have the same Kendall's tau ($\tau = 0.5$). The dependence structure of the four bivariate distributions can be compared.

Definition 9 (multivariate gaussian copula — MVN) *Let ρ be a symmetric, positive definite matrix with* $\operatorname{diag}\rho = \mathbf{1}$ *and $\mathbf{\Phi}_\rho$ the standardized multivariate normal distribution with correlation matrix ρ. The multivariate gaussian copula is then defined as follows:*

$$\mathbf{C}\left(u_1,\ldots,u_n,\ldots,u_N;\rho\right) = \mathbf{\Phi}_\rho\left(\mathbf{\Phi}^{-1}\left(u_1\right),\ldots,\mathbf{\Phi}^{-1}\left(u_n\right),\ldots,\mathbf{\Phi}^{-1}\left(u_N\right)\right) \tag{28}$$

[10] The short solid line corresponds to the values of λ.

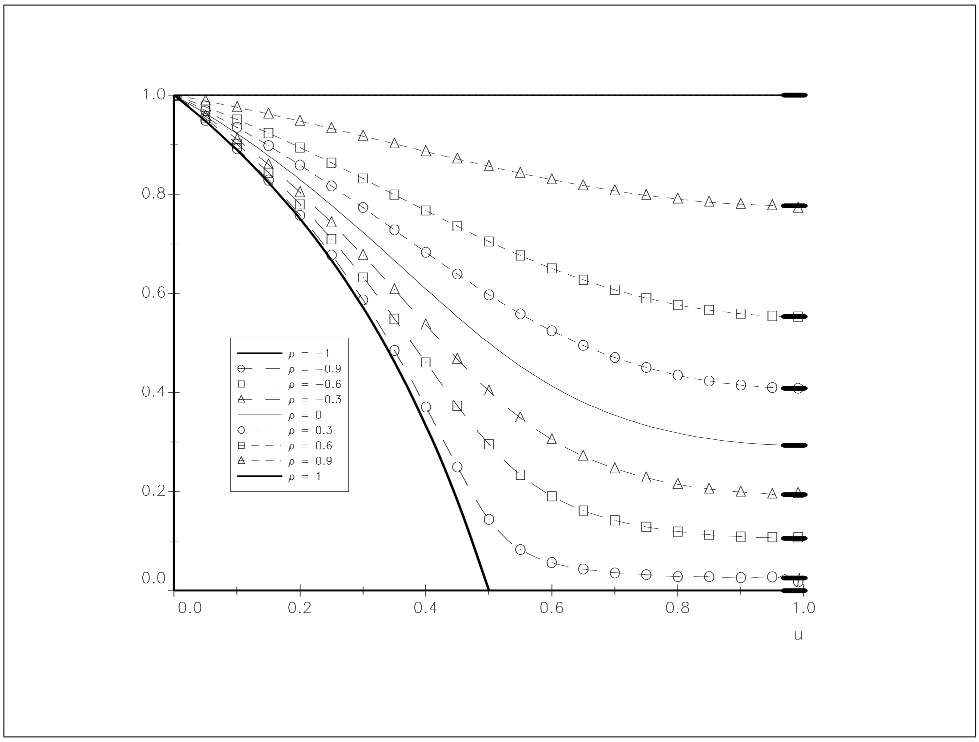

Figure 11: Quantile-dependent measure $\lambda(u)$ for the Student's copula ($\nu = 1$

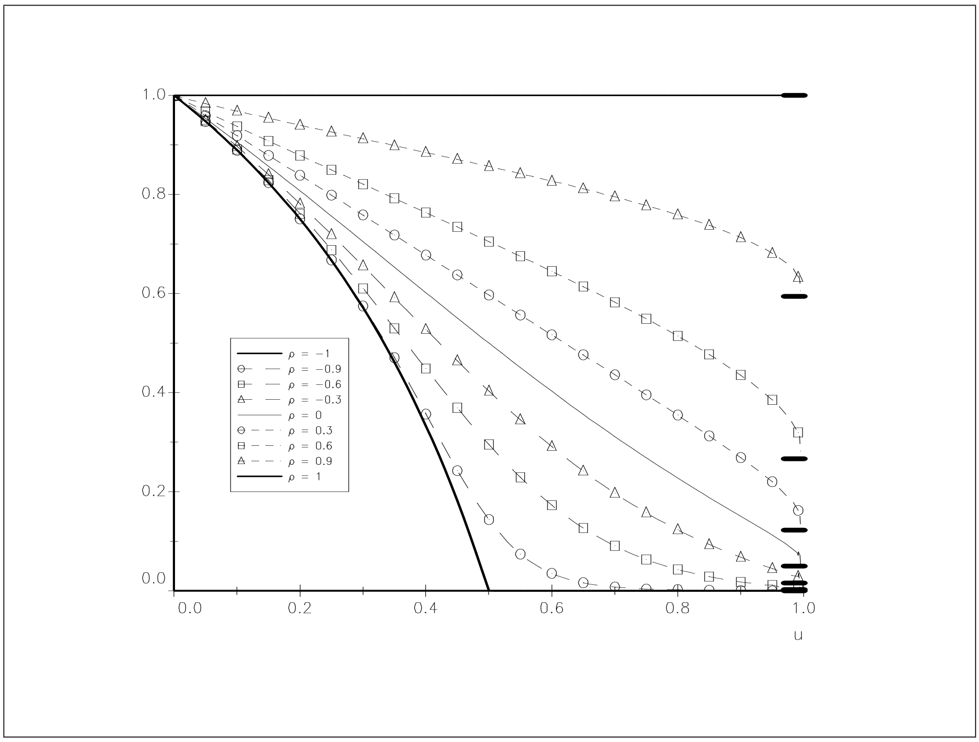

Figure 12: Quantile-dependent measure $\lambda(u)$ for the Student's copula ($\nu = 5$

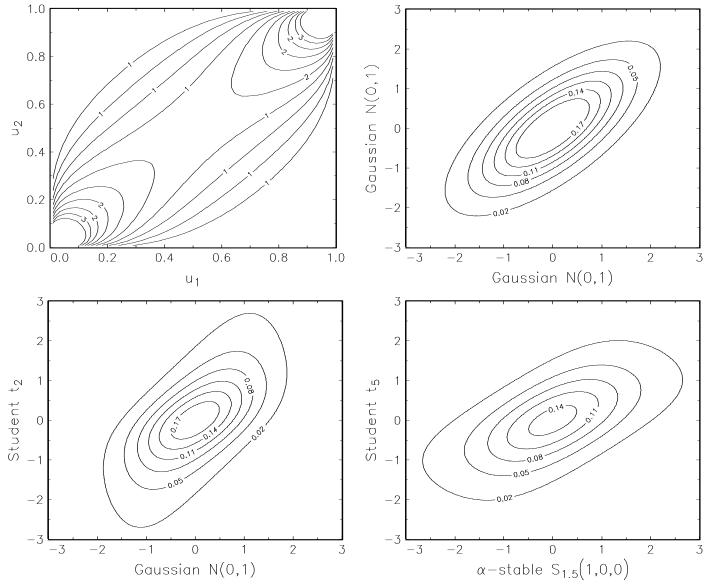

Figure 13: Contours of density for Gaussian copula

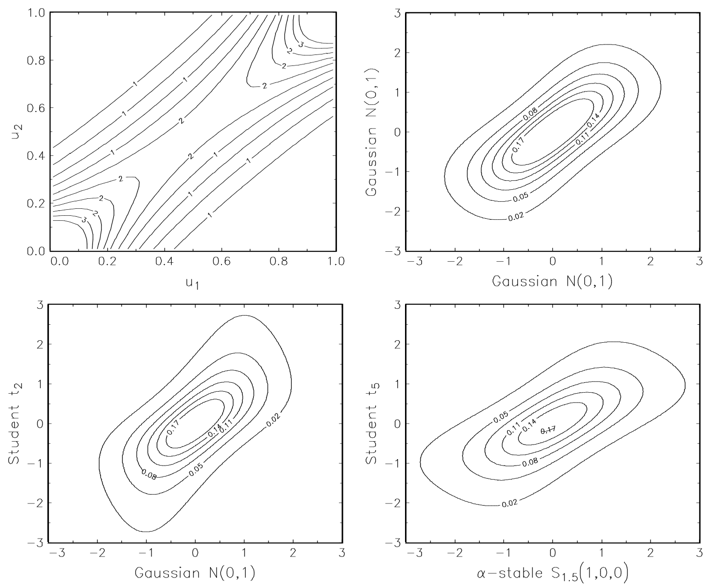

Figure 14: Contours of density for Frank copula

The corresponding density is[11]

$$\mathbf{c}\left(u_1,\ldots,u_n,\ldots,u_N;\rho\right)=\frac{1}{\left|\rho\right|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}\varsigma^{\top}\left(\rho^{-1}-\mathbb{I}\right)\varsigma\right) \tag{31}$$

with $\varsigma_n=\mathbf{\Phi}^{-1}\left(u_n\right)$.

Even if the MVN copula has not been extensively used in papers related to this topic, it permits tractable calculus like the MVN distribution. Let us consider the computation of conditional density. Let $\mathbf{U}=\left[\mathbf{U}_1^{\top},\mathbf{U}_2^{\top}\right]^{\top}$ denote a vector of uniform variates. In partitioned form, we have

$$\rho=\left[\begin{array}{cc}\rho_{11} & \rho_{12}\\ & \rho_{22}\end{array}\right] \tag{32}$$

We could also show that the conditional density of $\mathbf{U}_2$ given values of $\mathbf{U}_1$ is

$$\tag{33}$$

$$\mathbf{c}\left(\mathbf{U}_2|\mathbf{U}_1;\rho\right) \quad = \quad \frac{1}{\left|\rho_{22}-\rho_{12}^{\top}\rho_{11}^{-1}\rho_{12}\right|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}\varsigma\left(\left[\rho_{22}-\rho_{12}^{\top}\rho_{11}^{-1}\rho_{12}\right]^{-1}-\mathbb{I}\right)\varsigma\right) \tag{34}$$

with $\varsigma=\mathbf{\Phi}^{-1}\left(\mathbf{U}_2\right)-\rho_{12}^{\top}\rho_{11}^{-1}\mathbf{\Phi}^{-1}\left(\mathbf{U}_1\right)$. Using this formula and if the margins are specified, we could perform *quantile regressions* or calculate other interesting values like expected values.

Definition 10 (multivariate Student's copula — MVT) *Let ρ be a symmetric, positive definite matrix with* $\operatorname{diag}\rho=\mathbf{1}$ *and $T_{\rho,\nu}$ the standardized multivariate Student's distribution*[12] *with ν degrees of freedom and correlation matrix ρ. The multivariate Student's copula is then defined as follows:*

$$\mathbf{C}\left(u_1,\ldots,u_n,\ldots,u_N;\rho,\nu\right)=T_{\rho,\nu}\left(t_{\nu}^{-1}\left(u_1\right),\ldots,t_{\nu}^{-1}\left(u_n\right),\ldots,t_{\nu}^{-1}\left(u_N\right)\right) \tag{36}$$

with t_{ν}^{-1} the inverse of the univariate Student's distribution. The corresponding density is[13]

$$\mathbf{c}\left(u_1,\ldots,u_n,\ldots,u_N;\rho\right)=\left|\rho\right|^{-\frac{1}{2}}\frac{\Gamma\left(\frac{\nu+N}{2}\right)\left[\Gamma\left(\frac{\nu}{2}\right)\right]^N}{\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^N\Gamma\left(\frac{\nu}{2}\right)}\frac{\left(1+\frac{1}{\nu}\varsigma^{\top}\rho^{-1}\varsigma\right)^{-\frac{\nu+N}{2}}}{\prod\limits_{n=1}^{N}\left(1+\frac{\varsigma_n^2}{\nu}\right)^{-\frac{\nu+1}{2}}} \tag{37}$$

with $\varsigma_n=t_{\nu}^{-1}\left(u_n\right)$.

Remark 11 *Probability density function of MVN and MVT copulas are easy to compute. For cumulative density functions, the problem becomes harder. In this paper, we use the GAUSS procedure* `cdfmvn` *based on the algorithm developed by* Ford *and the FORTRAN subroutine* `mvtdst` *written by* Genz *and* Bretz *[1999a,1999b].*

---

[11] We have for the Multinormal distribution

$$\frac{1}{\left(2\pi\right)^{\frac{N}{2}}\left|\rho\right|^{\frac{1}{2}}}\exp\quad -\frac{1}{2}\mathbf{x}^{\top}\rho^{-1}\mathbf{x}\quad=\mathbf{c}\left(\mathbf{\Phi}\left(x_1\right),\ldots,\mathbf{\Phi}_n\left(x_n\right),\ldots,\mathbf{\Phi}_N\left(x_N\right)\right)\quad\prod_{n=1}^{N}\frac{1}{\sqrt{2\pi}}\exp\quad-\frac{1}{2}x_n^2\quad) \tag{29}$$

We deduce also

$$\mathbf{c}\left(u_1,\ldots,u_n,\ldots,u_N\right)=\frac{\frac{1}{\left(2\pi\right)^{\frac{N}{2}}\left|\rho\right|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\varsigma^{\top}\rho^{-1}\varsigma\right.}{\frac{1}{\left(2\pi\right)^{\frac{N}{2}}}\exp\left[-\frac{1}{2}\varsigma^{\top}\varsigma\right.} \tag{30}$$

[12] We have

$$T_{\rho,\nu}\left(x_1,\ldots,x_N\right)=\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_N}\frac{\Gamma\quad\frac{\nu+N}{2}\quad\left|\rho\right|^{-\frac{1}{2}}}{\Gamma\left[\frac{\nu}{2}\quad\left(\nu\pi\right)^{\frac{N}{2}}\right.}\quad1+\frac{1}{\nu}\mathbf{x}^{\top}\rho^{-1}\mathbf{x}\quad^{-\frac{\nu+N}{2}}\mathrm{d}\mathbf{x} \tag{35}$$

[13] We obtain this result by using the technique described in footnote 11.

There are few works that focus on *elliptical* copulas. However, they could be very attractive. SONG [2000] gives a multivariate extension of dispersion models with the Gaussian copula. JORGENSEN [1997] defines a dispersion model $Y \sim \mathcal{DM}\left(\mu, \sigma^2\right)$ with the following probability density form

$$f\left(y; \mu, \sigma^2\right) = a\left(y; \sigma^2\right) \exp\left(-\frac{1}{2\sigma^2} d\left(y; \mu\right)\right) \tag{38}$$

μ and σ^2 are called the *position* and *dispersion* parameters. d is the *unit deviance* with $d\left(y;\mu\right) \geq 0$ satisfying $d\left(y;y\right) = 0$. If $a\left(y;\sigma^2\right)$ takes the form $a_1\left(y\right) a_2\left(\sigma^2\right)$, the model is a *proper* dispersion model $\mathcal{PDM}$ (e.g. Simplex, Leipnik or von Mises distribution). With $d\left(y;\mu\right) = y d_1\left(\mu\right) + d_2\left(y\right) + d_3\left(\mu\right)$, we obtain an exponential dispersion model $\mathcal{EDM}$ (e.g. Gaussian, exponential, Gamma, Poisson, negative binomial, binomial or inverse Gaussian distributions). JORGENSEN and LAURITZEN [1998] propose a multivariate extension of the dispersion model defined by the following probability density form

$$f\left(\mathbf{y}; \mu, \Sigma\right) = a\left(\mathbf{y}; \Sigma\right) \exp\left(-\frac{1}{2} \operatorname{tr}\left\{\Sigma^{-1} t\left(\mathbf{y}; \mu\right) t\left(\mathbf{y}; \mu\right)^\top\right\}\right) \tag{39}$$

As noted by Song, this construction is not natural, because *their models are not marginally closed in the sense that the marginal distributions may not be in the given distribution class.* SONG [2000] proposes to define the multivariate dispersion model $\mathbf{Y} \sim \mathcal{MDM}\left(\mu, \sigma^2, \rho\right)$ as

$$f\left(\mathbf{y}; \mu, \sigma^2, \rho\right) = \frac{1}{\left|\rho\right|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \varsigma^\top \left(\rho^{-1} - \mathbb{I}\right) \varsigma\right) \prod_{n=1}^{N} f_n\left(y_n; \mu_n, \sigma_n^2\right) \tag{40}$$

with $\varsigma = \left(\varsigma_1, \ldots, \varsigma_N\right)^\top$, $\mu = \left(\mu_1, \ldots, \mu_N\right)^\top$, $\sigma^2 = \left(\sigma_1^2, \ldots, \sigma_N^2\right)^\top$ and $\varsigma_n = \mathbf{\Phi}^{-1}\left(\mathbf{F}_n\left(y_n; \mu_n, \sigma_n^2\right)\right)$. In this case, the univariate margins are effectively the dispersion model $\mathcal{DM}\left(\mu_n, \sigma_n^2\right)$. Moreover, these $\mathcal{MDM}$ distributions have many properties similar to the multivariate normal distribution.

We consider the example of the Weibull distribution. Let (b, c) two positive scalars. We have

$$f\left(y\right) = \frac{c y^{c-1}}{b^c} \exp\left(-\left(\frac{y}{b}\right)^c\right) \tag{41}$$

In the dispersion model framework, we have $d\left(y;\mu\right) = \frac{1}{2} y^c$, $a\left(y;\sigma^2\right) = \frac{c y^{c-1}}{\sigma^2}$, $\sigma^2 = b^c$ and $\mu = 0$. It is also both a proper and exponential distribution. A multivariate generalization is then given by

$$f\left(\mathbf{y}\right) = \frac{1}{\left|\rho\right|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \varsigma^\top \left(\rho^{-1} - \mathbb{I}\right) \varsigma\right) \prod_{n=1}^{N} \frac{c_n y^{c_n - 1}}{b_n^{c_n}} \exp\left(-\left(\frac{y_n}{b_n}\right)^{c_n}\right) \tag{42}$$

with $\varsigma_n = \mathbf{\Phi}^{-1}\left(\mathbf{1} - \exp\left(-\left(\frac{y_n}{b_n}\right)^{c_n}\right)\right)$.

2.3.2 Archimedean copulas

GENEST and MACKAY [1996] define Archimedean copulas as the following:

$$\mathbf{C}\left(u_1, \ldots, u_n, \ldots, u_N\right) = \begin{cases} \varphi^{-1}\left(\varphi\left(u_1\right) + \ldots + \varphi\left(u_n\right) + \ldots + \varphi\left(u_N\right)\right) & \text{if } \sum_{n=1}^{N} \varphi\left(u_n\right) \leq \varphi\left(0\right) \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

with $\varphi\left(u\right)$ a C^2 function with $\varphi\left(1\right) = 0$, $\varphi'\left(u\right) < 0$ and $\varphi''\left(u\right) > 0$ for all $0 \leq u \leq 1$. $\varphi\left(u\right)$ is called the *generator* of the copula. Archimedean copulas play an important role because they present several desired properties (**C** is symmetric, associative[14], etc.). Moreover, Archimedean copulas simplify calculus. For example, the Kendall's tau is given by

$$\tau = 1 + 4 \int_0^1 \frac{\varphi(u)}{\varphi'(u)} \,\mathrm{d}u \tag{44}$$

We have reported in the following table some classical bivariate Archimedean copulas[15]:

Copula	$\varphi(u)$	$\mathbf{C}(u_1, u_2)$
$\mathbf{C}^{\perp}$	$-\ln u$	$u_1 u_2$
Gumbel	$(-\ln u)^{\alpha}$	$\exp\left(-(\tilde{u}_1^{\alpha} + \tilde{u}_2^{\alpha})^{\frac{1}{\alpha}}\right)$
Joe	$(-\ln 1 - (1-u)^{\alpha})$	$1 - (\tilde{u}_1^{\alpha} + \tilde{u}_2^{\alpha} - \tilde{u}_1^{\alpha}\tilde{u}_2^{\alpha})^{\frac{1}{\alpha}}$
Kimeldorf-Sampson	$u^{-\alpha} - 1$	$\left(\tilde{u}_1^{-\alpha} + \tilde{u}_2^{-\alpha} - 1\right)^{-\frac{1}{\alpha}}$

Let $\varpi(u) = \exp(-\varphi(u))$. We note that the equation (43) could be written as

$$\varpi(\mathbf{C}(u_1, \ldots, u_n, \ldots, u_N)) = \prod_{n=1}^{N} \varpi(u_n) \tag{45}$$

By applying φ both to the joint distribution and the margins, the distributions *"become"* independent. We note also that Archimedean copulas are related to multivariate distributions generated by mixtures. Let γ be a vector of parameters generated by a joint distribution function $\mathbf{\Gamma}$ with margins $\mathbf{\Gamma}_n$ and $\mathbf{H}$ a multivariate distribution. We denote $\mathbf{F}_1, \ldots, \mathbf{F}_N$ N univariate distributions. Marshall and Olkin [1988] showed that

$$\mathbf{F}(x_1, \ldots, x_n, \ldots, x_N) = \int \cdots \int \mathbf{H}\left(H_1^{\gamma_1}(x_1), \ldots, H_n^{\gamma_n}(x_n), \ldots, H_N^{\gamma_N}(x_N)\right) \mathrm{d}\mathbf{\Gamma}(\gamma_1, \ldots, \gamma_n, \ldots, \gamma_N) \tag{46}$$

is a multivariate distribution with marginals $\mathbf{F}_1, \ldots, \mathbf{F}_N$. We have

$$H_n(x_n) = \exp\left(-\psi_n^{-1}(\mathbf{F}_n(x_n))\right) \tag{47}$$

with ψ_n the Laplace transform of the marginal distribution $\mathbf{\Gamma}_n$. Let ψ be the Laplace transform of the joint distribution $\mathbf{\Gamma}$. Another expression of (46) is

$$\mathbf{F}(x_1, \ldots, x_n, \ldots, x_N) = \psi\left(\psi_1^{-1}(\mathbf{F}_1(x_1)), \ldots, \psi_n^{-1}(\mathbf{F}_n(x_n)), \ldots, \psi_n^{-1}(\mathbf{F}_n(x_n))\right) \tag{48}$$

If the margins $\mathbf{\Gamma}_n$ are the same, $\mathbf{\Gamma}$ is the upper Fréchet bound and $\mathbf{H}$ the *product* copula, Marshall and Olkin [1988] remarked that

$$\mathbf{F}(x_1, \ldots, x_n, \ldots, x_N) = \psi\left(\psi^{-1}(\mathbf{F}_1(x_1)) + \ldots + \psi^{-1}(\mathbf{F}_n(x_n)) + \ldots + \psi^{-1}(\mathbf{F}_n(x_n))\right) \tag{49}$$

The inverse of the Laplace transform ψ^{-1} is then a *generator* for Archimedean copulas.

2.3.3 Extreme value copulas

Following Joe [1997], an extreme value copula $\mathbf{C}$ satisfies the following relationship[16]:

$$\mathbf{C}\left(u_1^t, \ldots, u_n^t, \ldots, u_N^t\right) = \mathbf{C}^t(u_1, \ldots, u_n, \ldots, u_N) \qquad \forall\, t > 0 \tag{50}$$

[14] $\mathbf{C}(u_1, \mathbf{C}(u_2, u_3)) = \mathbf{C}(\mathbf{C}(u_1, u_2), u_3)$.

[15] We use the notation of Joe [1997] to be more concise: $\bar{u} = 1 - u$ and $\tilde{u} = -\ln u$.

[16] This relationship will be explained in paragraph 4.3.2 page 45.

For example, the Gumbel copula is an extreme value copula:

$$\begin{aligned}
\mathbf{C}\left(u_1^t, u_2^t\right) &= \exp\left(-\left[\left(-\ln u_1^t\right)^\alpha + \left(-\ln u_2^t\right)^\alpha\right]^{\frac{1}{\alpha}}\right) \\
&= \exp\left(-\left(t^\alpha\left[(-\ln u_1)^\alpha + (-\ln u_1)^\alpha\right]\right)^{\frac{1}{\alpha}}\right) \\
&= \left[\exp\left(-\left[(-\ln u_1)^\alpha + (-\ln u_1)^\alpha\right]^{\frac{1}{\alpha}}\right)\right]^t \\
&= \mathbf{C}^t\left(u_1, u_2\right)
\end{aligned}$$

Let us consider the previous example of density contours with the Gumbel copula. We then obtain figure 15.

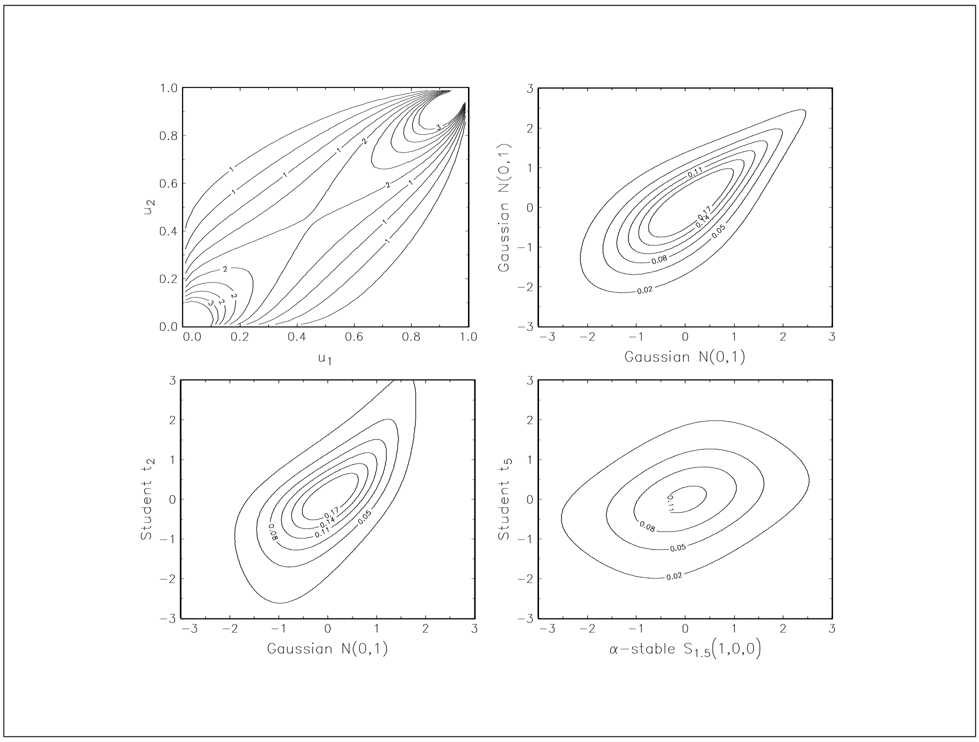

Figure 15: Contours of density for Gumbel copula

What is the link between extreme value copulas and the multivariate extreme value theory? The answer is straightforward. Let us denote $\chi_{n,m}^{+} = \max\left(X_{n,1}, \ldots, X_{n,k}, \ldots, X_{n,m}\right)$ with $\{X_{n,k}\}$ k *iid* random variables with the same distribution. Let $\mathbf{G}_n$ be the marginal distribution of the univariate extreme $\chi_{n,m}^{+}$. **Then, the joint limit distribution $\mathbf{G}$ of $\left(\chi_{1,m}^{+}, \ldots, \chi_{n,m}^{+}, \ldots, \chi_{N,m}^{+}\right)$ is such that**

$$\mathbf{G}\left(\chi_1^{+}, \ldots, \chi_n^{+}, \ldots, \chi_N^{+}\right) = \mathbf{C}\left(\mathbf{G}_1\left(\chi_1^{+}\right), \ldots, \mathbf{G}_n\left(\chi_n^{+}\right), \ldots, \mathbf{G}_N\left(\chi_N^{+}\right)\right) \tag{51}$$

where $\mathbf{C}$ is an extreme value copula and $\mathbf{G}_n$ a non-degenerate univariate extreme value distribution. The relation (51) gives us also a 'simple' way to construct multivariate extreme value distributions. In figure 16, we have plotted the contours of the density of bivariate extreme value distributions using a Gumbel copula and two GEV distributions[17].

[17]The parameters of the two margins are respectively $\mu = 0$, $\sigma = 1$, $\xi = 1$ and $.\mu = 0$, $\sigma = 1$, $\xi = 1.2$ (these parameters are defined in the formula 178 page 51).

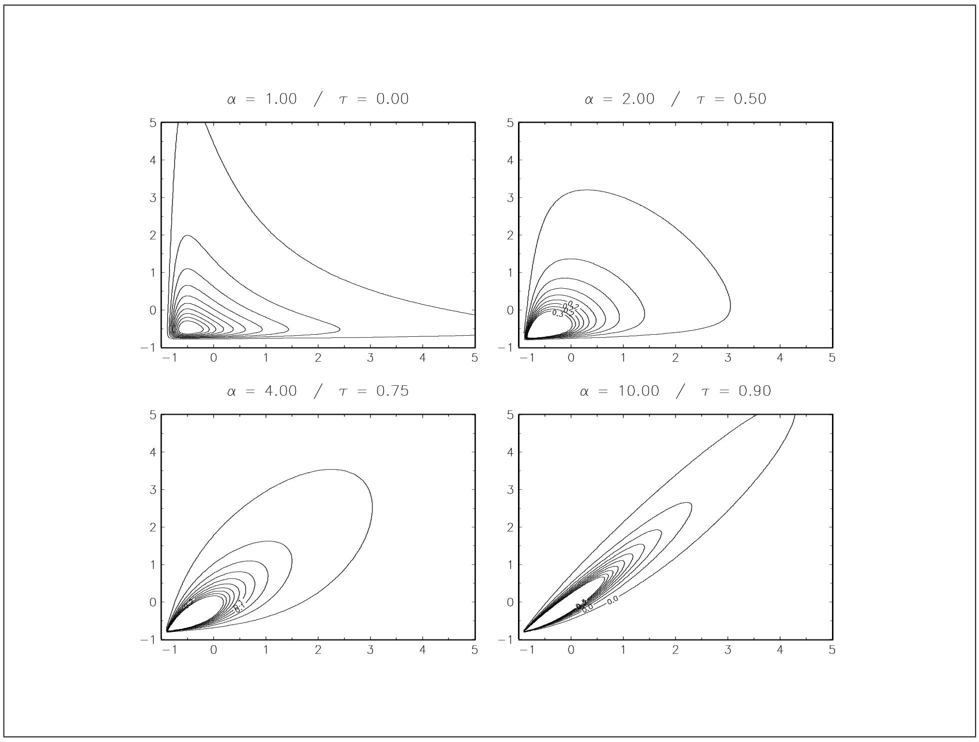

Figure 16: Example of bivariate extreme value distributions

3 Statistical inference of copulas

3.1 Simulation techniques

Simulations have an important role in statistical inference. They especially help to investigate properties of estimator. Moreover, they are necessary to understand the underlying multivariate distribution. For example, suppose that we generate a 5-dimensional distribution with a MVT copula, 2 generalized-pareto margins and three generalized-beta margins. You have to perform simulations to get an idea of the shape of the distribution.

The simulation of uniform variates for a given copula $\mathbf{C}$ can be accomplished with this following general algorithm:

1. Generate N independent uniform variates $(v_1, \ldots, v_n, \ldots, v_N)$;
2. Generate ***recursively*** the N variates as follows

$$u_n = \mathbf{C}^{-1}_{(u_1,\ldots,u_{n-1})}(v_n) \tag{52}$$

with

$$\begin{aligned}\mathbf{C}_{(u_1,\ldots,u_{n-1})}(u_n) &= \Pr\{U_n \le u_n \mid (U_1,\ldots,U_{n-1}) = (u_1,\ldots,u_{n-1})\} \\ &= \frac{\partial^{n-1}_{(u_1,\ldots,u_{n-1})}\mathbf{C}(u_1,\ldots,u_n,1,\ldots,1)}{\partial^{n-1}_{(u_1,\ldots,u_{n-1})}\mathbf{C}(u_1,\ldots,u_{n-1},1,\ldots,1)}\end{aligned} \tag{53}$$

The main idea of this algorithm is to simulate each u_n by using its conditional distribution. In the case of dimension 2, relation (52) becomes

$$\begin{aligned} u_1 &= v_1 \\ \partial_{u_1}\mathbf{C}(u_1,u_2) &= v_2 \end{aligned}$$

because $\partial_{u_1}\mathbf{C}(u_1,1)=1$. However, we have to note that random generation requires a root finding procedure. For classical copulas, there exist specific powerfull algorithms (DEVROYE [1986]). This is the case of Kimeldorf-Sampson, Gumbel, Marshall-Olkin, etc. and more generally the case of Archimedean copulas[18]. For MVN or MVT copulas, random uniforms are obtained by generating random numbers of the corresponding MVN or MVT distribution, and by applying the cdf of the corresponding univariate distribution.

Remark 12 *To obtain random numbers if the margins are not uniforms, we use the classical inversion method.*

3.2 Non-parametric estimation

3.2.1 Empirical copulas

Empirical copulas have been introduced by DEHEUVELS [1979]. Let $\mathcal{X}=\{(x_1^t,\ldots,x_N^t)\}_{t=1}^T$ denote a sample. The empirical copula distribution is given by

$$\hat{\mathbf{C}}\left(\frac{t_1}{T},\ldots,\frac{t_n}{T},\ldots,\frac{t_N}{T}\right)=\frac{1}{T}\sum_{t=1}^{T}\mathbf{1}_{x_1^t\leq x_1^{(t_1)},\ldots,x_n^t\leq x_n^{(t_n)},\ldots,x_N^t\leq x_N^{(t_N)}} \tag{56}$$

where $x_n^{(t)}$ are the order statistics and $1\leq t_1,\ldots,t_N\leq T$. The empirical copula frequency corresponds to $\hat{\mathbf{c}}\left(\frac{t_1}{T},\ldots,\frac{t_n}{T},\ldots,\frac{t_N}{T}\right)=\frac{1}{T}$ if $\left(x_1^{(t_1)},\ldots,x_N^{(t_N)}\right)$ belongs to $\mathcal{X}$ or 0 otherwise. The relationships between empirical copula distribution and frequency are

$$\hat{\mathbf{C}}\left(\frac{t_1}{T},\ldots,\frac{t_n}{T},\ldots,\frac{t_N}{T}\right)=\sum_{i_1=1}^{t_1}\cdots\sum_{i_N=1}^{t_N}\hat{\mathbf{c}}\left(\frac{i_1}{T},\ldots,\frac{i_n}{T},\ldots,\frac{i_N}{T}\right) \tag{57}$$

and[19]

$$\hat{\mathbf{c}}\left(\frac{t_1}{T},\ldots,\frac{t_n}{T},\ldots,\frac{t_N}{T}\right)=\sum_{i_1=1}^{2}\cdots\sum_{i_N=1}^{2}(-1)^{i_1+\cdots+i_N}\,\hat{\mathbf{C}}\left(\frac{t_1-i_1+1}{T},\ldots,\frac{t_n-i_n+1}{T},\ldots,\frac{t_N-i_N+1}{T}\right) \tag{58}$$

In figure 17, we have plotted the contours of the empirical copula for the assets AL and CU. We compare them with these computed with the gaussian distribution, i.e a gaussian copula with gaussian margins, and a gaussian copula (without assumptions on margins[20]). **We remark that the gaussian copula fits better the empirical copula than the gaussian distribution!**

Empirical copulas could be used to estimate dependence measures. For example, an estimation of the Spearman's ϱ is

$$\hat{\varrho}=\frac{12}{T^2-1}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\left(\hat{\mathbf{C}}\left(\frac{t_1}{T},\frac{t_2}{T}\right)-\frac{t_1t_2}{T^2}\right) \tag{59}$$

[18] We have also

$$\mathbf{C}_{(u_1,\ldots,u_{n-1})}(u_n)=\frac{\varphi_{(n-1)}\left(\varphi(u_1)+\ldots+\varphi(u_n)\right)}{\varphi_{(n-1)}\left(\varphi(u_1)+\ldots+\varphi(u_{n-1})\right)} \tag{54}$$

with

$$\varphi_{(n)}=\frac{\partial^n}{\partial\, u^n}\varphi^{-\mathbf{1}} \tag{55}$$

[19] Definition (1) of the copula requires that $\mathbf{C}$ is a N-increasing function (see footnote 3 page 4). This property means that there is a density associated to $\mathbf{C}$.

[20] The estimation methods of the parameters are described in the next chapter.

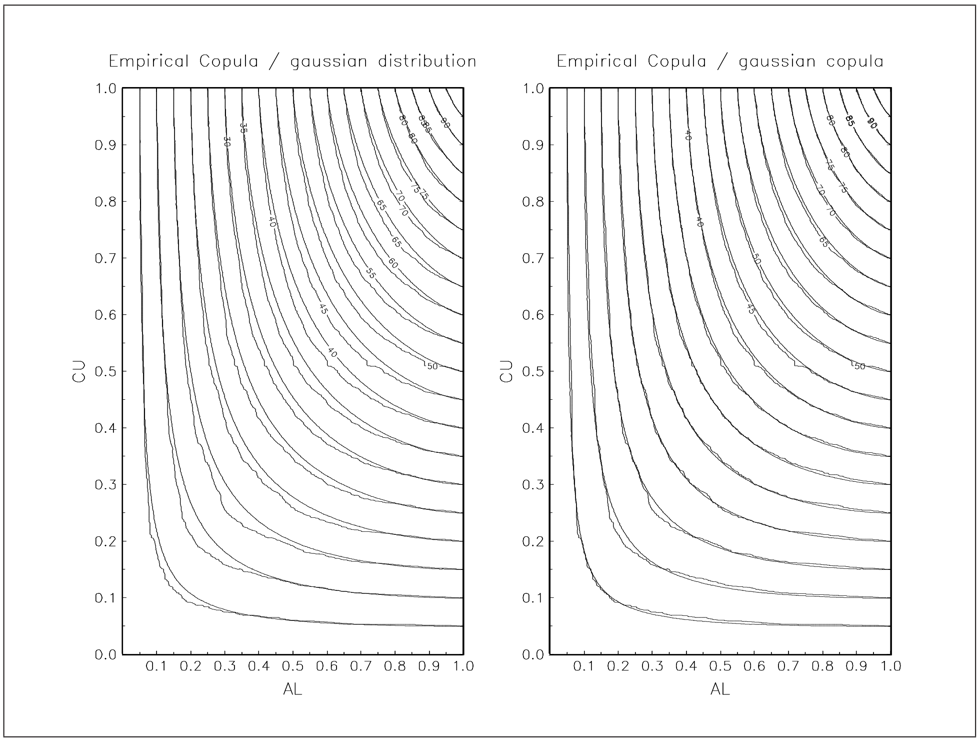

Figure 17: Comparison of empirical copula, Gaussian distribution and Gaussian copula

Using the relationship $\rho = 2\sin\left(\frac{\pi}{6}\varrho\right)$ between the parameter of gaussian copula and Spearman's correlation, we deduce an estimate of the correlation matrix of the assets for the gaussian copula (Table 2). If we compare it with the correlation given in table 1 page 5, we show that they are close but different.

	AL	AL-15	CU	NI	PB
AL	1.00	0.87	0.52	0.41	0.36
AL-15		1.00	0.45	0.36	0.32
CU			1.00	0.43	0.39
NI				1.00	0.34
PB					1.00

Table 2: Correlation matrix ρ of the LME data

3.2.2 Identification of an Archimedean copula

Genest and Rivest [1993] have developed an empirical method to identify the copula in the Archimedean case. Let $\mathbf{X}$ be a vector of N random variables, $\mathbf{C}$ the associated copula with generator φ and K the function defined by

$$K(u) = \Pr\{\mathbf{C}(U_1, \ldots, U_N) \leq u\} \tag{60}$$

Barbe, Genest, Ghoudi and Rémillard [1996] showed that

$$K(u) = u + \sum_{n=1}^{N} (-1)^n \frac{\varphi^n(u)}{n!} \varkappa_{n-1}(u) \tag{61}$$

with $\varkappa_n(u) = \frac{\partial_u \varkappa_{n-1}(u)}{\partial_u \varphi(u)}$ and $\varkappa_0(u) = \frac{1}{\partial_u \varphi(u)}$. In the bivariate case, this formula simplifies to

$$K(u) = u - \frac{\varphi(u)}{\varphi'(u)}$$

A non parametric estimate of K is given by

$$\hat{K}(u) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}_{[\vartheta_i \leq u]} \tag{62}$$

with

$$\vartheta_i = \frac{1}{T-1}\sum_{t=1}^{T} \mathbf{1}_{\left[x_1^t < x_1^i, \ldots, x_N^t < x_N^i\right]} \tag{63}$$

The idea is to fit $\hat{K}$ by choosing a parametric copula in the family of Archimedean copulas (see Figure 18 for assets AL and CU previously defined).

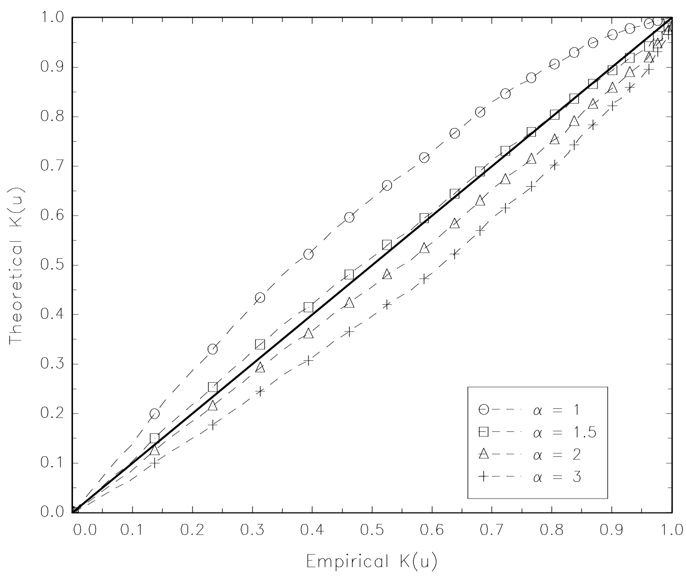

Figure 18: QQ-plot of the function $K(u)$ (Gumbel copula)

3.3 Parametric estimation

In "Multivariate Models and Dependence Concepts", Joe [1997] writes

> *Statistical modelling usually means that one comes up with a simple (or mathematically tractable) model without knowledge of the physical aspects of the situation. The statistical model needs to be 'real' and is not an end but a means of providing statistical inferences... My view of multivariate modelling, based on experience with multivariate data, is that models should try to capture important characteristics, such as the appropriate density shapes for the univariate margins and the*

appropriate dependence structure, and otherwise be as simple as possible. The parameters of the model should be in a form most suitable for easy interpretation (e.g., ***a parameter is interpreted as either a dependence parameter or a univariate parameter but not some mixture****)...*

In many applications, parametric models are useful in order to study the properties of the underlying statistical model. The idea to decompose the complex problems of multivariate modelling into two more simplified statistical problem is justified.

3.3.1 Maximum likelihood estimation

Let θ be the $K \times 1$ vector of parameters to be estimated and Θ the parameter space. The likelihood for observation t, that is the probability density of the observation t, considered as a function of θ, is denoted $L_t(\theta)$. Let $\ell_t(\theta)$ be the log-likelihood of $L_t(\theta)$. Given T observations, we get

$$\ell(\theta) = \sum_{t=1}^{T} \ell_t(\theta) \tag{64}$$

the log-likelihood function. $\hat{\theta}_{\mathrm{ML}}$ is the Maximum Likelihood estimator if

$$\ell\left(\hat{\theta}_{\mathrm{ML}}\right) \geq \ell(\theta) \qquad \forall\, \theta \in \Theta \tag{65}$$

We may show that $\hat{\theta}_{\mathrm{ML}}$ has the property of **asymptotic normality** (Davidson and MacKinnon [1993]) and we have

$$\sqrt{T}\left(\hat{\theta}_{\mathrm{ML}} - \theta_0\right) \longrightarrow \mathcal{N}\left(\mathbf{0}, \mathcal{J}^{-1}(\theta_0)\right) \tag{66}$$

with $\mathcal{J}(\theta_0)$ the **information matrix** of Fisher[21].

Applied to distribution (3), the expression of the log-likelihood becomes

$$\ell(\theta) = \sum_{t=1}^{T} \ln c\left(\mathbf{F}_1\left(x_1^t\right), \ldots, \mathbf{F}_n\left(x_n^t\right), \ldots, \mathbf{F}_N\left(x_N^t\right)\right) + \sum_{t=1}^{T}\sum_{n=1}^{N} \ln f_n\left(x_n^t\right) \tag{70}$$

If we assume uniform margins, we have

$$\ell(\theta) = \sum_{t=1}^{T} \ln c\left(u_1^t, \ldots, u_n^t, \ldots, u_N^t\right) \tag{71}$$

In the case of gaussian copula, we have

$$\ell(\theta) = -\frac{T}{2}\ln|\rho| - \frac{1}{2}\sum_{t=1}^{T} \varsigma_t^\top\left(\rho^{-1} - \mathbb{I}\right)\varsigma_t \tag{72}$$

[21] Let $J_{\hat{\theta}_{\mathrm{ML}}}$ be the $T \times K$ Jacobian matrix of $\ell_t(\theta)$ and $H_{\hat{\theta}_{\mathrm{ML}}}$ the $K \times K$ Hessian matrix of the likelihood function. The covariance matrix of $\hat{\theta}_{\mathrm{ML}}$ in finite sample can be estimated by the inverse of the negative Hessian

$$\mathrm{var}\ \hat{\theta}_{\mathrm{ML}} = -H_{\hat{\theta}_{\mathrm{ML}}}\ ^{-1} \tag{67}$$

or by the inverse of the OPG estimator

$$\mathrm{var}\ \hat{\theta}_{\mathrm{ML}} = J_{\hat{\theta}_{\mathrm{ML}}}^\top J_{\hat{\theta}_{\mathrm{ML}}}\ ^{-1} \tag{68}$$

Another estimator called the White (or "sandwich") estimator is defined by

$$\mathrm{var}\ \hat{\theta}_{\mathrm{ML}} = -H_{\hat{\theta}_{\mathrm{ML}}}\ ^{-1}\ J_{\hat{\theta}_{\mathrm{ML}}}^\top J_{\hat{\theta}_{\mathrm{ML}}}\ -H_{\hat{\theta}_{\mathrm{ML}}}\ ^{-1} \tag{69}$$

which takes into account of heteroskedasticity.

with $\varsigma_t = \left(\mathbf{\Phi}^{-1}\left(u_1^t\right), \ldots, \mathbf{\Phi}^{-1}\left(u_n^t\right), \ldots, \mathbf{\Phi}^{-1}\left(u_N^t\right)\right)$ and the ML estimate of ρ is also (Magnus and Neudecker [1988])

$$\hat{\rho}_{\mathrm{ML}} = \frac{1}{T} \sum_{t=1}^{T} \varsigma_t^\top \varsigma_t \tag{73}$$

Unlike for gaussian copula, the estimation of the parameters for other copulas may require numerical optimisation of the log-likelihood function. This is the case of Student's copula

$$\ell(\theta) \propto -\frac{T}{2} \ln|\rho| - \left(\frac{\nu+N}{2}\right) \sum_{t=1}^{T} \ln\left(1 + \frac{1}{\nu} \varsigma_t^\top \rho^{-1} \varsigma_t\right) + \left(\frac{\nu+1}{2}\right) \sum_{t=1}^{T} \sum_{n=1}^{N} \ln\left(1 + \frac{\varsigma_n^2}{\nu}\right) \tag{74}$$

The previous method, which we call the *exact maximum likelihood method* or EML, could be computational intensive in the case of high dimensional distribution, because it requires to jointly estimate the parameters of the margins and the parameters of the dependence structure. However, the copula representation splits the parameters into *specific* parameters for marginal distributions and *common* parameters for the dependence structure (or the parameters of the copula). The log-likelihood (70) could then be written as

$$\ell(\theta) = \sum_{t=1}^{T} \ln c\left(\mathbf{F}_1\left(x_1^t; \theta_1\right), \ldots, \mathbf{F}_n\left(x_n^t; \theta_n\right), \ldots, \mathbf{F}_N\left(x_N^t; \theta_N\right); \alpha\right) + \sum_{t=1}^{T} \sum_{n=1}^{N} \ln f_n\left(x_n^t; \theta_n\right) \tag{75}$$

with $\theta = (\theta_1, \ldots, \theta_N, \alpha)$. We could also perform the estimation of the univariate marginal distributions in a first step

$$\hat{\theta}_n = \arg\max \sum_{t=1}^{T} \ln f_n\left(x_n^t; \theta_n\right) \tag{76}$$

and then estimate α given the previous estimates

$$\hat{\alpha} = \arg\max \sum_{t=1}^{T} \ln c\left(\mathbf{F}_1\left(x_1^t; \hat{\theta}_1\right), \ldots, \mathbf{F}_n\left(x_n^t; \hat{\theta}_n\right), \ldots, \mathbf{F}_N\left(x_N^t; \hat{\theta}_N\right); \alpha\right) \tag{77}$$

This two-step method is called the method of *inference functions for margins* or IFM method. In general, we have

$$\hat{\theta}_{\mathrm{EML}} \neq \hat{\theta}_{\mathrm{IFM}} \tag{78}$$

In an unpublished thesis, Xu suggests "*that the IFM method is highly efficient compared with ML method*" (Joe [1997]). Using a close idea of the IFM method, we remark that the parameter vector α of the copula could be estimated without specifying the marginals. The method consists in transforming the data $(x_1^t, \ldots, x_N^t)$ into uniform variates $(\hat{u}_1^t, \ldots, \hat{u}_N^t)$, and in estimating the parameters in the following way:

$$\hat{\alpha} = \arg\max \sum_{t=1}^{T} \ln c\left(\hat{u}_1^t, \ldots, \hat{u}_n^t, \ldots, \hat{u}_N^t; \alpha\right) \tag{79}$$

In this case, $\hat{\alpha}$ could be viewed as the ML estimator given the *observed* margins (**without assumptions on the parametric form of the marginal distributions**). Because it is based on the empirical distributions, we call it the *canonical maximum likelihood method* or CML.

Example 13 *Let us consider two random variables X_1 and X_2 generated by a bivariate gaussian copula with exponential and standard gamma distributions. We have*

$$\mathbf{c}(x_1, x_2) = \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\left(\frac{\varsigma_1^2 + 2\varsigma_1\varsigma_2 + \varsigma_2^2}{1-\rho^2}\right) + \frac{1}{2}\left(\varsigma_1^2 + \varsigma_2^2\right)\right) \tag{80}$$

with

$$
\begin{aligned}
\varsigma_1 &= \mathbf{\Phi}^{-1}\left(1-\exp\left(-\lambda x_1\right)\right) \\
\varsigma_2 &= \mathbf{\Phi}^{-1}\left(\int_0^{x_2}\frac{x^{\gamma-1}e^{-x}}{\Gamma\left(\gamma\right)}\,\mathrm{d}x\right)
\end{aligned}
\tag{81}
$$

The vector of the parameters is also

$$
\theta = \begin{bmatrix} \lambda \\ \gamma \\ \rho \end{bmatrix} \tag{82}
$$

We perform a Monte Carlo study[22] *to investigate the properties of the three methods. In figure 19, we have reported the distributions of the estimators* $\hat{\rho}_{\mathrm{EML}}$, $\hat{\rho}_{\mathrm{IFM}}$ *and* $\hat{\rho}_{\mathrm{FML}}$ *for different sample sizes* $T=100$, $T=500$, $T=1000$ *and* $T=2500$. *We remark that the densities of the three estimators are very close.*

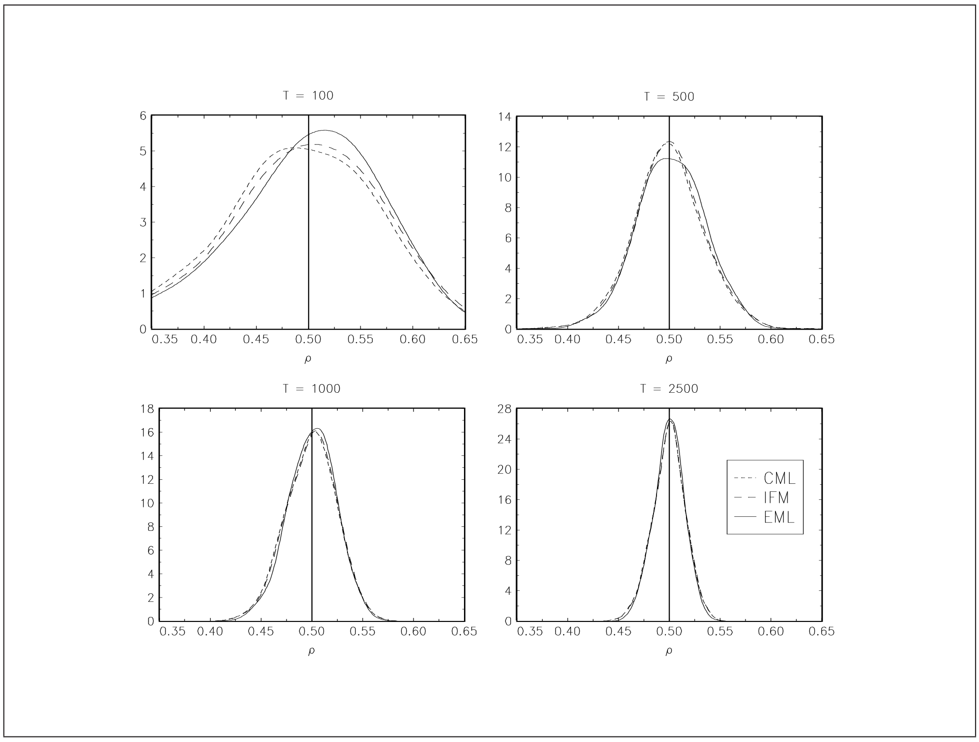

Figure 19: Density of the estimators

Remark 14 *To estimate the parameter* ρ *of the gaussian copula with the* CML *method*[23]*, we proceed as follows:*

1. *Transform the original data into gaussian data:*

   (a) *Estimate the empirical distribution functions (uniform transformation) using order statistics;*

[22] The number of replications is set to 1000. The value of the parameters are $\lambda = 2$, $\gamma = 1.5$ and $\rho = 0.5$.

[23] Note that this is **asymptotically** equivalent to compute Spearman's correlation and to deduce the correlation parameter using the relationship:

$$
\rho = 2\sin\frac{\pi}{6}\varrho \tag{83}
$$

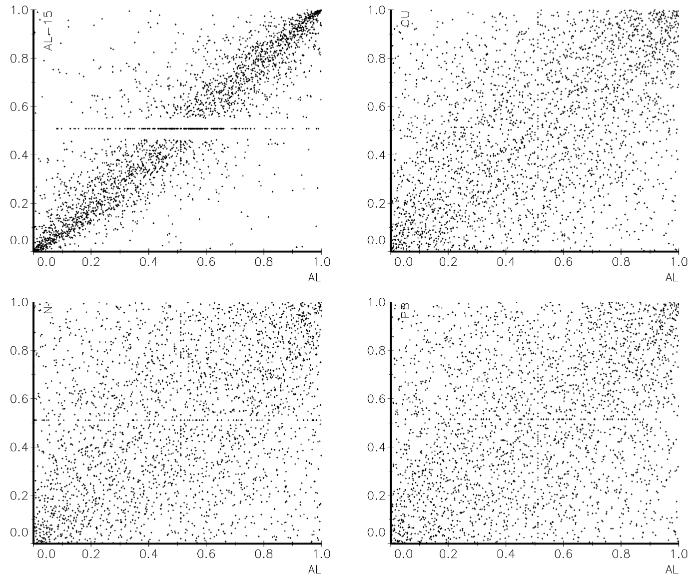

Figure 20: Empirical uniforms (or empirical distribution functions)

(b) *Then generate gaussian values by applying the inverse of the normal distribution to the empirical distribution functions.*

2. *Compute the correlation of the transformed data.*

Let us now use the LME example again. The correlation matrix estimated with the CML method is given by table (3).

	AL	AL-15	CU	NI	PB
AL	1.00	0.85	0.49	0.39	0.35
AL-15		1.00	0.43	0.35	0.32
CU			1.00	0.41	0.36
NI				1.00	0.33
PB					1.00

Table 3: Correlation matrix $\hat{\rho}_{\mathrm{CML}}$ of the gaussian copula for the LME data

3.3.2 Method of moments

We consider that the empirical moments $h_{t,i}(\theta)$ depend on the $K \times 1$ vector θ of parameters. T is the number of observations and m is the number of conditions or moments. Consider $\mathbf{h}_t(\theta)$ the row vector of the elements $h_{t,1}(\theta), \ldots, h_{t,m}(\theta)$ and $H(\theta)$ the $T \times m$ matrix with elements $h_{t,i}(\theta)$. Let $g(\theta)$ be a $m \times 1$ vector given by

$$g_i(\theta) = \frac{1}{T} \sum_{t=1}^{T} h_{t,i}(\theta) \tag{84}$$

The GMM[24] criterion function $Q(\theta)$ is defined by:

$$Q(\theta) = g(\theta)^\top W^{-1} g(\theta) \tag{85}$$

with W a symmetric positive definite $m \times m$ matrix. The GMM estimator $\hat{\theta}_{\text{GMM}}$ corresponds to

$$\hat{\theta}_{\text{GMM}} = \arg\min_{\theta \in \Theta} Q(\theta) \tag{86}$$

Like the ML estimators, we may show that $\hat{\theta}_{\text{GMM}}$ has the property of **asymptotic normality** and we have

$$\sqrt{T}\left(\hat{\theta}_{\text{GMM}} - \theta_0\right) \longrightarrow \mathcal{N}(\mathbf{0}, \Sigma) \tag{87}$$

In the case of **optimal weights** (W is the covariance matrix Φ of $H(\theta)$ – Hansen [1982]), we have

$$\operatorname{var}\left(\hat{\theta}_{\text{GMM}}\right) = \frac{1}{T}\left[D^\top \hat{\Phi}^{-1} D\right]^{-1} \tag{88}$$

with D the $m \times K$ Jacobian matrix of $g(\theta)$ computed for the estimate $\hat{\theta}_{\text{GMM}}$.

"*The ML estimator can be treated as a GMM estimator in which empirical moments are the components of the score vector*" (Davidson and MacKinnon [1993]). ML method is also a special case of GMM with $g(\theta) = \partial_\theta \ell(\theta)$ and $W = I_K$. That is why $\operatorname{var}\left(\hat{\theta}_{\text{GMM}}\right)$ is interpreted as an OPG estimator.

We may estimate the parameters of copulas with the method of moments. However, it requires in general to compute the moments. That could be done thanks to a symbolic software like Mathematica or Maple. When there does not exist analytical formulas, we could use numerical integration or simulation methods. Like the maximum likelihood method, we could use the method of moments in different ways in order to simplify the computational complexity of estimation.

4 Financial Applications

4.1 Credit scoring

Main idea

We use the theoretical background on scoring functions developed by Gouriéroux [1992]. We give a copula interpretation. Moreover, we discuss the dependence between scoring functions and show how to exploit it.

4.1.1 Theoretical background on scoring functions: a copula interpretation

4.1.2 Statistical methods with copulas

4.2 Asset returns modelling

Main idea

We consider some portfolio optimisation problems. In a first time, we fix the margins and analyze the impact of the copula. In a second time, we work with a given copula. Finally, we present some illustrations with different risk measures.
We then analyze the serial dependence with discrete stationary Markov chains constructed from a copula function (Fang, Hu and Joe [1994] and Joe [1996,1997]).
The third paragraph concerns continuous stochastic processes and their links with copulas (Darsow, Nguyen and Olsen [1992]).

[24] **G**eneralized **M**ethod of **M**oments.

4.2.1 Portfolio aggregation

4.2.2 Time series modelling

4.2.3 Markov processes

This paragraph is based on the seminal work of Darsow, Nguyen and Olsen [1992]. These authors define a product of copulas, which is noted the $*$ operation. They remark that "*the $*$ operation on copulas corresponds in a natural way to the operation on transition probabilities contained in the Chapman-Kolmogorov equations*".

4.2.3.1 The $*$ product and Markov processes

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two copulas of dimension 2. Darsow, Nguyen and Olsen [1992] define the product of $\mathbf{C}_1$ and $\mathbf{C}_2$ by the following function

$$\begin{array}{rccl} \mathbf{C}_1 * \mathbf{C}_2 : & \mathbf{I}^2 & \longrightarrow & \mathbf{I} \\ & (u_1, u_2) & \longmapsto & (\mathbf{C}_1 * \mathbf{C}_2)(u_1, u_2) = \int_0^1 \partial_2 \mathbf{C}_1 (u_1, u) \, \partial_1 \mathbf{C}_2 (u, u_2) \, \mathrm{d}u \end{array} \tag{89}$$

where $\partial_1 \mathbf{C}$ and $\partial_2 \mathbf{C}$ represent the first-order partial derivatives with respect to the first and second variable. This product has many properties (Darsow, Nguyen and Olsen [1992], Olsen, Darsow and Nguyen [1996]):

1. $\mathbf{C}_1 * \mathbf{C}_2$ is in $\mathcal{C}$ ($\mathbf{C}_1 * \mathbf{C}_2$ is a copula);
2. the $*$ product is right and left distributive over convex combinations;
3. the $*$ product is continuous in each place;
4. the $*$ product is associative;
5. $\mathbf{C}^{\perp}$ is the null element
$$\mathbf{C}^{\perp} * \mathbf{C} = \mathbf{C} * \mathbf{C}^{\perp} = \mathbf{C}^{\perp} \tag{90}$$
6. $\mathbf{C}^{+}$ is the identity
$$\mathbf{C}^{+} * \mathbf{C} = \mathbf{C} * \mathbf{C}^{+} = \mathbf{C} \tag{91}$$
7. the $*$ product is not commutative.

Using the fact that conditional probabilities of joint random variables (X_1, X_2) correspond to partial derivatives of the underlying copula $\mathbf{C}_{(X_1,X_2)}$, Darsow, Nguyen and Olsen [1992] investigate the interpretation of the $*$ product in the context of Markov processes. Karatzas and Shreve [1991] remind that $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ is said to be a Markov process with initial distribution μ if

(i) $\forall \mathcal{A}$ a σ-field of Borel sets in $\mathbb{R}$
$$\Pr\{X_0 \in \mathcal{A}\} = \mu(\mathcal{A}) \tag{92}$$

(ii) and for $t \geq s$
$$\Pr\{X_t \in \mathcal{A} \mid \mathcal{F}_s\} = \Pr\{X_t \in \mathcal{A} \mid X_s\} \tag{93}$$

Darsow, Nguyen and Olsen [1992] prove also the following theorem:

Theorem 15 *Let $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ be a stochastic process and let $\mathbf{C}_{s,t}$ denote the copula of the random variables X_s and X_t. Then the following are equivalent* *(i) The transition probabilities* $P_{s,t}(x,\mathcal{A}) = \Pr\{X_t \in \mathcal{A} \mid X_s = x\}$ *satisfy the Chapman-Kolmogorov equations*

$$P_{s,t}(x,\mathcal{A}) = \int_{-\infty}^{\infty} P_{s,\theta}(x,\mathrm{d}y)\, P_{\theta,t}(y,\mathcal{A}) \tag{94}$$

for all $s < \theta < t$ *and almost all* $x \in \mathbb{R}$.

(ii) For all $s < \theta < t$,

$$\mathbf{C}_{s,t} = \mathbf{C}_{s,\theta} * \mathbf{C}_{\theta,t} \tag{95}$$

As Darsow, Nguyen and Olsen [1992] remark, this theorem is a new approach to consider Markov processes:

> *In the conventional approach, one specifies a Markov process by giving the initial distribution* μ *and a family of transition probabilities* $P_{s,t}(x,\mathcal{A})$ *satisfying the Chapman-Kolmogorov equations. In our approach, one specifies a Markov process by giving all of the marginal distributions and a family of 2-copulas satisfying (95). Ours is accordingly an alternative approach to the study of Markov processes which is different in principle from the conventional one. Holding the transition probabilities of a Markov process fixed and varying the initial distribution necessarily varies all of the marginal distributions, but holding the copulas of the process fixed and varying the initial distribution does not affect any other marginal distribution.*

Darsow, Nguyen and Olsen [1992] recall that **satisfaction of the Chapman-Kolmogorov equations is a necessary but not sufficient condition for a Markov process**. They consider then the generalization of the $*$ product, which is denoted $\mathbf{C}_1 \star \mathbf{C}_2$

$$\begin{array}{ccc} \mathbf{I}^{N_1+N_2-1} & \longrightarrow & \mathbf{I} \\ (u_1,\ldots,u_{N_1+N_2-1}) & \longmapsto & (\mathbf{C}_1 \star \mathbf{C}_2)(\mathbf{u}) = \int_0^{u_{N_1}} \partial_{N_1}\mathbf{C}_1(u_1,\ldots,u_{N_1-1},u)\,\partial_1\mathbf{C}_2(u,u_{N_1+1},\ldots,u_{N_1+N_2-1})\,\mathrm{d}u \end{array} \tag{96}$$

with $\mathbf{C}_1$ and $\mathbf{C}_2$ two copulas of dimension N_1 and N_2. They also prove this second theorem:

Theorem 16 *Let* $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ *be a stochastic process. Let* $\mathbf{C}_{s,t}$ *and* $\mathbf{C}_{t_1,\ldots,t_N}$ *denote the copulas of the random variables* (X_s, X_t) *and* $(X_{t_1},\ldots,X_{t_N})$. X *is a Markov process if and only if for all positive integers* N *and for all* $(t_1,\ldots,t_N)$ *satisfying* $t_n < t_{n+1}$

$$\mathbf{C}_{t_1,\ldots,t_N} = \mathbf{C}_{t_1,t_2} \star \cdots \star \mathbf{C}_{t_n,t_{n+1}} \star \cdots \star \mathbf{C}_{t_{N-1},t_N} \tag{97}$$

4.2.3.2 An investigation of the Brownian copula

Revuz and Yor [1999] define the brownian motion as follows:

Definition 17 *There exists an almost-surely continuous process* W *with independent increments such that for each* t*, the random variable* $W(t)$ *is centered, Gaussian and has variance* t*. Such a process is called a standard linear Brownian motion.*

It comes from this definition that

$$P_{s,t}(x,(-\infty,y]) = \Phi\left(\frac{y-x}{\sqrt{t-s}}\right) \tag{98}$$

Moreover, we have by definition of the conditional distribution

$$P_{s,t}(x,(-\infty,y]) = \partial_1\mathbf{C}_{s,t}(\mathbf{F}_s(x),\mathbf{F}_t(y)) \tag{99}$$

with $\mathbf{F}_t$ the distribution of $W(t)$, i.e. $\mathbf{F}_t(y) = \Phi\left(y/\sqrt{t}\right)$. We have also

$$\mathbf{C}_{s,t}\left(\mathbf{F}_s(x), \mathbf{F}_t(y)\right) = \int_{-\infty}^{x} \Phi\left(\frac{y-z}{\sqrt{t-s}}\right) \mathrm{d}\mathbf{F}_s(z) \tag{100}$$

With the change of variables $u_1 = \mathbf{F}_s(x)$, $u_2 = \mathbf{F}_t(y)$ and $u = \mathbf{F}_s(z)$, we obtain

$$\mathbf{C}_{s,t}(u_1, u_2) = \int_{0}^{u_1} \Phi\left(\frac{\sqrt{t}\Phi^{-1}(u_2) - \sqrt{s}\Phi^{-1}(u)}{\sqrt{t-s}}\right) \mathrm{d}u \tag{101}$$

This copula has been first found by DARSOW, NGUYEN and OLSEN [1992], but nobody has studied it. However, Brownian motion plays an important role in diffusion process. That is why we try to understand the implied dependence structure in this paragraph.

We call the copula defined by (101) the *Brownian copula*. We have the following properties:

1. The conditional distribution $\Pr\{U_2 \leq u_2 \mid U_1 = u_1\}$ is given by

$$\partial_1 \mathbf{C}_{s,t}(u_1, u_2) = \Phi\left(\frac{\sqrt{t}\Phi^{-1}(u_2) - \sqrt{s}\Phi^{-1}(u_1)}{\sqrt{t-s}}\right) \tag{102}$$

   In the figure 22, we have reported the conditional probabilities for different s and t.

2. The conditional distribution $\Pr\{U_1 \leq u_1 \mid U_2 = u_2\}$ is given by

$$\partial_2 \mathbf{C}_{s,t}(u_1, u_2) = \sqrt{\frac{t}{t-s}} \frac{1}{\phi\left(\Phi^{-1}(u_2)\right)} \int_{0}^{u_1} \phi\left(\frac{\sqrt{t}\Phi^{-1}(u_2) - \sqrt{s}\Phi^{-1}(u)}{\sqrt{t-s}}\right) \mathrm{d}u \tag{103}$$

3. The density of the *Brownian* copula is

$$c_{s,t}(u_1, u_2) = \sqrt{\frac{t}{t-s}} \phi\left(\frac{\sqrt{t}\Phi^{-1}(u_2) - \sqrt{s}\Phi^{-1}(u)}{\sqrt{t-s}}\right) \frac{1}{\phi\left(\Phi^{-1}(u_2)\right)} \tag{104}$$

   In the figure 23, we have represented the density for different s and t.

4. The *Brownian copula* is symmetric.

5. We have $\mathbf{C}_{0,t} = \mathbf{C}^{\perp}$, $\mathbf{C}_{s,\infty} = \mathbf{C}^{\perp}$ and $\lim_{t \to s} \mathbf{C}_{s,t} = \mathbf{C}^{+}$.

6. The *Brownian copula* is not invariant by time translation

$$\mathbf{C}_{s,t} \neq \mathbf{C}_{s+\delta,t+\delta} \tag{105}$$

7. Let U_1 and U_2 be two **independent** uniform variables. Then, the copula of the random variables $\left(U_1, \Phi\left(\sqrt{t^{-1}s}\Phi^{-1}(U_1) + \sqrt{t^{-1}(t-s)}\Phi^{-1}(U_2)\right)\right)$ is the *Brownian copula* $\mathbf{C}_{s,t}$.

Let us now consider the relation between the marginal distributions in the context of stochastic processes. We have

$$F_t(X_t) = \Phi\left(\sqrt{t^{-1}s}\Phi^{-1}(F_s(X_s)) + \sqrt{t^{-1}(t-s)}\epsilon_t\right) \tag{106}$$

with ϵ_t a white noise process. It comes that

$$X_t = F_t^{-1}\left(\Phi\left(\sqrt{t^{-1}s}\Phi^{-1}(F_s(X_s)) + \sqrt{t^{-1}(t-s)}\epsilon_t\right)\right) \tag{107}$$

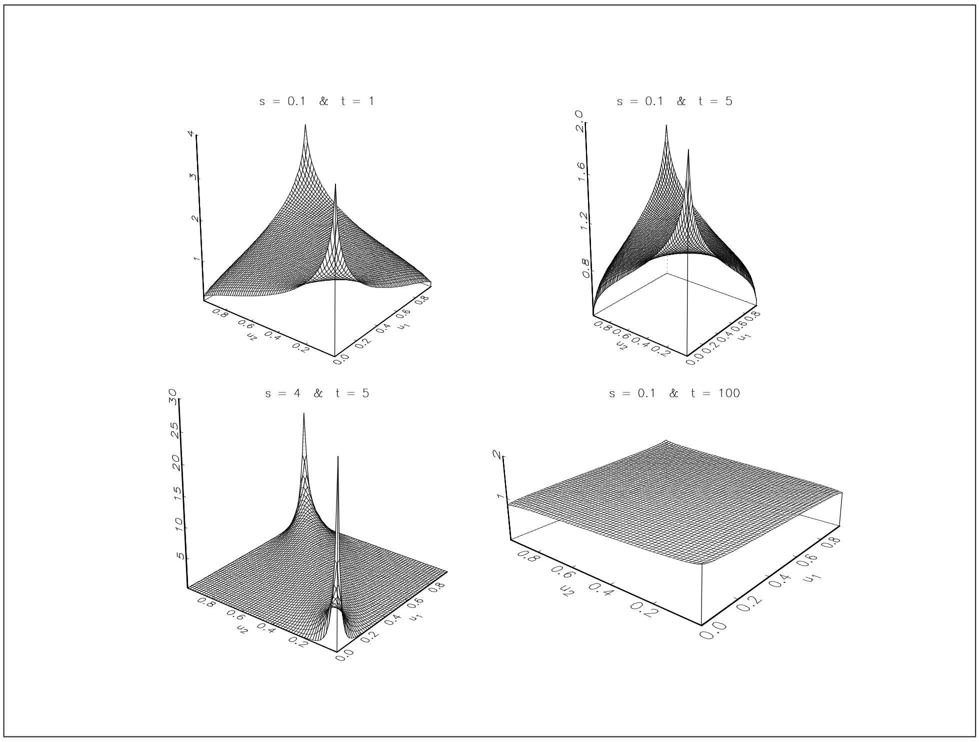

Figure 21: Probability density function of the Brownian copula

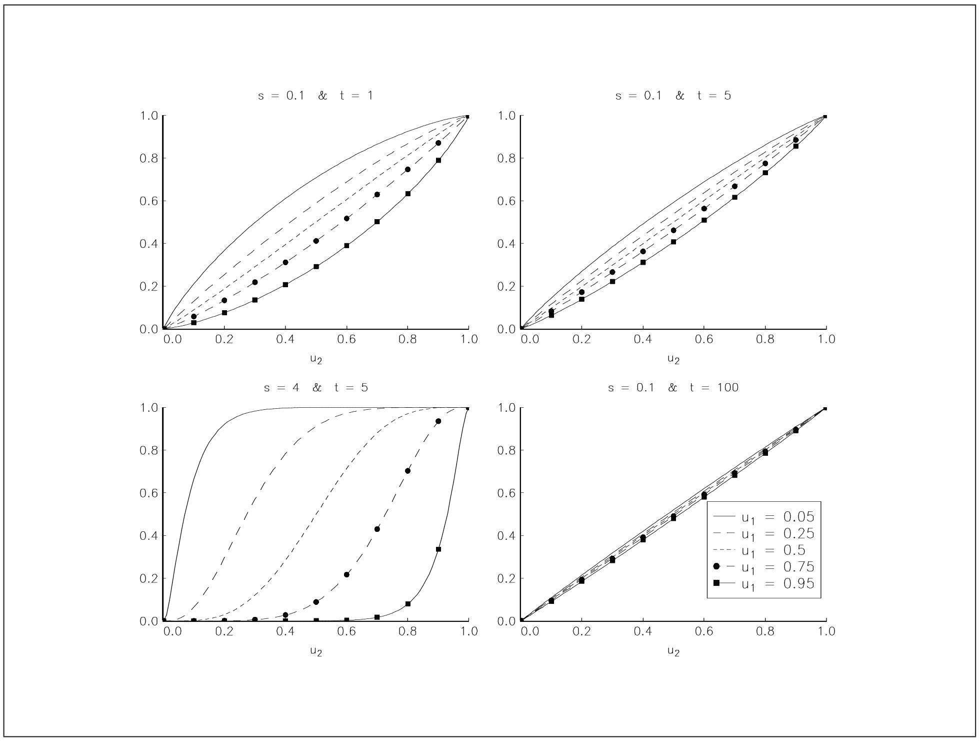

Figure 22: Plot of the conditional probabilities $\Pr\{U_2 \leq u_2 \mid U_1 = u_1\}$

In the case of the brownian motion, we retrieve the classical relation

$$\begin{aligned} X_t &= \sqrt{t}\Phi^{-1}\left(\Phi\left(\sqrt{t^{-1}s}\Phi^{-1}\left(\Phi\left(\frac{X_s}{\sqrt{s}}\right)\right)+\sqrt{t^{-1}(t-s)}\epsilon_t\right)\right) \\ &= X_s+\sqrt{t-s}\epsilon_t \end{aligned} \tag{108}$$

Let us now consider the following marginal distributions $\mathbf{F}_t(y) = t_\nu\left(y/\sqrt{\frac{vt}{v-2}}\right)$ with $\nu > 2$. By construction, we have this definition:

Definition 18 *There exists an almost-surely continuous process W^ν ~~with independent increments~~* ***with the same temporal dependence structure as the Brownian motion*** *such that for each t, the random variable $W^\nu(t)$ is centered, Student with variance t. Such a process is called a Student Brownian motion. $W^\nu(t)$ is then characterized by the marginals $\left(t_\nu\left(W^\nu/\sqrt{\frac{vt}{v-2}}\right)\right)_{t\geq 0}$ and the copula $\left(\int_0^{u_1}\Phi\left(\frac{\sqrt{t}\Phi^{-1}(u_2)-\sqrt{s}\Phi^{-1}(u)}{\sqrt{t-s}}\right)\mathrm{d}u\right)_{t>s\geq 0}$.*

Figure 23 presents some simulations of these process. We remark of course that when ν tends to infinity, W^ν tends to the Gaussian brownian motion. We now consider now a very simple illustration. Let $X(t)$ be defined by the following SDE:

$$\left\{\begin{array}{lcl} \mathrm{d}X(t) &=& \mu X(t)\,\mathrm{d}t+\mu X(t)\,\mathrm{d}W(t) \\ X(t_0) &=& x_0 \end{array}\right. \tag{109}$$

The solution is the Geometric Brownian motion and the diffusion process representation is

$$X(t) = x_0\exp\left(\left(\mu-\frac{1}{2}\sigma^2\right)(t-t_0)+\sigma\left(W(t)-W(t_0)\right)\right) \tag{110}$$

We consider now a new stochastic process $Y(t)$ defined by

$$Y(t) = x_0\exp\left(\left(\mu-\frac{1}{2}\sigma^2\right)(t-t_0)+\sigma\left(W^\nu(t)-W^\nu(t_0)\right)\right) \tag{111}$$

What is the impact of introducing these fat tailed distributions in the payoff of a KOC option ? The answer is not obvious, because there are two phenomenons:

1. First, we may suppose that the probability to be out of the barriers L and H is greater for the Student brownian motion than for the brownian motion

$$\Pr\{Y(t)\in[L,H]\} \leq \Pr\{X(t)\in[L,H]\} \tag{112}$$

2. But we have certainly an opposite influence on the terminal payoff (K is the strike)

$$\Pr\left\{(Y(T)-K)^+\geq g\right\} \geq \Pr\left\{(X(T)-K)^+\geq g\right\} \tag{113}$$

In figure 24, we have reported the probability density function of the KOC option with the following parameter values : $x_0 = 100$, $\mu = 0$, $\sigma = 0.20$, $L = 80$, $H = 120$ and $K = 100$. The maturity of the option is one year. For the student parameter, we use $\nu = 10$. In figure 25, we use $\nu = 4$.

4.2.3.3 Understanding the temporal dependence structure of diffusion processes

The Darsow-Nguyen-Olsen approach is very interesting to understand the temporal dependence structure of diffusion processes. In this paragraph, we review different diffusion processes and compare their copulas.

Theorem 19 *The copula of a Geometric Brownian motion is the Brownian copula.*

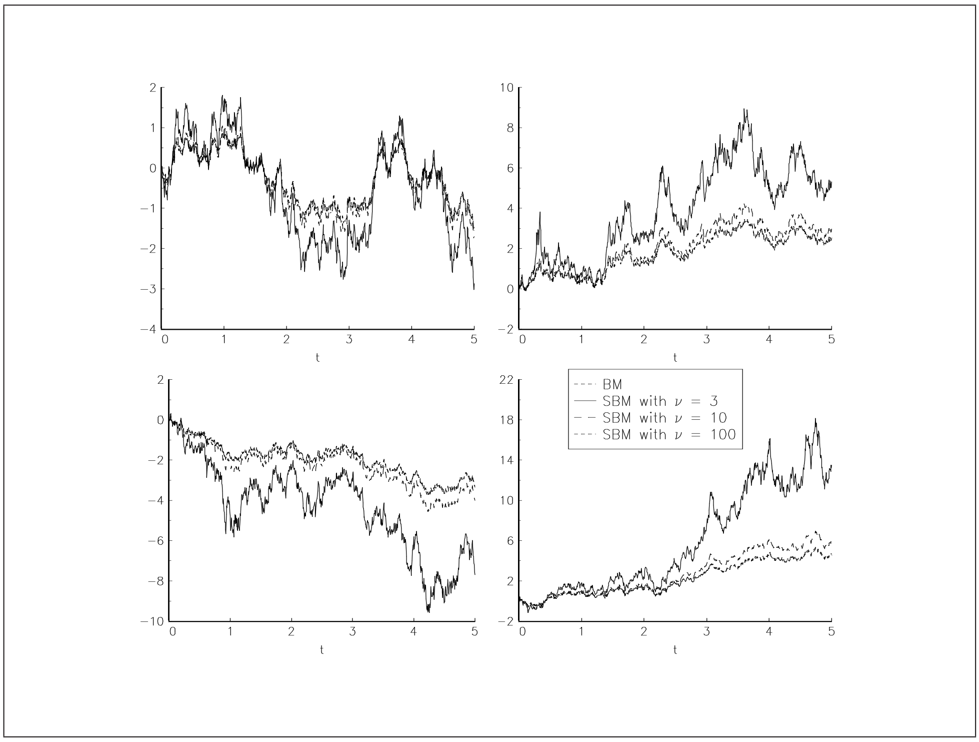

Figure 23: Simulations of a Brownian motion (BM) and of a Student Brownian motion (SBM

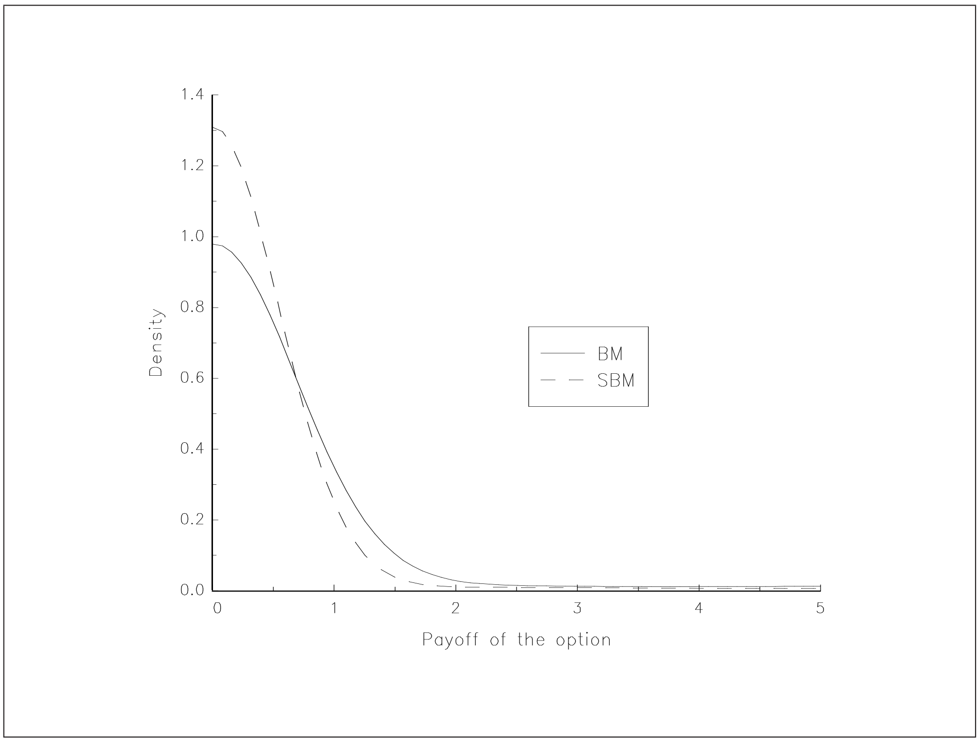

Figure 24: Density of the payoff of the KOC option ($\nu = 10$

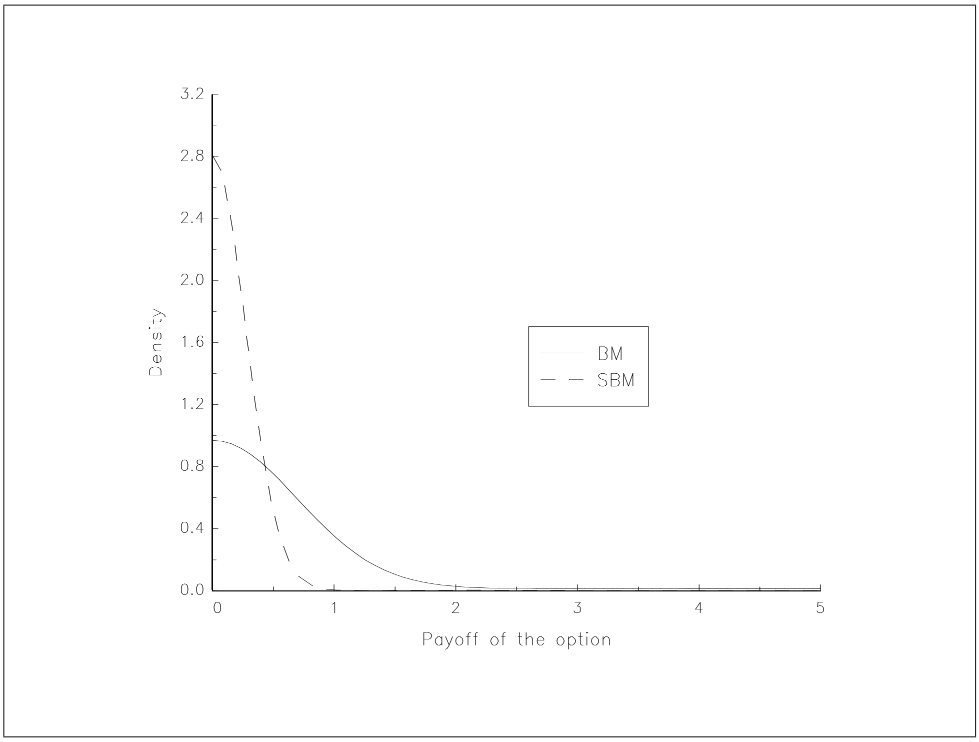

Figure 25: Density of the payoff of the KOC option ($\nu = 4$)

Proof. We have

$$P_{s,t}\left(x,\left(0,y\right]\right) = \Phi\left(\frac{\ln y - \ln x - \left(\mu - \frac{1}{2}\sigma^2\right)\left(t-s\right)}{\sigma\sqrt{t-s}}\right) \tag{114}$$

Then, it comes that

$$\mathbf{C}_{s,t}\left(\mathbf{F}_s\left(x\right),\mathbf{F}_t\left(y\right)\right) = \int_0^x \Phi\left(\frac{\ln y - \ln z - \left(\mu - \frac{1}{2}\sigma^2\right)\left(t-s\right)}{\sigma\sqrt{t-s}}\right)\,\mathrm{d}\mathbf{F}_s\left(z\right)$$

with $\mathbf{F}_t\left(y\right) = \Phi\left(\frac{\ln y - \ln x_0 - \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right)$. With the change of variables $u_1 = \mathbf{F}_s\left(x\right)$, $u_2 = \mathbf{F}_t\left(y\right)$ and $u = \mathbf{F}_s\left(z\right)$, we have

$$\begin{aligned}
\ln z - \ln x_0 - \left(\mu - \frac{1}{2}\sigma^2\right)s &= \sigma\sqrt{s}\Phi^{-1}\left(u\right) \\
\ln y - \ln x_0 - \left(\mu - \frac{1}{2}\sigma^2\right)t &= \sigma\sqrt{t}\Phi^{-1}\left(u_2\right) \\
\ln y - \ln z - \left(\mu - \frac{1}{2}\sigma^2\right)\left(t-s\right) &= \sigma\left[\sqrt{t}\Phi^{-1}\left(u_2\right) - \sqrt{s}\Phi^{-1}\left(u\right)\right]
\end{aligned} \tag{115}$$

The corresponding copula is then

$$\mathbf{C}_{s,t}\left(u_1,u_2\right) = \int_0^{u_1} \Phi\left(\frac{\sqrt{t}\Phi^{-1}\left(u_2\right) - \sqrt{s}\Phi^{-1}\left(u\right)}{\sqrt{t-s}}\right)\,\mathrm{d}u \tag{116}$$

Note that we could prove the previous theorem in another way. Let $\mathbf{C}_{s,t}$ be the copula of the random variables $W(s)$ and $W(t)$. Let us consider two functions $\alpha(x)$ and $\beta(x)$ defined as follows:

$$\begin{aligned}\alpha(x) &= x_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(s - t_0) + \sigma x\right) \\ \beta(x) &= x_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0) + \sigma x\right)\end{aligned} \tag{117}$$

Because these two functions are strictly increasing, the copula of the random variables $\alpha(W(s))$ and $\beta(W(t))$ is the same as the copula of the random variables $W(s)$ and $W(t)$. ■

Definition 20 *An Ornstein-Uhlenbeck process corresponds to the following SDE representation*

$$\begin{cases} \mathrm{d}X(t) = a(b - X(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t) \\ X(t_0) = x_0 \end{cases} \tag{118}$$

Using Ito calculus, we could show that the diffusion process representation is

$$X(t) = x_0 e^{-a(t-t_0)} + b\left(1 - e^{-a(t-t_0)}\right) + \sigma \int_{t_0}^{t} e^{a(\theta - t)}\,\mathrm{d}W(\theta) \tag{119}$$

By definition of the stochastic integral, it comes that the distribution $\mathbf{F}_t(x)$ is

$$\Phi\left(\frac{x - x_0 e^{-a(t-t_0)} + b\left(1 - e^{-a(t-t_0)}\right)}{\frac{\sigma}{\sqrt{2a}}\sqrt{1 - e^{-2a(t-t_0)}}}\right) \tag{120}$$

Theorem 21 *The Ornstein-Uhlenbeck copula is*

$$\mathbf{C}_{s,t}(u_1, u_2) = \int_0^{u_1} \Phi\left(\frac{\hbar(t_0, s, t)\,\Phi^{-1}(u_2) - \hbar(t_0, s, s)\,\Phi^{-1}(u)}{\hbar(s, s, t)}\right) \mathrm{d}u \tag{121}$$

with

$$\hbar(t_0, s, t) = \sqrt{e^{2a(t-s)} - e^{-2a(s-t_0)}} \tag{122}$$

Proof. We have

$$P_{s,t}(x, (-\infty, y]) = \Phi\left(\frac{y - x e^{-a(t-s)} + b\left(1 - e^{-a(t-s)}\right)}{\frac{\sigma}{\sqrt{2a}}\sqrt{1 - e^{-2a(t-s)}}}\right) \tag{123}$$

Then, it comes that

$$\mathbf{C}_{s,t}(\mathbf{F}_s(x), \mathbf{F}_t(y)) = \int_{-\infty}^{x} \Phi\left(\frac{y - z e^{-a(t-s)} + b\left(1 - e^{-a(t-s)}\right)}{\frac{\sigma}{\sqrt{2a}}\sqrt{1 - e^{-2a(t-s)}}}\right) \mathrm{d}\mathbf{F}_s(z) \tag{124}$$

Using the change of variables $u_1 = \mathbf{F}_s(x)$, $u_2 = \mathbf{F}_t(y)$ and $u = \mathbf{F}_s(z)$, it comes that

$$\begin{aligned} y - z e^{-a(t-s)} &= x_0 e^{-a(t-t_0)} - b\left(1 - e^{-a(t-t_0)}\right) + \frac{\sigma}{\sqrt{2a}}\sqrt{1 - e^{-2a(t-t_0)}}\Phi^{-1}(u_2) - \\ &\quad e^{-a(t-s)}\left[x_0 e^{-a(s-t_0)} - b\left(1 - e^{-a(s-t_0)}\right) + \frac{\sigma}{\sqrt{2a}}\sqrt{1 - e^{-2a(s-t_0)}}\Phi^{-1}(u)\right] \\ &= -b\left(1 - e^{-a(t-s)}\right) + \\ &\quad \frac{\sigma}{\sqrt{2a}}\left[\sqrt{1 - e^{-2a(t-t_0)}}\Phi^{-1}(u_2) - e^{-a(t-s)}\sqrt{1 - e^{-2a(s-t_0)}}\Phi^{-1}(u)\right] \end{aligned} \tag{125}$$

Then, we obtain a new expression of the copula

$$\mathbf{C}_{s,t}(u_1, u_2) = \int_0^{u_1} \Phi\left(\frac{\sqrt{1-e^{-2a(t-t_0)}}\Phi^{-1}(u_2) - e^{-a(t-s)}\sqrt{1-e^{-2a(s-t_0)}}\Phi^{-1}(u)}{\sqrt{1-e^{-2a(t-s)}}}\right) \mathrm{d}u \tag{126}$$

■

The Ornstein-Uhlenbeck copula presents the following properties:

1. The Brownian copula is a special case of the Ornstein-Uhlenbeck copula with the limit function
$$\lim_{a \longrightarrow 0} \hbar(t_0, s, t) = \sqrt{t - t_0} \tag{127}$$
2. The conditional distribution $\Pr\{U_2 \leq u_2 \mid U_1 = u_1\}$ is given by
$$\partial_1 \mathbf{C}_{s,t}(u_1, u_2) = \Phi\left(\frac{\hbar(t_0, s, t)\Phi^{-1}(u_2) - \hbar(t_0, s, s)\Phi^{-1}(u_1)}{\hbar(s, s, t)}\right) \tag{128}$$
3. The conditional distribution $\Pr\{U_1 \leq u_1 \mid U_2 = u_2\}$ is given by
$$\partial_2 \mathbf{C}_{s,t}(u_1, u_2) = \frac{\hbar(t_0, s, t)}{\hbar(s, s, t)} \frac{1}{\phi(\Phi^{-1}(u_2))} \int_0^{u_1} \phi\left(\frac{\hbar(t_0, s, t)\Phi^{-1}(u_2) - \hbar(t_0, s, s)\Phi^{-1}(u)}{\hbar(s, s, t)}\right) \mathrm{d}u \tag{129}$$
4. The density of the Ornstein-Uhlenbeck copula is
$$c_{s,t}(u_1, u_2) = \frac{\hbar(t_0, s, t)}{\hbar(s, s, t)} \phi\left(\frac{\hbar(t_0, s, t)\Phi^{-1}(u_2) - \hbar(t_0, s, s)\Phi^{-1}(u_1)}{\hbar(s, s, t)}\right) \frac{1}{\phi(\Phi^{-1}(u_2))} \tag{130}$$

We have represented the density of the Ornstein-Uhlenbeck copula for different values of s and t (t_0 is equal to 0). We could compare them with the previous ones obtained for the Brownian copula. We verify that

$$\lim_{a \longrightarrow \infty} \mathbf{C}_{s,t}(u_1, u_2) = \mathbf{C}^{\perp} \tag{131}$$

but we have

$$\lim_{a \longrightarrow -\infty} \mathbf{C}_{s,t}(u_1, u_2) = \mathbf{C}^{+} \tag{132}$$

In order to understand the temporal dependence of diffusion processes, we could investigate the properties of the associated copula in a deeper way. We have reported in figures 28 and 29 Spearman's rhos for Brownian and Ornstein-Uhlenbeck copulas.

Remark 22 *A new interpretation of the parameter a follows. For physicists, a is the mean-reverting coefficient. From a copula point of view, this parameter measures the dependence between the random variables of the diffusion process. The bigger this parameter, the less dependent the random variables.*

4.3 Risk measurement

Main idea

One of the most powerful application of copulas concerns the Risk Management. We consider four problems: loss aggregation, stress testing programs, default modelling and operational risk measurement.

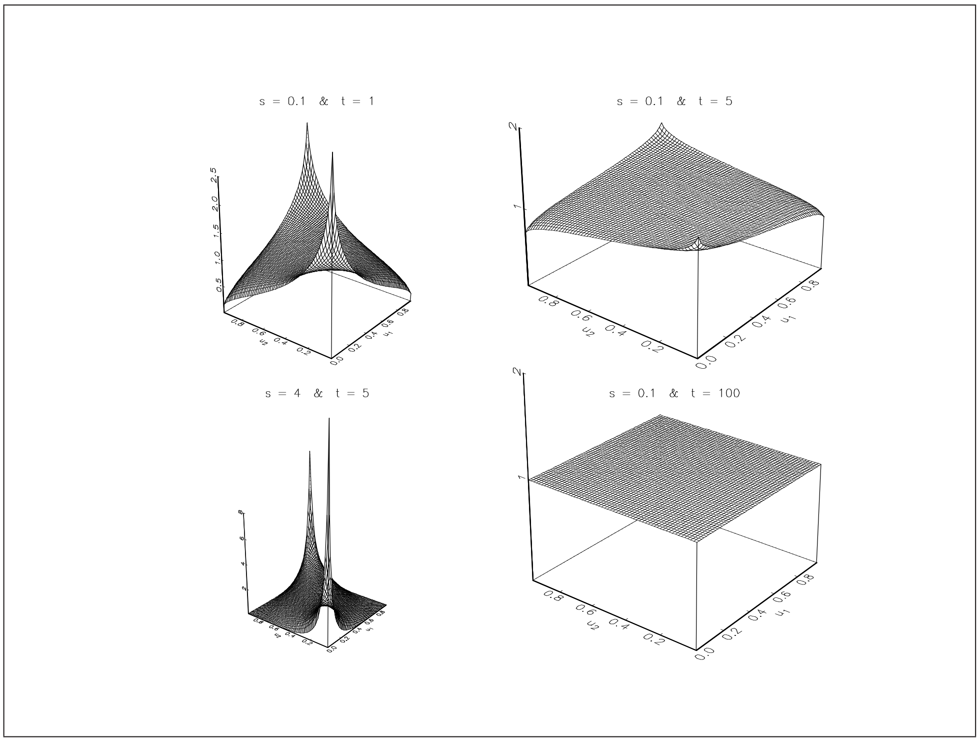

Figure 26: Probability density function of the Ornstein-Uhlenbeck copula ($a = \frac{1}{4}$)

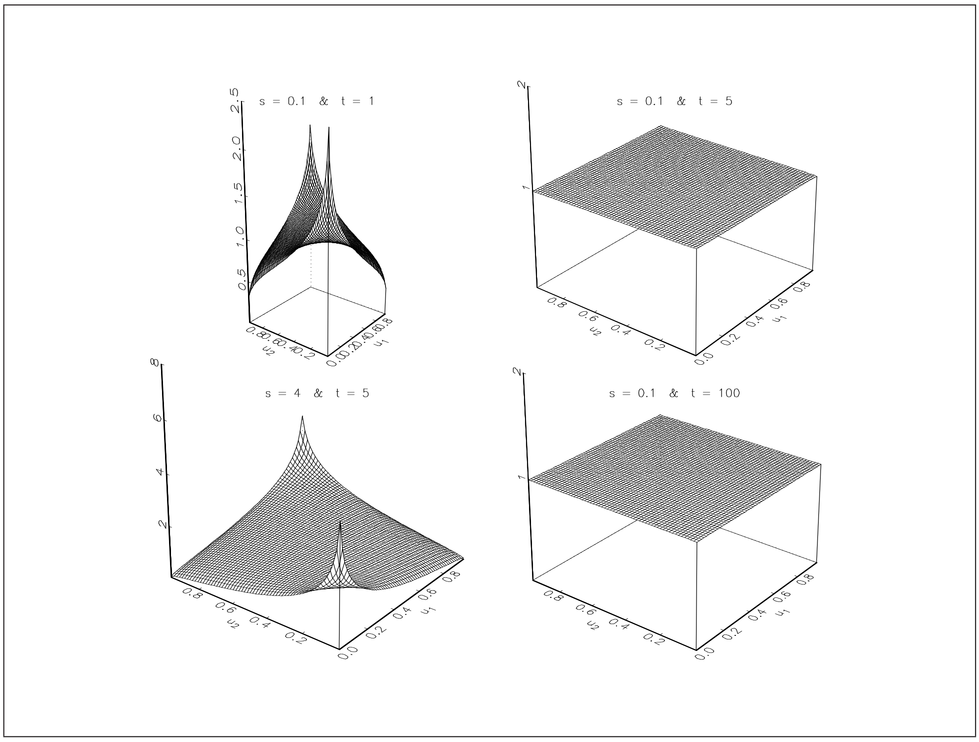

Figure 27: Probability density function of the Ornstein-Uhlenbeck copula ($a = 1$)

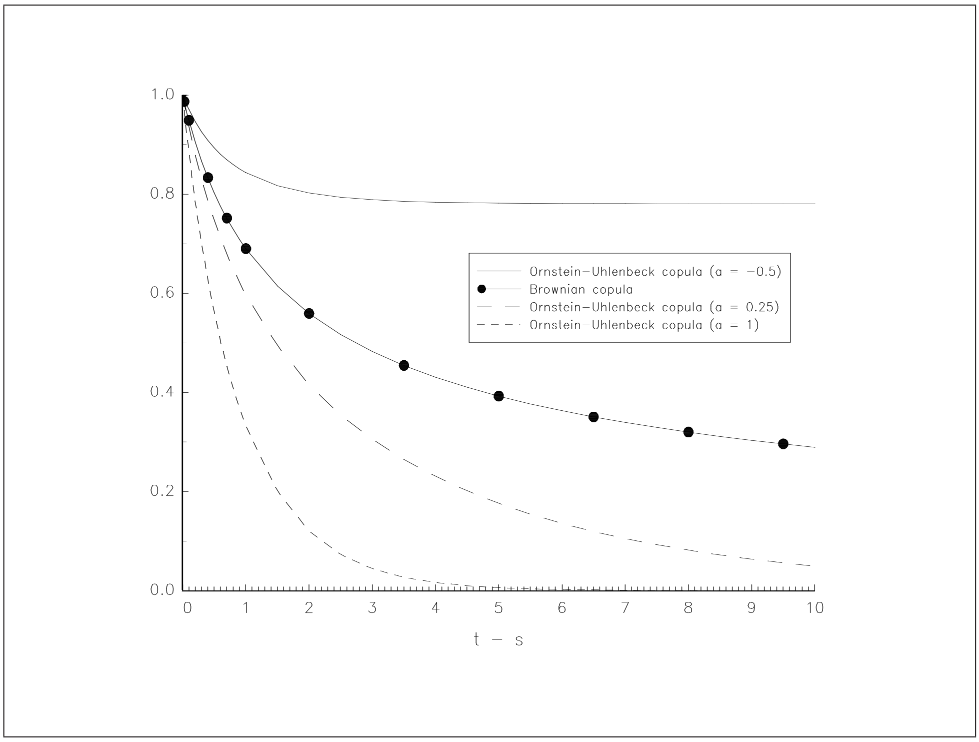

Figure 28: Spearman's rho ($s = 1$)

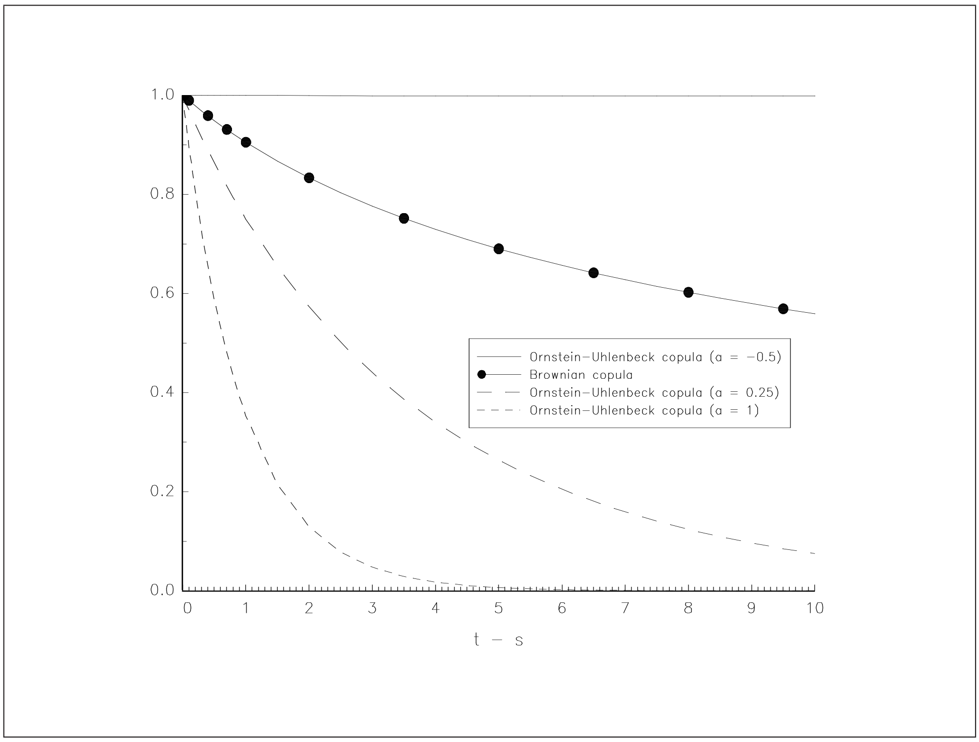

Figure 29: Spearman's rho ($s = 5$

4.3.1 Loss aggregation and Value-at-Risk analysis

In this paragraph, we are going to show how copulas could be used to aggregate loss distributions and to compute Value-at-Risk.

4.3.1.1 The discrete case

The individual risk model, which is intensively used in insurance, has been considered in further details by Wang [1999] and Marceau, Cossette, Gaillardetz and Rioux [1999]. Let X_n and X be the n^{th} risk and the aggregate loss. X_n is defined as follows

$$X_n = \begin{cases} A_n & \text{if} \quad B_n = 1 \\ 0 & \text{if} \quad B_n = 0 \end{cases} \tag{133}$$

where B_n is a Bernoulli random variable with parameter p_n. Marceau, Cossette, Gaillardetz and Rioux [1999] introduce the dependence for the joint distribution of the random variables B_n. Let $\mathbf{F}$ be the corresponding cumulative distribution function. We have

$$\mathbf{F}(e_1, \ldots, e_N) = \mathbf{C}\left(\mathbf{B}_{p_1}(e_1), \ldots, \mathbf{B}_{p_N}(e_N)\right) \tag{134}$$

As shown by Marceau, Cossette, Gaillardetz and Rioux [1999], the moment generating function (*mgf*) of $(X_1, \ldots, X_N)$ is

$$\mathcal{M}_{X_1,\ldots,X_N}(t_1, \ldots, t_N) = \sum_{e_1,\ldots,e_N \in \{0,1\}} c(e_1, \ldots, e_N) \prod_{n=1}^{N} \left[\mathcal{M}_{A_n}(t_n)\right]^{e_n} \tag{135}$$

where $\mathcal{M}_{A_n}(t) = \mathbb{E}\left[e^{tA_n}\right]$ represents the *mgf* of the random variable A_n. The distribution $\mathbf{G}$ of $X = \sum X_n$ is then obtained by inverting $\mathcal{M}_{X_1,\ldots,X_N}(t, \ldots, t)$ (because $\mathcal{M}_{X_1,\ldots,X_N}(t, \ldots, t)$ is the *mgf* of the sum — see Wang [1999]).

4.3.1.2 The continuous case

α) **Tractable copulas for high dimensions.**

Computational aspects is one of the main topic from an <u>industrial</u> point of view. It concerns the copula parameters estimation problem and the simulation issue. If the two problems can not be **easily** solved for high dimensions for a given copula, **the copula is not tractable to compute the Value-at-Risk**. Klugman and Parsa [2000] use for example the Frank copula to fit bivariate loss distributions. One of the reason is that they perform median regression which is very simple with this copula. However, **the Frank copula is not a good candidate for our problem**, because the estimation for higher dimensions is computationally difficult.

We have previously seen that the estimation of the ρ parameter of the gaussian copula is very easy with the IFM or CML algorithm (see remark 14 of the page 26). Moreover, simulations based on the gaussian copula are straighforward. **The gaussian copula is also a good candidate for our problem.**

For the Student copula, it is not possible to obtain an analytic expression of $\hat{\rho}_{\mathrm{ML}}$. However, we could derive an efficient algorithm which does not require optimization. Moreover, like the gaussian copula, simulations are straightforward. **The Student copula is then another good candidate for our problem.**

Proposition 23 *In the case of the Student copula, the ρ matrix may be estimated using the following algorithm:*

1. *Let $\hat{\rho}_0$ be the ML estimate of the ρ matrix for the gaussian copula;*

2. *$\hat{\rho}_{m+1}$ is obtained using the following equation*[25]

$$\hat{\rho}_{m+1} = \frac{1}{T}\left(\frac{\nu+N}{\nu}\right)\sum_{t=1}^{T}\frac{\varsigma_t^{\top}\varsigma_t}{1+\frac{1}{\nu}\varsigma_t^{\top}\hat{\rho}_m^{-1}\varsigma_t} \tag{139}$$

3. *Repeat the second step until convergence*[26] *— $\hat{\rho}_{m+1} = \hat{\rho}_m \; (:= \hat{\rho}_\infty)$.*

4. *The CML (or IFM) estimate of the ρ matrix for the Student copula is $\hat{\rho}_{\mathrm{CML}} = \hat{\rho}_\infty$.*

Remark 24 *We have now two copula functions which are tractable. Moreover, these copulas could be used in conjunction with any marginal distributions.* ***Many*** *multivariate distributions could then be used to compute the value-at-risk of a portfolio.*

Remark 25 *Let $\mathbf{P}(t)$ be the price vector of the assets at time t and $\mathbf{a}$ the portfolio. The one period value-at-risk with α confidence level is defined by $VaR = \mathbf{F}^{-1}(1-\alpha)$ with $\mathbf{F}$ the distribution of the random variate $\mathbf{a}^{\top}(\mathbf{P}(t+1)-\mathbf{P}(t))$. Generally, analytical value-at-risk is computed using the assumption of multivariate gaussian distribution. Using fat-tailed marginal distributions could then be done with the gaussian or the Student copula. Moreover, in some financial markets, there are some restrictions (for example arbitrage conditions in the forward market of the petroleum products). The one period value-at-risk must satisfy these restrictions. One possibility is to write these constraints in the form $h(\mathbf{P}(t+1), \mathbf{P}(t)) \geq 0$. Taking into account of these restrictions could then be done by constraining some random variates to be positive. Because of the copula representation, this could be done without difficulties (we have just to choose some margin distributions with positive real numbers support).*

---

[25] Recall that the density of the Student copula is given by the expression (37), it comes that the log-likelihood is

$$\begin{aligned}\ell(\theta) &= T \quad \ln\Gamma \quad \frac{\nu+N}{2} \quad -\Gamma \quad \frac{\nu}{2} \quad -NT \quad \Gamma \quad \frac{\nu+1}{2} \quad -\Gamma \quad \frac{\nu}{2} \\ &\quad -\frac{T}{2}\ln|\rho| - \quad \frac{\nu+N}{2} \quad \sum_{t=1}^{T}\ln \quad 1+\frac{1}{\nu}\varsigma_t^{\top}\rho^{-1}\varsigma_t \quad + \quad \frac{\nu+1}{2} \quad \sum_{t=1}^{T}\sum_{n=1}^{N}\ln \quad 1+\frac{\varsigma_n^2}{\nu}\end{aligned} \tag{136}$$

We then concentrate the log-likelihood

$$\frac{\partial\,\ell(\theta)}{\partial\,\rho^{-1}} = -\frac{T}{2}\rho - \quad \frac{\nu+N}{2} \quad \sum_{t=1}^{T}\frac{\frac{1}{\nu}\varsigma_t^{\top}\varsigma_t}{1+\frac{1}{\nu}\varsigma_t^{\top}\rho^{-1}\varsigma_t} \tag{137}$$

It comes also that the ML estimate must satisfy the following non-linear matrix equation

$$\hat{\rho}_{\mathrm{ML}} = \frac{1}{T} \quad \frac{\nu+N}{\nu} \quad \sum_{t=1}^{T}\frac{\varsigma_t^{\top}\varsigma_t}{1+\frac{1}{\nu}\varsigma_t^{\top}\hat{\rho}_{\mathrm{ML}}^{-1}\varsigma_t} \tag{138}$$

[26] For high dimension N, we may obtain a solution which is not a positive definite matrix because of computer roundoff errors. In this case, we suggest to use at each iteration a 'square root matrix decomposition' (Horn and Johnson [1991]) to adjust the correlation matrix

$$\hat{\rho}_{m+1} = \digamma^2 \tag{140}$$

with

$$\digamma = \digamma_1 + i\digamma_2 \tag{141}$$

where $\digamma_1$ and $\digamma_2$ are two definite positive matrices. Then, we obtain a new estimate $\hat{\rho}_{m+1}$ of the correlation matrix with

$$\hat{\rho}_{m+1} = \digamma_1^2 \tag{142}$$

The 'square root matrix decomposition' could be easily performed with a complex Schur decomposition approach (Golub and Van Loan [1989]). Note also that if the elements of the diagonal are note equal to one, we may rescale the matrix in the following way

$$(\hat{\rho}_{m+1})_{i,j} \longleftarrow \frac{(\hat{\rho}_{m+1})_{i,j}}{\sqrt{(\hat{\rho}_{m+1})_{i,i}^2}\times\sqrt{(\hat{\rho}_{m+1})_{j,j}^2}} \tag{143}$$

β) **A first illustration with gaussian margins.**

We remind that the correlation matrix estimated with the CML method for the gaussian copula is given by table 4. In the case of the Student copula, the estimates are different — see table 5. We now compute the *economic capital* measure for different portfolios. In the following table, we indicate the composition of the portfolios — a negative number corresponds to a short position.

	AL	AL-15	CU	NI	PB
P_1	1	1	1	1	1
P_2	-1	-1	-1	-1	-1
P_3	2	1	-3	4	5

	AL	AL-15	CU	NI	PB
AL	1.000	0.850	0.492	0.386	0.354
AL-15		1.000	0.429	0.346	0.316
CU			1.000	0.409	0.359
NI				1.000	0.333
PB					1.000

Table 4: Correlation matrix $\hat{\rho}_{\text{CML}}$ of the gaussian copula for the LME data

$\nu = 1$	AL	AL-15	CU	NI	PB
AL	1.000	0.820	0.326	0.254	0.189
AL-15		1.000	0.271	0.223	0.164
CU			1.000	0.267	0.219
NI				1.000	0.195
PB					1.000

$\nu = 2$	AL	AL-15	CU	NI	PB
AL	1.000	0.875	0.429	0.351	0.282
AL-15		1.000	0.371	0.314	0.253
CU			1.000	0.364	0.302
NI				1.000	0.278
PB					1.000

$\nu = 3$	AL	AL-15	CU	NI	PB
AL	1.000	0.888	0.466	0.387	0.321
AL-15		1.000	0.410	0.349	0.291
CU			1.000	0.399	0.336
NI				1.000	0.312
PB					1.000

$\nu = 5$	AL	AL-15	CU	NI	PB
AL	1.000	0.893	0.493	0.410	0.350
AL-15		1.000	0.437	0.373	0.320
CU			1.000	0.423	0.362
NI				1.000	0.337
PB					1.000

$\nu = 25$	AL	AL-15	CU	NI	PB
AL	1.000	0.878	0.503	0.408	0.364
AL-15		1.000	0.446	0.370	0.331
CU			1.000	0.425	0.371
NI				1.000	0.345
PB					1.000

$\nu = 100$	AL	AL-15	CU	NI	PB
AL	1.000	0.861	0.496	0.394	0.358
AL-15		1.000	0.436	0.354	0.322
CU			1.000	0.415	0.363
NI				1.000	0.337
PB					1.000

Table 5: Correlation matrix $\hat{\rho}_{\text{CML}}$ of the Student copula for the LME data

We assume that the *economic capital* measure is given by the one day *value-at-risk* of the portfolio[27]. In practice, the *VaR* with a 99% confidence level is estimated using an historical approach, because the analytical approach based on the multinormal distribution leads to many exceptions in the backtesting procedure. Nevertheless, the actual databases are too small to compute the historical *VaR* with higher confidence level. This is a key point because the *economic capital allocation* projects are based on higher confidence level. To get an idea, the classical confidence levels by ratings are reported below:

rating	BBB	A	AA	AAA
α confidence level	99.75%	99.9%	99.95%	99.97%

[27]The actual prices are set to 100.

We have computed the *VaR* by assuming that the margins are gaussian but with different copula functions. The results are sumarized by tables 6 and 7. We remark that the *economic capital* with a 99.9% confidence level is lower with the gaussian copula than with the student copula[28]. **It indicates that the dependence structure — or the copula function — has a great influence on the *value-at-risk* computation.**

	90%	95%	99%	99.5%	99.9%
P_1	7.265	9.297	13.15	14.59	17.54
P_2	7.263	9.309	13.17	14.59	17.65
P_3	13.92	17.92	25.24	27.98	33.40

Table 6: Economic capital measure with gaussian copula

$\nu = 1$	90%	95%	99%	99.5%	99.9%
P_1	5.696	7.977	13.22	15.46	20.16
P_2	5.709	7.980	13.19	15.46	20.15
P_3	13.43	19.34	34.32	40.65	55.12

$\nu = 2$	90%	95%	99%	99.5%	99.9%
P_1	6.476	8.751	13.79	15.90	20.26
P_2	6.470	8.753	13.70	15.77	20.12
P_3	13.20	18.27	30.53	36.00	49.47

$\nu = 3$	90%	95%	99%	99.5%	99.9%
P_1	6.834	9.077	13.85	15.73	20.31
P_2	6.812	9.082	13.84	15.83	20.15
P_3	13.18	17.86	28.69	33.56	45.01

$\nu = 5$	90%	95%	99%	99.5%	99.9%
P_1	7.081	9.283	13.88	15.68	19.58
P_2	7.109	9.351	13.94	15.72	19.57
P_3	13.31	17.63	27.18	31.18	40.78

$\nu = 25$	90%	95%	99%	99.5%	99.9%
P_1	7.318	9.453	13.56	15.10	18.18
P_2	7.308	9.401	13.37	14.94	18.41
P_3	13.64	17.60	25.38	28.36	34.90

$\nu = 100$	90%	95%	99%	99.5%	99.9%
P_1	7.293	9.387	13.27	14.79	17.56
P_2	7.297	9.427	13.42	14.78	17.56
P_3	13.85	17.81	25.32	28.11	34.07

Table 7: Economic capital measure with Student copula

γ) **A second illustration with fat-tailed margins.**

We consider the case of fat-tailed margins, which is a more realistic assumption in finance. We assume that the dependence structure is given by a Student copula with 2 degrees of freedom. In the previous paragraph, the margins of the standardized asset returns were gaussians. We could now suppose that the distribution of the standardized returns for one asset is a Student with ν degrees of freedom — the other distributions remains gaussians. For the AL asset, we then obtain the left quadrants of figure 30. The middle and right quadrants correspond respectively to the CU and PB assets. We remark also the great influence of the fat tails on the value-at-risk for high values of the confidence level.

More realistic margins could of course be used to describe the asset returns (see paragraph 4.2). We consider for example the generalized hyperbolic distribution (Eberlein and Keller [1995], Eberlein [1999], Prause [1999]). The corresponding density function is given by

$$\begin{aligned} f(x) &= a(\lambda,\alpha,\beta,\delta)\left(\delta^2+(x-\mu)^2\right)^{\frac{1}{4}(2\lambda-1)} \\ &\quad \times \mathbf{K}_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)\exp\left(\beta(x-\mu)\right) \end{aligned} \tag{144}$$

where $\mathbf{K}$ denotes the modified Bessel function of the third kind and

$$a(\lambda,\alpha,\beta,\delta) = \frac{\left(\alpha^2-\beta^2\right)^{\frac{1}{2}\lambda}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}\mathbf{K}_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \tag{145}$$

[28]To understand this result, remember that 'the extremes are correlated' in the case of the Student copula.

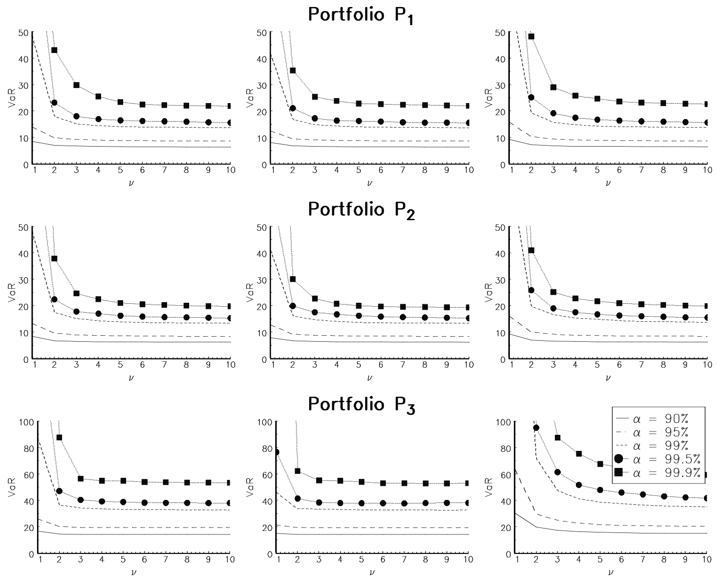

Figure 30: Value-at-risk with Student margins

In a recent paper, EBERLEIN [2000] applies the hyperbolic model to market risk management. Nevertheless, he considers only the univariate case. The multivariate case is treated in PRAUSE [1999]. The distribution of the multivariate generalized hyperbolic distribution has the following form

$$\begin{aligned} f(\mathbf{x}) &= A(\lambda,\alpha,\beta,\delta,\rho)\left(\delta^2+(\mathbf{x}-\mu)^\top\rho^{-1}(\mathbf{x}-\mu)\right)^{\frac{1}{4}(2\lambda-N)} \\ &\quad \times \mathbf{K}_{\lambda-\frac{N}{2}}\left(\alpha\sqrt{\delta^2+(\mathbf{x}-\mu)^\top\rho^{-1}(\mathbf{x}-\mu)}\right)\exp\left(\beta^\top(\mathbf{x}-\mu)\right) \end{aligned} \tag{146}$$

where

$$A(\lambda,\alpha,\beta,\delta,\rho)=\frac{\left(\alpha^2-\beta^\top\rho\beta\right)^{\frac{1}{2}\lambda}}{(2\pi)^{\frac{N}{2}}\alpha^{\lambda-\frac{N}{2}}\delta^\lambda\mathbf{K}_\lambda\left(\delta\sqrt{\alpha^2-\beta^\top\rho\beta}\right)} \tag{147}$$

However, this distribution leads to big computational problems for estimation and simulation steps. This is avoided if we build a multivariate distribution with univariate hyperbolic distributions and a copula. With gaussian copula, the density of the distribution is

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{|\rho|^{\frac{1}{2}}}A(\lambda,\alpha,\beta,\delta,\rho)\left[\prod_{n=1}^{N}\left(\delta_n^2+(x_n-\mu_n)^2\right)^{\frac{1}{4}(2\lambda_n-1)}\right] \\ &\quad \times\left[\prod_{n=1}^{N}\mathbf{K}_{\lambda_n-\frac{1}{2}}\left(\alpha_n\sqrt{\delta_n^2+(x_n-\mu_n)^2}\right)\right]\exp\left(\beta^\top(\mathbf{x}-\mu)-\frac{1}{2}\varsigma^\top\left(\rho^{-1}-\mathbb{I}\right)\varsigma\right) \end{aligned} \tag{148}$$

where

$$A(\lambda,\alpha,\beta,\delta,\rho)=\frac{1}{(2\pi)^{\frac{N}{2}}}\prod_{n=1}^{N}\frac{\left(\alpha_n^2-\beta_n^2\right)^{\frac{1}{2}\lambda_n}}{\alpha_n^{\lambda_n-\frac{1}{2}}\delta_n^{\lambda_n}\mathbf{K}_{\lambda_n}\left(\delta_n\sqrt{\alpha_n^2-\beta_n^2}\right)} \tag{149}$$

and $\varsigma = (\varsigma_1, \ldots, \varsigma_N)^\top$, $\varsigma_n = \mathbf{\Phi}^{-1}(\mathbf{F}_n(x_n))$ and $\mathbf{F}_n$ the univariate GH distribution. Note that this distribution has $3(N-1)$ more parameters by comparison with the multivariate generalized hyperbolic distribution.

4.3.2 Multivariate extreme values and market risk

4.3.2.1 Topics on extreme value theory

α) **The univariate case.**

Let us first consider m independent random variables $X_1, \ldots, X_k, \ldots, X_m$ with the same probability function $\mathbf{F}$. The distribution of the extremes $\chi_m^+ = \left(\bigwedge_{k=1}^m X_k\right)$ is also given by Fisher-Tippet theorem (Embrechts, Klüppelberg and Mikosch (1997)):

Theorem 26 *If there exist some constants a_m and b_m and a non-degenerate limit distribution $\mathbf{G}$ such that*

$$\lim_{m\to\infty} \Pr\left\{\frac{\chi_m^+ - b_m}{a_m} \le x\right\} = \mathbf{G}(x) \qquad \forall x \in \mathbb{R} \tag{150}$$

then $\mathbf{G}$ is one of the following distribution:

(Fréchet)	$\mathbf{G}(x) = \mathbf{\Upsilon}_\alpha(x) = \begin{cases} 0 \\ \exp(-x^{-\alpha}) \end{cases}$	$\begin{matrix} x \le 0 \\ x > 0 \end{matrix}$	$g(x) = \begin{cases} 0 \\ \alpha x^{-(1+\alpha)} \exp(-x^{-\alpha}) \end{cases}$
(Weibull)	$\mathbf{G}(x) = \mathbf{\Psi}_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha) \\ 1 \end{cases}$	$\begin{matrix} x \le 0 \\ x > 0 \end{matrix}$	$g(x) = \begin{cases} \alpha(-x)^{\alpha-1}\exp(-(-x)^\alpha) \\ 0 \end{cases}$
(Gumbel)	$\mathbf{G}(x) = \mathbf{\Lambda}(x) = \exp(-e^{-x})$	$x \in \mathbb{R}$	$g(x) = \exp(-x - e^{-x})$

In this case, we say that $\mathbf{F}$ belongs to the maximum domain of attraction[29] *of $\mathbf{G}$ — $\mathbf{F} \in \mathrm{MDA}(\mathbf{G})$.*

The relation (150) can be written as follows:

$$\mathbf{F}^m(a_m x + b_m) \longrightarrow \mathbf{G}(x) \tag{151}$$

We remark that if we could specify $\{a_m\}$ and $\{b_m\}$ such that

$$\lim_{m\to\infty} m\left[1 - \mathbf{F}(a_m x + b_m)\right] = -\ln \mathbf{G}(x) \tag{152}$$

then $\mathbf{F} \in \mathrm{MDA}(\mathbf{G})$.

β) **The multivariate case.**

The theory of multivariate extremes is presented in Galambos [1987] and Resnick [1987]. In the multidimensional case, we are interested in characterizing the distribution of the extremes χ_m^+:

$$\chi_m^+ = \left(\chi_{1,m}^+, \ldots, \chi_{n,m}^+, \ldots, \chi_{N,m}^+\right) = \left(\bigwedge_{k=1}^m X_{1,k}, \ldots, \bigwedge_{k=1}^m X_{n,k}, \ldots, \bigwedge_{k=1}^m X_{N,k}\right) \tag{153}$$

Like the univarariate case, we study the limit distribution of the *normalized* extremes:

$$\begin{aligned} \lim_{m\to\infty} \Pr\left\{\frac{\chi_{1,m}^+ - b_{1,m}}{a_{1,m}} \le x_1, \ldots, \frac{\chi_{n,m}^+ - b_{n,m}}{a_{n,m}} \le x_n, \ldots, \frac{\chi_{N,m}^+ - b_{N,m}}{a_{N,m}} \le x_N\right\} &= \mathbf{G}(x) \\ \forall (x_1, \ldots, x_n, \ldots, x_N) &\in \mathbb{R}^N \end{aligned} \tag{154}$$

[29] This concept is equivalent to the concept of domain of attraction of sums, but applied to maxima.

Proposition 27 (Resnick (1987), page 264) *The class of multivariate extreme value distributions is the class of max-stable distribution functions with nondegenerate marginals.*

Resnick [1987] does not use the copula concept to specify MEV distributions. However, he transforms the distribution $\mathbf{G}$ into another distribution $\mathbf{G}_*$ such that $\mathbf{G}_*$ has marginals $\boldsymbol{\Upsilon}$:

> *To characterize max-stable distributions with nondegenerate marginals, it is an enormous help to standardize the problem so that* $\mathbf{G}$ *has specified marginals.*

In fact, the distribution $\mathbf{G}_*$ of Resnick is a copula. We prefer to adopt the point of view presented in the chapter 6 of Joe [1997], which is easier to understand.

Theorem 28 *The class of multivariate extreme value distribution is the class of extreme copulas with nondegenerate marginals.*

We have previously noted that an extreme copula satisfy

$$\mathbf{C}\left(u_1^t,\ldots,u_n^t,\ldots,u_N^t\right)=\mathbf{C}^t\left(u_1,\ldots,u_n,\ldots,u_N\right) \qquad \forall\, t>0 \tag{155}$$

This relation is explained in details in Joe [1997]. The idea is the following. Suppose that $\mathbf{C}$ is an extreme copula and that the marginals are univariate extreme distributions. In this case, $\mathbf{F}\left(x_1,\ldots,x_n,\ldots,x_N\right)=\mathbf{C}\left(\mathbf{F}_1\left(x_1\right),\ldots,\mathbf{F}_n\left(x_n\right),\ldots,\mathbf{F}_N\left(x_N\right)\right)$ is a MEV distribution. From univariate extreme value theory, $\mathbf{C}^k\left(\mathbf{F}_1\left(x_1\right),\ldots,\mathbf{F}_n\left(x_n\right),\ldots,\mathbf{F}_N\left(x_N\right)\right)$ and $\mathbf{C}\left(\mathbf{F}_1^k\left(x_1\right),\ldots,\mathbf{F}_n^k\left(x_n\right),\ldots,\mathbf{F}_N^k\left(x_N\right)\right)$ must have the same limit distribution for each integer k. We have

$$\mathbf{C}\left(u_1^k,\ldots,u_n^k,\ldots,u_N^k\right)=\mathbf{C}^k\left(u_1,\ldots,u_n,\ldots,u_N\right) \qquad \forall\, k\in\mathbb{N}$$

This can be extended to the real case. In this case, extreme copulas generate *max-infinitely divisible* distributions.

We now follow Joe [1987] pages 174-175 in order to obtain the Pickands representation of MEV distributions. Let $\mathbf{D}$ be a multivariate distribution with unit exponential survival margins and $\mathbf{C}$ an extreme copula. Using the relation

$$\begin{aligned}\mathbf{C}\left(u_1,\ldots,u_n,\ldots,u_N\right) &= \mathbf{C}\left(\mathrm{e}^{-\tilde{u}_1},\ldots,\mathrm{e}^{-\tilde{u}_n},\ldots,\mathrm{e}^{-\tilde{u}_N}\right)\\ &= \mathbf{D}\left(\tilde{u}_1,\ldots,\tilde{u}_n,\ldots,\tilde{u}_N\right)\end{aligned} \tag{156}$$

we have

$$\mathbf{D}^t\left(\tilde{\mathbf{u}}\right)=\mathbf{D}\left(t\tilde{\mathbf{u}}\right) \tag{157}$$

and then $\mathbf{D}$ is a *min-stable multivariate exponential* (MSMVE) distribution.

Theorem 29 (Pickands (1981) representation of MSMVE distributions) *Let* $\mathbf{D}\left(\tilde{\mathbf{u}}\right)$ *be a survival function with exponential margins.* $\mathbf{D}$ *satisfies*

$$-\ln\mathbf{D}\left(t\tilde{\mathbf{u}}\right)=-t\ln\mathbf{D}\left(\tilde{\mathbf{u}}\right) \qquad \forall\, t>0 \tag{158}$$

iff the representation of $\mathbf{D}$ *is*

$$-\ln\mathbf{D}\left(\tilde{\mathbf{u}}\right)=\int\cdots\int_{\mathcal{S}_N}\max_{1\leq n\leq N}\left(q_n\tilde{u}_n\right)\,\mathrm{d}S\left(\mathbf{q}\right) \qquad \tilde{\mathbf{u}}\geq\mathbf{0} \tag{159}$$

where $\mathcal{S}_N$ *is the* N*-dimensional unit simplex and* S *a finite measure on* $\mathcal{S}_N$*.*

This is the formulation given by JOE [1997]. Note that it is similar to the proposition 5.11 of RESNICK [1987], although the author does not use copulas. Sometimes, the Pickands representation is presented using a *dependence function* $B(\mathbf{w})$ defined by

$$\begin{aligned} \mathbf{D}(\tilde{\mathbf{u}}) &= \exp\left[-\left(\sum_{n=1}^{N}\tilde{u}_n\right)B(w_1,\ldots,w_n,\ldots,w_N)\right] && (160)\\ B(\mathbf{w}) &= \int\cdots\int_{\mathcal{S}_N}\max_{1\leq n\leq N}(q_n w_n)\,\mathrm{d}S(\mathbf{q}) && (161)\end{aligned}$$

with $w_n = \tilde{u}_n/\sum_1^N\tilde{u}_n$. B is a convex function and

$$\max(w_1,\ldots,w_n,\ldots,w_N)\leq B(w_1,\ldots,w_n,\ldots,w_N)\leq 1 \tag{162}$$

This is the formulation of TAWN [1990]. It comes necessarily that an extreme copula verifies

$$\mathbf{C}^{\perp}\prec\mathbf{C}\prec\mathbf{C}^{+} \tag{163}$$

Let $\mathbf{F}$ be a N-variate distribution with margins $\mathbf{F}_1,\ldots,\mathbf{F}_N$ and an associated copula $\mathbf{C}$. We assume that the limit distribution exists and so $\mathbf{F}$ belongs to the maximum domain of attraction of a distribution $\mathbf{G}$. Copulas will then help us to solve the problem of the characterization of $\mathbf{G}$. Let us denote $\mathbf{G}_1,\ldots,\mathbf{G}_N$ the margins of $\mathbf{G}$ and $\mathbf{C}_\star$ its corresponding copula.

Theorem 30 $\mathbf{F}\in\mathrm{MDA}(\mathbf{G})$ *iff*

1. $\mathbf{F}_n\in\mathrm{MDA}(\mathbf{G}_n)$ *for all* $n=1,\ldots,N$*;*
2. $\mathbf{C}\in\mathrm{MDA}(\mathbf{C}_\star)$.

Remark 31 $\mathbf{G}_n$ *is necessary one of the three univariate extreme distributions and* $\mathbf{C}_\star$ *is an extreme copula.* $\mathbf{F}_n\in\mathrm{MDA}(\mathbf{G}_n)$ *could be checked with condition (152). The normalized coefficients* $\{a_{n,m}\}$ *and* $\{b_{n,m}\}$ *only depend on the marginals.* $\mathbf{C}\in\mathrm{MDA}(\mathbf{C}_\star)$ *if* $\mathbf{C}$ *satisfies*

$$\lim_{m\to\infty}\mathbf{C}^m\left(u_1^{1/m},\ldots,u_n^{1/m},\ldots,u_N^{1/m}\right)=\mathbf{C}_\star(u_1,\ldots,u_n,\ldots,u_N) \tag{164}$$

The following theorem is due to ABDOUS, GHOUDI and KHOUDRAJI [1999]:

Theorem 32 $\mathbf{C}\in\mathrm{MDA}(\mathbf{C}_\star)$ *iff*

$$\lim_{u\to 0}\frac{1-\mathbf{C}\left((1-u)^{w_1},\ldots,(1-u)^{w_n},\ldots,(1-u)^{w_N}\right)}{u}=B(w_1,\ldots,w_n,\ldots,w_N) \tag{165}$$

This theorem is important because MEV distributions are generally specified *via* the dependence function.

γ) The bivariate case.

In the bivariate case, the theory of extremes is easier because convexity and property (162) become necessary and sufficient conditions for (159) — TAWN [1988]. We have

$$\begin{aligned}\mathbf{C}(u_1,u_2) &= \mathbf{D}(\tilde{u}_1,\tilde{u}_2)\\ &= \exp\left[-(\tilde{u}_1+\tilde{u}_2)B\left(\frac{\tilde{u}_1}{\tilde{u}_1+\tilde{u}_2},\frac{\tilde{u}_2}{\tilde{u}_1+\tilde{u}_2}\right)\right]\\ &= \exp\left[\ln(u_1u_2)B\left(\frac{\ln u_1}{\ln(u_1u_2)},\frac{\ln u_2}{\ln(u_1u_2)}\right)\right]\\ &= \exp\left[\ln(u_1u_2)A\left(\frac{\ln u_1}{\ln(u_1u_2)}\right)\right] && (166)\end{aligned}$$

with $A(w) = B(w, 1-w)$. Of course, A is convex with $A(0) = A(1) = 1$ and verifies $\max(w, 1-w) \le A(w) \le 1$.

We consider the Gumbel copula defined page 18. We have also $-\ln \mathbf{D}(\tilde{\mathbf{u}}) = (\tilde{u}_1^\alpha + \tilde{u}_2^\alpha)^{\frac{1}{\alpha}}$, $B(w_1, w_2) = (\tilde{u}_1^\alpha + \tilde{u}_2^\alpha)^{\frac{1}{\alpha}} / (\tilde{u}_1 + \tilde{u}_2) = (w_1^\alpha + w_2^\alpha)^{\frac{1}{\alpha}}$ and $A(w) = [w^\alpha + (1-w)^\alpha]^{\frac{1}{\alpha}}$. The specification of the extreme copula using the dependence measure $A(w)$ simplifies calculus. For example, we have (CAPÉRAÀ, FOUGÈRES and GENEST [1997]):

$$\begin{aligned} \tau &= 4\int_{\mathbf{I}} w(1-w)\,\mathrm{d}\ln A(w) \\ \varrho &= 12\int_{\mathbf{I}} \frac{1}{[A(w)+1]^2}\,\mathrm{d}w - 3 \end{aligned} \tag{167}$$

When the extremes are independent, we have $A(w) = 1$ ($\alpha = 1$ for the Gumbel copula), and so $\tau = \varrho = 0$. The perfect dependent case corresponds to $A(w) = \max(w, 1-w)$ that leads to the upper Fréchet copula — $\mathbf{C}(u_1, u_2) = \exp\left[\ln(u_1 u_2)\max\left(\frac{\ln u_1}{\ln(u_1 u_2)}, \frac{\ln u_2}{\ln(u_1 u_2)}\right)\right] = \min(u_1, u_2)$. Equation (165) can be used to verify the independence of extremes. For example, if we consider the Kimeldorf-Sampson copula $\mathbf{C}(u_1, u_2) = \left(u_1^{-\alpha} + u_2^{-\alpha} - 1\right)^{-\frac{1}{\alpha}}$ with $\alpha \ge 0$, we have $\lim_{u\to 0} \frac{1-\mathbf{C}\left((1-u)^w, (1-u)^{1-w}\right)}{u} = \lim_{u\to 0} \frac{1-\left((1-u)^{-\alpha w} + (1-u)^{-\alpha(1-w)} - 1\right)^{-\frac{1}{\alpha}}}{u} = \lim_{u\to 0} \frac{1-(1+\alpha u + \mathsf{o}(u))^{-\frac{1}{\alpha}}}{u} = \lim_{u\to 0} \frac{u + \mathsf{o}(u)}{u} = 1$. $A(w)$ then equals to 1, and so Kimeldorf-Sampson copulas belong to the domain of attraction of $\mathbf{C}^\perp$.

δ) **Parametric family of extreme copulas.**

In the next paragraph, we will discuss about non-parametric extreme copulas. However, parametric copulas play a central role, because of Monte Carlo issues for finance. The following table contains the most known extreme copulas and their dependence function (GHOUDI, KHOUDRAJI and RIVEST [1998]):

Family	α	$\mathbf{C}(u_1, u_2)$	$A(w)$
$\mathbf{C}^\perp$		$u_1 u_2$	1
Gumbel	$[1, \infty)$	$\exp\left[-(\tilde{u}_1^\alpha + \tilde{u}_2^\alpha)^{\frac{1}{\alpha}}\right]$	$[w^\alpha + (1-w)^\alpha]^{\frac{1}{\alpha}}$
Gumbel II	$[0, 1]$	$u_1 u_2 \exp\left[\alpha \frac{\tilde{u}_1 \tilde{u}_2}{\tilde{u}_1 + \tilde{u}_2}\right]$	$\alpha w^2 - \alpha w + 1$
Galambos	$[0, \infty)$	$u_1 u_2 \exp\left[\left(\tilde{u}_1^{-\alpha} + \tilde{u}_2^{-\alpha}\right)^{-\frac{1}{\alpha}}\right]$	$1 - \left[w^{-\alpha} + (1-w)^{-\alpha}\right]^{-\frac{1}{\alpha}}$
Hüsler-Reiss	$[0, \infty)$	$\exp[-\tilde{u}_1 \vartheta(u_1, u_2; \alpha) - \tilde{u}_2 \vartheta(u_2, u_1; \alpha)]$	$w\xi(w; \alpha) + (1-w)\xi(1-w; \alpha)$
Marshall-Olkin	$[0, 1]^2$	$u_1^{1-\alpha_1} u_2^{1-\alpha_2} \min(u_1^{\alpha_1}, u_2^{\alpha_2})$	$\max(1 - \alpha_1 w, 1 - \alpha_2(1-w))$
$\mathbf{C}^+$		$\min(u_1, u_2)$	$\max(w, 1-w)$

with $\vartheta(u_1, u_2; \alpha) = \Phi\left(\frac{1}{\alpha} + \frac{1}{2}\alpha \ln \frac{\ln u_1}{\ln u_2}\right)$ and $\xi(w; \alpha) = \Phi\left(\frac{1}{\alpha} + \frac{1}{2}\alpha \ln \frac{\ln w}{\ln(1-w)}\right)$. These dependence functions are plotted in figure 31.

The Gumbel copulas have been studied by TAWN [1988]. We note that these copulas are *symmetric*[30]. It implies that the random variables are exchangeable, which could be a discutable assumption. To obtain more flexible parametric copulas, we could use the *asymmetrization* technique (GENEST, GHOUDI and RIVEST [1998]). Let A_1 and A_2 be two dependence functions and $(p_1, p_2) \in [0, 1]^2$. Then, the following formula defines a new dependence function A:

$$\begin{aligned} A(w) &= (p_1 w + p_2(1-w))\,A_1\left(\frac{p_1 w}{p_1 w + p_2(1-w)}\right) + \\ &\quad ((1-p_1)w + (1-p_2)(1-w))\,A_2\left(\frac{(1-p_1)w}{(1-p_1)w + (1-p_2)(1-w)}\right) \end{aligned} \tag{168}$$

[30] that is $\mathbf{C}(u_1, u_2) = \mathbf{C}(u_2, u_1)$.

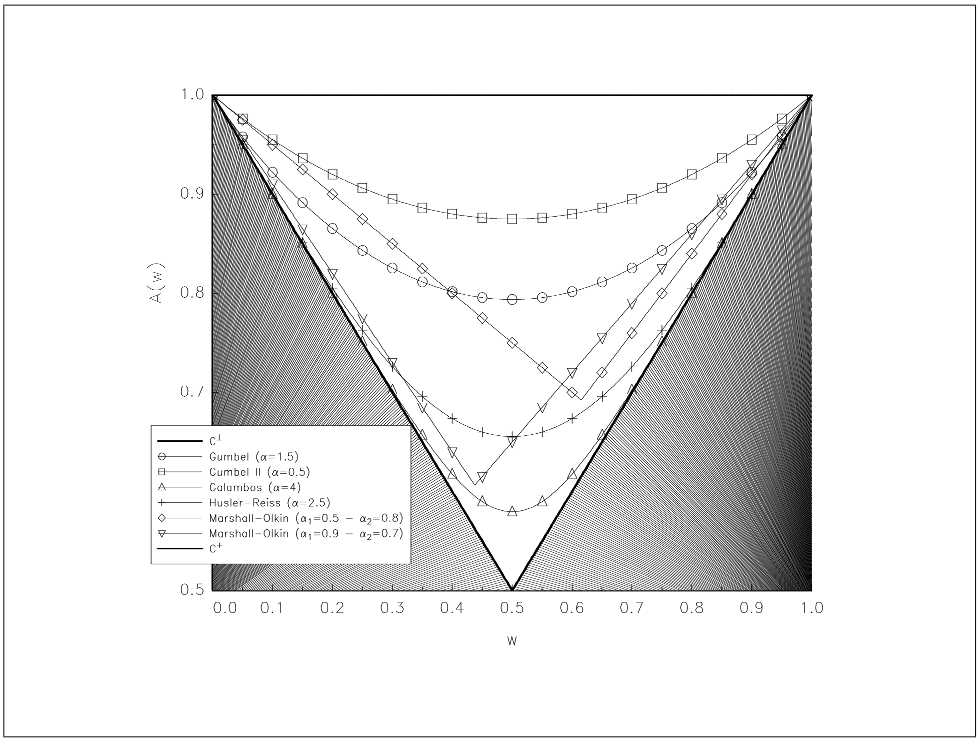

Figure 31: Dependence functions $A(w)$

Note that for $p_1 = p_2 = p$, we obtain $A(w) = pA_1(w) + (1-p)A_2(w)$ (TAWN [1988]). In this case, the copula defined by $A(w)$ remains symmetric. With A_1 and A_2 the dependence functions of the Gumbel and product copulas, we obtain

$$\begin{aligned} A(w) &= (p_1 w + p_2(1-w)) \frac{\left[p_1^\alpha w^\alpha + p_2^\alpha (1-w)^\alpha\right]^{\frac{1}{\alpha}}}{(p_1 w + p_2(1-w))} + \\ &\quad ((1-p_1)w + (1-p_2)(1-w)) \\ &= \left[p_1^\alpha w^\alpha + p_2^\alpha (1-w)^\alpha\right]^{\frac{1}{\alpha}} + (p_2 - p_1)w + (1-p_2) \end{aligned} \tag{169}$$

It comes that the corresponding copula is such that

$$\begin{aligned} \mathbf{C}(u_1, u_2) &= \exp\left[-\left(p_1^\alpha \tilde{u}_1^\alpha + p_2^\alpha \tilde{u}_2^\alpha\right)^{\frac{1}{\alpha}} - (1-p_1)\tilde{u}_1 - (1-p_2)\tilde{u}_2\right] \\ &= \mathbf{C}^{\mathsf{G}}\left(u_1^{p_1}, u_2^{p_2}\right)\mathbf{C}^{\perp}\left(u_1^{1-p_1}, u_2^{1-p_2}\right) \end{aligned} \tag{170}$$

In the extreme value litterature, it corresponds to the *asymmetric logistic model*. Relation (170) can be generalized and it appears that the copula associated with (168) is

$$\mathbf{C}(u_1, u_2) = \mathbf{C}_1\left(u_1^{p_1}, u_2^{p_2}\right)\mathbf{C}_2\left(u_1^{1-p_1}, u_2^{1-p_2}\right) \tag{171}$$

For example, the Marshall-Olkin copula is a combination of the product copula and the upper Fréchet copula:

$$\mathbf{C}(u_1, u_2) = \mathbf{C}^{\perp}\left(u_1^{1-\alpha_1}, u_2^{1-\alpha_2}\right)\mathbf{C}^{+}\left(u_1^{\alpha_1}, u_2^{\alpha_2}\right) \tag{172}$$

We then verify that the dependence function is

$$\begin{aligned} A(w) &= (\alpha_1 w + \alpha_2 (1-w)) \max\left(\frac{\alpha_1 w}{(\alpha_1 w + \alpha_2 (1-w))}, \frac{\alpha_2 (1-w)}{(\alpha_1 w + \alpha_2 (1-w))}\right) \\ &\quad + ((1-\alpha_1) w + (1-\alpha_2)(1-w)) \\ &= \max(1-\alpha_1 w, 1-\alpha_2 (1-w)) \end{aligned} \tag{173}$$

and we remark that it is the limit of the *asymmetric logistic* copula as α tends to ∞.

We now consider the multivariate case. We start with the Gumbel copula. A natural generalization for $N \geq 3$ is given by

$$\mathbf{C}(u_1, \ldots, u_n, \ldots, u_N) = \exp\left[-\left(\tilde{u}_1^\alpha + \ldots + \tilde{u}_n^\alpha + \ldots + \tilde{u}_N^\alpha\right)^{\frac{1}{\alpha}}\right] \tag{174}$$

The corresponding dependence function is also $B(\mathbf{w}) = \left[\sum_1^N w_n^\alpha\right]^{\frac{1}{\alpha}}$. This first generalization comes from the definition of multivariate archimedean copulas. However, it is not very interesting because this multivariate extension has a single parameter. It follows that the tail index is the same for all the $\frac{N(N-1)}{2}$ bivariate margins. One possible extension is to use *compound* methods. For example, $\mathbf{C}(u_1, u_2, u_3) = \exp\left[-\left(\left(\tilde{u}_1^{\alpha_2} + \tilde{u}_2^{\alpha_2}\right)^{\frac{\alpha_1}{\alpha_2}} + \tilde{u}_3^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\right]$ is an extreme copula if $\alpha_2 > \alpha_1 \geq 1$. The dependence function is $B(\mathbf{w}) = \left(\left(w_1^{\alpha_2} + w_2^{\alpha_2}\right)^{\frac{\alpha_1}{\alpha_2}} + w_3^{\alpha_1}\right)^{\frac{1}{\alpha_1}}$, that is mentioned in Tawn [1990]. Nevertheless, the parameters are difficult to understand in this second generalization. A more *interpretable* copula is the family MM1 of Joe [1997]

$$B(\mathbf{w}) = \left[\sum_{i=1}^N \sum_{j=i+1}^N \left[\left(p_i w_i^\eta\right)^{\alpha_{i,j}} + \left(p_j w_j^\eta\right)^{\alpha_{i,j}}\right]^{\frac{1}{\alpha_{i,j}}} + \sum_{i=1}^N \nu_i p_i w_i^\eta\right]^{\frac{1}{\eta}} \tag{175}$$

where $p_i = (\nu_i + N - 1)^{-1}$. This copula comes from *mixtures of max-id distributions* (Joe and Hu [1996]). The parameters ν_i control the bivariate and multivariate asymmetries, $\alpha_{i,j}$ are the pairwise coefficients and η is the common parameter. The bivariate margins are

$$\mathbf{C}(u_1, u_2) = \exp\left(-\left[\left[\left(p_i \tilde{u}_i^\eta\right)^{\alpha_{i,j}} + \left(p_j \tilde{u}_j^\eta\right)^{\alpha_{i,j}}\right]^{\frac{1}{\alpha_{i,j}}} + \frac{(\nu_i + N - 2)}{(\nu_i + N - 1)} \tilde{u}_i^\eta + \frac{(\nu_j + N - 2)}{(\nu_j + N - 1)} \tilde{u}_i^\eta\right]^{\frac{1}{\eta}}\right) \tag{176}$$

and the associated tail dependence is $\lambda = 2 - \left[\left(p_i^{\alpha_{i,j}} + p_j^{\alpha_{i,j}}\right)^{\frac{1}{\alpha_{i,j}}} + \frac{(\nu_i+N-2)}{(\nu_i+N-1)} + \frac{(\nu_j+N-2)}{(\nu_j+N-1)}\right]^{\frac{1}{\eta}}$. However, the parametric form of both bivariate and multivariate copulas is *not well tractable*, and the same problem arises with the generalization of other bivariate copulas. As pointed by Embrechts, de Haan and Huang [2000], *it is important to stress at this point the fact that current multivariate extreme value theory, from an applied point of view, only allows for a treatment of fairly low-dimensional problems.*

4.3.2.2 Estimation methods

α) The non-parametric approach.

The two-dimensional non-parametric approach is considered into several papers (Pickands [1981], Deheuvels [1991], Capéraà, Fougères and Genest [1997], Abdous, Ghoudi and Khoudraji [2000]). These papers are generally based on the fact that the distribution of $Z = \frac{\ln U_1}{\ln U_1 + \ln U_2}$ satisfies $\mathbf{F}(z) = z + (1-z) A^{-1}(z) \partial_z A(z)$ where the joint distribution of U_1 and U_2 is given by the extreme copula $\mathbf{C}_\star(u_1, u_2) = \exp\left[\ln(u_1 u_2) A\left(\frac{\ln u_1}{\ln(u_1 u_2)}\right)\right]$ (Ghoudi, Khoudraji and Rivest [1998]). Because $A(0) = A(1) = 1$, it comes that $A(w) = \exp \int_0^w \frac{\mathbf{F}(z) - z}{1-z} \, \mathrm{d}z$

or $A(w) = \exp - \int_w^1 \frac{\mathbf{F}(z) - z}{1 - z} \, \mathrm{d}z$. We obtain also two non-parametric estimators of $A(w)$ by using the empirical distribution $\hat{\mathbf{F}}$ in place of the theoretical distribution $\mathbf{F}$. Using relationship (165), ABDOUS, GHOUDI and KHOUDRAJI [1999] consider another estimator directly based on the sample of the original variables (X_1, X_2), not on the sample of the maximum variables $\left(\chi_1^+, \chi_2^+\right)$.

β) The parametric approch.

In the form of componentwise maxima, inference with ML method is applied to the distribution $\mathbf{G}$

$$\mathbf{G}\left(\chi_1^+, \ldots, \chi_n^+, \ldots, \chi_N^+\right) = \mathbf{C}_\star\left(\mathbf{G}_1\left(\chi_1^+\right), \ldots, \mathbf{G}_n\left(\chi_n^+\right), \ldots, \mathbf{G}_N\left(\chi_N^+\right)\right) \tag{177}$$

where $\mathbf{C}_\star$ is an extreme copula and $\mathbf{G}_n$ a GEV distribution $\mathcal{GEV}(\mu, \sigma, \xi)$ defined by

$$\mathbf{G}(x) = \exp\left\{-\left[1 + \xi\left(\frac{x - \mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\} \tag{178}$$

defined on the support $\Delta = \left\{x : 1 + \xi\left(\frac{x-\mu}{\sigma}\right) > 0\right\}$. The three types of non-degenerate univariate distributions are then combined into a generalized extreme value family and we have the correspondances $\xi = \alpha^{-1} > 0$, $\xi = -\alpha^{-1} < 0$ and $\xi \longrightarrow 0$ for the Fréchet, Weibull and Gumbel distributions respectively. We note that the IFM or CML methods could be used to estimate the parameters of (177) in order to reduce computational time[31]. A very important point concerns the starting values for the estimation of the copula. They could be obtained by 'inverting' the upper tail dependence[32] (see CURRIE [1999]) or other concordance measures like Kendall's tau (GHOUDI, KHOUDRAJI and RIVEST [1998]).

Family	τ
$\mathbf{C}^\perp$	0
Gumbel	$1 - \alpha^{-1}$
Gumbel II	$8\alpha^{-\frac{1}{2}}(4 - \alpha)^{-\frac{1}{2}} \arctan\sqrt{\alpha(4 - \alpha)^{-1}} - 2$
Galambos	complicated form
Hüsler-Reiss	no analytical form
Marshall-Olkin	$\alpha_1\alpha_2(\alpha_1 - \alpha_1\alpha_2 + \alpha_2)^{-1}$
$\mathbf{C}^+$	1

γ) The point processes approch.

[31] In this case, the individual log-likelihood of the univariate margins is as follows

$$\ell\left[\chi_n^+; \theta\right] = -\ln\sigma - \frac{1+\xi}{\xi} \ln\left(1 + \xi\left(\frac{\chi_n^+ - \mu}{\sigma}\right)\right) - \left[1 + \xi\left(\frac{\chi_n^+ - \mu}{\sigma}\right)\right]^{-\frac{1}{\xi}} \tag{179}$$

and the score vector is

$$\nabla_\theta \, \ell\left[\chi_n^+; \theta\right] = \begin{bmatrix} \frac{1+\xi-\omega_{n,m}^{-\frac{1}{\xi}}}{\sigma\omega_{n,m}} \\ \frac{\left((1+\xi)-\omega_{n,m}^{-\frac{1}{\xi}}\right)(\chi_n^+-\mu)-\sigma\omega_{n,m}}{\sigma^2\omega_{n,m}} \\ \left(1 - \omega_{n,m}^{-\frac{1}{\xi}}\right)\left(\frac{1}{\xi^2}\ln\omega_{n,m} - \frac{(\chi_n^+-\mu)}{\xi\sigma\omega_{n,m}}\right) - \frac{(\chi_n^+-\mu)}{\sigma\omega_{n,m}} \end{bmatrix} \tag{180}$$

with $\omega_{n,m} = 1 + \xi\left(\frac{\chi_n^+-\mu}{\sigma}\right)$.

[32] λ could be estimated using the sample of componentwise maxima or directly using the entire sample, because the extreme value limit has the same upper tail dependence under some assumptions (theorem 6.8 of JOE [1997]).

Instead of using componentwise maxima, it is sometimes more efficient to work with higher frequency data. Point process characterization is also a natural way to perform the estimation (Coles and Tawn [1991], Joe [1997]). Let $\{(X_{1,t},\ldots,X_{n,t},\ldots,X_{N,t}), t=1,\ldots,T\}$ be a sample of length T with margins $\mathbf{F}_n$, $Y_{n,t}$ the Fréchet transform of the variate $X_{n,t}$ and $\mathbf{Y}_t$ the corresponding vector $(Y_{n,t})_1^N$. We consider the point process $\mathbf{N}_T = (\mathbf{Y}_1,\ldots,\mathbf{Y}_T)$ on $\mathbb{R}_+^N$. Under some hypothesis, it comes that $\mathbf{N}_T$ converges to a non-homogeneous Poisson process $\mathbf{N}$ with intensity measure Λ, which satisfies the homogeneity property $\Lambda([0,\mathbf{y}]^c) = t\Lambda([0,t\mathbf{y}]^c)$ with $[0,\mathbf{y}]^c = \mathbb{R}_+^N \setminus [0,\mathbf{y}]$. We have also $\Pr\left\{\frac{1}{T}\mathbf{Y}_t \notin [0,\mathbf{y}]^c\right\} \rightarrow \exp(-\Lambda([0,\mathbf{y}]^c))$. The log-likelihood is then

$$\ell(\theta) = -\Lambda([0,\mathbf{y}]^c) + \sum_{t=1}^{T} \mathbf{1}_{\left[\frac{1}{T}\mathbf{y}_t\in[0,\mathbf{y}]^c\right]} \lambda\left(\frac{1}{T}\mathbf{y}_t\right) \tag{181}$$

with λ the associated intensity function[33] and $\{\mathbf{y}_t, t=1,\ldots,T\}$ a sample of observed data of the process $\mathbf{Y}_t$. By assuming that the upper tails of X_n are Generalized Pareto $\mathcal{GP}(\sigma_n,\xi_n)$, the Fréchet transformed data take the form

$$y_{n,t} = \begin{cases} t_n^-(x_{n,t}) = -\ln(F_n(x_{n,t}))^{-1} & \text{if } x_{n,t} \leq \bar{x}_n \\ t_n^+(x_{n,t}) = -\ln\left(1-(1-F_n(\bar{x}_n))\left(1+\xi_n\left(\frac{x_{n,t}-\bar{x}_n}{\sigma_n}\right)\right)_+^{-\frac{1}{\xi}}\right)^{-1} & \text{if } x_{n,t} > \bar{x}_n \end{cases} \tag{182}$$

Thus, the log-likelihood becomes (Coles and Tawn [1991])

$$\ell(\theta) = -\Lambda([0,\mathbf{y}]^c) + \sum_{t=1}^{T} \mathbf{1}_{\left[\frac{1}{T}\mathbf{y}_t\in[0,\mathbf{y}]^c\right]} \left[\lambda\left(\frac{1}{T}\mathbf{y}_t\right)\zeta_t\right] \tag{183}$$

with $[0,\mathbf{y}]^c = \left([0,t_1^-(\bar{x}_1)] \times \ldots \times [0,t_N^-(\bar{x}_N)]\right)^c$ and

$$\zeta_t = \frac{1}{T}\prod_{n=1}^{N}\frac{1}{\sigma_n}(1-F_n(x_{n,t}))^{-\xi_n} y_{n,t}^2 \left(1-\exp\left(-\frac{1}{y_{n,t}}\right)\right)^{1+\xi_n} \exp\left(\frac{1}{y_{n,t}}\right) \tag{184}$$

The choice of Λ can be done in the class of extreme copulas with

$$\Lambda([0,\mathbf{y}]^c) = -\ln\mathbf{C}\left(\exp\left(-\frac{1}{y_{(1)}}\right),\ldots,\exp\left(-\frac{1}{y_{(n)}}\right),\ldots,\exp\left(-\frac{1}{y_{(N)}}\right)\right) \tag{185}$$

For example, the intensity measure associated with the Gumbel copula is

$$\begin{aligned} \Lambda([0,\mathbf{y}]^c) &= -\ln\left(\exp\left[-\left(\left(-\ln\left(\exp\left(-\frac{1}{y_{(1)}}\right)\right)\right)^\alpha + \left(-\ln\left(\exp\left(-\frac{1}{y_{(2)}}\right)\right)\right)^\alpha\right)^{\frac{1}{\alpha}}\right]\right) \\ &= \left(y_{(1)}^{-\alpha} + y_{(2)}^{-\alpha}\right)^{\frac{1}{\alpha}} \end{aligned} \tag{186}$$

With $\mathbf{C}(u_1,u_2,u_3) = \exp\left[-\left((\tilde{u}_1^{\alpha_2}+\tilde{u}_2^{\alpha_2})^{\frac{\alpha_1}{\alpha_2}} + \tilde{u}_3^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\right]$, $\Lambda([0,\mathbf{y}]^c)$ is equal to $\left(\left(y_{(1)}^{-\alpha_2}+y_{(2)}^{-\alpha_2}\right)^{\frac{\alpha_1}{\alpha_2}} + y_{(3)}^{-\alpha_1}\right)^{\frac{1}{\alpha_1}}$ and the intensity measure associated with the dependence function (175) is

$$\Lambda([0,\mathbf{y}]^c) = \left[\sum_{i=1}^{N}\sum_{j=i+1}^{N}\left[\left(\frac{p_i}{y_{(i)}^\eta}\right)^{\alpha_{i,j}} + \left(\frac{p_j}{y_{(j)}^\eta}\right)^{\alpha_{i,j}}\right]^{\frac{1}{\alpha_{i,j}}} + \sum_{i=1}^{N}\nu_i\frac{p_i}{y_{(i)}^\eta}\right]^{\frac{1}{\eta}} \tag{187}$$

Other parametric intensity measures could be found in Coles and Tawn [1991].

[33] $\lambda(\mathbf{y}) = (-1)^N\,\partial^N_{y_{(1)},\ldots,y_{(N)}}\,\Lambda([0,\mathbf{y}]^c)$ with $\mathbf{y} = \left[y_{(1)},\ldots,y_{(N)}\right.$.

4.3.2.3 Applications

α) **The LME data.**

We have estimated the GEV distributions both for the maxima and minima[34]. For the opposite of the minima $-\chi^-$, we obtain the following estimated parameters[35]:

Parameters	AL	AL-15	CU	NI	PB
μ	0.022	0.018	0.028	0.031	0.033
σ	0.007	0.006	0.013	0.012	0.012
ξ	0.344	$0.275^{(\star)}$	$0.095^{(\star\star\star)}$	0.363	0.424

The corresponding distributions are plotted in figure 32. Using the estimated values, one can then build univariate stress-testing programs by choosing the worst case scenario for a given return period (Legras [1999]).

Risk scales (daily variation) for long positions (in %)					
Waiting time (in years)	AL	AL-15	CU	NI	PB
5	-6.74	-4.82	-8.03	-11.01	-12.25
10	-8.57	-5.90	-9.35	-14.27	-16.35
25	-11.75	-7.68	-11.23	-20.02	-23.96
50	-14.91	-9.36	-12.76	-25.83	-32.03
75	-17.14	-10.49	-13.70	-29.97	-37.97
100	-18.92	-11.38	-14.39	-33.29	-42.85

In the case of maxima χ^+, we obtain the following results:

Parameters	AL	AL-15	CU	NI	PB
μ	0.024	0.018	0.029	0.034	0.032
σ	0.010	0.008	0.011	0.015	0.011
ξ	$0.188^{(\star\star\star)}$	$0.239^{(\star)}$	$0.142^{(\star\star)}$	$0.026^{(\star\star\star)}$	0.405

Risk scales (daily variation) for short positions (in %)					
Waiting time (in years)	AL	AL-15	CU	NI	PB
5	6.96	5.79	7.53	8.46	11.13
10	8.35	7.10	8.82	9.59	14.65
25	10.46	9.20	10.73	11.10	21.07
50	12.32	11.12	12.34	12.26	27.76
75	13.52	12.39	13.36	12.95	32.64
100	14.43	13.38	14.12	13.45	36.63

We now consider the bivariate case. We assume that the dependence is given by the Gumbel copula. Using the IFM method[36], the estimated values of the α parameter and the corresponding Kendall's tau are given

[34] The estimation is performed with ML method on componentwise data with m equal to 44 trading days that corresponds to a 2 months period.

[35] $^{(\star)}$, $^{(\star\star)}$ and $^{(\star\star\star)}$ mean that the significance test is rejected respectively at 1%, 5% and 10%.

[36] Data are transformed into uniforms with the $\mathcal{GEV}$ cdf. The corresponding individual log-likelihood is then

$$\begin{aligned} \ell(u_1, u_2; \alpha) &= -(\tilde{u}_1^\alpha + \tilde{u}_2^\alpha)^{\frac{1}{\alpha}} + \tilde{u}_1 + \tilde{u}_2 + (\alpha - 1)\ln(\tilde{u}_1\tilde{u}_2) + \Big[\alpha^{-1} - 2 \ \ln(\tilde{u}_1^\alpha + \tilde{u}_2^\alpha) \\ &\quad + \ln\Big[(\tilde{u}_1^\alpha + \tilde{u}_2^\alpha)^{\frac{1}{\alpha}} + \alpha - 1\Big] \end{aligned} \tag{188}$$

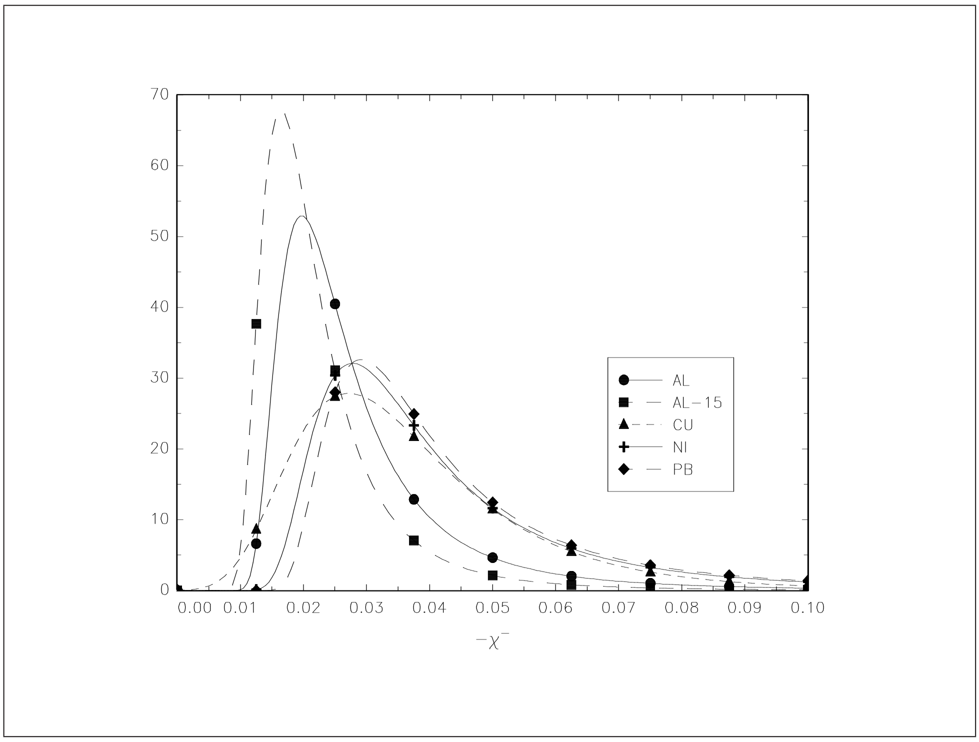

Figure 32: $\mathcal{GEV}$ densities of the minima

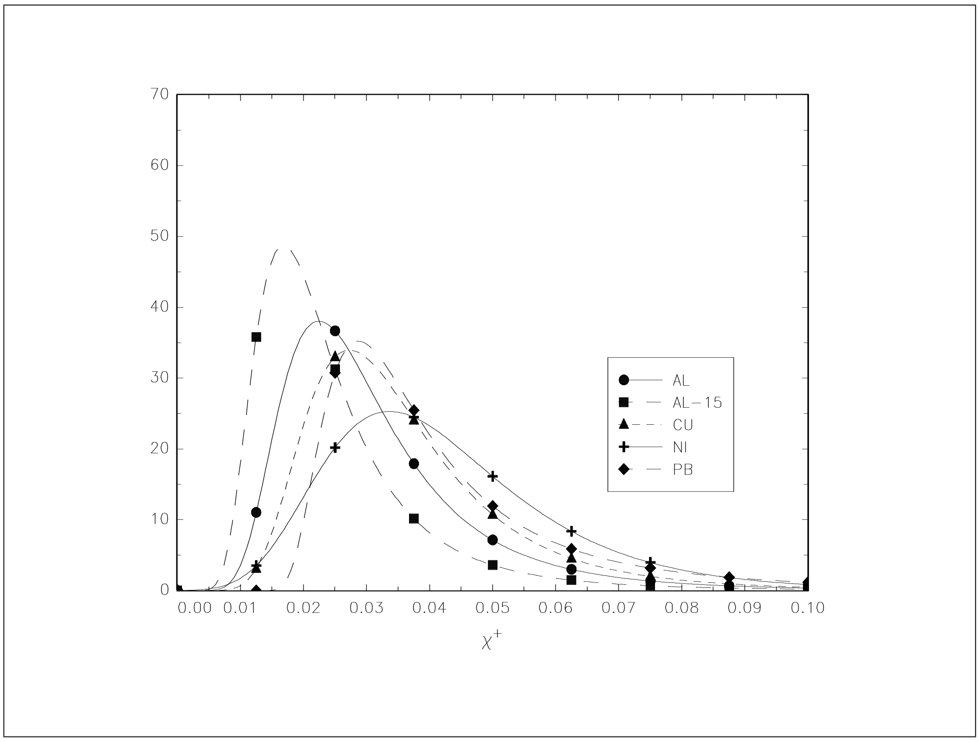

Figure 33: $\mathcal{GEV}$ densities of the maxima

						results for $-\chi_1^-, -\chi_2^-$									
$\hat{\alpha}_{\text{ML}}$	AL-15	CU	NI	PB	LR	AL-15	CU	NI	PB	τ	AL-15	CU	NI	PB
AL	2.397	1.159	1.462	1.425	AL	62.09	3.003	18.583	15.19	AL	0.583	0.137	0.316	0.298
AL-15		1.092	1.356	1.191	AL-15		1.011	13.080	4.432	AL-15		0.084	0.263	0.160
CU			1.212	1.004	CU			4.290	0.002	CU			0.175	0.004
NI				1.500	NI				19.49	NI				0.333
						results for χ_1^+, χ_2^+								
$\hat{\alpha}_{\text{ML}}$	AL-15	CU	NI	PB	LR	AL-15	CU	NI	PB	τ	AL-15	CU	NI	PB
AL	3.015	1.065	1.291	1.116	AL	86.39	0.485	7.755	1.893	AL	0.668	0.061	0.226	0.104
AL-15		1.040	1.175	1.033	AL-15		0.160	3.105	0.163	AL-15		0.039	0.149	0.032
CU			1.107	1.116	CU			1.260	1.423	CU			0.097	0.104
NI				1.194	NI				5.126	NI				0.162

Table 8: Bivariate estimation

in table 8. Moreover, we have reported the likelihood ratio where the null hypothesis is the product copula. With 99% confidence level, we do not reject independence between the maxima couples (AL,CU), (AL,PB), (AL-15,CU), (AL-15,NI), (AL-15,PB), (CU,NI), (CU,PB) and (NI,PB). For the minima, the dependence is not rejected for (AL,AL-15), (AL,NI), (AL,PB), (AL-15,NI) and (NI,PB).

Remark 33 *We observe that there is a contrast between the minima and the maxima cases. The minima appear more dependent than the maxima. From an economic point of view, it means that bear markets are 'more correlated' than bull markets.*

Stress scenario in the bivariate case could be viewed as "*a failure area*" (see DE HAAN, PENG, SINHA and DRAISMA [1997] for more details on this topic and on *exceedence probability*). We have

$$\begin{aligned}
\Pr\left\{\chi_1^+ > \chi_1, \chi_2^+ > \chi_2\right\} &= 1 - \Pr\left\{\chi_1^+ \leq \chi_1\right\} - \Pr\left\{\chi_2^+ \leq \chi_2\right\} + \Pr\left\{\chi_1^+ \leq \chi_1, \chi_2^+ \leq \chi_2\right\} \\
&= 1 - \mathbf{F}_1(\chi_1) - \mathbf{F}_2(\chi_2) + \mathbf{C}(\mathbf{F}_1(\chi_1), \mathbf{F}_2(\chi_2)) \\
&= \bar{\mathbf{C}}(\mathbf{F}_1(\chi_1), \mathbf{F}_2(\chi_2)) \qquad (189)
\end{aligned}$$

with $\bar{\mathbf{C}}$ the joint survival function. Let $\mathfrak{t}$ be the waiting time (measured in term of the componentwise period). The failure area is the set defined by $\left\{(\chi_1, \chi_2) \in \mathbb{R}^2 \mid u_1 = \mathbf{F}_1(\chi_1), u_2 = \mathbf{F}_2(\chi_2), \bar{\mathbf{C}}(\mathbf{F}_1(\chi_1), \mathbf{F}_2(\chi_2)) < \frac{1}{\mathfrak{t}}\right\}$. We have represented in figures 34 and 35 the failure area of the (AL,AL-15) and (AL,CU) maxima, that is the set of the bivariate scenarios for a portfolio short in the two assets[37]. We remark that the two figures give different results because of the value of the α parameter. Note that the same methodology can be applied to others portfolios, for example a portfolio short in the first asset and long in the second asset. By computing the failure area for the four quadrants, we obtain figure 36 for the pair (AL,AL-15). A 5 years return period is assumed. The obtained failure area is compared with the independent case (product copula)[38].

We finish this paragraph with some remarks on multivariate extreme value modelling. First, the Gumbel copula is not always the best choice. Sometimes, asymmetric copulas appear more appropriate. For example, the asymmetric logistic copula[39] gives a better fit for the minima pair (AL,AL-15) (see figure 37). Second,

[37] The solid lines represent the univariate risk scales.

[38] The solid line corresponds to the failure area with the Gumbel copula, whereas the dotted line corresponds to the independence.

[39] The corresponding individual log-likelihood is then

$$\ell(u_1, u_2; \alpha) = -\left(\acute{u}_1^\alpha + \acute{u}_2^\alpha\right)^{\frac{1}{\alpha}} + \acute{u}_1 + \acute{u}_2 + \left[\alpha^{-1} - 1 \;\; \ln\left(\acute{u}_1^\alpha + \acute{u}_2^\alpha\right) + \ln \varkappa(u_1, u_2; \alpha) \qquad (190)$$

with $\acute{u}_i = p_i \tilde{u}_i$, $\tilde{u}_i = -\ln u_i$ and

$$\begin{aligned}
\varkappa(u_1, u_2; \alpha) \quad = \quad & (p_1 p_2)(\acute{u}_1 \acute{u}_2)^{\alpha-1}\left(\acute{u}_1^\alpha + \acute{u}_2^\alpha\right)^{-1} \;\; \left(\acute{u}_1^\alpha + \acute{u}_2^\alpha\right)^{\alpha^{-1}} + \alpha - 1 \;\; + \\
& (1 - p_1) p_2 \acute{u}_2^{\alpha-1} + \\
& p_1 (1 - p_2) \acute{u}_1^{\alpha-1} + \\
& (1 - p_1)(1 - p_2)\left(\acute{u}_1^\alpha + \acute{u}_2^\alpha\right)^{1-\alpha^{-1}} \qquad (191)
\end{aligned}$$

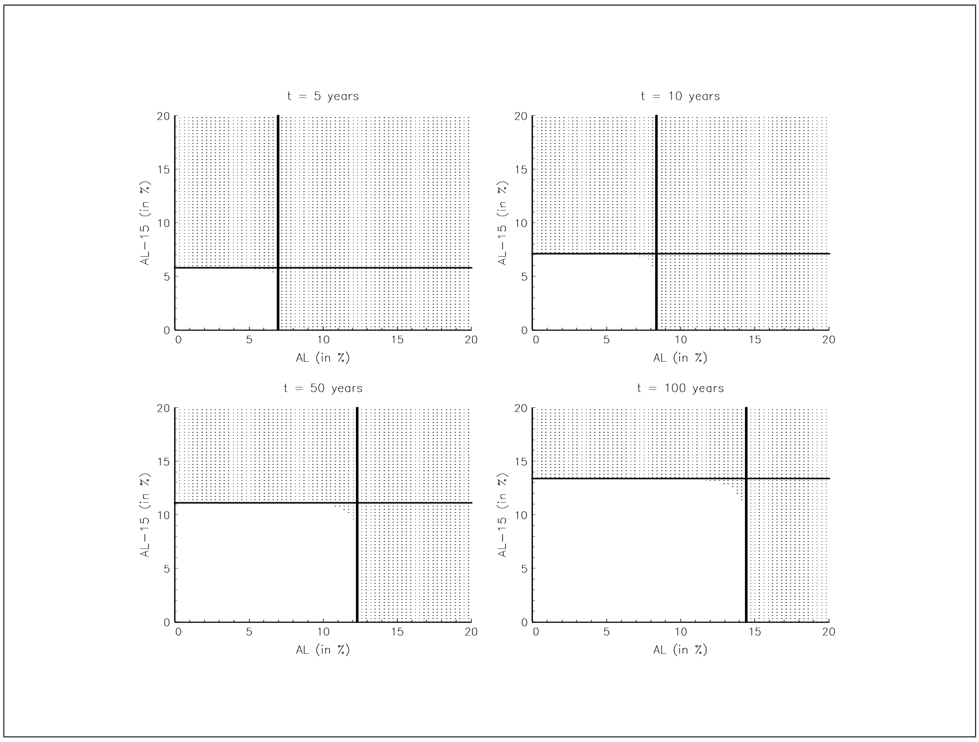

Figure 34: Failure area for the (AL,AL-15) maxima

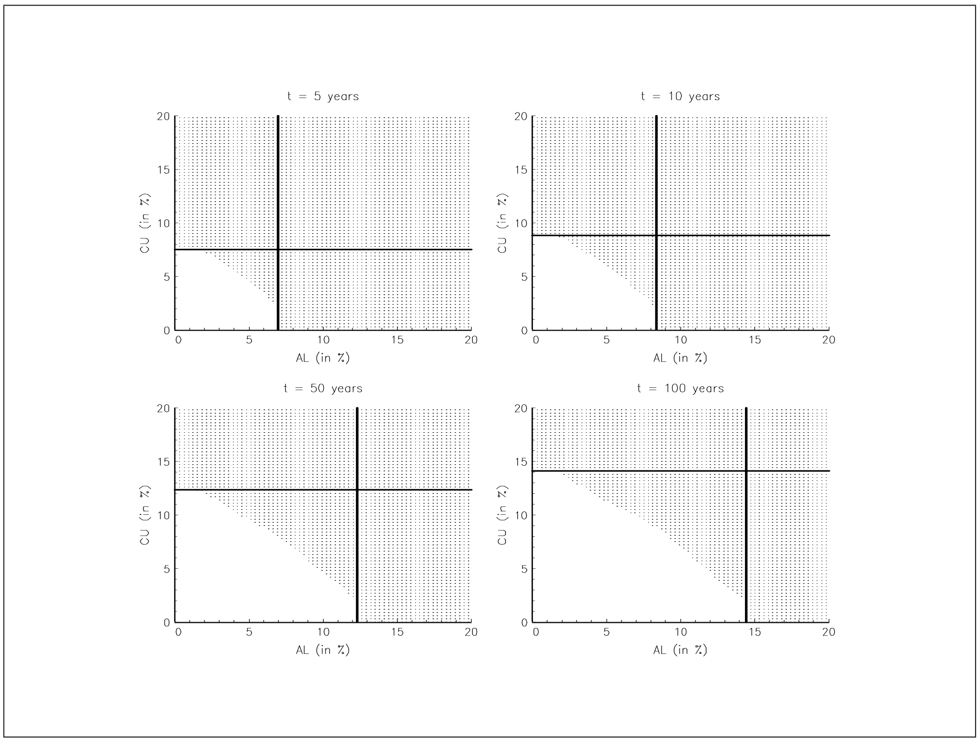

Figure 35: Failure area for the (AL,CU) maxima

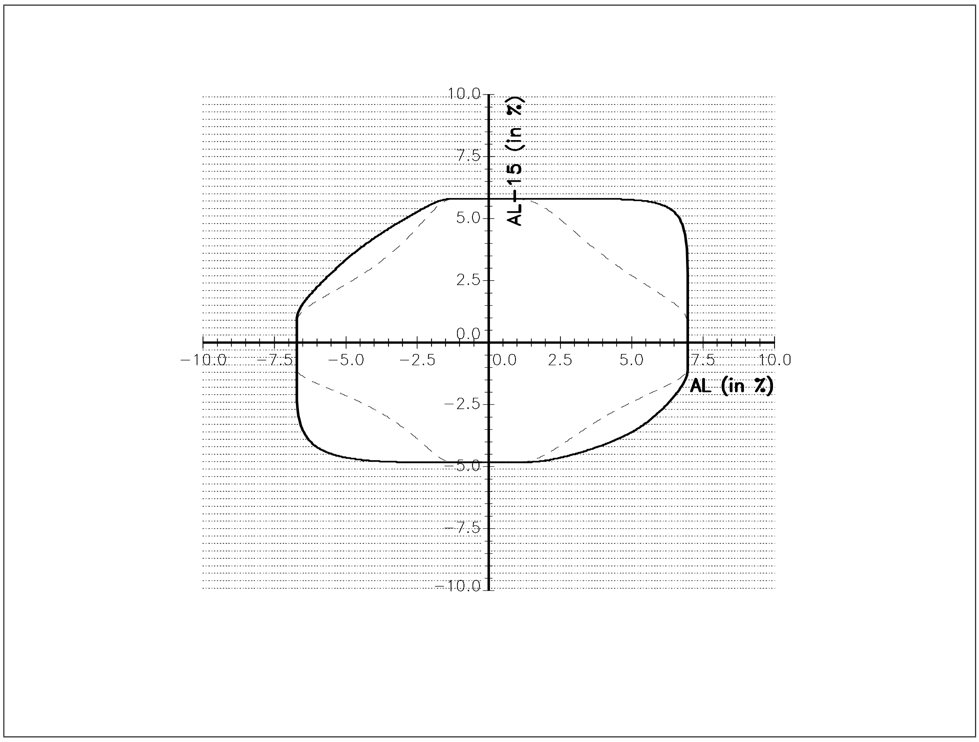

Figure 36: Failure area for the pair (AL,AL-15) and a 5 years waiting time

the extension to multivariate copulas is generally done with the '*clustering*' method (Joe [1997]). It implies some restrictions on the dependence structure. Let us consider the modelling of the 5 maxima pairs. Because of the bivariate results, we could assume that the multivariate copula has the form $\mathbf{C}^{\aleph}(u_1,u_2,u_3,u_4,u_5) = \exp\left[-\left(\left(\tilde{u}_1^{\alpha_2}+\tilde{u}_2^{\alpha_2}\right)^{\frac{\alpha_1}{\alpha_2}}+\tilde{u}_4^{\alpha_1}\right)^{\frac{1}{\alpha_1}}-\tilde{u}_3-\tilde{u}_5\right]$. The ML estimates[40] are $\hat{\alpha}_1 = 1.049$ and $\hat{\alpha}_2 = 3.043$, and the LR test of the product copula hypothesis is rejected. Nevertheless, this copula implies that the bivariate margins of (AL,NI) and (AL-15,NI) are $\mathbf{C}^{\aleph}(u_1,1,1,u_4,1) = \exp\left[-\left(\tilde{u}_1^{\alpha_1}+\tilde{u}_4^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\right]$ and $\mathbf{C}^{\aleph}(1,u_2,1,u_4,1) = \exp\left[-\left(\tilde{u}_2^{\alpha_1}+\tilde{u}_4^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\right]$, and so the dependence structure of (AL,NI) and (AL-15,NI) must be the same. This is a direct implication of the compound copula methodology.

[40] The individual log-likelihood is

$$\begin{aligned}\ell(u_1,u_2,u_3,u_4,u_5;\alpha_1,\alpha_2) &= -\left[\tilde{u}_1^{\alpha_2}+\tilde{u}_2^{\alpha_2}{}^{\frac{\alpha_1}{\alpha_2}}+\tilde{u}_4^{\alpha_1}{}^{\frac{1}{\alpha_1}}+\tilde{u}_1+\tilde{u}_2+\tilde{u}_4+(\alpha_2-1)\ln(\tilde{u}_1\tilde{u}_2)\right. \\ &\quad +(\alpha_1-1)\ln(\tilde{u}_4)+\ \alpha_1\alpha_2^{-1}-2\ \ln A+\ \alpha_1^{-1}-2\ \ln B+\ln\varkappa(u_1,u_2,u_3,u_4,u_5;\alpha_1,\alpha_2)\end{aligned} \tag{192}$$

with

$$\begin{aligned}\varkappa(u_1,u_2,u_3,u_4,u_5;\alpha_1,\alpha_2) &= A^{\frac{\alpha_1}{\alpha_2}}B^{\frac{2}{\alpha_1}-1}+(\alpha_2-\alpha_1)B^{\frac{1}{\alpha_1}}+(\alpha_1-1)A^{\frac{\alpha_1}{\alpha_2}}B^{\frac{1}{\alpha_1}-1} \\ &\quad +(\alpha_1-1)(2\alpha_1-1)A^{\frac{\alpha_1}{\alpha_2}}B^{-1}+(\alpha_1-1)(\alpha_2-\alpha_1) \\ A &= \tilde{u}_1^{\alpha_2}+\tilde{u}_2^{\alpha_2} \\ B &= A^{\frac{\alpha_1}{\alpha_2}}+\tilde{u}_4^{\alpha_1}\end{aligned} \tag{193}$$

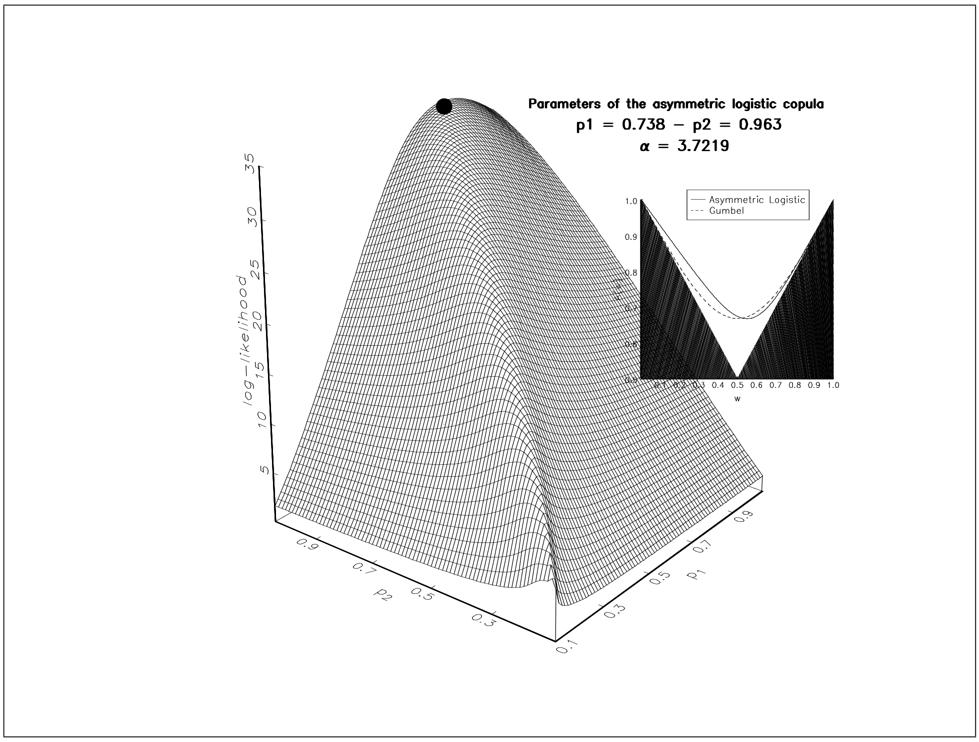

Figure 37: ML estimation of the asymmetric logistic copula for the minima pair (AL,AL-15)

β) **Estimating the severity of crisis and multivariate stress tests.**

In the previous paragraph, we have introduced the notion of *failure area*. In the multivariate case, it is defined as the following set

$$\left\{ (\chi_1, \ldots, \chi_n, \ldots, \chi_N) \in \mathbb{R}^N \mid u_1 = \mathbf{F}_1(\chi_1), \ldots, u_n = \mathbf{F}_n(\chi_n), \ldots, u_N = \mathbf{F}_N(\chi_N), \bar{\mathbf{C}}(u_1, \ldots, u_n, \ldots, u_N) < \frac{1}{\mathfrak{t}} \right\} \tag{194}$$

with

$$\bar{\mathbf{C}}(u_1, \ldots, u_n, \ldots, u_N) = \sum_{n=0}^{N} \left[(-1)^n \sum_{\mathbf{u} \in \mathcal{Z}(N-n,N)} \mathbf{C}(\mathbf{u}) \right] \tag{195}$$

where $\mathcal{Z}(M,N)$ denotes the set $\left\{ \mathbf{u} \in [0,1]^N \mid \sum_{n=1}^{N} \mathcal{X}_{\{1\}}(u_n) = M \right\}$. It is possible to compute the implicit return period $\mathfrak{t}$ for a given vector $(\chi_1, \ldots, \chi_n, \ldots, \chi_N)$. We have then

$$\mathfrak{t}(\chi_1, \ldots, \chi_n, \ldots, \chi_N) = \bar{\mathbf{C}}^{-1}(\mathbf{F}_1(\chi_1), \ldots, \mathbf{F}_n(\chi_n), \ldots, \mathbf{F}_N(\chi_N)) \tag{196}$$

Remark 34 *The strength of a crisis* $(\chi_1, \ldots, \chi_n, \ldots, \chi_N)$ *is generally a subjective notion. However, the implied return period* $\mathfrak{t}(\chi_1, \ldots, \chi_n, \ldots, \chi_N) = \bar{\mathbf{C}}^{-1}(\mathbf{F}_1(\chi_1), \ldots, \mathbf{F}_n(\chi_n), \ldots, \mathbf{F}_N(\chi_N))$ *could be viewed as a measure of the severity. Moreover, we could compare the strength of two crisis by directly comparing the waiting times.*

Remark 35 *The implied return period measure can be used to quantify the stress tests provided by the economists for the stress testing program of a bank. Univariate stress tests could be done with statistical tools like the extreme value theory (*Legras *[1999],* Costinot, Riboulet *and* Roncalli *[2000a]). The extension to bivariate case is not obvious (*Legras *and* Soupé *[2000],* Costinot, Riboulet *and* Roncalli *[2000b]). Building multivariate stress tests with copulas could be done using the failure area concept. However, it produces a set and it* *is difficult to* ***choose one scenario in particular****. Nevertheless, the implied return period is a useful measure to quantify a given stress test and to verify its consistency. Some examples will be given further.*

Before considering some illustrations, we give more explicit representations of the joint survivor copula $\bar{\mathbf{C}}$. In the case $N = 2$, we have $\bar{\mathbf{C}}(u_1, u_2) = \sum_{n=0}^{2}\left[(-1)^n \sum_{\mathbf{u}\in\mathcal{Z}(2-n,2)} \mathbf{C}(\mathbf{u})\right] = \mathbf{C}(1,1) - \mathbf{C}(u_1, 1) - \mathbf{C}(1, u_2) + \mathbf{C}(u_1, u_2) = \mathbf{1} - u_1 - u_2 + \mathbf{C}(u_1, u_2)$. We obtain the previous expression In the case, $N = 3$, we have $\bar{\mathbf{C}}(u_1, u_2, u_3) = \sum_{n=0}^{3}\left[(-1)^n \sum_{\mathbf{u}\in\mathcal{Z}(3-n,3)} \mathbf{C}(\mathbf{u})\right] = \mathbf{C}(1,1,1) - \mathbf{C}(u_1,1,1) - \mathbf{C}(1,u_2,1) - \mathbf{C}(1,1,u_3) + \mathbf{C}(u_1,u_2,1) + \mathbf{C}(u_1,1,u_3) + \mathbf{C}(1,u_2,u_3) - \mathbf{C}(u_1,u_2,u_3) = \mathbf{1} - u_1 - u_2 - u_3 + \mathbf{C}(u_1,u_2) + \mathbf{C}(u_1,u_3) + \mathbf{C}(u_2,u_3) - \mathbf{C}(u_1,u_2,u_3)$. The extension to higher dimensions is straightforward.

For the LME dataset, the dimension is $N = 5$. We assume that the portfolio is short in each asset. Let consider a first stress test

$$\chi^{(1)} = \begin{bmatrix} 0.05 \\ 0.05 \\ 0.05 \\ 0.05 \\ 0.05 \end{bmatrix}$$

The associated return period using the previous estimated copula $\mathbf{C}^{\aleph}$ is equal to 209 years[41]. If we suppose that the copula is $\mathbf{C}^{\perp}$ or $\mathbf{C}^{+}$, the return period becomes 2317 and 3 years respectively. Let us consider a second stress program

$$\chi^{(2)} = \begin{bmatrix} 0.02 \\ 0.03 \\ 0.03 \\ 0.01 \\ 0.10 \end{bmatrix}$$

Is this second test harder than the first one ? If we compute the waiting times, we obtain $\mathsf{t}_{(2)}^{\aleph} = 27$, $\mathsf{t}_{(2)}^{\perp} = 34$, and $\mathsf{t}_{(2)}^{+} = 4$. This example is very interesting, because we have $\mathsf{t}_{(2)}^{\aleph} < \mathsf{t}_{(1)}^{\aleph}$, $\mathsf{t}_{(2)}^{\perp} < \mathsf{t}_{(1)}^{\perp}$ but $\mathsf{t}_{(2)}^{+} > \mathsf{t}_{(1)}^{+}$. For $\chi_n^{(1)} < \chi_n^{(2)}$ and for all $n = 1, \ldots, N$, we could check that $\mathsf{t}_{(2)} > \mathsf{t}_{(1)}$ for every copula. If the inequalities are not verified for all the components, the comparison is less obvious. The concept of the return period then becomes a very useful tool. We note that we have necessarily $\mathsf{t}^{+} < \mathsf{t} < \mathsf{t}^{\perp}$ because of the properties induced by the order $\prec$.

A question arises: what is the link between the univariate and the multivariate stress scenarios ? To answer this question, we consider an univariate stress test with a 5 years waiting time :

$$\chi^{(3)} = \begin{bmatrix} 0.0696 \\ 0.0579 \\ 0.0753 \\ 0.0846 \\ 0.1113 \end{bmatrix}$$

We then obtain $\mathsf{t}_{(3)}^{\aleph} = 49939$, $\mathsf{t}_{(3)}^{\perp} = 3247832$ and $\mathsf{t}_{(3)}^{+} = 5$! This very simple example shows that we have to be careful to build multivariate tests from univariate tests. There is no problem if we assume a perfect dependence. Otherwise, we could obtain meaningless stress tests for the Risk Management of the bank.

[41] In order to help the reader to reproduce this result, we indicate some intermediary calculus. Using the estimated parameters of the univariate margins $\mathcal{GEV}$, we have $u_1 = 0.889$, $u_2 = 0.943$, $u_3 = 0.836$, $u_4 = 0.713$, and $u_5 = 0.744$. The value of $\bar{\mathbf{C}}(u_1, u_2, u_3, u_4, u_5)$ is then 8.434×10^{-4}. We assume that the number of trading days in one year is 250. Because we have set the componentwise period to 44 days, the return period (expressed in years) is given by the formula

$$\mathsf{t}(u_1, u_2, u_3, u_4, u_5) = \frac{1}{8.434 \times 10^{-4}} \times \frac{44}{250}$$

The previous example gives an idea of **about how the dependence structure contributes on the stress test**. One possible measure could be

$$\pi = \frac{\mathfrak{t}^{\perp}(\chi_1,\ldots,\chi_n,\ldots,\chi_N) - \mathfrak{t}(\chi_1,\ldots,\chi_n,\ldots,\chi_N)}{\mathfrak{t}^{\perp}(\chi_1,\ldots,\chi_n,\ldots,\chi_N) - \mathfrak{t}^{+}(\chi_1,\ldots,\chi_n,\ldots,\chi_N)} \tag{197}$$

We have $\pi \in [0,1]$. π is equal to 0 if the estimated copula is the product copula, whereas it is 1 in the case of the upper Fréchet copula. With the above examples, we have $\pi^{\aleph}_{(1)} = 91\%$, $\pi^{\aleph}_{(2)} = 24\%$ and $\pi^{\aleph}_{(3)} = 98\%$. The impact of the dependence structure is smaller for the second test than for the other tests.

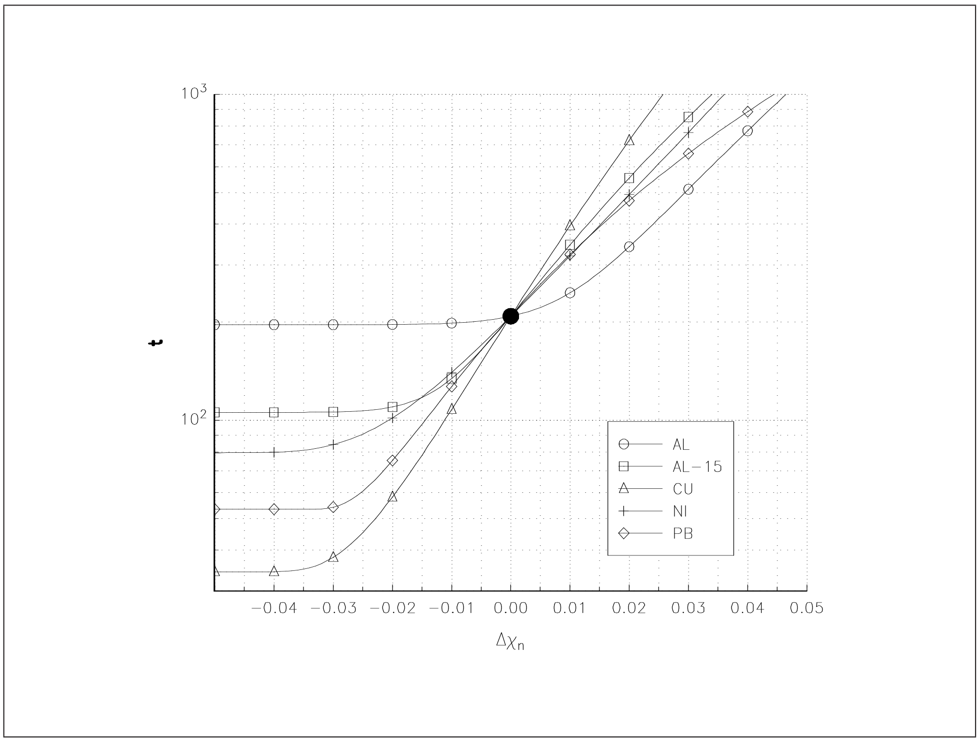

Figure 38: Univariate stress scenario contribution with the copula $\mathbf{C}^{\aleph}$

Let us now consider the consistency problem of a multivariate stress test. The underlying idea is the following. Suppose that the dependence structure is given by the upper Fréchet copula. Then, the return period is defined by the maximum return period of the univariate stress tests

$$\mathfrak{t}(\chi_1,\ldots,\chi_n,\ldots,\chi_N) = \max_n \mathfrak{t}(\chi_n) \tag{198}$$

The return period of the multivariate scenario is only determined by one of the univariate scenario. The contribution of this univariate scenario is also maximal. For a general dependence structure, there is a univariate scenario that will have the bigger contribution. However, one may think that this contribution might not be justified. Indeed, univariate stress scenarios with small waiting times could produce a higher return period for the multivariate stress scenario, because of the dependence structure. In fact, if we are able to understand the computed value of the waiting time, we can understand the consistency of the multivariate stress scenario. Nevertheless, this exercise is difficult from a pratical point of view. Costinot, Riboulet and Roncalli [2000c] have done this analysis on a stress testing program with eight factors (two indices, two exchange rates and four factors of interest rates). We illustrate this point with LME data and stress scenario $\chi^{(1)}$. Remind that

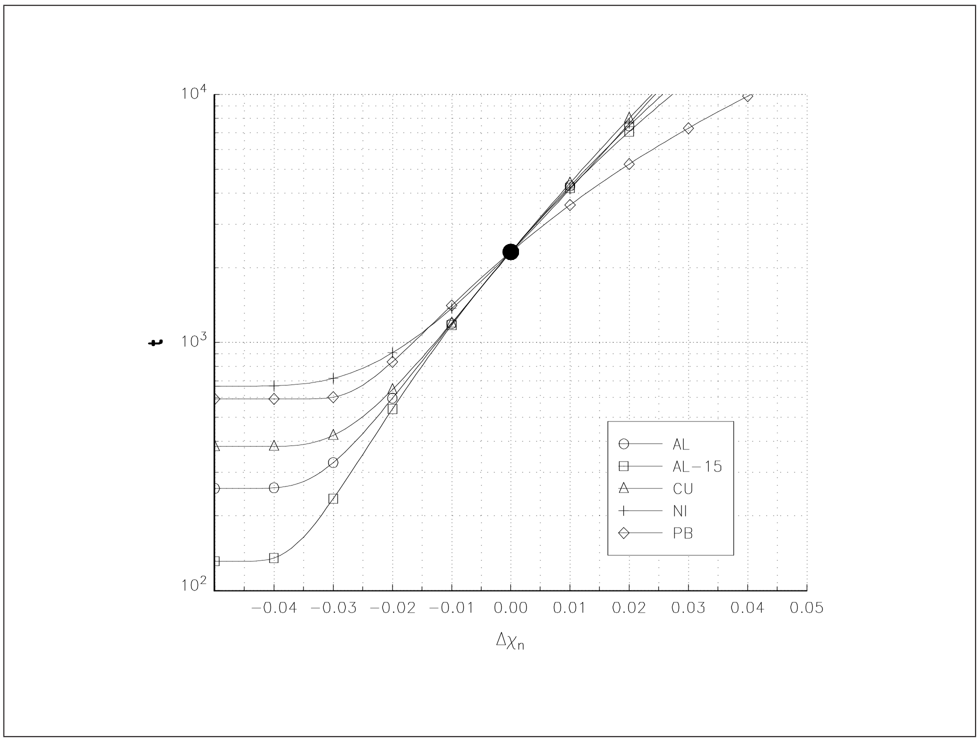

Figure 39: Univariate stress scenario contribution with the copula $\mathbf{C}^{\perp}$

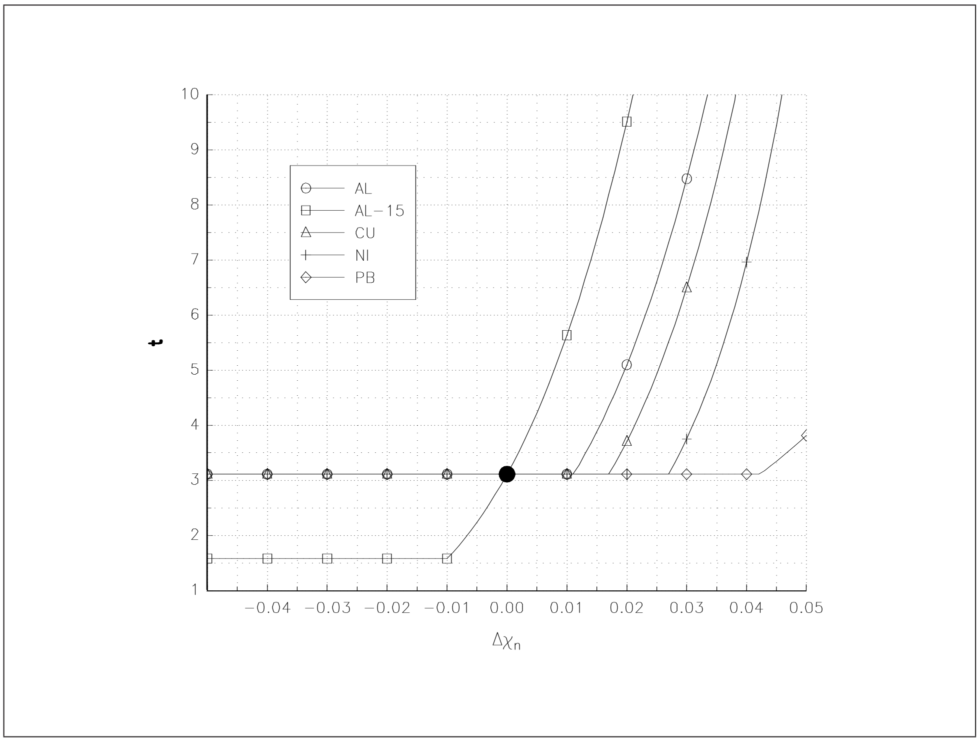

Figure 40: Univariate stress scenario contribution with the copula $\mathbf{C}^{+}$

$\mathfrak{t}^{\aleph}(5\%, 5\%, 5\%, 5\%, 5\%)$ is 209 years. In figure 38, we have plotted $\mathfrak{t}^{\aleph}(\chi_1, \ldots, \chi_n + \Delta\chi_n, \ldots, \chi_N)$, the waiting time with a change in one component, caeteris paribus[42]. We remark that CU is the univariate stress scenario with the higher contribution. If we would have a more plausible scenario, we could set $\chi_3^{(1)}$ equal to 4%. In this case, the return period $\mathfrak{t}^{\aleph}(5\%, 5\%, 4\%, 5\%, 5\%)$ is 108 years that is a more realistic waiting time. We note that the contribution of the n^{th} univariate stress test can be measured by computing the return period without it, that is $\mathfrak{t}(\chi_1, \ldots, \chi_n = -\infty, \ldots, \chi_N)$. We obtain the following results

	AL	AL-15	CU	NI	PB
$\mathfrak{t}(\chi_1, \ldots, \chi_n = -\infty, \ldots, \chi_N)$	196	106	34	80	53

A deeper analysis could show that the high value of 209 years is explained by the dependence structure of CU and PB — $\mathfrak{t}^{\aleph}(0.05, 0.05, -\infty, 0.05, -\infty)$ is equal to 9 years!

4.3.3 Survival copulas and credit risk

4.3.4 *Correlated* frequencies and operational risk

The *standard* measurement methodology for operational risk with internal data is the following (Georges and Roncalli [1999]):

- Let ζ be the random variable that describes the severity of loss. We define also $\zeta^k(t)$ as the random process of ζ for each operational risk k ($k = 1, \ldots, K$).
- For each risk, we assume that the number of events at time t is a random variable $N_k(t)$.
- The loss process $\varrho(t)$ is also defined as

$$\begin{aligned} \varrho(t) &= \sum_{k=1}^{K} \varrho^k(t) \\ &= \sum_{k=1}^{K} \sum_{j=1}^{N_k(t)} \zeta_j^k(t) \end{aligned} \tag{199}$$

- The *Economic Capital* with an α confidence level is usually defined as

$$\mathsf{EC} = \mathbf{F}^{-1}(\alpha) \tag{200}$$

Even if the methodology is simple, many problems arise in practice[43], but they are out of concern in this paper. We focus on another issue of operational risk : the correlation between frequencies of different types of risk

$$\mathbb{E}[N_{k_1}(t) N_{k_2}(t)] \neq \mathbb{E}[N_{k_1}(t)] \times \mathbb{E}[N_{k_2}(t)] \tag{201}$$

$N_k(t)$ is generally assumed to be a Poisson variable $\mathcal{P}$ with mean λ_k. The idea is also to use a multivariate extension of the Poisson distribution. However, multivariate poisson distributions are relatively complicated for dimensions higher than two (Johnson, Kotz and Balakrishnan [1997]). Let N_{11}, N_{12} and N_{22} be three **independent** Poisson variates with means λ_{11}, λ_{12} and λ_{22}. In the bivariate case, the joint distribution is

[42] Figures 39 and 40 correspond respectively to the copulas $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$.

[43] For example, the estimation of the distributions of $\zeta^k(t)$ and $N^k(t)$, the availability of exhaustive data, the modelling of the right tail and the adequacy of the data for the **extreme events**, the fact that the database of severity losses could contain both events and aggregated events.

based on the variables $N_1 = N_{11} + N_{12}$ and $N_2 = N_{22} + N_{12}$. We have of course $N_1 \sim \mathcal{P}(\lambda_1 = \lambda_{11} + \lambda_{12})$ and $N_2 \sim \mathcal{P}(\lambda_2 = \lambda_{22} + \lambda_{12})$. Moreover, the joint probability function is

$$\Pr\{N_1 = n_1, N_2 = n_2\} = \sum_{n=0}^{\min(n_1,n_2)} \frac{\lambda_{11}^{n_1-n}\lambda_{12}^{n_2-n}\lambda_{12}^{n}e^{-(\lambda_{11}+\lambda_{22}+\lambda_{12})}}{(n_1-n)!\,(n_2-n)!n!} \tag{202}$$

The Pearson correlation between N_1 and N_2 is $\rho = \lambda_{12}\left[(\lambda_{11}+\lambda_{12})(\lambda_{22}+\lambda_{12})\right]^{-\frac{1}{2}}$ and it comes that

$$\rho \in \left[0, \min\left(\sqrt{\frac{\lambda_{11}+\lambda_{12}}{\lambda_{22}+\lambda_{12}}}, \sqrt{\frac{\lambda_{22}+\lambda_{12}}{\lambda_{11}+\lambda_{12}}}\right)\right] \tag{203}$$

With this construction, we have only positive dependence. In an operational risk context, it is equivalent to say that the two risks are affected by specific and systemic risks. Nevertheless, people in charge of operational risk in a bank have little experience with this approach and are more familiar with correlation concepts. To use this approach, it is also necessary to invert the previous relationships. In this case, we have

$$\begin{aligned} \lambda_{12} &= \rho\sqrt{\lambda_1\lambda_2} \\ \lambda_{11} &= \lambda_1 - \rho\sqrt{\lambda_1\lambda_2} \\ \lambda_{22} &= \lambda_2 - \rho\sqrt{\lambda_1\lambda_2} \end{aligned} \tag{204}$$

In dimension K, there is a generalisation of the bivariate case by considering more than K independent Poisson variates. However, the corresponding multivariate Poisson distribution is not *tractable* because the correlation coefficients have not an easy expression.

A possible alternative is to use a copula $\mathbf{C}$. In this case, the probability mass function is given by the Radon-Nikodym density of the distribution function:

$$\Pr\{N_1 = n_1, \ldots, N_k = n_k, \ldots, N_K = n_K\} = \sum_{i_1=1}^{2}\cdots\sum_{i_K=1}^{2}(-1)^{i_1+\cdots+i_K}\,\mathbf{C}\left(\sum_{n=0}^{n_1}\frac{\lambda_1^{n+1-i_1}e^{-\lambda_1}}{n!}, \ldots, \sum_{n=0}^{n_k}\frac{\lambda_k^{n+1-i_k}e^{-\lambda_k}}{n!}, \ldots, \sum_{n=0}^{n_K}\frac{\lambda_K^{n+1-i_K}e^{-\lambda_K}}{n!}\right) \tag{205}$$

Assuming a gaussian copula, we note $\mathcal{P}(\lambda, \rho)$ the multivariate Poisson distribution generated by the gaussian copula with parameter ρ and univariate Poisson distribution $\mathcal{P}(\lambda_k)$ (to illustrate this distribution, we give an example in the following footnote[44]). We have to remark that the parameter of the gaussian copula ρ is not equal to the Pearson correlation matrix, but is generally very close (see the figure 41). The *Economic Capital* with an α confidence level for operational risk could then be calculated by assuming that $\mathbf{N} = \{N_1, \ldots, N_k, \ldots, N_K\}$ follows a multivariate Poisson distribution $\mathcal{P}(\lambda, \rho)$. Moreover, there are no computational difficulties, because

[44]The next table contains the probability mass function $p_{i,j} = \Pr\{N_1 = i, N_2 = j\}$ of the bivariate Poisson distribution $P(\lambda_1 = 1, \lambda_2 = 1, \rho = 0.5)$.

$p_{i,,j}$	0	1	2	3	4	5	$\cdots$	$p_{i,\cdot}$
0	0.0945	0.133	0.0885	0.0376	0.0114	0.00268		0.368
1	0.0336	0.1	0.113	0.0739	0.0326	0.0107		0.368
2	0.00637	0.0312	0.0523	0.0478	0.0286	0.0123		0.184
3	0.000795	0.00585	0.0137	0.0167	0.013	0.0071		0.0613
4	7.28E-005	0.000767	0.00241	0.00381	0.00373	0.00254		0.0153
5	5.21E-006	7.6E-005	0.000312	0.000625	0.000759	0.000629		0.00307
$\vdots$								
$p_{\cdot,j}$	0.135	0.271	0.271	0.18	0.0902	0.0361		1

If $\rho = -0.5$, we obtain the following values for $p_{i,j}$.

the estimation of the parameters λ and ρ is straightforward, and the distribution can be easily obtained with Monte Carlo methods[45].

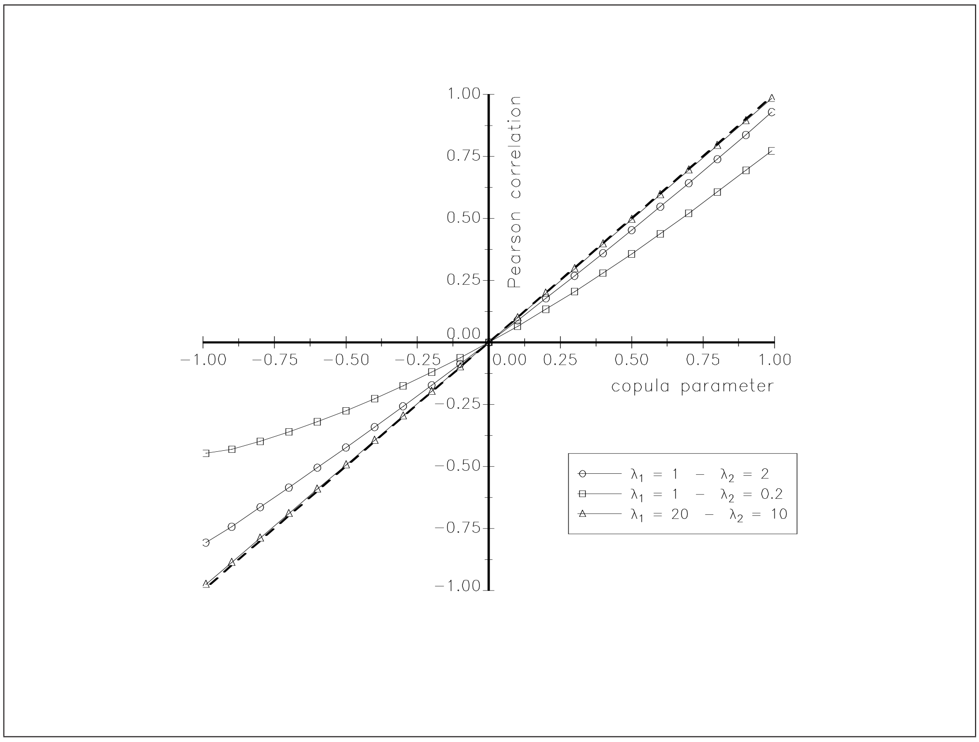

Figure 41: Relationship between the copula parameter and the Pearson correlation

5 Conclusion

The aim of this paper was to present the concept of copula and how it could be used in finance. The copula is in fact the dependence structure of the model. Copulas reveal to be a **very powerful tool** in the finance profession, more especially in the modelling of assets and in the risk management. Nevertheless, the finance industry needs more works on copula and their applications. Even if it is an old notion, there are many research directions to explore. Moreover, many pedagogical works have to be done in order to familiarize the finance industry with copulas.

$p_{i,,j}$	0	1	2	3	4	5	$\cdots$	$p_{i,\cdot}$
0	0.0136	0.0617	0.101	0.0929	0.058	0.027		0.368
1	0.0439	0.112	0.111	0.0649	0.026	0.00775		0.368
2	0.0441	0.0683	0.0458	0.0188	0.00548	0.00121		0.184
3	0.0234	0.0229	0.0109	0.00331	0.000733	0.000126		0.0613
4	0.00804	0.00505	0.00175	0.000407	7.06E-005	9.71E-006		0.0153
5	0.002	0.00081	0.000209	3.79E-005	5.26E-006	5.89E-007		0.00307
$\vdots$								
$p_{\cdot,j}$	0.135	0.271	0.271	0.18	0.0902	0.0361		1

[45] The simulation of 20 millions draw of the loss distribution **F** with 5 types of risk takes less than one hour with a Pentium III 750 Mhz and the GAUSS programming langage.

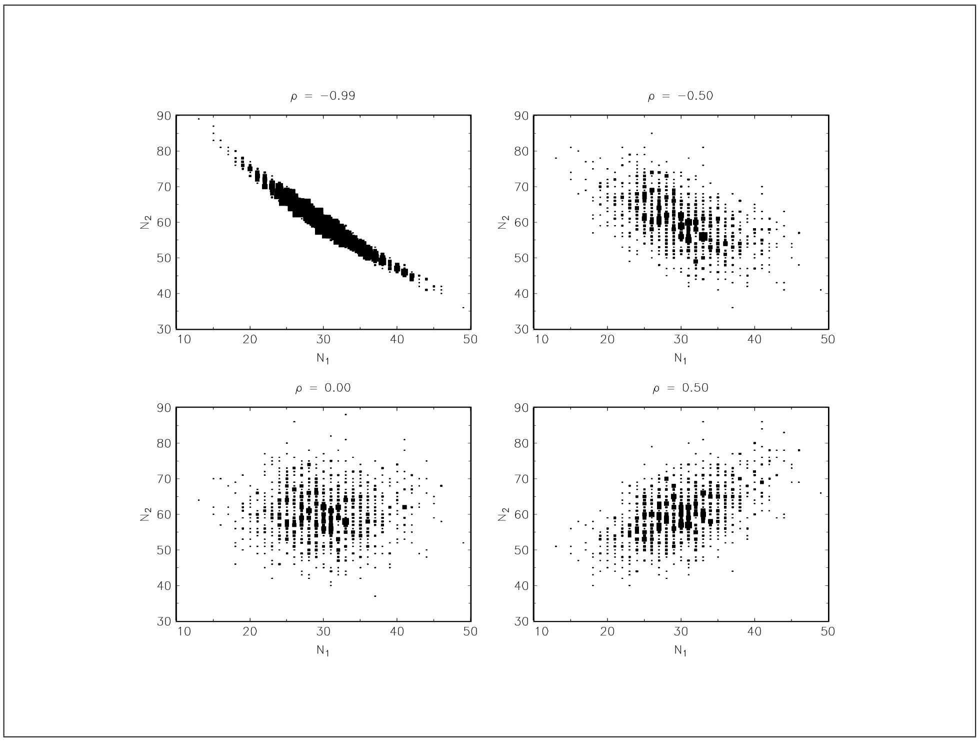

Figure 42: Random generation of bivariate Poisson variates $\mathcal{P}(30)$ and $\mathcal{P}(60)$

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